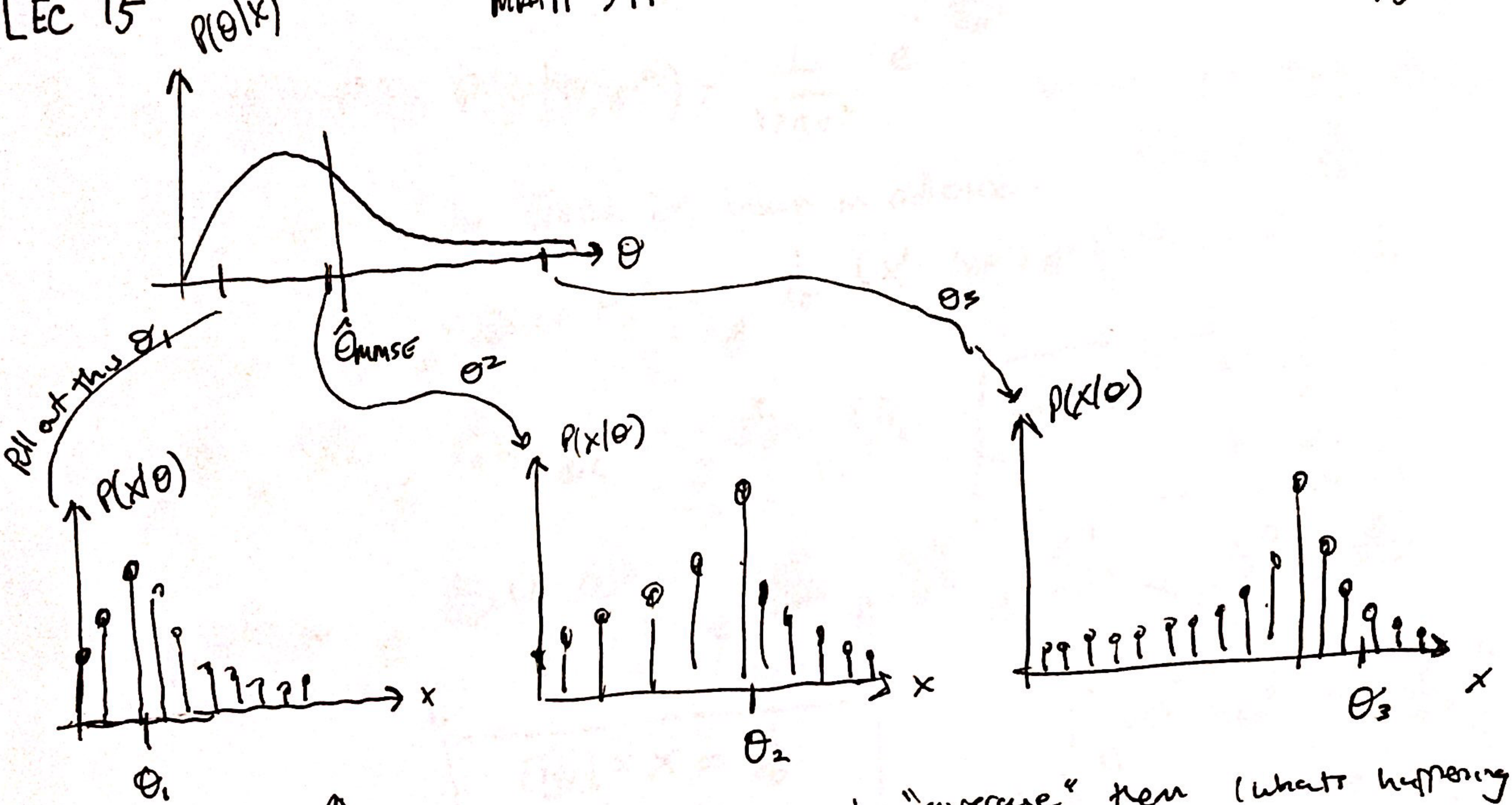
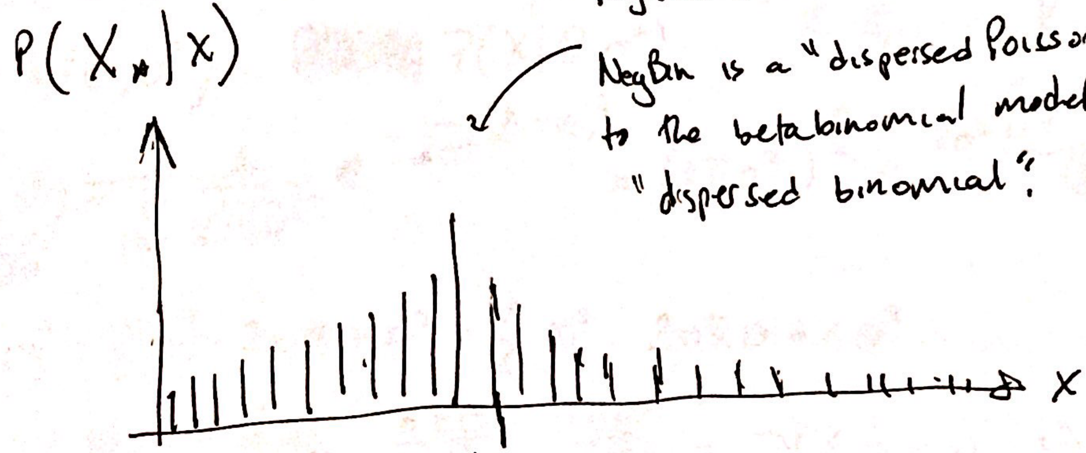


