

# BIOSTAT 704 - Homework 1

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## Problem 1

**BE Exercise 7.1:** Consider a random sample of size  $n$  from a distribution with CDF  $F(x) = 1 - 1/x$  if  $1 \leq x < \infty$ , and zero otherwise.

a) Derive the CDF of the smallest order statistic,  $X_{1:n}$

$$\begin{aligned} G_n(y) &= P(Y_n \leq y) \\ &= 1 - P(Y_n > y) \\ &= 1 - P(X_{1:n} > y) \\ &= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y) \\ &= 1 - P(X_1 > y)P(X_2 > y) \dots P(X_n > y) \\ &= 1 - [P(X_i > y)]^n \\ &= 1 - [1 - P(X_i \leq y)]^n \\ &= 1 - [1 - F_X(x)]^n \\ &= 1 - [1 - (1 - 1/x)]^n \\ &= 1 - (1/x)^n \\ G_{X_{1:n}}(x) &= \begin{cases} 1 - \frac{1}{x^n}, & \text{for } x \geq 1 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

b) Find the limiting distribution of  $X_{1:n}$

For  $x \geq 1$ ,

$$\lim_{n \rightarrow \infty} 1 - 1/x^n = 1$$

Thus the limiting distribution of  $G_{X_{1:n}}(x)$  is,

$$G(y) = \begin{cases} 1, & \text{for } x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

c) Find the limiting distribution of  $X_{1:n}^n$

Let  $Y_n = X_{1:n}^n$ . For  $y \geq 1$ ,

$$\begin{aligned}
 G_n(y) &= P(Y_n \leq y) \\
 &= P(X_{1:n}^n \leq y) \\
 &= P(X_{1:n} \leq y^{1/n}) \\
 &= 1 - \left(\frac{1}{y^{1/n}}\right)^n \\
 &= 1 - \frac{1}{y} \\
 \lim_{n \rightarrow \infty} 1 - \frac{1}{y} &= 1 - \frac{1}{y}
 \end{aligned}$$

Thus, the limiting distribution of  $X_{1:n}^n$  is given as follows:

$$G(y) = \begin{cases} 1 - \frac{1}{y}, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

**BE Exercise 7.18:** In exercise 1 (above), find the limiting distribution of  $n \ln(X_{1:n})$

Let  $Y_n = n \ln(X_{1:n})$ . For  $y \geq 1$ ,

$$\begin{aligned}
 G_n(y) &= P(Y_n \leq y) \\
 &= P(n \ln(X_{1:n}) \leq y) \\
 &= P(\ln(X_{1:n}) \leq \frac{y}{n}) \\
 &= P(X_{1:n} \leq e^{y/n}) \\
 &= 1 - \left(\frac{1}{e^{y/n}}\right)^n \\
 &= 1 - e^{-y}
 \end{aligned}$$

The limiting distribution of  $n \ln(X_{1:n})$  is given below:

$$G(y) = \begin{cases} 1 - e^{-y}, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

## Problem 2

**BE Exercise 7.2:** Consider a random sample of size  $n$  from a distribution with CDF  $F(x) = (1 + e^{-x})^{-1}$  for all real  $x$ .

a) Does the largest order statistic,  $X_{n:n}$ , have a limiting distribution?

Let  $Y_n = X_{n:n}$ . For all real  $x$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(X_{n:n} \leq y) \\
&= P(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y) \\
&= P(X_1 \leq y)P(X_2 \leq y) \dots P(X_n \leq y) \\
&= [P(X_i \leq y)]^n \\
&= [(1 - e^{-y})^{-1}]^n \\
&= (1 - e^{-y})^{-n}
\end{aligned}$$

NOTE: THIS PROBLEM IS NOT FINISHED. I NEED TO DETERMINE HOW TO TAKE THE LIMIT AS  $N \rightarrow \infty$ .

