$$P(X_\alpha|X) \propto \int_0^\infty \int_{\mathbb{R}} K(X_\alpha|\theta,\sigma^2)\, K(\theta,\sigma^2|X)\, d\theta\, d\sigma^2$$

Assume Jeffrey's Prior $P(\theta,\sigma^2) \propto \frac{1}{\sigma^2}$

$$= \int_0^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_\alpha-\theta)^2} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} e^{-\frac{n}{2\sigma^2}(\theta-\bar{x})^2}\, d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{1}{2}} (\sigma^2)^{-\frac{n}{2}-1} e^{-\frac{(n-1)s^2/2}{\sigma^2}} \int_{\mathbb{R}} e^{-\frac{1}{2\sigma^2}\left(\underbrace{(X_\alpha-\theta)^2 + n(\theta-\bar{x})^2}_{X_\alpha^2 - 2X_\alpha\theta + (n+1)\theta^2 - 2n\theta\bar{x} + n\bar{x}^2}\right)} d\theta\, d\sigma^2$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{(n-1)s^2/2 + X_\alpha^2/2 + n\bar{x}^2/2}{\sigma^2}} \int_{\mathbb{R}} \overbrace{e^{\frac{X^\alpha + n\bar{x}}{\sigma^2}\theta - \frac{n+1}{2\sigma^2}\theta^2}}^{e^{a\theta - b\theta^2}} d\theta\, d\sigma^2$$

$$\int_{\mathbb{R}} e^{a\theta - b\theta^2} d\theta = \sqrt{\frac{\pi}{b}}\, e^{a^2/4b}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} \sqrt{\frac{\pi}{\frac{n+1}{2\sigma^2}}}\, e^{\left(\frac{X^\alpha + n\bar{x}}{\sigma^2}\right)^2 / \cancel{4}^{\,2} \frac{n+1}{\cancel{2}\sigma^2}}\, d\sigma^2$$

$$\rightarrow \frac{(X^\alpha + n\bar{x})^2}{2\sigma^2(n+1)}$$

$$= \int_0^\infty (\sigma^2)^{-\frac{n+1}{2}-1} e^{-\frac{\overbrace{(n-1)s^2/2 + X_\alpha^2/2 + n\bar{x}^2/2 - (X^\alpha+n\bar{x})^2/(2(n+1))}^{\beta}}{\sigma^2}} (\sigma^2)^{\frac{1}{2}} \sqrt{\frac{2\pi}{n+1}}\, d\sigma^2$$

$$\propto \int_0^\infty (\sigma^2)^{-\overbrace{\frac{n}{2}}^{\alpha}-1} e^{-\frac{\beta}{\sigma^2}} d\sigma^2 = \Gamma(\alpha)\beta^{-\alpha} = \Gamma\left(\frac{n}{2}\right)\beta^{-\alpha} \propto \beta^{-\alpha}$$

$$= \left(\frac{(n-1)s^2}{2} + \frac{X_\alpha^2}{2} + \frac{n\bar{x}^2}{2} - \frac{\overbrace{(X_\alpha + n\bar{x})^2}^{X_\alpha^2 + 2nX_\alpha\bar{x} + n^2\bar{x}^2}}{2n+2}\right)^{-\frac{n}{2}} = \left(aX_\alpha^2 + bX_\alpha + c\right)^{-\frac{n}{2}}$$

$$a = \frac{1}{2} - \frac{1}{2n+2} = \frac{1}{2}\left(1 - \frac{1}{n+1}\right) = \frac{1}{2}\frac{n}{n+1}$$

$$b = -\frac{2n\bar{x}}{2n+2} = -\frac{n\bar{x}}{n+1}, \quad c = \frac{(n-1)s^2}{2} + \frac{n\bar{x}^2}{2} - \frac{n^2\bar{x}^2}{2n+2} = \frac{1}{2}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)$$

$$= \left(\frac{1}{a}\right)^{n/2}\left(\frac{1}{a}\right)^{-n/2}\left(aX_\alpha^2 + bX_\alpha + c\right)^{-\frac{n}{2}} = \left(\frac{1}{a}\right)^{-n/2}\left(X_\alpha^2 + \frac{b}{a}X_\alpha + \frac{c}{a}\right)^{-n/2}$$

$$\propto \left(X_\alpha^2 + \frac{b}{a}X_\alpha + \frac{c}{a}\right)^{-n/2} \propto \left(\left(X^\alpha + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}\right)^{-n/2} \left(\frac{1}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-n/2}$$

$$\propto \left(1 + \frac{\left(X^\alpha + \frac{b}{2a}\right)^2}{\frac{c}{a} - \frac{b^2}{4a^2}}\right)^{-\frac{\overbrace{(n-1)}^{v}+1}{2}} = \left(1 + \frac{1}{\underbrace{n-1}_{v}} \frac{\left(X^\alpha - \overbrace{\left(-\frac{b}{2a}\right)}^{\mu}\right)^2}{\underbrace{\left(\frac{c}{a} - \frac{b^2}{4a^2}\right)/_{n-1}}_{s_0^2}}\right)^{-\frac{(n-1)+1}{2}}$$