Keep doing this ↑ over and over again and "average" them (whats happening in the integral) to get:

Negative Binomial.

NegBin is a "dispersed Poisson" analogous to the betabinomial model being a "dispersed binomial".



$E[X_* | x] = \hat{\theta}_{MMSE}$

$$VAR[X_* | x] = r\frac{1-p}{p^2} = \hat{\theta}_{MMSE} \cdot \frac{1}{p} = \underbrace{\frac{\beta+n+1}{\beta+n}}_{\in[1,2]} \hat{\theta}_{MMSE}$$

Thus the posterior predictive distribution has up to 2x the poisson (i.e. more spread out, or less sure where the realization will be).

$\mathcal{F}$: one $N(\theta,\sigma^2) = P(X|\theta,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\theta)^2}$

can represent as theta vec

$\vec{\theta} = \begin{bmatrix}\theta \\ \sigma^2\end{bmatrix}$. Let σ^2 be fixed i.e. known in advance.

$P(X|\theta,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{1}{2\sigma^2}(x^2-2x\theta+\theta^2)}$

$= e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \propto \underline{e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}}} = \boxed{e^{ax-bx^2} \quad a=\frac{\theta}{\sigma^2} \quad b=\frac{1}{2\sigma^2}}$

$\boxed{E[X] = \theta = \frac{a}{2b} \quad VAR[X] = \sigma^2 = \frac{1}{2b}}$

$P(\theta|x,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(x-\theta)^2} = e^{-\frac{x^2}{2\sigma^2}} e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \propto \underline{e^{\frac{x\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}}$

now mean

$= \boxed{e^{a\theta-b\theta^2} \quad a=\frac{x}{\sigma^2} \quad b=\frac{1}{2\sigma^2}}$

$\boxed{E[\theta] = x = \frac{a}{2b} \quad VAR[\theta] = \sigma^2 = \frac{1}{2b}}$

$\mathcal{F}$: $\overset{iid}{\sim} N(\theta,\sigma^2)$. ~~$P(X|\theta)$~~ $P(X|\theta,\sigma^2)$ $X_1, X_2, \ldots, X_n$

$P(X|\underbrace{\theta,\sigma^2}_{**}) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_i-\theta)^2} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\underbrace{\sum(x_i-\theta)^2}_{*}} =$

