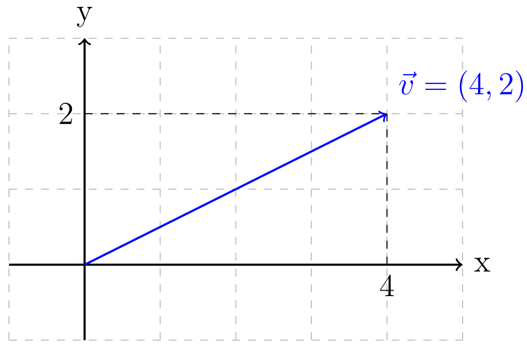
Notes on Vectors, Matrices and Linear Transformations

Vectors and Their Properties

Definition of a Vector

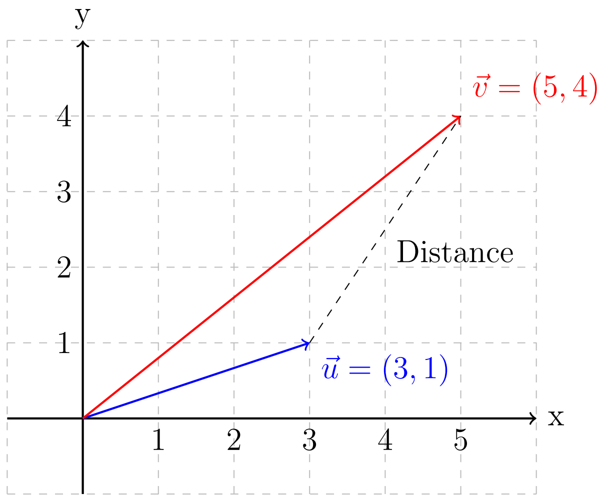
A vector is a mathematical object that has both magnitude and direction. Vectors can be represented graphically as arrows in a coordinate space.



Vector Distance

The distance between two vectors $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$ is calculated using:

$$\text{Distance} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The vector distance represents the straight-line path between two points or vectors in space, as shown in the plot. It is essentially the magnitude of the vector connecting $\vec{u}$ and $\vec{v}$.

Example: Calculate the distance between $(3, 1)$ and $(5, 4)$:

$$\text{Distance} = \sqrt{(5-3)^2 + (4-1)^2} = \sqrt{4+9} = \sqrt{13}$$

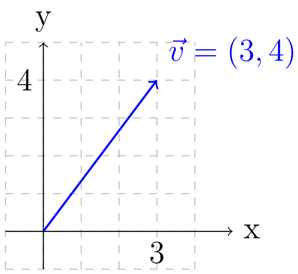
Norms of a Vector

The norm of a vector is a measure of its magnitude. There are different types of norms, commonly used are:

Euclidean Norm (L_2)

$\|\vec{v}\|_2 = \sqrt{x^2 + y^2}$

Represents the straight-line distance from the origin to the vector.

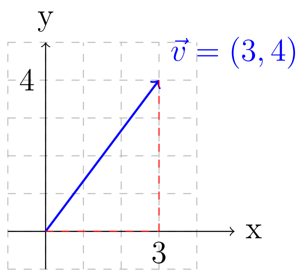


$\|\vec{v}\|_2 = \sqrt{3^2 + 4^2} = 5$

Manhattan Norm (L_1)

$\|\vec{v}\|_1 = |x| + |y|$

Represents the distance if you can only move along grid lines.

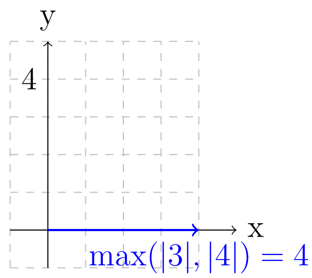


$\|\vec{v}\|_1 = |3| + |4| = 7$

Infinity Norm (L_∞)

$\|\vec{v}\|_\infty = \max(|x|, |y|)$

Represents the maximum magnitude of any component of the vector.



$\|\vec{v}\|_\infty = \max(|3|, |4|) = 4$

The difference between a norm and a distance is that a norm measures the magnitude of a single vector, while distance measures the separation between two vectors.

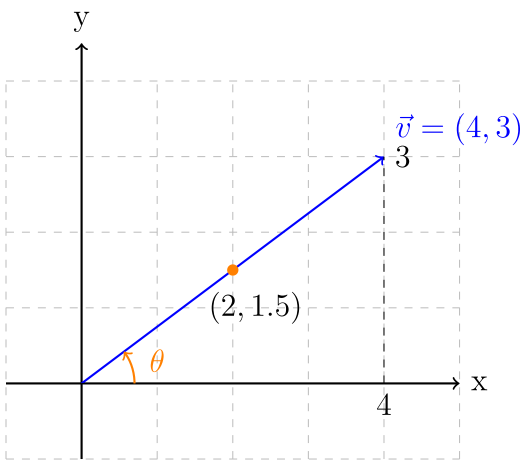
Direction of a Vector

The direction of a vector is given by the angle it makes with the x-axis. This angle θ is calculated using:

$$\tan\theta = \frac{y}{x}$$

Thus, θ can be determined as:

$$\theta = \arctan\left(\frac{y}{x}\right)$$

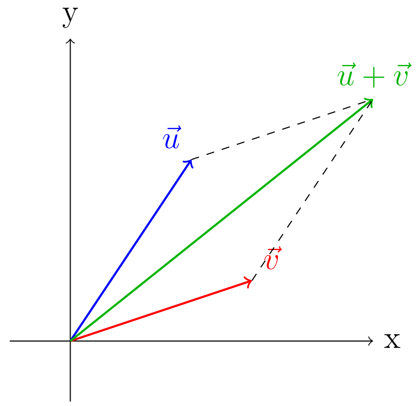


For example, if $\vec{v} = (4, 3)$:

$$\tan(\theta) = \frac{3}{4},$$
$$\theta = \arctan\left(\frac{3}{4}\right) \approx 0.64 \text{ radians} = 36.87°.$$

Transformations

Sum of Vectors

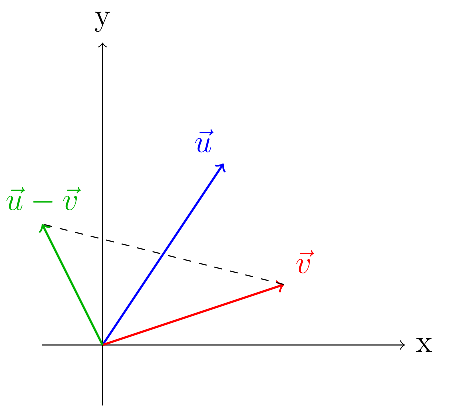


The vector sum $\vec{u} + \vec{v}$ is obtained by placing the tail of $\vec{v}$ at the head of $\vec{u}$.

$$\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$$

Given $\vec{u} = (2, 3)$ and $\vec{v} = (3, 1)$:
$\vec{u} + \vec{v} = (2 + 3, 3 + 1) = (5, 4)$

Difference of Vectors

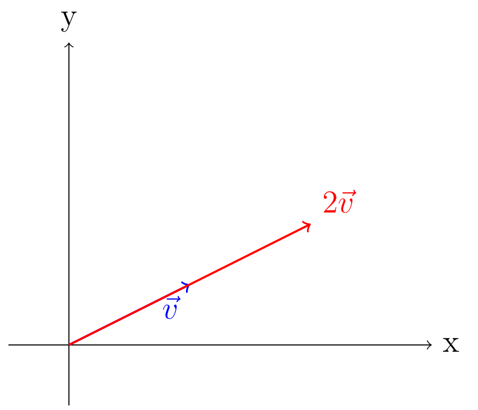


The difference $\vec{u} - \vec{v}$ represents the vector from the tip of $\vec{v}$ to the tip of $\vec{u}$.

$$\vec{u} - \vec{v} = (x_1 - x_2, y_1 - y_2)$$

Given $\vec{u} = (2, 3)$ and $\vec{v} = (3, 1)$:
$\vec{u} - \vec{v} = (2 - 3, 3 - 1) = (-1, 2)$

