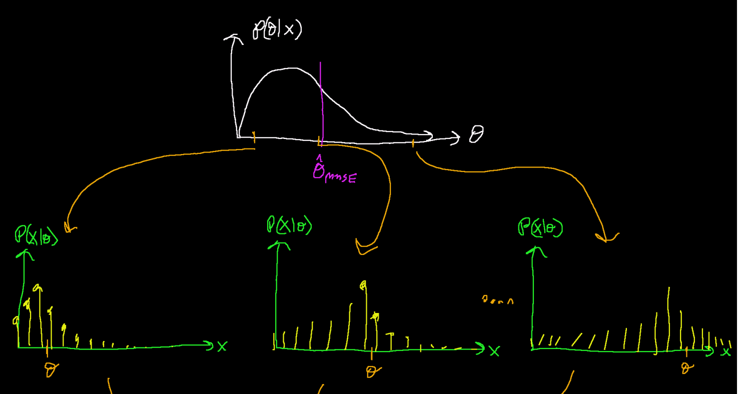

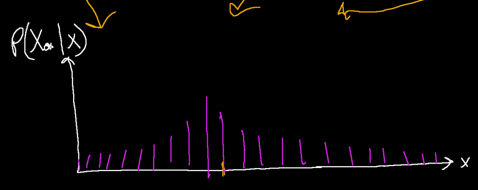

The negative binomial model is a "dispersed Poisson" analagous to the betabinomial model being a "dispersed binomial".

$E[X_*|X] = \hat{\theta}_{MMSE}$

$$Var[X_*|X] = r\frac{1-p}{p^2} = \hat{\theta}_{MMSE} \cdot \frac{1}{p} = \underbrace{\frac{\beta+n+1}{\beta+n}}_{\in[1,2]} \hat{\theta}_{MMSE}$$

Thus, the posterior predictive distribution has variance up to 2x the Poisson (i.e. more spread out or less sure of where the realization will be).

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curlyF: one N(θ, σ^2) $= P(X \mid \underbrace{\theta, \sigma^2}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X-\theta)^2}$

$\vec{\theta} = \begin{bmatrix}\theta \\ \sigma^2\end{bmatrix}$

let σ^2 be fixed / known in advance.

$$P(X|\theta,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(X-\theta)^2} = e^{-\frac{1}{2\sigma^2}(X^2-2X\theta+\theta^2)}$$

$$= e^{-\frac{X^2}{2\sigma^2}} e^{\frac{X\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} \propto e^{-\frac{X^2}{2\sigma^2}} e^{\frac{X\theta}{\sigma^2}} = e^{ax-bx^2}, \quad a = \frac{\theta}{\sigma^2}, \quad b = \frac{1}{2\sigma^2}$$

$$P(\theta|X,\sigma^2) \propto e^{-\frac{1}{2\sigma^2}(X-\theta)^2} = e^{-\frac{X^2}{2\sigma^2}} e^{\frac{X\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}}$$

$$\propto e^{\frac{X\theta}{\sigma^2}} e^{-\frac{\theta^2}{2\sigma^2}} = e^{a\theta-b\theta^2}, \quad a = \frac{X}{\sigma^2}, \quad b = \frac{1}{2\sigma^2}$$

$$E[\theta] = X = \frac{a}{2b}, \quad Var[\theta] = \sigma^2 = \frac{1}{2b}$$

$$E[X] = \theta = \frac{a}{2b} = \frac{\frac{\theta}{\cancel{\sigma^2}}}{\cancel{2}\left(\frac{1}{\cancel{2\sigma^2}}\right)} = \theta$$

$$Var[X] = \sigma^2 = \frac{1}{2b} = \frac{1}{\cancel{2}\left(\frac{1}{\cancel{2}\sigma^2}\right)} = \sigma^2$$

$\mathcal{F}: \overset{iid}{\sim} N(\theta,\sigma^2) \quad X_1, \ldots, X_n$

$$P(X|\theta,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(X_i-\theta)^2} = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum(X_i-\theta)^2}$$

$$\sum(X_i-\theta)^2 = \sum X_i^2 - 2X_i\theta + \theta^2 = \sum X_i^2 - 2n\bar{X}\theta + n\theta^2$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum X_i^2}{2\sigma^2}} e^{\frac{n\bar{X}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} = \mathcal{L}(\theta; X, \sigma^2)$$

$$\propto e^{-\frac{\sum X_i^2}{2\sigma^2}} e^{\frac{n\bar{X}\theta}{\sigma^2}}$$

$$P(\theta|X,\sigma^2) \propto e^{\frac{n\bar{X}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} = e^{a\theta-b\theta^2}, \quad a = \frac{n\bar{X}}{\sigma^2}, \quad b = \frac{n}{2\sigma^2} \Rightarrow 2b = \frac{n}{\sigma^2}$$

$$\propto N\left(\frac{a}{2b}, \frac{1}{2b}\right) = N\left(\frac{\frac{n\bar{X}}{\sigma^2}}{\frac{n}{\sigma^2}}, \frac{1}{\frac{n}{\sigma^2}}\right) = N\left(\bar{X}, \frac{\sigma^2}{n}\right)$$

