If form of posterior and prior are the same that is a conjugate prior.

<u>Laplace's Prior</u>: prior of indifference/uniform. In the Poisson model θ is in the set $(0,\infty)$, so we need a distribution which is uniform on that set. A distribution would look like $P(\theta) = c > 0$, $\int_0^\infty P(\theta)\,d\theta = \int_0^\infty c\,d\theta = c\int_0^\infty d\theta = \infty$ does not exist. There cannot be a proper laplace prior, but there is an <u>improper</u> laplace prior:

diverges

$P(\theta)\propto 1 \Rightarrow$ laplaces Idea

$$P(\theta|x) \propto P(x|\theta)P(\theta) = e^{-n\theta}\theta^{\sum x_i}\,p(\theta) = e^{-n\theta}\theta^{\sum x_i + 1 - 1} \propto \text{Gamma}($$

$$\propto \text{Gamma}(\underbrace{1+\sum x_i}_{\alpha = 1},\ \underbrace{0+n}_{\beta=0}) \Rightarrow P(\theta) = \text{Gamma}(1,0), \text{ an improper prior.}$$

$\Downarrow$

1 success in no trials, nonsense!

Is the posterior proper? α, β must both be > 0.

Yes always since $\sum x_i >= 0$ its first parameter is always $\geq 1 > 0$ and since $n \geq 1$ its second parameter is always $\geq 1 > 0$.

Haldane's Prior of complete ignorance. Setting all pseudodata to be zero i.e. $x_0 = 0$, $n_0 = 0 \Rightarrow P(\theta) = \text{Gamma}(0,0)$ obviously improper. $\Rightarrow$

$P(\theta|x) = \text{Gamma}(0+\sum x_i, 0+n) = \text{Gamma}(\sum x_i, n)$. Is the posterior proper? <u>Only if</u> $\sum x_i > 0$. If you don't get at least one success it blows up.

$$\hat{\theta}_{MMSE} = \frac{\sum x_i}{n} = \bar{x} = \hat{\theta}_{MLE}$$

. Now on to Jeffrey's.

Jeffrey's prior, we need fisher Information. $P_J(\theta) \propto \sqrt{I(\theta)}$

From previous notes: $\ell'(\theta) = -n + \frac{\sum x_i}{\theta} \Rightarrow \ell''(\theta) = -\frac{\sum x_i}{\theta^2}$, then

$$I(\theta) = E_X[-\ell''(\theta)] = \frac{E[\sum x_i]}{\theta^2} \overset{iid}{=} \frac{n E[X_i]}{\theta^2}.$$ Need expectation of Poisson

$$X \sim \text{Poisson}(\theta).\ E[X] = \sum_{x \in \text{Supp}[X]} x\, p(x) = \underbrace{\sum_{x=0}^{\infty} x \frac{e^{-\theta}\theta^x}{x!} = e^{-\theta}\sum_{x=1}^{\infty}}_{\text{first term } x=0 \text{ is } 0 \text{ in the sum}} \frac{x\theta^x}{x!} =$$

(cancel)

re-index trick $y = x-1 \Leftrightarrow x = y+1$

$$= e^{-\theta}\sum_{x=1}^{\infty}\frac{\theta^x}{(x-1)!} = e^{-\theta}\sum_{y=0}^{\infty}\frac{\theta^{y+1}}{y!} = \theta e^{-\theta}\underbrace{\sum_{y=0}^{\infty}\frac{\theta^y}{y!}}_{\text{taylor series for } e^\theta} = \theta e^{-\theta}e^{\theta} = \theta$$

$$\boxed{E[X] = \theta \Rightarrow I(\theta) = \frac{n\theta}{\theta^2} = \frac{n}{\theta}}$$

Free variable change at will
constant - you cannot change

$$P_J(\theta) \propto \sqrt{I(\theta)} = \sqrt{\frac{n}{\theta}} \propto \theta^{-\frac{1}{2}+1-1} \propto \text{Gamma}\left(\tfrac{1}{2}, 0\right) \text{ improper}$$

$\Downarrow$ Posterior

$$p(\theta|X) = \text{Gamma}\left(\sum x_i + \tfrac{1}{2},\ 0+n\right)$$

Is Jeffrey's prior proper, yes, always same reason for laplace
n always > 0.