Multiplying a Vector by a Scalar



Scaling a vector $\vec{v}$ by a scalar k stretches or shrinks its length by $|k|$. The direction remains unchanged if $k > 0$ and reverses if $k < 0$.

$$\vec{v} = (x, y), \; k\vec{v} = (kx, ky)$$

Given $\vec{v} = (2, 1)$ and scalar $k = 2$:
$k\vec{v} = 2(2, 1) = (4, 2)$

Transpose of a Vector and Matrix

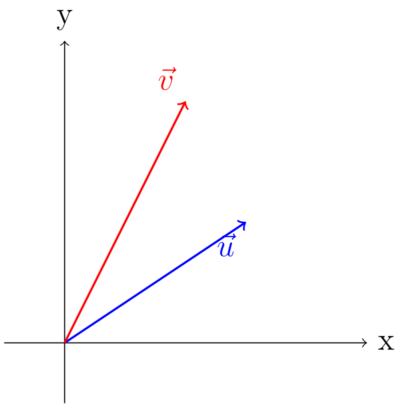
- A vector transpose converts a column vector into a row vector and vice versa.
- For a matrix A, the transpose A^T swaps rows with columns.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Dot Products

Algebraic Definition



The dot product of two vectors is a scalar value obtained by multiplying their corresponding components and summing the results. For $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$:

$$\vec{u} \cdot \vec{v} = x_1 * x_2 + y_1 * y_2 + ... + x_n * y_n$$

Given $\vec{u} = (3, 2)$ and $\vec{v} = (2, 4)$:

$$\vec{u} \cdot \vec{v} = (3)(2) + (2)(4) = 6 + 8 = 14$$

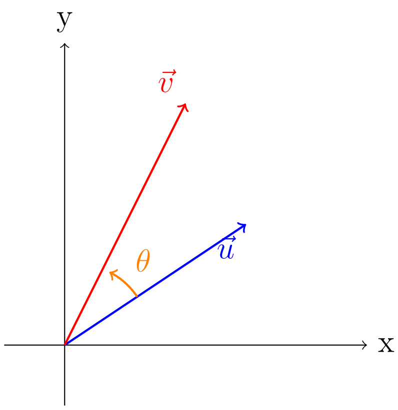
Norm of One Vector

The norm of a vector $\vec{v}$ can be computed as: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Geometric Definition

The dot product measures the alignment of two vectors. Geometrically, it can also be calculated using the magnitudes of the vectors and the cosine of the angle θ between them:

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$



Given $\vec{u} = (3, 2)$ and $\vec{v} = (2, 4)$:

$$\|\vec{u}\| = \sqrt{(3)^2 + (2)^2} = \sqrt{9 + 4} = \sqrt{13}$$

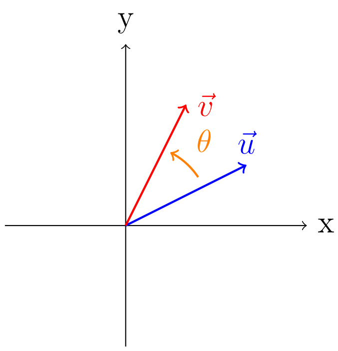
$$\|\vec{v}\| = \sqrt{(2)^2 + (4)^2} = \sqrt{4 + 16} = \sqrt{20}$$

$$\vec{u} \cdot \vec{v} = (3)(2) + (2)(4) = 6 + 8 = 14$$

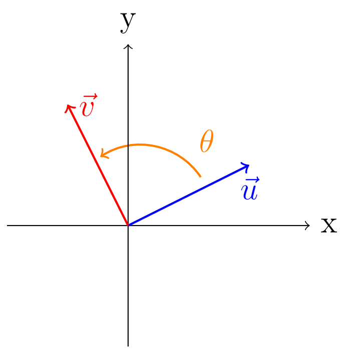
$$14 = \sqrt{13}\sqrt{20} \cos \theta$$

$$\cos \theta = \frac{14}{\sqrt{13} \cdot \sqrt{20}} = \frac{14}{\sqrt{260}} \approx 0.868$$

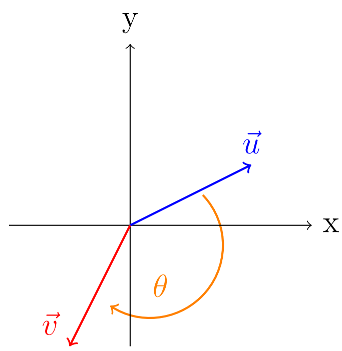
θ: $\theta = \arccos(0.868) \approx 29.85°$



$\vec{u} \cdot \vec{v} > 0$
The angle θ is acute
$(0° \leq \theta < 90°)$
The vectors point in a similar direction.



$\vec{u} \cdot \vec{v} = 0$
The angle θ is 90°
(90°)
The vectors are orthogonal (perpendicular).



$\vec{u} \cdot \vec{v} < 0$
The angle θ is obtuse
$(90° < \theta \leq 180°)$
The vectors point in opposite directions.

Multiplying a Matrix by a Vector

Multiplying a matrix A by a vector $\vec{v}$ applies a linear transformation to $\vec{v}$. This operation may involve scaling, rotating, or shearing the vector depending on the properties of A.
For a matrix A and vector $\vec{v}$:

$$A\vec{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

Example: Given $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$A\vec{v} = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 2 \\ 4 \cdot 1 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

Dot Product as Linear Equations

The dot product can be interpreted as solving a system of linear equations. When a matrix A multiplies a vector $\vec{x}$, the result is a vector where each element represents the dot product of a row in A with $\vec{x}$.

$$A\vec{x} = \vec{b}, \quad \text{where} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \ \vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \ \vec{b} = \begin{bmatrix} 10 \\ 15 \\ 12 \\ 13 \end{bmatrix}.$$

$$\begin{aligned} a + b + c &= 10, \\ a + 2b + c &= 15, \\ a + b + 2c &= 12. \end{aligned}$$

Example:
To compute the first element of $\vec{b}$:

$$b_1 = (1 \cdot a) + (1 \cdot b) + (1 \cdot c)$$

Given $a = 3$, $b = 4$, $c = 3$:

$$b_1 = (1 \cdot 3) + (1 \cdot 4) + (1 \cdot 3) = 3 + 4 + 3 = 10$$

$$b_2 = (1 \cdot a) + (2 \cdot b) + (1 \cdot c) = (1 \cdot 3) + (2 \cdot 4) + (1 \cdot 3) = 3 + 8 + 3 = 15$$

Key Points:

- The number of columns in A must equal the length of $\vec{x}$ for the multiplication to be valid.
- The resulting vector $\vec{b}$ has a length equal to the number of rows in A.
- This process is equivalent to solving a system of linear equations.

$$\underset{4 \times 3}{A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}} \qquad \underset{3 \times 1}{\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}} \qquad \underset{4 \times 1}{\vec{b} = \begin{bmatrix} 10 \\ 15 \\ 12 \\ 13 \end{bmatrix}}$$

Matrices as Linear Transformations

A matrix A represents a linear transformation that maps vectors from one space to another. Each column of A corresponds to the image of the basis vectors under the transformation.

