$\lambda \in (0,\infty)$, $Y \sim \text{Poisson}(\lambda)$, $\text{Supp}[Y] = \{0,1,2,\dots\} = \mathbb{N}_0$ $E[Y] = \lambda$

$X =$ [illegible] $n=1$ Poisson(θ). θ is our unknown parameter of interest. It is not the same θ in the binomial. Let's try to find the conjugate prior for this parametric model (likelihood).

$P(\theta|X) \propto P(X|\theta)P(\theta)$. We want to find $P(\theta)$ of the same distribution as $P(\theta|X)$

$$P(\theta|x) \propto P(x|\theta)P(\theta) = \frac{e^{-\theta}\theta^x}{x!}P(\theta) \propto e^{-\theta}\theta^x K(\theta) = \underbrace{e^{-\theta}\theta^x e^{-\theta b}\theta^a}_{\text{same kernel}} = \underbrace{e^{-\theta(b+1)}\theta^{x+a}}_{\text{same kernel}}$$

Lets figure out $P(\theta)$ from $K(\theta)$.

$$\int_0^\infty K(\theta)\,d\theta = \int_0^\infty e^{-\theta b}\theta^a\,d\theta = \int_0^\infty \theta^{a+1-1}e^{-b\theta}\,d\theta = \text{u.sub} =$$

$$\Rightarrow P(\theta) = \frac{b^{a+1}}{\Gamma(a+1)}\theta^{a+1-1}e^{-b\theta} = \text{Gamma}(a+1, b) \text{ r.v.}$$

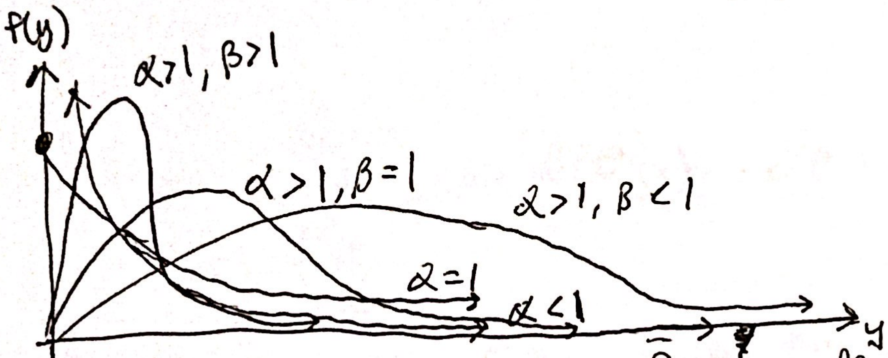
If $Y \sim \text{Gamma}(\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\beta y} \propto \boxed{y^{\alpha-1}e^{-\beta y}}$ kernel, $\text{Supp}[Y] = (0,\infty)$

Parameter space $\alpha > 0$, $\beta > 0$ $E[Y] = \int_0^\infty y f(y)\,dy = \text{u-sub} = \frac{\alpha}{\beta}$ $\boxed{\hat{\theta}_{MMSE}}$

$\text{Mode}[Y] = \text{calc} = \frac{\alpha-1}{\beta}$ if $\alpha \geq 1$ $\boxed{\hat{\theta}_{MAP}}$

$\text{MED}[Y] = q$ s.t. $\int_0^q f(y)\,dy = \frac{1}{2}$ no closed form expression. use computer to do numerical integration.

$= \text{qgamma}(\frac{1}{2}, \alpha, \beta)$ $\boxed{\hat{\theta}_{MMAE}}$



Use shape / rate for Gamma parameterization

Back to inference on Poisson Model.

$\mathbb{X}$: iid Poisson(θ)

$$P(\theta|X) \propto P(X|\theta)P(\theta) = P(\theta)\prod_{i=1}^{n}\frac{e^{-\theta}\theta^{x_i}}{x_i!} = P(\theta)\frac{e^{-n\theta}\theta^{\sum x_i}}{\prod x_i!} \propto e^{-n\theta}\theta^{\sum x_i} k(\theta)$$

$$= e^{-n\theta}\theta^{\sum x_i}\underbrace{\theta^{\alpha-1}e^{-\beta\theta}}_{\text{gamma kernel}} = \underbrace{\theta^{\alpha+\sum x_i-1}e^{-(\beta+n)\theta}}_{\text{gamma kernel}} \propto \text{Gamma}(\alpha+\sum x_i, \beta+n)$$

psuedo events, # of events in all data, # of observations, # of psuedo observations

$$\Rightarrow \hat{\theta}_{MMSE} = \frac{\alpha+\sum x_i}{\beta+n}, \quad \hat{\theta}_{MAP} = \frac{\sum x_i+\alpha-1}{\beta+n}, \quad \hat{\theta}_{MMAE} = \text{qgamma}\left(\tfrac{1}{2}, \alpha+\sum x_i, \beta+n\right)$$

only if $(\alpha+\sum x_i \geq 1)$

$$CR_{\theta,1-\alpha_0} = \left[\text{qgamma}\left(\frac{\alpha_0}{2}, \alpha+\sum x_i, \beta+n\right), \text{qgamma}\left(1-\frac{\alpha_0}{2}, \alpha+\sum x_i, \beta+n\right)\right]$$

$H_a: \theta > \theta_0$ vs $H_0: \theta \leq \theta_0$, $P_{val} =$ ~~pgamma~~ $P(H_0|X) = P(\theta \leq \theta_0 | X)$

$$= \cancel{\text{pgamma}} \int_0^{\theta_0} P(\theta|X)\,d\theta = \text{pgamma}(\theta_0, \alpha+\sum x_i, \beta+n)$$

Derive MLE.

$$\mathcal{L}(\theta; x) = \frac{e^{-n\theta}\,\theta^{\sum x_i}}{\prod x_i!} \Rightarrow \ell(\theta; x) = -n\theta + \sum x_i \ln(\theta) - \ln\left(\prod x_i!\right)$$

$$\Rightarrow \ell'(\theta; x) = -n + \frac{\sum x_i}{\theta}$$ ← set this equal to 0.

$$\frac{\sum x_i}{\theta} = n \Rightarrow \hat{\theta}_{MLE} = \frac{\sum x_i}{n} = \bar{x}$$. Lets Derive shrinkage.

$$\hat{\theta}_{MMSE} = \frac{\alpha + \sum x_i}{\beta + n} = \frac{\alpha}{\beta+n}\cdot\frac{\beta}{\beta} + \frac{\sum x_i}{\beta+n}\cdot\frac{n}{n} = \overbrace{\frac{\beta}{\beta+n}}^{\rho}\,\overbrace{\frac{\alpha}{\beta}}^{E[\theta]} + \overbrace{\frac{n}{\beta+n}}^{1-\rho}\,\overbrace{\frac{\sum x_i}{n}}^{\hat{\theta}_{MLE}}$$

$$\lim_{n\to\infty} \rho = 0$$

ρ = how much are you we

degree to which your prior expectation counts