Bayesian two-sided testing

$H_a : \theta \notin [\theta_0 \pm \delta] \longrightarrow H_0 : \theta \in [\theta_0 \pm \delta]$ δ = "margin of equivalence" You choose this.

EXAMPLE:

$n=100$ $x=43$ Beta(1,1) $\Rightarrow P(\theta|x) = \text{Beta}(44, 58)$

$\underset{\text{rejection region}}{\alpha_0 = 5\%}$ $\delta = 1\%$ $\Rightarrow H_0 : \theta \in [.49, .51]$

$\Rightarrow \text{pval} = P(H_0|x) = \int_{.49}^{.51} P(\theta|x)\, d(\theta) = \int_{.49}^{.51} \text{Beta}(44, 58)\, d\theta = \text{pbeta}(.51, 44, 58) - \text{pbeta}(.49, 44, 58)$

$\|$

$F_{\theta|x}(\theta_0 + \delta) - F_{\theta|x}(\theta_0 - \delta)$ $= 0.06 \Rightarrow$ Retain H_0

(If it was less than 5% you would reject)

$\boxed{\text{not less than } \alpha_0 = 5\%}$

There is insufficient evidence to prove this coin is unfair.

$X : \text{Bin}(n, \theta)$ with n fixed, $P(\theta) = \text{Beta}(\alpha, \beta) \Rightarrow P(\theta|x) = \text{Beta}(\alpha + x, \beta + n - x)$

Laplace Prior $P(\theta) = \text{Beta}(1,1) \Rightarrow n_0 = 2$ psuedo trials $x_0 = 1$ psuedo successes

Laplace's uniform prior is "flat" in an effort to be objective, i.e. let the data speak for itself, not to be subjective, not allow personal biases to be part of your inference or conclusion.

Can we be more objective? How about $n_0 = 0$, $\alpha = \beta = 0$ $\underline{P(\theta) = \text{Beta}(0,0)}$

problem with this

Not a PDF.

Since its integral over the support diverges

$\boxed{\text{parameter space is } \alpha > 0,\ \beta > 0}$

this is an improper prior since it is not a true random variable.

Do we care? Churning through the math we get posterior:

$P(\theta|X) = \text{Beta}(x, n-x) \leftarrow$ the posterior is proper as long as $x<n$ and $x>0$ which means you need to have at least one success and at least one failure in your data. IF its proper you have full Bayesian inference: point estimates, CR's, etc.

However you always have $\hat{\theta}_{MMSE} = \frac{x}{n} = \hat{\theta}_{MLE}$

Also, $\rho = 0$ (no shrinkage), This prior was first introduced by Haldane in 1932 - so we call it Haldane prior. $n_0 = 0$

END OF MIDTERM 1 MATERIAL

- (MONDAY is a Review session) Bring more questions

(go to problem #2 on HW 3 to prep for exam)

NEW TOPIC:

Lets talk about mixture / compound dist.

EXAMPLE. Lets say $X \sim \begin{cases} N(0, 1^2) & wp\ \frac{1}{2} \\ \underbrace{N(10, 2^2)}_{\text{model}} & \underbrace{wp\ \frac{1}{2}}_{\text{mixture}} \end{cases}$

~~[illegible]~~

$P(X) = \int_{\Theta} P(X, \vec{\theta})\, d\theta$ if $\vec{\theta}$ is cts. In this case Θ is discrete (Bern(θ))

so you can get sum $\sum_{\theta \in \Theta} P(X, \vec{\theta}) = \sum_{\vec{\theta} \in \Theta} P(X|\theta) P(\theta)$

$$= \left(\frac{1}{\sqrt{2\pi 1^2}} e^{-\frac{1}{2\cdot 1^2}(x-0)^2}\right)(0.5) + \left(\frac{1}{\sqrt{2\pi 2^2}} e^{-\frac{1}{2\cdot 2^2}(x-10)^2}\right)(0.5)$$

= PDF for X above.