Basic Linear Transformation

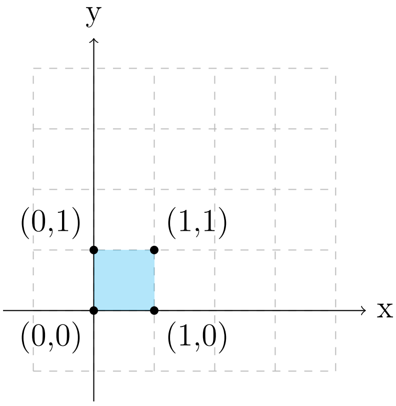
Given:

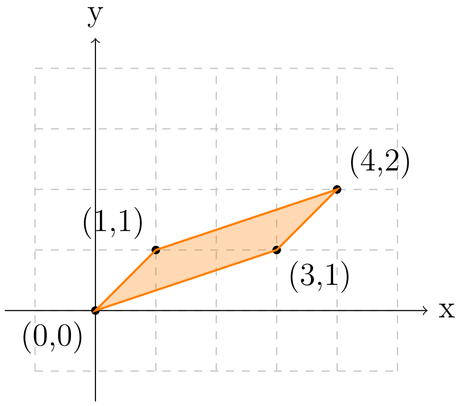
$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} a \\ b \end{bmatrix},$$

the matrix A maps points as:

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 0 + 1 \cdot 1 \\ 1 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 1 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$$A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 0 + 1 \cdot 0 \\ 1 \cdot 0 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 + 1 \cdot 0 \\ 1 \cdot 1 + 1 \cdot 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$





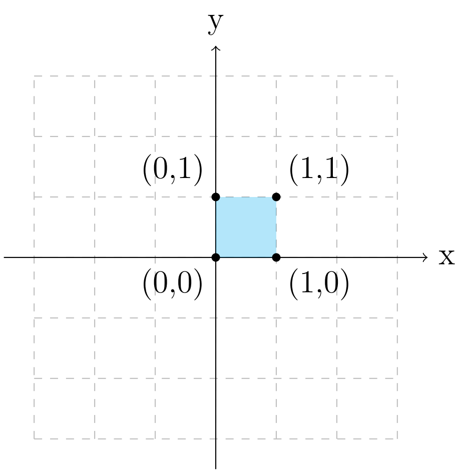
The matrix A transforms the unit square defined by $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ into a parallelogram.

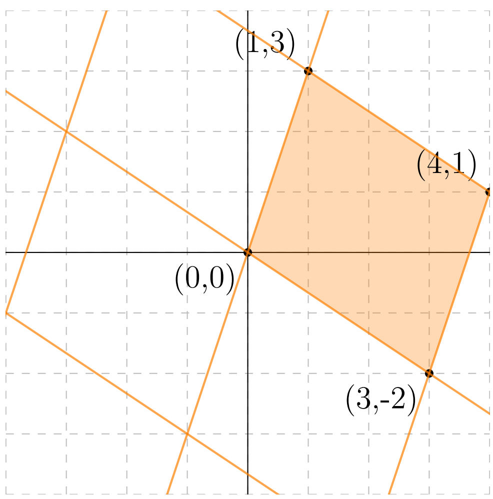
Skewed Transformation

Given:

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 3 \end{bmatrix},$$

the matrix A applies a shearing and stretching effect on the grid. We map each grid line and region from the original space to the transformed space. The parallelogram is distorted.





The matrix A maps the original unit square to a parallelogram, accurately showing the transformation. The grid lines are restricted to the coordinate system boundaries.

Combining Linear Transformations

Linear transformations can be combined by multiplying their matrices. This allows us to apply multiple transformations in sequence using a single combined matrix.

Example: Combining Two Transformations

Given two transformation matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix},$$

we can combine them by calculating the product $C = B \cdot A$:

$$C = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}.$$

$C_{11} = 2 \cdot 3 + (-1) \cdot 1 = 6 - 1 = 5$

$C_{12} = 2 \cdot 1 + (-1) \cdot 2 = 2 - 2 = 0$

$C_{21} = 0 \cdot 3 + 2 \cdot 1 = 0 + 2 = 2$

$C_{22} = 0 \cdot 1 + 2 \cdot 2 = 0 + 4 = 4$

Step-by-Step Transformation

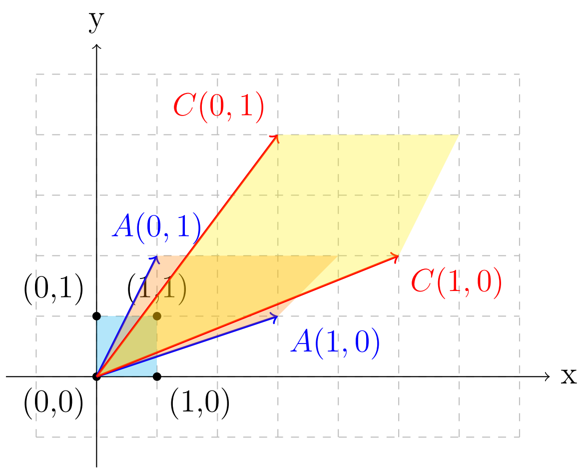
1. Start with a unit square defined by points $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$.

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, B \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, C \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, B \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, C \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$A \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, B \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, C \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, B \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, C \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

4. The final shape is a parallelogram defined by the combined transformation matrix C:

$$C = \begin{bmatrix} 5 & 0 \\ 2 & 4 \end{bmatrix}.$$



The matrix A skews and stretches the unit square into an orange parallelogram. The matrix B applies another transformation to the orange parallelogram, creating a final yellow parallelogram. The combined transformation matrix C directly maps the original unit square to the yellow parallelogram.

Dimensions of Matrices and Multiplication

Matrix multiplication is defined only when the number of columns in the first matrix matches the number of rows in the second matrix. The resulting matrix has dimensions corresponding to the rows of the first matrix and the columns of the second matrix.

Matrix Multiplication Rules

Compatibility: Columns of the first matrix must match rows of the second matrix.

$$\text{If } A \text{ is } m \times n \text{ and } B \text{ is } n \times p, \text{ then } A \cdot B \text{ is defined.}$$

Dimensions of the Result: The resulting matrix has dimensions $m \times p$, where:
- m: Number of rows from the first matrix.
- p: Number of columns from the second matrix.

Example

Given matrices:

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -1 & 2 & 0 & 3 \end{bmatrix},$$

A has dimensions 2×3, and B has dimensions 3×4. Since the number of columns in A matches the number of rows in B, we can multiply $A \cdot B$. The result will be a matrix C with dimensions 2×4.

$$C = A \cdot B = \begin{bmatrix} 3 & 1 & 4 \\ 2 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 & 1 & -2 \\ 1 & 5 & 2 & 0 \\ -1 & 2 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 9 & 5 & 6 \\ 3 & -3 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} (3 \cdot 3 + 1 \cdot 1 + 4 \cdot (-1)) & (3 \cdot 0 + 1 \cdot 5 + 4 \cdot 2) & (3 \cdot 1 + 1 \cdot 2 + 4 \cdot 0) & (3 \cdot (-2) + 1 \cdot 0 + 4 \cdot 3) \\ (2 \cdot 3 + (-1) \cdot 1 + 2 \cdot (-1)) & (2 \cdot 0 + (-1) \cdot 5 + 2 \cdot 2) & (2 \cdot 1 + (-1) \cdot 2 + 2 \cdot 0) & (2 \cdot (-2) + (-1) \cdot 0 + 2 \cdot 3) \end{bmatrix}$$

- The columns of A correspond to the weights applied to the rows of B.
- The final matrix C encapsulates the combined effect of rows from A interacting with columns of B.

Identity Matrix

The **identity matrix** is a square matrix that serves as the multiplicative identity in matrix algebra. This means that when any matrix is multiplied by the identity matrix, the result is the original matrix. An identity matrix I of size $n \times n$ is a square matrix with:
- All diagonal entries equal to 1.
- All off-diagonal entries equal to 0.

For example:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Multiplicative Property

Let A be a matrix of size $m \times n$, and I be an identity matrix: - When A is multiplied by I_n (on the right), the result is A:

$$A \cdot I_n = A.$$

- When I_m is multiplied by A (on the left), the result is A:

$$I_m \cdot A = A.$$

Given:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

Multiplication with I_2:

$$A \cdot I_2 = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 \cdot 1 + 3 \cdot 0 & 2 \cdot 0 + 3 \cdot 1 \\ 4 \cdot 1 + 5 \cdot 0 & 4 \cdot 0 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Multiplication from the left:

$$I_2 \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 \cdot 2 + 0 \cdot 4 & 1 \cdot 3 + 0 \cdot 5 \\ 0 \cdot 2 + 1 \cdot 4 & 0 \cdot 3 + 1 \cdot 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}.$$

Matrix Inverse

The **inverse** of a square matrix A (if it exists) is a matrix A^{-1} such that:

$$A \cdot A^{-1} = A^{-1} \cdot A = I,$$

where I is the identity matrix of the same dimension as A.