**b) Does  $X_{n:n} - \ln(n)$  have a limiting distribution?**

Let  $Y_n = X_{n:n} - \ln(n)$ . For all real  $x$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(X_{n:n} - \ln(n) \leq y) \\
&= P(X_{n:n} \leq y + \ln(n)) \\
&= P(X_1 \leq y + \ln(n), X_2 \leq y + \ln(n), \dots, X_n \leq y + \ln(n)) \\
&= P(X_1 \leq y + \ln(n))P(X_2 \leq y + \ln(n)) \dots P(X_n \leq y + \ln(n)) \\
&= [P(X_i \leq y + \ln(n))]^n \\
&= [(1 - e^{-y - \ln(n)})^{-1}]^n \\
&= (1 - e^{-y - \ln(n)})^{-n} \\
&= (1 - e^{-y}e^{-\ln(n)})^{-n} \\
&= (1 - \frac{e^{-y}}{n})^{-n} \\
\lim_{n \rightarrow \infty} (1 - \frac{e^{-y}}{n})^{-n} &= e^{-e^{-y}}
\end{aligned}$$

To verify if the function  $e^{-e^{-y}}$  is a CDF, we need to check three requirements:

1. The limit as  $y \rightarrow \infty$  is 1, and the limit as  $y \rightarrow -\infty$  is 0
2. It is a non-decreasing function
3. It is continuous over all its domain, so it is right continuous

All three conditions are met, so we can conclude that it is a CDF. Thus,  $X_{n:n} - \ln(n)$  does have a limiting distribution, and  $G(y) = e^{-e^{-y}}$ .

**BE Exercise 7.19: In exercise 2 (above), find the limiting distribution of  $(1/n)e^{X_{n:n}}$**

Let  $(1/n)e^{X_{n:n}}$ . For all real  $x$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(1/ne^{X_{n:n}} \leq y) \\
&= P(X_{n:n} \leq \ln(ny)) \\
&= [(1 + e^{-\ln(ny)})^{-1}]^n \\
&= (1 + \frac{1}{ny})^{-n} \\
&= e^{-1/y}
\end{aligned}$$

Note that this function does not meet the requirements of a CDF for  $y < 0$ , so the limiting distribution,  $G^*(y)$ , is as follows:

$$\begin{cases} e^{-1/y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

**BE Exercise 7.3:** Consider a random sample of size  $n$  from a distribution with CDF  $F(x) = 1 - x^{-2}$  if  $x > 1$ , and 0 otherwise. Determine whether each of the following sequences has a limiting distribution; if so, then give the limiting distribution.

a)  $X_{1:n}$

Let  $Y_n = X_{1:n}$ . For  $x > 1$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(X_{1:n} \leq y) \\
&= 1 - P(X_{1:n} > y) \\
&= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y) \\
&= 1 - P(X_1 > y)P(X_2 > y) \dots P(X_n > y) \\
&= 1 - [P(X_i > y)]^n \\
&= 1 - [1 - P(X_i \leq y)]^n \\
&= 1 - [1 - (1 - x^{-2})]^n \\
&= 1 - [y^{-2}]^n \\
&= 1 - y^{-2n} \\
\lim_{n \rightarrow \infty} 1 - y^{-2n} &= 1
\end{aligned}$$

We have a small problem in that we need to adjust the support so that the function is right continuous at  $y = 1$ , such that the limiting distribution of  $Y_n$  is

$$G^*(y) = \begin{cases} 1, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

b)  $X_{n:n}$

Let  $Y_n = X_{n:n}$ . For  $x > 1$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(X_{n:n} \leq y) \\
&= P(X_1 \leq y, X_2 \leq y, \dots, X_n \leq y) \\
&= P(X_1 \leq y)P(X_2 \leq y) \dots P(X_n \leq y) \\
&= [P(X_i \leq y)]^n \\
&= [1 - y^{-2}]^n \\
\lim_{n \rightarrow \infty} (1 - x^{-2})^n &= 0
\end{aligned}$$

This sequence does not have a limiting distribution.

