# Notes on Linear Algebra for Machine Learning

## System of Linear Equations

A system of linear equations can be expressed as:

$$\begin{aligned}
w_1x_1^{(1)} + w_2x_2^{(1)} + \cdots + w_nx_n^{(1)} + b &= y^{(1)} \\
w_1x_1^{(2)} + w_2x_2^{(2)} + \cdots + w_nx_n^{(2)} + b &= y^{(2)} \\
w_1x_1^{(3)} + w_2x_2^{(3)} + \cdots + w_nx_n^{(3)} + b &= y^{(3)} \\
&\vdots \\
w_1x_1^{(m)} + w_2x_2^{(m)} + \cdots + w_nx_n^{(m)} + b &= y^{(m)}
\end{aligned}$$

### Matrix Form

The above system can be represented compactly in matrix form as:

$$A\mathbf{x} + \mathbf{b} = \mathbf{y},$$

where $A$ is the feature matrix, $\mathbf{x}$ contains the weights, $\mathbf{b}$ is the bias, and $\mathbf{y}$ represents the outputs.

**x: n-dimensional vector of weights**

$$\mathbf{x} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$$

**b: bias vector**

$$\mathbf{b} = \begin{bmatrix} b \\ b \\ \vdots \\ b \end{bmatrix}$$

**y: target vector**

$$\mathbf{y} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

## Consider the system:

$$\begin{aligned}
2x + y &= 5 \\
x - y &= 1
\end{aligned}$$

Matrix representation:

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

- The first row of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$ corresponds to the coefficients of $2x + y = 5$:
  - 2 is the coefficient of $x$.
  - 1 is the coefficient of $y$.
- The second row corresponds to the coefficients of $x - y = 1$:
  - 1 is the coefficient of $x$.
  - $-1$ is the coefficient of $y$.
- The vector on the right-hand side $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$ represents the constants 5 and 1 from the respective equations.

### Lines and Planes

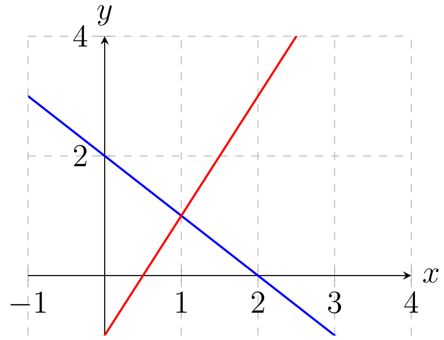
- A single equation in two variables represents a line.
- A single equation in three variables represents a plane.

### Singularity

A matrix is singular if its determinant is zero, meaning it does not have an inverse. Constants do not affect singularity because they only scale the determinant.

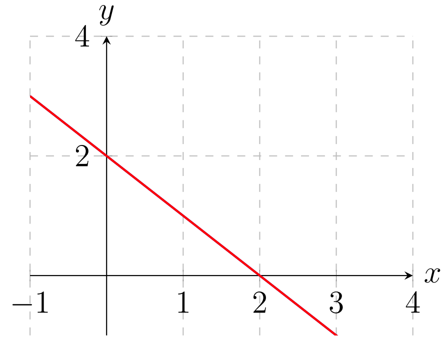
### Types of Systems

**Complete**
**Non-Singular**
$x + y = 2$
$2x - y = 1$
Unique solution
Lines intersect



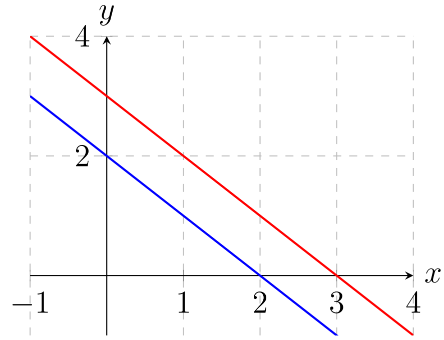

$A = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$
$\det \neq 0$

**Redundant**
**Singular**
$x + y = 2$
$2x + 2y = 4$
Infinite solutions
Lines overlap




Matrix:
$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$
$\det = 0$

**Contradictory**
**Singular**
$x + y = 2$
$x + y = 3$
No solutions
Lines are parallel




Matrix:
$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
$\det = 0$

## Determinants

The determinant helps determine if a matrix is singular (non-invertible). If $\det(A) = 0$, the matrix is singular.