Not all matrices have an inverse. A matrix that has an inverse is called **invertible** or **non-singular**. If a matrix does not have an inverse, it is called **singular**. The matrix A is invertible if and only if $\det(A) \neq 0$.

Properties of Matrix Inverses

1. Only square matrices can have inverses. 2. The inverse is unique (if it exists). 3. $(A^{-1})^{-1} = A$. 4. $(AB)^{-1} = B^{-1}A^{-1}$ (note the reversed order). 5. If A is invertible, then A^{-1} is also invertible.

How to Check if a Matrix is Invertible

To determine if a square matrix A is invertible: 1. Compute $\det(A)$. 2. If $\det(A) = 0$, the matrix is singular (not invertible). 3. If $\det(A) \neq 0$, the matrix is invertible.

Applications of Matrix Inverses

Matrix inverses are used in: - Solving systems of linear equations: For $A\vec{x} = \vec{b}$, the solution is $\vec{x} = A^{-1}\vec{b}$ (if A is invertible). - Transformations in linear algebra. - Calculating transition matrices in Markov chains.

Finding the Inverse of a 2x2 Matrix

For a 2×2 matrix:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the inverse (if it exists) is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

where $\det(A) = ad - bc$ is the determinant of A.

Example: Given:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix},$$

$$\det(A) = (4)(6) - (7)(2) = 24 - 14 = 10.$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}.$$

Finding the Inverse Using Matrix Augumentation

The inverse of a matrix A can also be found by solving a system of linear equations. This method involves augmenting A with the identity matrix and performing row operations to reduce A to the identity matrix. The resulting augmented matrix will contain the inverse of A.

Example: Given:

$$A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix},$$

find A^{-1} by solving:

$$A \cdot A^{-1} = I \quad \text{or} \quad \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 1: Augment the matrix A with the identity matrix.

$$\begin{bmatrix} 4 & 7 & 1 & 0 \\ 2 & 6 & 0 & 1 \end{bmatrix}.$$

Step 2: Perform row operations to reduce A to the identity matrix.

1. Divide the first row by 4 to make the pivot 1:

$$\begin{bmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 2 & 6 & 0 & 1 \end{bmatrix}.$$

2. Subtract 2 times the first row from the second row:

$$\begin{bmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & \frac{10}{4} & -\frac{1}{2} & 1 \end{bmatrix}.$$

3. Divide the second row by $\frac{10}{4} = 2.5$:

$$\begin{bmatrix} 1 & \frac{7}{4} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

4. Subtract $\frac{7}{4}$ times the second row from the first row:

$$\begin{bmatrix} 1 & 0 & \frac{3}{5} & -\frac{7}{10} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

Step 3: The right-hand side is the inverse of A:

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{7}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

Finding the Inverse Using Linear Equations

To find the inverse of a matrix A using linear equations, we solve:

$$A \cdot A^{-1} = I,$$

where $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and I is the identity matrix: $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example: Given: $A = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$, find $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by solving the equations derived from:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Step 1: Write out the equations.

$$\begin{aligned} 4a + 7c &= 1, \qquad & 4b + 7d &= 0, \\ 2a + 6c &= 0, \qquad & 2b + 6d &= 1. \end{aligned}$$

Step 2: Solve the system of equations.

1. From $4a + 7c = 1$: $c = \frac{1-4a}{7}$,
2. From $2a + 6c = 0$: Substitute c:

$$2a + 6\left(\frac{1-4a}{7}\right) = 0 \implies 2a + \frac{6-24a}{7} = 0.$$

$$14a + 6 - 24a = 0 \implies -10a + 6 = 0 \implies a = \frac{3}{5}.$$

3. Substitute $a = \frac{3}{5}$ into $c = \frac{1-4a}{7}$:

$$c = \frac{1 - 4\left(\frac{3}{5}\right)}{7} = \frac{1 - \frac{12}{5}}{7} = \frac{\frac{-7}{5}}{7} = -\frac{1}{5}.$$

4. From $4b + 7d = 0$: $d = -\frac{4b}{7}$,
5. From $2b + 6d = 1$: Substitute $d = -\frac{4b}{7}$:

$$2b + 6\left(-\frac{4b}{7}\right) = 1 \implies 2b - \frac{24b}{7} = 1.$$

$$14b - 24b = 7 \implies -10b = 7 \implies b = -\frac{7}{10}.$$

6. Substitute $b = -\frac{7}{10}$ into $d = -\frac{4b}{7}$:

$$d = -\frac{4\left(-\frac{7}{10}\right)}{7} = \frac{28}{70} = \frac{2}{5}.$$

Step 3: Write the inverse matrix.

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \frac{3}{5} & -\frac{7}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

Verification: Multiply $A \cdot A^{-1}$ to ensure it equals the identity matrix:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{7}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Singularity and Rank of Linear Transformations

Linear transformations, represented by matrices, can be classified as singular or non-singular based on their properties. The rank of a matrix is a fundamental concept that relates to the linear independence of its rows or columns.

Singular and Non-Singular Transformations

- A matrix A is **non-singular** if it is invertible. This occurs when $\det(A) \neq 0$.
- A matrix A is **singular** if it is not invertible. This occurs when $\det(A) = 0$)

Example: Consider the matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

$\det(A) = (2)(3) - (1)(1) = 6 - 1 = 5 \neq 0. \quad \Rightarrow \quad$ Therefore, A is **non-singular**.

$\det(B) = (1)(4) - (2)(2) = 4 - 4 = 0. \quad \Rightarrow \quad$ Therefore, B is **singular**.

Rank of Linear Transformations

The **rank** of a matrix A is the maximum number of linearly independent rows or columns in A. It provides information about the dimension of the image of the linear transformation.

- If the rank of A equals the number of rows (or columns), A is **full rank**.
- If the rank of A is less than the number of rows or columns, A is **rank-deficient**.
- For a square matrix A, if the rank of A equals its size, A is non-singular.

Example: Consider the matrices:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

- For C, the rank is 2 because only the first two rows are linearly independent. The third row is a linear combination of the first two.
- For D, the rank is 2 because both rows are linearly independent. Since D is a 2×2 matrix and has full rank, it is non-singular.

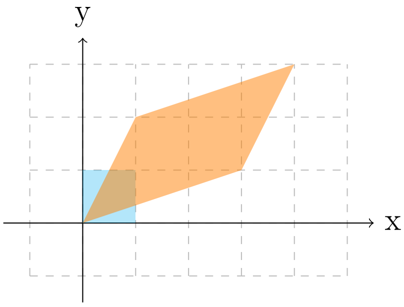
Geometric Interpretation: The rank of a matrix corresponds to the dimension of the subspace to which it maps the input space. For instance:

- A rank-2 matrix maps a 2D plane in $\mathbb{R}^3$ to a 2D plane in $\mathbb{R}^2$.
- A rank-1 matrix collapses all input vectors onto a line.

Determinant as an Area

The determinant of a matrix can be interpreted geometrically as the area (or volume in higher dimensions) of the parallelogram (or parallelepiped) defined by the column vectors of the matrix. It gives a measure of how a linear transformation changes the area or volume of a shape.

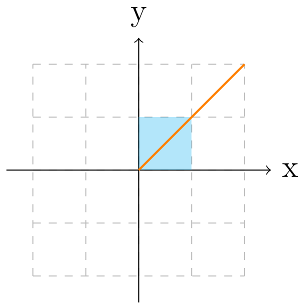
Rank 2 (Full Rank)
Non-Singular Transformation: The matrix is invertible, and the determinant is non-zero, the transformed shape is a parallelogram, and its area is given by the absolute value of the determinant.



Matrix: $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$
Non-Singular
Dimension: 2
Determinant: 5
Area: 5

Rank 1 (Deficient)
Singular Transformation:

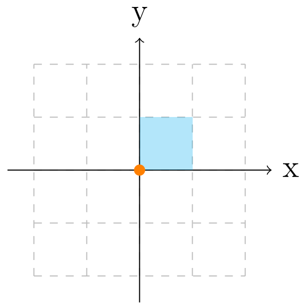
The determinant is zero, and the transformation collapses the area into a line.