<u>Another Example</u>: $X \sim \begin{cases} Bin(10, .1) & wp\ \frac{1}{4} \\ Bin(10, .8) & wp\ \frac{3}{4} \end{cases}$

$$p(X) = \left(\binom{10}{x} .1^x (.9)^{10-x}\right) \cdot \left(\frac{1}{4}\right) + \binom{10}{x} (.8)^x (.2)^{10-x} \left(\frac{3}{4}\right)$$

Have we seen p(X) before that's the result of a marginalizing making?

p(X) a mixture / compound distribution.

$$p(\theta|X) = \frac{\overbrace{p(X|\theta)}^{\text{likelihood}} \overbrace{p(\theta)}^{\text{Prior}}}{p(X)} = Beta(\alpha + x, \beta + n - x)$$

$$p(X) = \int_{\Theta} p(X|\theta) p(\theta) \, d\theta$$

computed from likelihood and prior you supply.

from parametric model you pick

you chose

If : Bin (n, θ) n fixed, $P(\theta) = Beta(\alpha, \beta)$

$$P(X) = \int_0^1 \binom{n}{x} \theta^x (1-\theta)^{n-x} \frac{1}{B(\alpha,\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \, d\theta$$

$$= \binom{n}{x} \frac{1}{B(\alpha,\beta)} \underbrace{\int_0^1 \theta^{(x+\alpha)-1} (1-\theta)^{(\beta+n-x)-1} \, d\theta}_{\text{Beta Function}}$$

Question on EXAM

$$= \frac{\binom{n}{x}}{B(\alpha,\beta)} B(x+\alpha, \beta + n - x)$$

← n, α, β are constants supplied by me

$$= \text{Beta Binomial}(n, \alpha, \beta) \text{ r.v.}$$

$Y \sim$ Beta Binomial (n, α, β) $\quad Supp[Y] = \{0, 1, \ldots, n\}$ $\quad \boxed{n \in \mathbb{N}, \ \alpha > 0, \ \beta > 0}$

$E[Y] = n \frac{\alpha}{\alpha+\beta}$ $\quad VAR[Y] = n \cdot \frac{\alpha\beta(\alpha+\beta+n)}{(\alpha+\beta)^2(\alpha+\beta+1)}$

Since the beta function is not available in closed form the PMF/CDF are not available in closed form, need computer.
We will use R notation: $P(Y=y) =$ dbetabinom (y, n, α, β)
$P(Y \leq y) =$ pbetabinom (y, n, α, β)

Beta binomial is sometimes called "over dispersed binomial"
(See prof notes for pictures)

Let $\theta := \frac{\alpha}{\alpha+\beta} \Rightarrow \theta\alpha + \theta\beta = \alpha \Rightarrow (\theta - 1)\alpha = -\theta\beta \Rightarrow \beta = \alpha \frac{1-\theta}{\theta}$

$\Downarrow$

$E[X] = n\theta$, intuitive formula, they same as Binomial Expectation.

Let $\alpha \to \infty$, keep $\theta = \frac{\alpha}{\alpha+\beta}$ constant

$$\lim_{\alpha\to\infty} VAR[X] = \lim_{\alpha\to\infty} n \cdot \frac{\alpha\left(\alpha\frac{1-\theta}{\theta}\right)\left(\alpha + \alpha\frac{1-\theta}{\theta} + n\right)}{\left(\alpha + \alpha\frac{1-\theta}{\theta}\right)^2\left(\alpha + \alpha\frac{1-\theta}{\theta} + 1\right)}$$

$$= n \cdot \lim \frac{\frac{1-\theta}{\theta}}{\left(1 + \frac{1-\theta}{\theta}\right)^2} \cdot \lim \frac{\cancel{\alpha + \alpha\frac{1-\theta}{\theta} + n}\ 1}{\cancel{\alpha + \alpha\frac{1-\theta}{\theta} + 1}} =$$

$$= n\theta(1-\theta)$$