$$\hat{\theta}_{MMSE} = \frac{\frac{n\bar{x}}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} + \frac{\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$= \frac{\frac{\sigma^2}{n}}{\frac{\sigma^2}{n}} \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} \hat{\theta}_{MLE} + \frac{\tau^2}{\tau^2} \overbrace{\frac{\frac{1}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}^{\rho} E[\theta \mid \sigma^2]$$

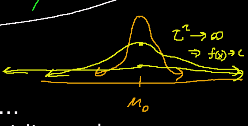
$$= \frac{n\tau^2}{n\tau^2} \frac{1}{1+\frac{\sigma^2}{n\tau^2}} \hat{\theta}_{MLE} + \frac{\sigma^2}{\sigma^2} \frac{1}{\frac{n\tau^2}{\sigma^2}+1} E[\theta \mid \sigma^2] = \overbrace{\frac{n\tau^2}{n\tau^2+\sigma^2}}^{1-\rho} \hat{\theta}_{MLE} + \overbrace{\frac{\sigma^2}{n\tau^2+\sigma^2}}^{\rho} E[\theta \mid \sigma^2]$$

What is Laplace's Prior? $P(\theta \mid \sigma^2) \propto 1$

see lec. 15

$$P(\theta \mid X, \sigma^2) \propto P(X \mid \theta, \sigma^2) P(\theta \mid \sigma^2) \propto P(X \mid \theta, \sigma^2) \propto N\left(\bar{x}, \frac{\sigma^2}{n}\right)$$

$$= N\left(\underbrace{\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}_{\theta_p}, \underbrace{\frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}}_{\sigma_p^2}\right) \Rightarrow \frac{\sigma^2}{n} = \frac{1}{\frac{n}{\sigma^2}+\cancel{\frac{1}{\tau^2}}} \Rightarrow \tau^2 = \infty$$



$$\frac{\frac{n\bar{x}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}} = \bar{x} \Rightarrow \mu_0$$ could be any value...
by convention we let it equal zero

$$\Rightarrow P(\theta \mid \sigma^2) = N(0, \infty)$$ is Laplace's prior (improper)

But is the posterior proper? Yes, always!

Jeffrey's prior: $P_J(\theta \mid \sigma^2) \propto \sqrt{I(\theta \mid \sigma^2)} = \sqrt{\frac{n}{\sigma^2}} \propto 1 \propto N(0, \infty)$

same as Laplace

$$\ell'(\theta; x, \sigma^2) = \frac{n\bar{x}}{\sigma^2} - \frac{n\theta}{\sigma^2} \Rightarrow \ell''(\theta; x, \sigma^2) = -\frac{n}{\sigma^2}$$

$$\Rightarrow I(\theta \mid \sigma^2) = E_X[-\ell''(\theta; x, \sigma^2)] = E_X\left[\frac{n}{\sigma^2}\right] = \frac{n}{\sigma^2}$$

We want a pseudocount interpretation of the hyperparameters mu_0 and tau^2. The best way to do this, is to do a small reparameterization of the prior's tau^2. Recall that we know sigsq.

$$\tau^2 := \frac{\sigma^2}{n_0} \Rightarrow P(\theta \mid X, \sigma^2) = N\left(\frac{\frac{n\bar{x}}{\sigma^2}+\frac{n_0\mu_0}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{n_0}{\sigma^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{n_0}{\sigma^2}}\right)$$

$$= N\left(\frac{\overbrace{n\bar{x}}^{\Sigma x_i} + \overbrace{n_0\mu_0}^{\Sigma y_i}}{n+n_0}, \frac{\sigma^2}{n+n_0}\right)$$

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{n}{n+n_0}\bar{x} + \overbrace{\frac{n_0}{n+n_0}}^{\rho}\mu_0$$

So n_0 represents number of pseudoobservations. What does mu_0 represent?

Let $Y_1, Y_2, \ldots, Y_{n_0}$ be the "pseudodata". Let $\mu_0 = \bar{y}$,

be the sample average of the pseudodata and n_0 mu_0 is the sum of the pseudodata.

What's the Haldane prior of total ignorance?

$$n_0 = 0 \Rightarrow N\left(\mu_0, \frac{\sigma^2}{n_0}\right) = N(0, \infty) \propto 1$$

This means all three objective priors we studied are the same.