All we've done thus far is probability theory and we seemingly just made random computations for fun. Now we'll do Bayesian...

$\mathcal{F}: X_1, \ldots, X_n \overset{iid}{\sim} N(\theta,\sigma^2)$ with σ^2 known. Let's find posterior.

$$P(\theta|X,\sigma^2) \propto P(X|\theta,\sigma^2) P(\theta|\sigma^2)$$

$$= (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum X_i^2}{2\sigma^2}} e^{\frac{n\bar{X}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} P(\theta|\sigma^2)$$

$$\propto e^{\frac{n\bar{X}\theta}{\sigma^2}} e^{-\frac{n\theta^2}{2\sigma^2}} \underbrace{P(\theta|\sigma^2)}$$

$$\propto e^{a\theta-b\theta^2} e^{\alpha\theta-\beta\theta^2}$$

$\Rightarrow P(\theta|\sigma^2) = N\left(\frac{\alpha}{2\beta}, \frac{1}{2\beta}\right)$

$$= e^{(a+\alpha)\theta-(b+\beta)\sigma^2}$$

$$\propto N\left(\frac{a+\alpha}{2(b+\beta)}, \frac{1}{2(b+\beta)}\right)$$

Traditionally...

$\alpha = \frac{\mu_0}{\tau^2}, \quad \beta = \frac{1}{2\tau^2}$ "tau"

$P(\theta|\sigma^2) = N(\mu_0, \tau^2)$

$$= N\left(\frac{\frac{n\bar{X}}{\sigma^2}+\alpha}{\frac{n}{\sigma^2}+2\beta}, \frac{1}{\frac{n}{\sigma^2}+2\beta}\right) = N\left(\frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)$$

The normal-normal conjugate model (where sigsq is assumed fixed).

$$P(\theta|\sigma^2) = N(\mu_0,\tau^2) \Rightarrow P(\theta|X,\sigma^2) = N\left(\frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right)$$

Point estmation

$$\hat{\theta}_{MMSE} = E[\theta|X,\sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$\hat{\theta}_{MMAE} = Med[\theta|X,\sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$\hat{\theta}_{MAP} = Mode[\theta|X,\sigma^2] = \frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}$$

$$\hat{\theta}_{MMSE} = \hat{\theta}_{MMAE} = \hat{\theta}_{MAP}$$

Credible Regions

$$CR_{\theta,1-\alpha_0} = \left[qnorm\left(\frac{\alpha_0}{2}, \frac{\frac{n\bar{X}}{\sigma^2}+\frac{\mu_0}{\tau^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}, \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau^2}}\right), qnorm\left(1-\frac{\alpha_0}{2}, mean, var\right)\right]$$

Hypothesis Tests

$H_a: \theta < \theta_0 \Rightarrow H_0: \theta \geq \theta_0$

$$p_{val} = P(H_0|X,\sigma^2) = \int_{\theta_0}^{\infty} P(\theta|X,\sigma^2)\,d\theta = 1 - pnorm(\theta_0, mean, var)$$

Let's calculate the MLE (this was on HW1)

$$\mathcal{L}(\theta;X,\sigma^2) = (2\pi\sigma^2)^{-n/2} e^{-\frac{\sum X_i^2}{2\sigma^2}+\frac{n\bar{X}\theta}{\sigma^2}-\frac{n\theta^2}{2\sigma^2}}$$

$$\ell(\theta;X,\sigma^2) = \ln(\ ) - \frac{\sum X_i^2}{2\sigma^2} + \frac{n\bar{X}\theta}{\sigma^2} - \frac{n\theta^2}{2\sigma^2}$$

$$\ell'(\theta;X,\sigma^2) = \frac{n\bar{X}}{\cancel{\sigma^2}} - \frac{n\theta}{\cancel{\sigma^2}} \overset{set}{=} 0 \Rightarrow \cancel{n}\bar{X} = \cancel{n}\theta \Rightarrow \hat{\theta}_{MLE} = \bar{X}$$