Determinant of a 2x2 Matrix

$$\text{For } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det(A) = ad - bc$$

Example

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \quad \Rightarrow \quad \det(A) = (2)(4) - (3)(1) = 5$$

Diagonals for 3x3 Matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$\det(A) = aei + bfg + cdh - ceg - bdi - afh$$

# Linear Dependence and Independence in Systems of Equations

A system of equations can be described as linearly dependent or independent based on the relationship between its rows or columns.

## Linearly Independent Rows or Columns

Rows or columns of a matrix are linearly independent if no row (or column) can be expressed as a linear combination of the others.

- A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is linearly independent if:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0} \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_n = 0$$

- Example:

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ has linearly independent rows because neither row is a scalar multiple of the other.

## Linearly Dependent Rows or Columns

Rows or columns of a matrix are linearly dependent if at least one row (or column) can be expressed as a linear combination of the others.

- Example:

$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ has linearly dependent rows because the second row is twice the first.

## Operations Preserving Linearity

The following operations do not change the linear dependence or independence of rows or columns:

- **Addition or Subtraction:** Adding or subtracting one row (or column) to/from another.
- **Scalar Multiplication:** Multiplying a row (or column) by a nonzero scalar.
- **Row Switching:** Interchanging two rows or columns.

Visual Representation Consider two vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$. These vectors are linearly dependent because $\mathbf{v}_2 = 2\mathbf{v}_1$.Graphically, the vectors lie on the same line.

## Determining Dependence Using Determinants

- For a square matrix $A$, if $\det(A) = 0$, the rows and columns are linearly dependent.
- If $\det(A) \neq 0$, the rows and columns are linearly independent.

**Linearly Independent Rows**
Consider the matrix:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$\det(A) = (1)(4) - (2)(3) = -2 \neq 0$. Conclusion: The rows and columns are linearly independent.

**Linearly Dependent Rows**
Consider the matrix:

$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$\det(B) = (1)(4) - (2)(2) = 0$. Conclusion: The rows and columns are linearly dependent.

### Determining Dependence Using Row Reduction

By reducing a matrix to its row echelon form:

- If a row of all zeros exists, the rows are linearly dependent.
- If no such row exists, the rows are linearly independent.

### Row Echelon Form

Row echelon form is a simplified form of a matrix obtained through row operations, where:

- All nonzero rows are above any rows of all zeros.
- The leading coefficient (pivot) of a nonzero row is always to the right of the leading coefficient of the row above it.
- The pivot positions contain a 1, and all entries below the pivot are zeros.

**Original Matrix**

$$M = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 1 \\ -1 & -2 & 5 \end{bmatrix}$$

**Row Echelon Form**

$$M' = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}$$

Explanation:

- Subtract 2 times the first row from the second row.
- Add the first row to the third row.
- Scale rows to simplify.

## Solving Singular Systems of Linear Equations

**Complete System**

$x + y = 2$
$2x - y = 1$

Matrix representation:

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solve using elimination:
Multiply the first equation by 2:
$2x + 2y = 4$.
Subtract $2x + y = 1$:
$y = 1$, $x = 1$.

**Redundant System**

$x + y = 2$
$2x + 2y = 4$

Matrix representation:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Row reduction:
Subtract row 1 from row 2:
$[0, 0, 0]$.
Infinite solutions since the second equation is dependent on the first.

**Contradictory System**

$x + y = 2$
$x + y = 3$

Matrix representation:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Row reduction:
Subtract row 1 from row 2:
$[0, 0, -1]$.
Contradiction as $0 \neq -1$, no solutions exist.

## Elimination Method for Systems with More Variables

Consider:

$$\begin{aligned} x + 2y - z &= 4 \\ 2x - y + 3z &= 5 \\ -x + y + z &= 3 \end{aligned}$$

Using elimination:

- Add the first and third equations to eliminate $x$.
- Substitute back to solve for $y$ and $z$.

## Solving the System using Matrix Row Reduction

The above system can be written as:

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 2 & -1 & 3 & | & 5 \\ -1 & 1 & 1 & | & 3 \end{bmatrix}$$

Using row operations, transform this matrix into **row echelon form**.