Posterior Predictive distribution. You see n observations and want to know the distribution of n^* future observations.

$X_1\ X_2\ X_3 \ldots X_n$ ~IID Poisson | ~~X_{n+1}~~ want $X_* | X \sim ?$

$n^* = 1$

past | future

$$P(X_* \mid X) = \int_{\Theta} \underset{\substack{\uparrow \\ \text{Poisson}}}{P(X_* \mid \theta)} \underset{\substack{\uparrow \\ \text{Gamma}}}{P(\theta \mid X)}\, d\theta = \int_0^\infty \frac{e^{-\theta}\theta^{x_*}}{X_*!} \frac{(\beta+n)^{\alpha+\sum x_i}}{\Gamma(\alpha+\sum x_i)} \theta^{\alpha+\sum x_i - 1} e^{-(\beta+n)\theta}\, d\theta$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{X_*!\,\Gamma(\alpha+\sum x_i)} \int_0^\infty e^{-\theta}\theta^{x_*}\theta^{\alpha+\sum x_i-1} e^{-(\beta+n)\theta}\, d\theta$$

$$= \int_0^\infty \theta^{x_*+\alpha+\sum x_i - 1} e^{-(\beta+n+1)\theta}\, d\theta$$

now we do a substitution. Let $t = (\beta+n+1)\theta \Rightarrow \theta = \frac{t}{\beta+n+1} \Rightarrow$

$\frac{d\theta}{dt} = \frac{1}{\beta+n+1} \Rightarrow d\theta = \frac{dt}{\beta+n+1}$. If $\theta = 0 \Rightarrow t = 0$ if $\theta = \infty \Rightarrow t = \infty$

So nothing chages in Bonds of integral.

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{X_*!\,\Gamma(\alpha+\sum x_i)} \int_0^\infty \frac{t^{x_*+\alpha+\sum x_i+1}}{(\beta+n+1)^{x_*+\alpha+\sum x_i - 1}} \cdot e^{-t} \frac{1}{(\beta+n+1)}\, dt$$

cancels

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{X_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}} \underbrace{\int_0^\infty t^{(x_*+\alpha+\sum x_i)-1} e^{-t}\, dt}_{\text{gamma integral}}$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}\,\Gamma(x_*+\alpha+\sum x_i)}{X_*!\,\Gamma(\alpha+\sum x_i)(\beta+n+1)^{x_*+\alpha+\sum x_i}}$$

$$= \frac{(\beta+n)^{\alpha+\sum x_i}}{(\beta+n+1)^{\alpha+\sum x_i}} \cdot \frac{1}{(\beta+n+1)^{x_*}} \cdot \frac{\Gamma(x_* + \alpha + \sum x_i)}{x_*!\,\Gamma(\alpha + \sum x_i)}$$

$$= \left(\frac{\beta+n}{\beta+n+1}\right)^{\alpha+\sum x_i} \left(\frac{1}{\beta+n+1}\right)^{x_*} \frac{\Gamma(x_* + \alpha + \sum x_i)}{x_*!\,\Gamma(\alpha + \sum x_i)}$$

re-write using this

Let $p = \frac{\beta+n}{\beta+n+1} \in (0,1)$

Because $\beta > 0$, $n > 0$

$1-p = \frac{1}{\beta+n+1} \in (0,1)$

Let $r = \sum x_i + \alpha > 0$

$$= p^{r} (1-p)^{x_*} \frac{\Gamma(x_* + r)}{x_*!\,\Gamma(r)} =$$ Extended Negative Binomial

$=$ ExtNegBin(r, p) an r.v.

If $\alpha \in \{0,1,2,\dots\}$ then

$$= \binom{x_* + r - 1}{r} p^{r} (1-p)^{x_*} = \text{NegBin}(r,p)$$

From 368 ~~the~~ the neg binomial is the sum of iid geometric random variables. Since the expectation of the geometric rv is $\frac{(1-p)}{p}$, the expectation of the neg bin by linearity is

$$P(X_* \mid X) = \text{ExtNegBin}(r,p) \Rightarrow E[X_* \mid X] = r\left(\frac{1-p}{p}\right)$$

$$= \frac{(\sum x_i + \alpha)}{n+\beta} = \hat{\theta}_{MMSE}$$