Laplace's Prior: the prior of indifference / uniformity. In the Poisson model θ in (0, infinity). We need a distribution that is uniform on that set. A distribution would look like:

$$P(\theta) = c > 0, \int_0^\infty P(\theta)\,d\theta = \int_0^\infty c\,d\theta = c\int_0^\infty 1\,d\theta = \infty \Rightarrow P(\theta) = c \text{ d.n.e.}$$

There cannot be a proper Laplace prior. But there is an improper Laplace prior:

$P(\theta) \propto 1 \Rightarrow$ Laplace's idea

$$P(\theta|X) \propto P(X|\theta)P(\theta) = e^{-n\theta}\theta^{\sum x_i}P(\theta) \propto e^{-n\theta}\theta^{\sum x_i + 1 - 1}$$

$$\propto \text{Gamma}(\underbrace{1+\sum x_i}_{\alpha}, \underbrace{0+n}_{\beta})$$

$$\Rightarrow P(\theta) = \text{Gamma}(1, 0),$$ an improper prior

$$\Downarrow$$

$x_0 = 1, \; n_0 = 0$ nonsense!

Is the posterior proper? Yes. ALWAYS. Since sum x_i >= 0,
its first parameter is always >=1 > 0 and since n >= 1, its second parameter is always >=1 > 0.

Haldane's prior of complete ignorance. Setting all pseudodata to be zero i.e. x_0 = 0, n_0 = 0 => $P(\theta) = \text{Gamma}(0,0)$ improper!

$$\Rightarrow P(\theta|X) = \text{Gamma}\left(\sum x_i, n\right) \Rightarrow \hat{\theta}_{MMSE} = \frac{\sum x_i}{n} = \bar{x} = \hat{\theta}_{MLE}$$

Is this posterior proper? Only if sum x_i > 0.

Jeffrey's prior. $P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \theta^{\overbrace{-\frac{1}{2}+1-1}^{\frac{1}{2}-1}} \propto \text{Gamma}\left(\frac{1}{2}, 0\right)$ improper

$$\Downarrow$$

$$P(\theta|X) = \text{Gamma}\left(\tfrac{1}{2} + \sum x_i, 0+n\right)$$

$$\dots \; \ell'(\theta) = -n + \frac{\sum x_i}{\theta} \Rightarrow \ell''(\theta) = -\frac{\sum x_i}{\theta^2}$$

$$I(\theta) = E_X[\ell''(\theta)] = \frac{E[\sum x_i]}{\theta^2} = \frac{nE[X_i]}{\theta^2} = \frac{n\theta}{\theta^2} = \frac{n}{\theta}$$

Is Jeffrey's prior proper? YES. ALWAYS!

$X \sim \text{Poisson}(\theta)$

if $x = 0$, term $= 0$

$$E[X] = \sum_{x \in Supp[X]} x\,p(x) = \sum_{x=0}^{\infty} x\frac{e^{-\theta}\theta^x}{x!} = e^{-\theta}\sum_{x=1}^{\infty}\frac{x\theta^x}{x!} = e^{-\theta}\sum_{x=1}^{\infty}\frac{\theta^x}{(x-1)!}$$

$y = x-1 \Leftrightarrow x = y+1$

$$= e^{-\theta}\sum_{y=0}^{\infty}\frac{\theta^{y+1}}{y!} = \theta e^{-\theta}\underbrace{\sum_{y=0}^{\infty}\frac{\theta^y}{y!}}_{e^{\theta}} = \theta\cancel{e^{-\theta}e^{\theta}} = \theta$$

Posterior predictive distribution. You see n observations and you want to know the distribution of n* future observation(s). For our case here, we let n* = 1.

$$\underbrace{X_1 \; X_2 \; \dots \; X_n}_{n} \overset{iid}{\sim} \text{Poisson}(\theta) \quad \Big| \quad X_* | X \sim ?$$

$n_* = 1$

← past → ← future →

$$P(X_*|X) = \int_{\Theta} \underbrace{P(X_*|\theta)}_{= \text{Poisson}(\theta)} \underbrace{P(\theta|X)}_{= \text{Gamma}(\alpha+\sum x_i, \beta+n)} d\theta$$

$$= \int_0^{\infty} \frac{e^{-\theta}\theta^{x_*}}{x_*!} \frac{(\beta+n)^{\alpha+\sum x_i}}{\Gamma(\alpha+\sum x_i)} \theta^{\alpha+\sum x_i - 1} e^{-(\beta+n)\theta}\,d\theta$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)} \int_0^{\infty} e^{-\theta}\theta^{x_*}\theta^{\alpha+\sum x_i - 1} e^{-(\beta+n)\theta}\,d\theta$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)} \int_0^{\infty} \theta^{x_*+\alpha+\sum x_i - 1} e^{-(\beta+n+1)\theta}\,d\theta$$

let $t = (\beta+n+1)\theta \Rightarrow \theta = \frac{t}{\beta+n+1} \Rightarrow \frac{d\theta}{dt} = \frac{1}{\beta+n+1} \Rightarrow d\theta = \frac{1}{\beta+n+1}dt$

if $\theta = 0 \Rightarrow t = 0$, if $\theta = \infty \Rightarrow t = \infty$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)} \int_0^{\infty} \frac{t^{x_*+\alpha+\sum x_i - 1}}{(\beta+n+1)^{x_*+\alpha+\sum x_i \cancel{-1}}} e^{-t} \frac{1}{\cancel{(\beta+n+1)^1}}\,dt$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{x_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}} \underbrace{\int_0^{\infty} t^{(x_*+\alpha+\sum x_i)-1} e^{-t}\,dt}_{\text{gamma integral}}$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}\,\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}}$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{(\beta+n+1)^{\alpha+\sum x_i}} \frac{1}{(\beta+n+1)^{x_*}} \frac{\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)}$$

$$= \left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum x_i} \left(\frac{1}{\beta+n+1}\right)^{x_*} \frac{\Gamma(x_*+\alpha+\sum x_i)}{x_*!\,\Gamma(\alpha+\sum x_i)}$$

let $p := \frac{\beta+n}{\beta+n+1} \in (0,1)$, $1-p = \frac{1}{\beta+n+1} \in (0,1)$, $r := \sum x_i + \alpha > 0$

$$= p^r(1-p)^{x_*} \frac{\Gamma(x_*+r)}{x_*!\,\Gamma(r)} = \text{ExtNegBin}(r, p)$$

extended negative binomial random variable model

if $\alpha \in \{0, 1, 2, \dots\}$

$$= \binom{x_*+r-1}{r} p^r(1-p)^{x_*} = \text{NegBin}(r, p)$$

From 368.... the negative binomial is the sum of r iid geometric random variables. Since the expectation of the geometric rv is (1-p)/p, the expectation of the negative binomial by linearity is

$$P(X_*|X) = \text{ExtNegBin}(r, p) \Rightarrow E[X_*|X] = r\frac{1-p}{p}$$

$$= \left(\sum x_i + \alpha\right)\frac{\frac{1}{\cancel{n+\beta+1}}}{\frac{n+\beta}{\cancel{n+\beta+1}}}$$

$$= \frac{\sum x_i + \alpha}{n+\beta} = \hat{\theta}_{MMSE}.$$