1. Eliminate the first entry of Row 2 and Row 3: Subtract $2 \times$ Row 1 from Row 2 and add $1 \times$ Row 1 to Row 3:

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 0 & -5 & 5 & | & -3 \\ 0 & 3 & 0 & | & 7 \end{bmatrix}$$

2. Eliminate the second entry of Row 3: Add $\frac{3}{5} \times$ Row 2 to Row 3:

$$\begin{bmatrix} 1 & 2 & -1 & | & 4 \\ 0 & -5 & 5 & | & -3 \\ 0 & 0 & 3 & | & 5 \end{bmatrix}$$

At this stage, the matrix is in upper triangular form, and pivoting has ensured numerical stability. Further row reduction can be performed to reach reduced row echelon form if necessary.

## Reduced Row Echelon Form (RREF)

To continue, we transform the matrix into reduced row echelon form by ensuring that:

- Each pivot (leading entry in a row) is 1.
- All entries above and below each pivot are 0.

1. **Normalize the Pivot in Row 3:** Divide Row 3 by $\frac{44}{15}$:

$$\begin{bmatrix} 3 & -2 & 1 & | & 1 \\ 0 & \frac{5}{3} & -\frac{4}{3} & | & \frac{5}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

2. **Eliminate Entries Above the Pivot in Column 3:** Subtract $1 \times$ Row 3 from Row 1 and add $\frac{4}{3} \times$ Row 3 to Row 2:

$$\begin{bmatrix} 3 & -2 & 0 & | & 0 \\ 0 & \frac{5}{3} & 0 & | & \frac{1}{3} \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

3. **Normalize the Pivot in Row 2:** Divide Row 2 by $\frac{5}{3}$:

$$\begin{bmatrix} 3 & -2 & 0 & | & 0 \\ 0 & 1 & 0 & | & \frac{1}{5} \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

4. **Eliminate Entries Above the Pivot in Column 2:** Add $2 \times$ Row 2 to Row 1:

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{2}{3} \\ 0 & 1 & 0 & | & \frac{1}{5} \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

The matrix is now in reduced row echelon form.

### Using RREF to Solve Linear Equations

In reduced row echelon form, the augmented matrix corresponds directly to the solution of the system of linear equations:

$$x = \frac{2}{3}$$
$$y = \frac{1}{5}$$
$$z = 1$$

Each row of the RREF represents an equation where the variable corresponding to the pivot is isolated. For example:

- Row 1: $x = \frac{2}{3}$
- Row 2: $y = \frac{1}{5}$
- Row 3: $z = 1$

## Rank of a Matrix

The rank of a matrix is the maximum number of linearly independent rows or columns.

**Example 1:**

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

The rank of $A$ is 2 because one row is linearly dependent.

**Example 2:**

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The rank of $B$ is 3 because all rows and columns are linearly independent.

### Using Row Echelon Form to Solve

The rank of a matrix determines if the system has a solution:

- If rank $A =$ rank of augmented matrix, the system is consistent.
- If rank $A =$ rank of augmented matrix, but the number of variables exceeds the rank, there are infinitely many solutions.
- If rank $A <$ rank of augmented matrix, no solution exists.

# Gaussian Elimination

## Steps:

- Write the system as an augmented matrix.
- Use row operations to transform the matrix into row echelon form.
- Solve using back substitution.
- Stop if you encounter a row of 0s

## Augmented Matrix

An augmented matrix includes the coefficients and constants:

$$\begin{bmatrix} 2 & -1 & 3 & | & 5 \\ 1 & 1 & -1 & | & 2 \\ 3 & -2 & 1 & | & 1 \end{bmatrix}$$

## Pivoting

Pivoting ensures numerical stability by swapping rows. Specifically:

- Choose the row with the largest absolute value in the pivot column to reduce numerical errors.
- Swap this row with the current row to make the largest pivot value appear in the diagonal position.
- Continue this process for subsequent columns to maintain numerical stability.