(*) $\sum(x_i-\theta)^2 = \sum x_i^2 - 2x_i\theta + \theta^2 = \sum x_i^2 - 2n\bar{x}\theta + n\theta^2$

$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} = \mathcal{L}(\theta; x, \sigma^2)$ (AKA Joint Density)

$\propto e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}}$ ← kernel of likelihood (switch given/variables) (**) eliminate θ, σ^2

$P(\theta|x,\sigma^2) \propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} = e^{a\theta-b\theta^2} \quad a=\frac{n\bar{x}}{\sigma^2} \quad b=\frac{n}{2\sigma^2}$

$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N\left(\bar{x}, \frac{\sigma^2}{n}\right)$

Do Bayesian inference on this!

$X : X_1, \dots, X_n \overset{iid}{\sim} N(\theta, \sigma^2)$ with σ^2 known. (looking for inference on the mean)

condition on it everywhere

So we need posterior. Lets find it.

$$\cancel{\quad} P(\theta \mid X, \sigma^2) \propto \underbrace{P(X \mid \theta, \sigma^2)}_{\text{(we have this)}} P(\theta \mid \sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum x_i^2}{2\sigma^2}} e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} P(\theta \mid \sigma^2)$$

$$\propto e^{\frac{n\bar{x}\theta}{\sigma^2}} e^{\frac{-n\theta^2}{2\sigma^2}} P(\theta \mid \sigma^2) \Rightarrow N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right)$$

$$\boxed{\text{Prior} = P(\theta \mid \sigma^2) = N(\mu_0, \tau^2)}$$

Traditionally $\alpha = \frac{\mu_0}{\tau^2}$ $\beta = \frac{1}{2\tau^2}$

$$\propto e^{a\theta - b\theta^2} e^{\alpha\theta - \beta\theta^2}$$

$$= e^{(a+\alpha)\theta - (b+\beta)\theta^2} \propto N\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right)$$

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \alpha}{\frac{n}{\sigma^2} + 2\beta}, \frac{1}{\frac{n}{\sigma^2} + 2\beta}\right)$$

Normal-Normal conjugate model where sigsq is assumed fixed

$$= N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$$

$$P(\theta \mid \sigma^2) = \text{Prior} = N(\mu_0, \tau^2) \implies P(\theta \mid X, \sigma^2) = N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right)$$

Point estimation:

$$\hat{\theta}_{MMSE} = E[\theta \mid X, \sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MMAE} = MED[\theta \mid X, \sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAP} = Mode[\theta \mid X, \sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$\hat{\theta}_{MMSE} = \hat{\theta}_{MMAE} = \hat{\theta}_{MAP}$$

Credible Regions:

$$CR_{\theta, 1-\alpha_0} = \left[qnorm\left(\frac{\alpha_0}{2}, \underbrace{\frac{\frac{n\bar{X}}{\sigma^2} + \frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\mathrm{I}}, \underbrace{\frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}}_{\mathrm{II}}\right), qnorm\left(1-\frac{\alpha_0}{2}, \mathrm{I}, \mathrm{II}\right) \right]$$

Hypothesis TEST:

$$H_a : \theta < \theta_0 \Rightarrow H_0 : \theta \geq \theta_0$$

$$P_{val} = P(H_0 \mid X, \sigma^2) = \int_{\theta_0}^{\infty} f(\theta \mid X, \sigma^2)\, d\theta = 1 - pnorm(\theta_0, \mathrm{I}, \mathrm{II})$$

MLE: $$\mathcal{L}(\theta; x, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}}$$

$$\ell(\theta; x, \sigma^2) = \ln(\quad) - \frac{\sum x_i^2}{2\sigma^2} + \frac{n\bar{x}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2} \longrightarrow$$

$$\ell'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \leftarrow \text{set} = 0 \text{ to find MLE}$$

$$n\bar{x} = n\theta \Rightarrow \hat{\theta}_{MLE} = \bar{x}$$