Matrix: $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
Singular
Dimension: 1
Determinant: 0
Area: 0

Rank 0 (Singular)
Singular Transformation:

The determinant is zero, and the transformation collapses the area into a single point.



Matrix: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Singular
Dimension: 0
Determinant: 0
Area: 0

Geometric Interpretation of Negative Determinants

The determinant of a matrix not only scales the area but also encodes the orientation of the transformation. A **negative determinant** indicates that the transformation flips the orientation of the basis vectors and implies that the transformation reverses the orientation of the coordinate system.
The original counterclockwise orientation becomes clockwise.
The signed area of the transformed parallelogram is negative.

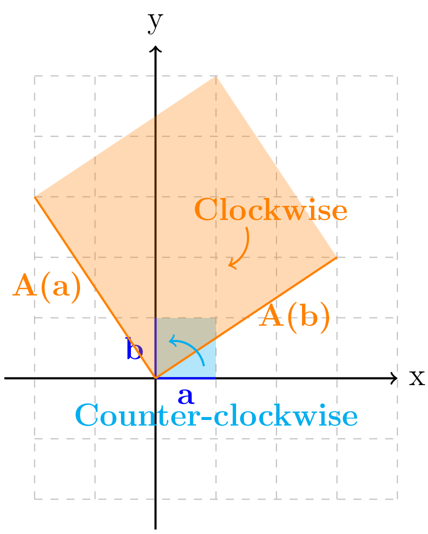
Given the matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix},$$

the determinant is calculated as:

$$\text{Det}(A) = (1)(1) - (3)(2) = 1 - 6 = -5.$$

Explanation:
- The original unit square is transformed into a parallelogram.
- The negative determinant shows that the orientation changes from counterclockwise to clockwise.
- The signed area of the parallelogram is -5.



Determinant of a Product

The determinant of a product of matrices is the product of their determinants:

$$\text{If } C = B \cdot A, \text{ then } \det(C) = \det(B) \cdot \det(A).$$

If A is non-singular ($\det(A) \neq 0$) and B is singular ($\det(B) = 0$), then:

$$\det(B \cdot A) = \det(B) \cdot \det(A) = 0.$$

Therefore, $B \cdot A$ is singular.

Example:

Given two matrices:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix},$$

$$\det(A) = (3)(2) - (1)(1) = 6 - 1 = 5, \quad \det(B) = (1)(1) - (1)(-2) = 1 + 2 = 3.$$

Compute the product $C = B \cdot A$:

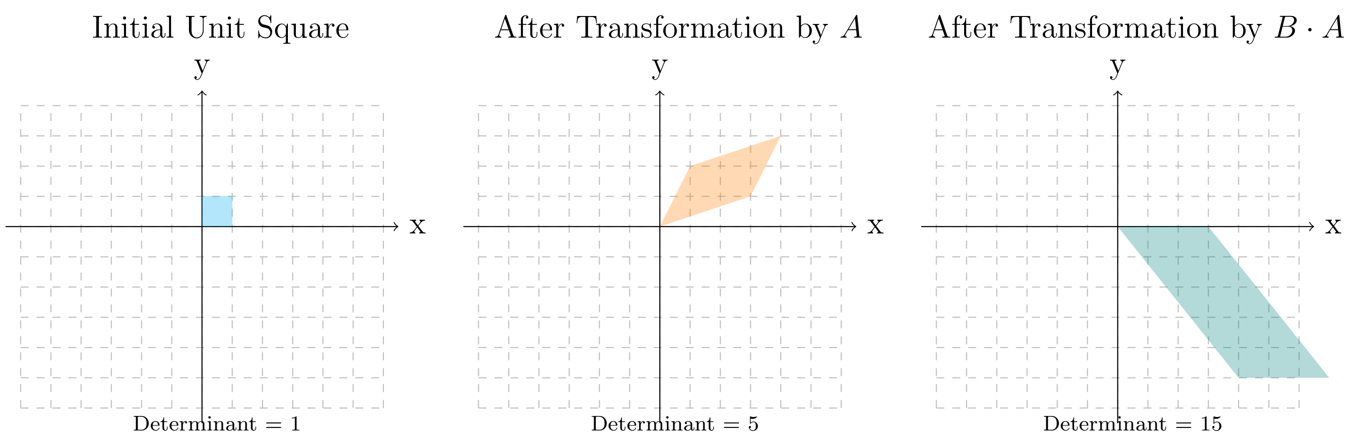
$$C = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -5 & 0 \end{bmatrix}.$$

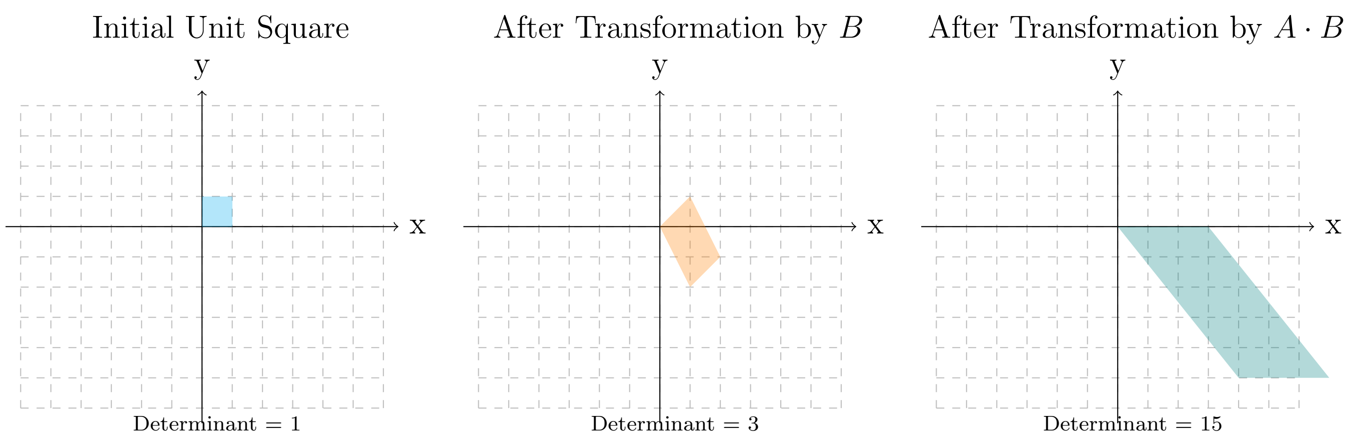
The determinant of C is:

$$\det(C) = (4)(0) - (3)(-5) = 0 + 15 = 15.$$

Verification:

$$\det(C) = \det(B) \cdot \det(A) = 3 \cdot 5 = 15.$$





Conclusion

The determinant of the final parallelogram is the same regardless of the order of the transformations ($A \cdot B$ or $B \cdot A$). This demonstrates that the determinant of a product of matrices is invariant under the order of multiplication:

$$\det(A \cdot B) = \det(B \cdot A).$$

Determinant of an Inverse Matrix

The determinant of an inverse matrix has a well-defined relationship with the determinant of the original matrix. Specifically, for a square matrix A:

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

This relationship only holds if A is invertible, meaning $\det(A) \neq 0$. If $\det(A) = 0$, the matrix A is singular and does not have an inverse. If A is singular ($\det(A) = 0$), then: A has no inverse and $\det(A^{-1})$ is undefined because division by zero is not possible.

Why is this true?