Here is the step-by-step pivoting process for the given matrix:
1. **Initial Matrix:**

$$\begin{bmatrix} 2 & -1 & 3 & | & 5 \\ 1 & 1 & -1 & | & 2 \\ 3 & -2 & 1 & | & 1 \end{bmatrix}$$

2. **Pivot on Column 1** (Largest absolute value in column 1 is 3): Swap row 1 with row 3:

$$\begin{bmatrix} 3 & -2 & 1 & | & 1 \\ 1 & 1 & -1 & | & 2 \\ 2 & -1 & 3 & | & 5 \end{bmatrix}$$

3. Eliminate Entries Below the Pivot in Column 1: Subtract $\frac{1}{3} \times$ Row 1 from Row 2 and $\frac{2}{3} \times$ Row 1 from Row 3:

$$\begin{bmatrix} 3 & -2 & 1 & | & 1 \\ 0 & \frac{5}{3} & -\frac{4}{3} & | & \frac{5}{3} \\ 0 & \frac{1}{3} & \frac{8}{3} & | & \frac{13}{3} \end{bmatrix}$$

4. **Pivot on Column 2** (Largest absolute value in column 2 is $\frac{5}{3}$, ignoring the rows above the pivot position, i.e -2)): No row swap needed.
5. Eliminate Entries Below the Pivot in Column 2: Subtract $\frac{1}{5} \times$ Row 2 from Row 3:

$$\begin{bmatrix} 3 & -2 & 1 & | & 1 \\ 0 & \frac{5}{3} & -\frac{4}{3} & | & \frac{5}{3} \\ 0 & 0 & \frac{44}{15} & | & \frac{44}{15} \end{bmatrix}$$

At this stage, the matrix is in upper triangular form, and pivoting has ensured numerical stability. Further row reduction can be performed to reach reduced row echelon form if necessary.

## Back Substitution

After reducing the augmented matrix to an upper triangular form, back substitution is used to solve for the variables starting from the last row upward.

For the triangular form obtained in the pivoting process:

$$\begin{bmatrix} 3 & -2 & 1 & | & 1 \\ 0 & \frac{5}{3} & -\frac{4}{3} & | & \frac{5}{3} \\ 0 & 0 & \frac{44}{15} & | & \frac{44}{15} \end{bmatrix}$$

The steps for back substitution are: 1. Solve the last equation for $z$:

$$\frac{44}{15}z = \frac{44}{15} \quad \Rightarrow \quad z = 1.$$

2. Substitute $z = 1$ into the second equation:

$$\frac{5}{3}y - \frac{4}{3}(1) = \frac{5}{3} \quad \Rightarrow \quad y = 1.$$

3. Substitute $y = 1$ and $z = 1$ into the first equation:

$$3x - 2(1) + 1 = 1 \quad \Rightarrow \quad 3x = 2 \quad \Rightarrow \quad x = \frac{2}{3}.$$

Thus, the solution is:

$$x = \frac{2}{3}, \quad y = 1, \quad z = 1.$$

## Rows of Zeros

A row of zeros in a matrix can indicate key properties about the solutions of a system of linear equations, depending on the associated augmented matrix. They appear in echelon, reduced echelon, and even in unreduced matrices.

| **Scenario** | **Example Matrix** | **Explanation** |
|---|---|---|
| **Infinitely Many Solutions:** A row of zeros with a zero in the constant column. | $\begin{bmatrix} 1 & 2 & -1 & \| & 5 \\ 0 & 0 & -7 & \| & 9 \\ 0 & 0 & 0 & \| & 0 \end{bmatrix}$ | The last row implies $0 = 0$, which is consistent. The system has infinitely many solutions. |
| **No Solutions:** A row of zeros with a non-zero constant. | $\begin{bmatrix} 1 & 2 & -1 & \| & 5 \\ 0 & 0 & -7 & \| & 9 \\ 0 & 0 & 0 & \| & 4 \end{bmatrix}$ | The last row implies $0 = 4$, which is a contradiction. The system has no solutions. |

## The Identity Matrix

The identity matrix, denoted as $I$, is a square matrix with ones on the main diagonal and zeros elsewhere. It acts as the multiplicative identity in matrix operations, meaning:

$$AI = IA = A,$$

where $A$ is any compatible matrix. In back substitution, the identity matrix represents the fully reduced form where all variables have been isolated.

**Examples:** For a 2x2 matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For a 3x3 matrix:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$