What is the posterior predictive distribution for n_* = 1 observ?

$$P(X_* \mid X, \sigma^2) = \int_{\Theta} \overbrace{P(X_* \mid \theta, \sigma^2)}^{N(\theta, \sigma^2)} \overbrace{P(\theta \mid X, \sigma^2)}^{N(\theta_p, \sigma_p^2)} d\theta$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x_* - \theta)^2} \frac{1}{\sqrt{2\pi\sigma_p^2}} e^{-\frac{1}{2\sigma_p^2}(\theta - \theta_p)^2} d\theta$$

$$\propto \int_{\mathbb{R}} e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} e^{-\frac{\theta^2}{2\sigma_p^2}} e^{\frac{\theta\theta_p}{\sigma_p^2}} e^{-\frac{\theta_p^2}{2\sigma_p^2}} d\theta$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \int_{\mathbb{R}} \overbrace{e^{a\theta - b\theta^2}}^{\propto \text{Normal}} d\theta \qquad a = \frac{x_*}{\sigma^2} + \frac{\theta_p}{\sigma_p^2}, \quad b = \frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)$$

$$P(\theta) = N\left(\frac{a}{2b}, \frac{1}{2b}\right) = \frac{1}{\sqrt{2\pi\left(\frac{1}{2b}\right)}} e^{-\frac{1}{2\left(\frac{1}{2b}\right)}\left(\theta - \frac{a}{2b}\right)^2} = \sqrt{\frac{b}{\pi}} e^{-b\left(\theta^2 - \frac{\theta a}{b} + \frac{a^2}{4b^2}\right)}$$

$$= \sqrt{\frac{b}{\pi}} e^{-b\theta^2 + \theta a - \frac{a^2}{4b}} = \overbrace{\sqrt{\frac{b}{\pi}} e^{-\frac{a^2}{4b}}}^{c} e^{a\theta - b\theta^2}$$

$$= e^{-\frac{x_*^2}{2\sigma^2}} \frac{1}{c} \int_{\mathbb{R}} \overbrace{c e^{a\theta - b\theta^2} d\theta}^{1} = e^{-\frac{x_*^2}{2\sigma^2}} \sqrt{\frac{\pi}{b}} e^{\frac{a^2}{4b}}$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} \left(\frac{1}{2}\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)\right)^{-\frac{1}{2}} e^{\frac{\left(\frac{x_*}{\sigma^2} + \frac{\theta_p}{\sigma_p^2}\right)^2}{2\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)}} \qquad \text{let } A = \left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right)$$

$$\propto e^{-\frac{x_*^2}{2\sigma^2}} e^{\frac{x_*^2}{2A\sigma^4}} e^{\frac{x_*\theta_p}{A\sigma^2\sigma_p^2}} e^{\frac{\theta_p^2}{2A\sigma_p^4}} \propto e^{\overbrace{\frac{\theta_p}{A\sigma^2\sigma_p^2}}^{u} x_* - \overbrace{\frac{1}{2}\left(\frac{1}{\sigma^2} - \frac{1}{A\sigma^4}\right)}^{v} x_*^2}$$

NEG

$$A\sigma^2\sigma_p^2 = \sigma^2\sigma_p^2\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right) = \sigma_p^2 + \sigma^2, \quad A\sigma^4 = \sigma^4\left(\frac{1}{\sigma^2} + \frac{1}{\sigma_p^2}\right) = \sigma^2 + \frac{\sigma^4}{\sigma_p^2}$$

$$\propto N\left(\frac{u}{2v}, \frac{1}{2v}\right) = N\left(\frac{\frac{\theta_p}{\cancel{\sigma_p^2 + \sigma^2}}}{\frac{1}{\cancel{\sigma_p^2 + \sigma^2}}}, \sigma_p^2 + \sigma^2\right) = N(\theta_p, \sigma_p^2 + \sigma^2)$$

$$2v = \frac{1}{\sigma^2} - \frac{1}{\sigma^2 + \frac{\sigma^4}{\sigma_p^2}} = \frac{1}{\sigma^2} - \frac{\sigma_p^2}{\sigma^2\sigma_p^2 + \sigma^4} = \frac{\cancel{\sigma^2\sigma_p^2} + \sigma^4 - \cancel{\sigma_p^2\sigma^2}}{\cancel{\sigma^2}(\sigma^2\sigma_p^2 + \sigma^4)}$$

$$= \frac{\sigma^2}{\sigma^2(\sigma_p^2 + \sigma^2)} \Rightarrow \frac{1}{2v} = \sigma_p^2 + \sigma^2$$