To understand this, recall the property of determinants for the product of two matrices:

$$\det(A \cdot B) = \det(A) \cdot \det(B).$$

1. If A is invertible, then $A \cdot A^{-1} = I$, where I is the identity matrix.
2. Applying the determinant property:

$$\det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) = \det(I).$$

3. Since the determinant of the identity matrix is 1 ($\det(I) = 1$), we have:

$$\det(A) \cdot \det(A^{-1}) = 1.$$

4. Rearranging gives:

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Example

Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$. The determinant of A is:

$$\det(A) = (3)(4) - (1)(2) = 12 - 2 = 10.$$

The inverse of A is:

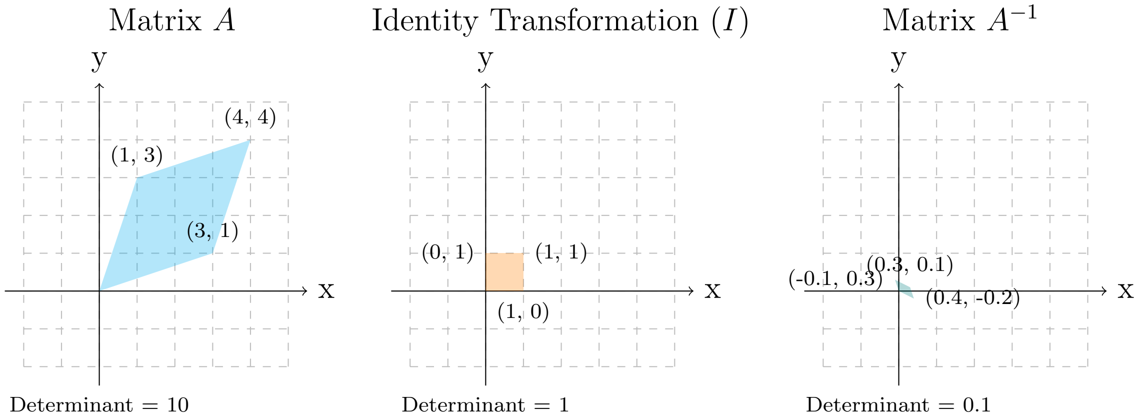
$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}.$$

The determinant of A^{-1} is:

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{10}.$$

Geometric Interpretation

The determinant of a matrix represents the scaling factor of the area (or volume in higher dimensions) caused by the linear transformation. When taking the inverse: - The transformation reverses the scaling effect. - The inverse transformation shrinks the area/volume by the same factor the original matrix scaled it by.



Summary

- The determinant of an inverse matrix is the reciprocal of the determinant of the original matrix: $\det(A^{-1}) = \frac{1}{\det(A)}$. - Geometrically, the inverse transformation scales the area/volume inversely. - If A is singular, no inverse exists.

Span of Vectors

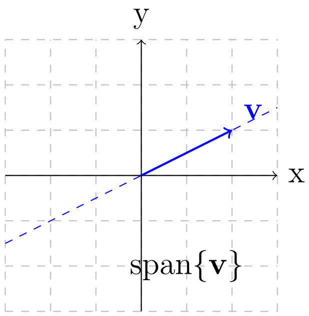
The **span** of a set of vectors is the collection of all possible linear combinations of those vectors. It represents the subspace of the vector space that the vectors can reach through scaling and addition.

Formally, for a set of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in $\mathbb{R}^m$:

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \mid c_1, c_2, \dots, c_n \in \mathbb{R}\}.$$

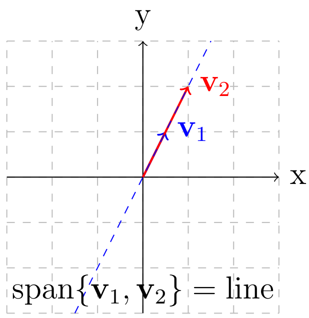
- The span of a set of vectors forms a **subspace**.
- If the span of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ equals the entire vector space (e.g., $\mathbb{R}^m$), the vectors are said to **span the space**.
- The dimension of the span is the number of **linearly independent** vectors in the set.

Span of a Single Vector



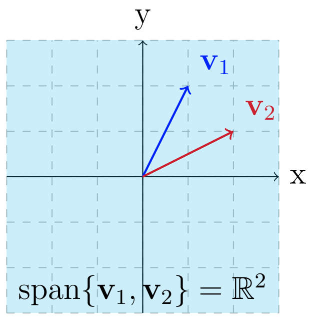
A single vector spans a line passing through the origin.

Span of Two Linearly Dependent Vectors



Two linearly dependent vectors span a line, as they point in the same direction.

Span of Two Linearly Independent Vectors



Two linearly independent vectors span the entire plane $\mathbb{R}^2$.

Bases in Linear Algebra

A **basis** is a set of vectors that spans a vector space **and is linearly independent**. The concept of a basis is fundamental in linear algebra as it provides a framework for representing vectors and understanding the structure of a vector space.

- The number of vectors in a basis is equal to the **dimension** of the vector space.
- Bases are not unique; there are infinitely many possible bases for any vector space.

Definition

Let V be a vector space over a field $\mathbb{R}$ (or $\mathbb{C}$). A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a **basis** of V if: 1. The vectors are **linearly independent**, i.e., the equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = 0$$

implies $c_1 = c_2 = \cdots = c_n = 0$. 2. The vectors **span** the vector space V, i.e., any vector $\vec{v} \in V$ can be written as a linear combination:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n.$$

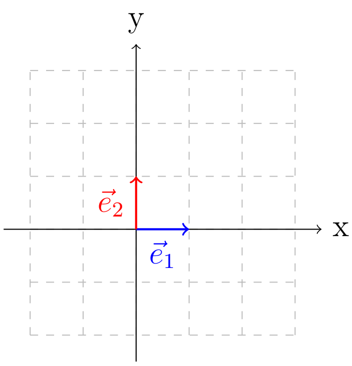
Geometric Interpretation

The basis vectors define the coordinate system for the vector space. For example:

- In $\mathbb{R}^2$, the standard basis defines the familiar Cartesian plane.
- A non-standard basis can tilt or stretch the coordinate grid, representing the same space in a different way.

Examples of Bases

Standard Basis for $\mathbb{R}^2$

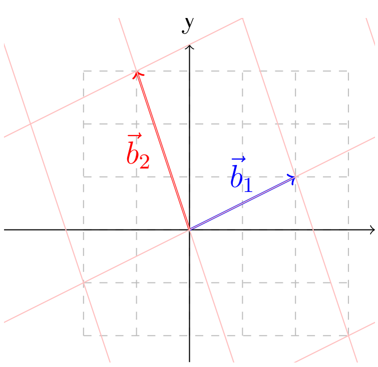


The standard basis of $\mathbb{R}^2$ consists of the vectors

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix} \text{ and } \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}.$$

Any vector $\vec{v} = \begin{bmatrix}x\\y\end{bmatrix}$ can be written as $\vec{v} = x\vec{e}_1 + y\vec{e}_2$.

Non-Standard Basis for $\mathbb{R}^2$

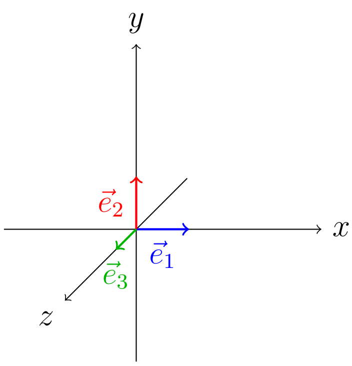


A valid non-standard basis for $\mathbb{R}^2$ is

$$\vec{b}_1 = \begin{bmatrix}2\\1\end{bmatrix} \text{ and } \vec{b}_2 = \begin{bmatrix}-1\\3\end{bmatrix}.$$

These vectors are linearly independent and span $\mathbb{R}^2$.

Standard Basis for $\mathbb{R}^3$



The standard basis for $\mathbb{R}^3$ consists of

$$\vec{e}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \vec{e}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \vec{e}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Any vector in $\mathbb{R}^3$ can be expressed as a linear combination of these basis vectors.