NOTE: I HAVE NO IDEA HOW TO CALCULATE THIS LIMIT. I GOT THIS FROM DESMOS, SO I'D LIKE HELP KNOWING HOW TO CALCULATE IT

c)  $n^{-1/2}X_{n:n}$

Let  $Y_n = n^{-1/2}X_{n:n}$ . For  $x > 1$ ,

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(n^{-1/2}X_{n:n} \leq y) \\
&= P(X_{n:n} \leq n^{1/2}y) \\
&= P(X_1 \leq n^{1/2}y, X_2 \leq n^{1/2}y, \dots, X_n \leq n^{1/2}y) \\
&= P(X_1 \leq n^{1/2}y)P(X_2 \leq n^{1/2}y) \dots P(X_n \leq n^{1/2}y) \\
&= [P(X_i \leq n^{1/2}y)]^n \\
&= [1 - (n^{1/2}y)^{-2}]^n \\
&= [1 - \frac{1}{ny^2}]^n \\
\lim_{n \rightarrow \infty} [1 - \frac{1}{ny^2}]^n &= e^{-1/y^2}
\end{aligned}$$

The function is not right continuous at  $y = 1$ , but the other two conditions are met. That leads us to give the following limiting distribution:

$$G^*(y) = \begin{cases} e^{-1/y^2}, & y \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

### Problem 3

Consider Example 7.2.2 in the textbook, on pages 233. Let  $X_1, X_2, \dots, X_n$  be a random sample from an exponential distribution,  $X_i \sim EXP(\theta)$  and let  $Y_n = X_{1:n}$  be the smallest order statistic.

a) Using the definition of a CDF (Theorem 2.2.3), show that  $G_n(Y)$  is a proper CDF. That is, show that the conditions (2.2.8 - 2.2.11) hold for  $G_n(y)$ .

I BELIEVE THIS IS ASKING TO VERIFY THAT  $G_n(y)$  IS A PROPER CDF AS OPPOSED TO  $G(y)$ . I NEED TO CONFIRM THIS IN OFFICE HOURS ON MONDAY.

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= P(X_{1:n} \leq y) \\
&= 1 - P(X_{1:n} > y) \\
&= 1 - P(X_1 > y, X_2 > y, \dots, X_n > y) \\
&= 1 - P(X_1 > y)P(X_2 > y) \dots P(X_n > y) \\
&= 1 - [P(X_i > y)]^n \\
&= 1 - [1 - P(X_i \leq y)]^n \\
&= 1 - [1 - (1 - e^{-y})]^n \\
&= 1 - e^{-ny}
\end{aligned}$$

The following conditions must be met for  $G_n(y)$  to be a proper CDF. I will evaluate each one in turn:

1.  $\lim_{y \rightarrow \infty} G_n(y)$  must equal 1, and  $\lim_{y \rightarrow -\infty} G_n(y)$  must equal 0 ‘- Note that  $n$  is required to be a positive integer for this to be a valid sequence. This being the case, no matter what arbitrary value we may have for  $n$ , the limit as  $y$  goes to positive infinity is 1, and the limit as  $y$  goes to negative infinity is 0. This condition holds
2. The function must not be decreasing
  - Again, because  $n$  must be a positive integer, any value we choose for  $n$  over  $G_n(y)$ ’s domain will not alter the fact that the function is not decreasing
3. The function must be right continuous over its domain
  - The function is continuous for all  $y > 0$ .

b) Sketch the graph of the limiting function  $G(Y)$ .

c) Explain why the authors state that  $G(y)$  being discontinuous (and not even right-continuous) at  $y = 0$  “is not a problem”. How does this connect with our explanations in the lecture notes on pages 1–2?

#### Problem 4

(Example 7.2.6 in the textbook, on page 235)

Consider  $X_1, \dots, X_n$  a random sample where  $X_i \sim EXP(\theta)$ . Define the random sequence  $Y_n = (1/\theta)X_{n:n} - \ln(n)$ . The purpose of this exercise is to demonstrate that  $Y_n$  converges in distribution

a) Prove that the CDF of  $Y_n$  is

$$G_n(Y) = \begin{cases} [1 - \frac{1}{n}e^{-y}]^n, & \text{when } y > \ln(n) \\ 0, & \text{otherwise} \end{cases}$$

b) Show that  $\lim_{n \rightarrow \infty} G_n(Y) = G(Y)$ , with  $e^{-e^{-y}}$ , where  $-\infty < y < \infty$ .

c) Show that  $G(Y)$  is a CDF.

#### Problem 5

BE Exercise 7.5