$$\propto T_v(\mu, s_0^2) = T_{n-1}\left(\bar{x}, \frac{n+1}{n}s^2\right) \overset{n \text{ large}}{\approx} N(\bar{x}, s^2)$$

$$-\frac{b}{2a} = \frac{\frac{n\bar{x}}{\cancel{n+1}}}{\frac{n}{\cancel{n+1}}} = \bar{x}$$

$$\frac{c}{a} = \frac{\cancel{\frac{1}{2}}\left((n-1)s^2 + n\bar{x}^2 - \frac{n^2\bar{x}^2}{n+1}\right)}{\cancel{\frac{1}{2}}\frac{n}{n+1}} = \frac{(n-1)(n+1)}{n}s^2 + \overbrace{(n+1)\bar{x}^2}^{\cancel{n}\bar{x}^2 + \bar{x}^2} - \cancel{n\bar{x}^2}$$

$$-\frac{b^2}{4a^2} = -\left(\frac{b}{2a}\right)^2 = -\left(-\frac{b}{2a}\right)^2 = -\bar{x}^2 \quad \Rightarrow \frac{c}{a} - \frac{b^2}{4a^2} = \frac{(n-1)(n+1)}{n}s^2$$

$$s_0^2 = \frac{\frac{(\cancel{n-1})(n+1)s^2}{n}}{\cancel{n-1}} = \frac{n+1}{n}s^2$$

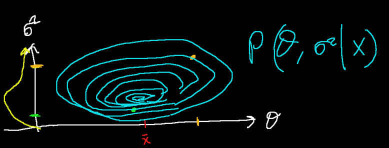
Under Jeffrey's prior...

$$\underbrace{P(\theta,\sigma^2|X)}_{} = \underbrace{P(\theta|X,\sigma^2)}_{}\,\underbrace{P(\sigma^2|X)}_{}$$

By Def of cond. probability

$$\text{NormInvGamma}(\cdot,\cdot,\cdot,\cdot) = N\left(\bar{x}, \frac{\sigma^2}{n}\right) \cdot \text{InvGamma}\left(\frac{n-1}{2}, \frac{(n-1)s^2}{2}\right)$$

You can think of a normal inverse gamma as first sampling from an InverseGamma((n-1)/2, (n-1)s^2/2) to get a sigsq value and then you use that value of sigsq to draw a theta from N(xbar, sigsq/n) and return the two-dimensional point [theta sigsq].



This can also be done the other way...

$$P(\theta,\sigma^2|x) = \underbrace{P(\sigma^2|X,\theta)}_{\text{InvGamma}\left(\frac{n}{2}, \frac{n\hat{\sigma}^2_{MLE}}{2}\right)}\,\underbrace{P(\theta|x)}_{T_{n-1}\left(\bar{x}, \frac{s^2}{n}\right)}$$

If we decompose the first way, we draw theta from N(xbar, sigsq/n) and thus sigsq must be known. What if we break this by instead of using the Jeffrey's prior, use

$$P(\theta) = N(\mu_0, \tau^2) \quad \text{and} \quad P(\sigma^2) = \text{InvGamma}\left(\frac{n_0}{2}, \frac{n_0\sigma_0^2}{2}\right)$$

These were the two priors we began with when we started investigating the normal likelihood model. However, it's important to note we are not allowing $\tau^2 = \sigma^2/n_0$

What happens? The two priors are disconnected completely.

$$P(\theta,\sigma^2) = P(\theta)P(\sigma^2) \text{ not } P(\theta|\sigma^2)P(\sigma^2)$$

Let's derive the posterior under this two-dimensional prior.

$$P(\theta,\sigma^2|x) \propto P(X|\theta,\sigma^2)P(\theta,\sigma^2) = P(X|\theta,\sigma^2)P(\theta)P(\sigma^2)$$

$$\propto K(X|\theta,\sigma^2)\,K(\theta)\,K(\sigma^2)$$

$$n\bar{x}^2 - 2n\bar{x}\theta + n\theta^2 \qquad \theta^2 - 2\theta\mu_0 + \mu_0^2$$

$$= (\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{x}-\theta)^2\right)} e^{-\frac{1}{2\tau^2}(\theta-\mu_0)^2} (\sigma^2)^{-\frac{n_0}{2}-1} e^{-\frac{n_0\sigma_0^2/2}{\sigma^2}}$$

$$= (\sigma^2)^{-\frac{n}{2}-\frac{n_0}{2}-1} e^{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n_0\sigma_0^2 + n\bar{x}^2\right)} e^{\left(\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}\right)\theta - \left(\frac{n}{2\sigma^2} + \frac{1}{2\tau^2}\right)\theta^2}$$