Changing Basis

To express a vector in a new basis: 1. Compute the transformation matrix P, whose columns are the new basis vectors expressed in the original basis. 2. Transform the vector $\vec{v}$ using:

$$\vec{v}_{\text{new}} = P^{-1}\vec{v}.$$

Example: Changing Basis

Let the standard basis be $\{\vec{e}_1, \vec{e}_2\}$ and the new basis be:

$$\vec{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

The transformation matrix is:

$$P = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}.$$

To find the coordinates of $\vec{v} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in the new basis:

$$\vec{v}_{\text{new}} = P^{-1}\vec{v}.$$

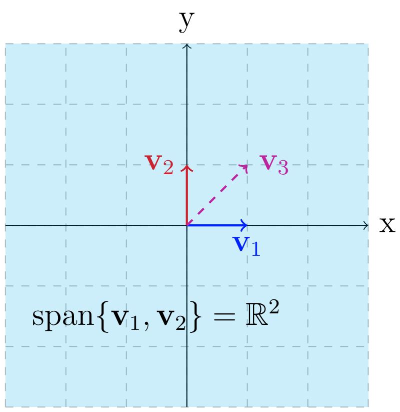
Span vs Basis

The **span** and **basis** are closely related concepts:

- The **span** of a set of vectors is the set of all linear combinations of those vectors.
- A **basis** of a vector space is a minimal set of linearly independent vectors whose span equals the entire vector space.
- The span of a set of vectors can contain infinitely many vectors, but a basis contains only the minimal number of vectors needed to span the space.
- Every vector in the span can be written uniquely as a linear combination of the basis vectors.

Example:

- In $\mathbb{R}^2$, the vectors $\mathbf{v}_1 = (1, 0)$ and $\mathbf{v}_2 = (0, 1)$ form a basis because:
  - They are linearly independent.
  - Their span is $\mathbb{R}^2$, covering the entire plane.
    Since the vectors are linearly independent and their span covers $\mathbb{R}^2$, they form a basis.
- Any other vector, such as $\mathbf{v}_3 = (1, 1)$, can be expressed as a combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, making it redundant.
  Therefore, $\mathbf{v}_3$ is part of the span of $\mathbf{v}_1$ and $\mathbf{v}_2$, but does not extend the basis.



Linear Independence and Dependence

Vectors in a vector space can either be **linearly independent** or **linearly dependent**:

- **Linearly Independent:** A set of vectors is linearly independent if no vector in the set can be written as a linear combination of the others. The only solution to:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

  is $c_1 = c_2 = \cdots = c_n = 0$ (trivial solution).

- **Linearly Dependent:** A set of vectors is linearly dependent if at least one vector in the set can be written as a linear combination of the others.

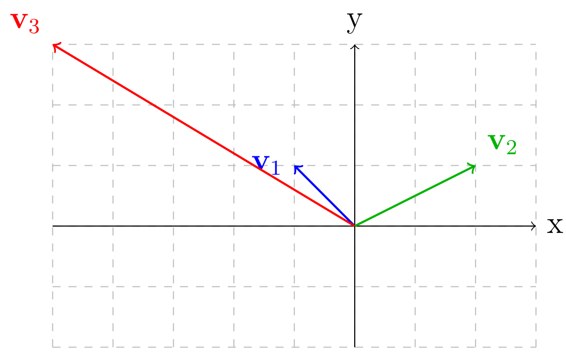
Checking for Linear Dependence

To check if a vector $\mathbf{v}_3$ is linearly dependent on $\mathbf{v}_1$ and $\mathbf{v}_2$, solve for α and β in:

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{v}_3$$

If a solution exists, the vectors are linearly dependent.

Example:



Given:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

Check if $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
Write the equations:

$$-\alpha + 2\beta = -5 \quad (1), \quad \alpha + \beta = 3 \quad (2)$$

Solution: Add equations (1) and (2): $3\beta = -2 \implies \beta = -\frac{2}{3}$
Substitute $\beta = -\frac{2}{3}$ into (2): $\alpha - \frac{2}{3} = 3 \implies \alpha = \frac{11}{3}$ Thus: $\mathbf{v}_3 = \frac{11}{3}\mathbf{v}_1 - \frac{2}{3}\mathbf{v}_2$
The vectors are **linearly dependent**.

Geometric Interpretation

- **Linearly Independent Vectors:** Span a unique subspace. For example:
  - In $\mathbb{R}^2$, two independent vectors span the plane.
  - In $\mathbb{R}^3$, three independent vectors span the entire 3D space.
- **Linearly Dependent Vectors:** At least one vector lies in the span of the others. The set does not span a higher-dimensional space than the span of its independent vectors.

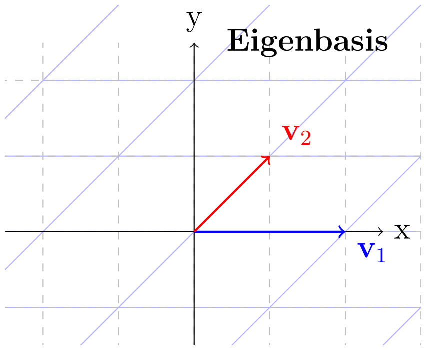
Eigenbasis

An **eigenbasis** is a basis of a vector space formed by the eigenvectors of a matrix A, provided that A has enough linearly independent eigenvectors to span the space. When the eigenbasis is used, the matrix A can be transformed into a diagonal matrix.
For an $n \times n$ square matrix, if there are n linearly independent eigenvectors, they form a basis for $\mathbb{R}^n$.

- An eigenbasis allows A to act as simple scaling along the eigenvector directions.
- For a matrix A with eigenbasis $\{\mathbf{v}_1, \mathbf{v}_2\}$:

$$A = P\Lambda P^{-1}, \quad \text{where } P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

Example:



Consider the vector space $\mathbb{R}^2$ with a transformation matrix A. Using an eigenbasis:

- The standard grid is transformed along the eigenvector directions.
- The resulting grid remains aligned with the eigenvectors, scaled by their eigenvalues.
- Any vector can be written as a linear combination of the eigenbasis vectors.

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are defined for **square matrices** (i.e., matrices with the same number of rows and columns). An **eigenvector** of a matrix A is a non-zero vector $\mathbf{v}$ such that:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is the **eigenvalue** corresponding to $\mathbf{v}$.

Eigenvectors point in directions that remain unchanged under the transformation A, except for scaling. Eigenvalues represent the factor by which the eigenvector is scaled during the transformation.

Example:

Consider the matrix:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

1. For $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$:

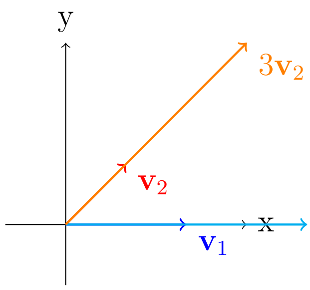
$$A\mathbf{v}_1 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\mathbf{v}_1$$

Thus, $\lambda_1 = 2$ and $\mathbf{v}_1$ is an eigenvector.

2. For $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$:

$$A\mathbf{v}_2 = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3\mathbf{v}_2$$

Thus, $\lambda_2 = 3$ and $\mathbf{v}_2$ is an eigenvector.



Eigenvectors define invariant directions under the transformation.
Eigenvalues scale the eigenvectors during the transformation.
For example:

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$

Applications of Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors have many practical uses in linear algebra and beyond. Below are a few examples using the eigenvectors and eigenvalues of the matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$, with:

$$\lambda_1 = 2, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \lambda_2 = 3, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Example 1: Vector Transformation

Let $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Express $\mathbf{u}$ in terms of eigenbasis:

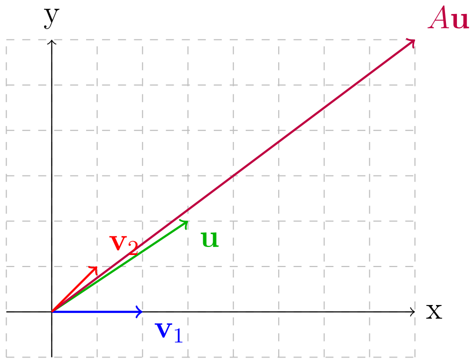
$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2, \quad c_1 = 1, c_2 = 2.$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The transformation $A\mathbf{u}$ becomes:

$$A\mathbf{u} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = 1 \cdot 2\mathbf{v}_1 + 2 \cdot 3\mathbf{v}_2.$$

$$A\mathbf{u} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}.$$



Example 2: Diagonalization

Diagonalization simplifies operations such as computing powers of A.
Instead of multiplying A repeatedly, we use the diagonal matrix $\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, constructed using the eigenvalues, to compute powers efficiently:

$$A^n = P\Lambda^n P^{-1}.$$

The matrix P aligns the standard basis with the eigenvectors of A. The transformation $P^{-1}AP = \Lambda$ maps A to a diagonal form in the eigenbasis, where $\mathbf{v}_1$ and $\mathbf{v}_2$ define the rotated axes.
Eigenvectors form the columns of P. P^{-1} is the inverse of P:

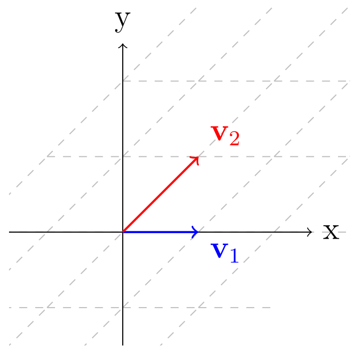
$$P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

Diagonalize A:

$$A = P \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} P^{-1}.$$

To compute A^n:

$$A^n = P \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} P^{-1}.$$



Example 3: Principal Component Analysis (PCA)

Eigenvectors are crucial in data reduction and analysis:

- Suppose A represents a covariance matrix of a dataset. Eigenvectors indicate the principal directions of variation.
- Eigenvalues measure the magnitude of variation along these directions.
- Use the eigenvector with the largest eigenvalue (principal component) to project data into a lower-dimensional space.

Finding Eigenvalues and Eigenvectors Using Determinants

Find the eigenvalues and eigenvectors of the matrix: $A = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$.

Step 1: Eigenvalues

Eigenvalues are solutions to the characteristic equation:

$$\det(A - \lambda I) = 0,$$

where I is the identity matrix and λ is the eigenvalue.

$$A - \lambda I = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix}.$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} = (9-\lambda)(3-\lambda) - 4 \cdot 4.$$

$$(9-\lambda)(3-\lambda) - 16 = 27 - 9\lambda - 3\lambda + \lambda^2 - 16.$$

The characteristic polynomial: $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1) = 0.$

Eigenvalues: $\lambda_1 = 11, \quad \lambda_2 = 1.$

Step 2: Eigenvectors

For each eigenvalue λ, solve the equation:

$$(A - \lambda I)\mathbf{v} = 0,$$

where $\mathbf{v}$ is the eigenvector.

For $\lambda_1 = 11$:

$$A - 11I = \begin{bmatrix} 9-11 & 4 \\ 4 & 3-11 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}.$$

Solve:

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From $-2x + 4y = 0$, we get $y = \frac{1}{2}x$.
Let $x = 2$, $y = 1$:

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1$:

$$A - I = \begin{bmatrix} 9-1 & 4 \\ 4 & 3-1 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}.$$

Solve:

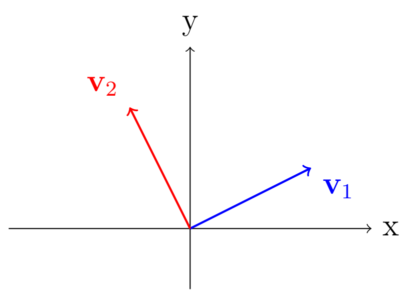
$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From $8x + 4y = 0$, we get $y = -2x$.
Let $x = -1$, $y = 2$:

$$\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Solution:

- **Eigenvalues:** $\lambda_1 = 11$, $\lambda_2 = 1$.
- **Eigenvectors:** $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.



Verification:

Verify that $A\mathbf{v} = \lambda\mathbf{v}$ for both eigenvalues and eigenvectors.

Finding Eigenvalues and Eigenvectors Using Determinants

Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}.$$

Step 1: Eigenvalues

$$A - \lambda I = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix}.$$

Compute the determinant:

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix}.$$

Using cofactor expansion:

$$(2-\lambda)\det \begin{bmatrix} -\lambda & -3 \\ -3 & -\lambda \end{bmatrix} - 1\det \begin{bmatrix} 1 & -3 \\ -1 & -\lambda \end{bmatrix} + (-1)\det \begin{bmatrix} 1 & -\lambda \\ -1 & -3 \end{bmatrix}.$$

$$\det(A-\lambda I) = (2-\lambda)((-\lambda)(-\lambda) - (-3)(-3)) - (1)((1)(-\lambda) - (-1)(-3)) - ((1)(-3) - (-1)(-\lambda)).$$

After simplifying, the characteristic polynomial is: $\lambda^3 - \lambda^2 - 11\lambda + 15 = 0$.

Eigenvalues: $\lambda_1 = 4, \quad \lambda_2 = 1, \quad \lambda_3 = -3.$

Step 2: Eigenvectors

For each eigenvalue λ, solve $(A - \lambda I)\mathbf{v} = 0$.

For $\lambda_1 = 4$:

$$A - 4I = \begin{bmatrix} -2 & 1 & -1 \\ 1 & -4 & -3 \\ -1 & -3 & -4 \end{bmatrix}.$$

Row reduce to find $\mathbf{v}_1$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 1$:

$$A - I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -3 \\ -1 & -3 & -1 \end{bmatrix}.$$

Row reduce to find $\mathbf{v}_2$:

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = -3$:

$$A + 3I = \begin{bmatrix} 5 & 1 & -1 \\ 1 & 3 & -3 \\ -1 & -3 & 3 \end{bmatrix}.$$

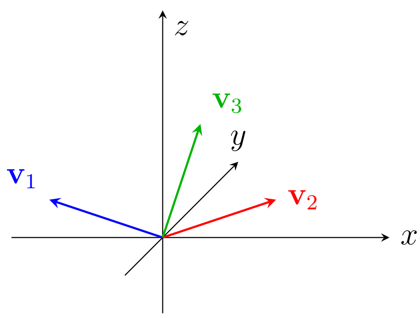
Row reduce to find $\mathbf{v}_3$:

$$\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solution:

- **Eigenvalues:** $\lambda_1 = 4, \lambda_2 = 1, \lambda_3 = -3.$
- **Eigenvectors:**

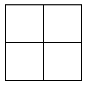
$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$



Eigenvalues, Eigenvectors, and Their Multiplicity

The number of eigenvalues and eigenvectors of a square matrix depends on its dimensions, characteristic polynomial, the structure of the matrix, the multiplicity of its eigenvalues and the matrix's diagonalizability. The eigenvectors determine whether the transformation represented by the matrix has distinct or overlapping directions of scaling.

2x2 Matrices - 2 cases



Distinct Eigenvalues $(\lambda_1 \neq \lambda_2)$

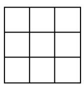
- The matrix has **2 linearly independent eigenvectors**, each corresponding to a distinct eigenvalue and a different direction.
-The eigenvectors span $\mathbb{R}^2$ (or $\mathbb{C}^2$ for complex eigenvalues).

Repeated Eigenvalue $(\lambda_1 = \lambda_2)$

- One eigenvalue with algebraic multiplicity 2.
- The number of eigenvectors depends on the matrix's structure:

- If there is only **1 eigenvector**, the matrix is **defective** and cannot be diagonalized.
- If there are **2 eigenvectors**, the eigenvectors span $\mathbb{R}^2$.

3x3 Matrices - 3 cases



Distinct Eigenvalues

$$(\lambda_1 \neq \lambda_2 \neq \lambda_3)$$

The matrix has **3 linearly independent eigenvectors**.
The eigenvectors span $\mathbb{R}^3$.
Three eigenvectors corresponding to three different directions.

One Repeated Eigenvalue

$$(\lambda_1 = \lambda_2 \neq \lambda_3)$$

The matrix may have **2 or 3 linearly independent eigenvectors**:
- If 2 eigenvectors, the matrix is defective.
- If 3 eigenvectors, the eigenvectors span $\mathbb{R}^3$.

All Eigenvalues Are Equal

$$(lambda_1 = \lambda_2 = \lambda_3)$$

The matrix may have **1, 2, or 3 linearly independent eigenvectors**:
- If 1 eigenvector, the matrix is highly defective.
- If 2 eigenvectors, the matrix is partially defective.
- If 3 eigenvectors, the eigenvectors span $\mathbb{R}^3$.