BIOSTAT 704 - Homework 1

Austin Allen

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Problem 1

BE Exercise 7.1: Consider a random sample of size n from a distribution with CDF $F(x) = 1 - 1/x$ if $1 \leq x < \infty$, and zero otherwise.

a) Derive the CDF of the smallest order statistic, $X_{1:n}$

$$\begin{aligned}
G_n(y) &= P(Y_n \leq y) \\
&= 1 - P(Y_n > y) \\
&= 1 - P(X_{1:n} > y) \\
&= 1 - P(X_1 > y, X_2 > y, ..., X_n > y) \\
&= 1 - P(X_1 > y)P(X_2 > y)...P(X_n > y) \\
&= 1 - [P(X_i > y)]^n \\
&= 1 - [1 - P(X_i \leq y)]^n \\
&= 1 - [1 - F_X(x)]^n \\
&= 1 - [1 - (1 - 1/x)]^n \\
&= 1 - (1/x)^n \\
G_{X_{1:n}}(x) &= \begin{cases} 1 - \frac{1}{x^n}, & \text{for } x \geq 1 \\ 0, & \text{o.w.} \end{cases}
\end{aligned}$$

b) Find the limiting distribution of $X_{1:n}$ For $x \geq 1$,

$$\lim_{n \to \infty} 1 - 1/x^n = 1$$

Thus the limiting distribution of $G_{X_{1:n}}(x)$ is,

$$G(y) = \begin{cases} 1, & \text{for } x \geq 1 \\ 0, & \text{o.w.} \end{cases}$$

c) Find the limiting distribution of $X_{1:n}^n$ Let $Y_N = X_{1:n}^n$. For $y \geq 1$,

$$
\begin{aligned}
G_n(y) &= P(Y_n \le y) \\
&= P(X_{1:n}^n \le y) \\
&= P(X_{1:n} \le y^{1/n}) \\
&= 1 - (\frac{1}{y^{1/n}})^n \\
&= 1 - \frac{1}{y} \\
G(y) &= \lim_{n\to\infty} 1 - \frac{1}{y} \\
&= 1 - \frac{1}{y}
\end{aligned}
$$

BE Exercise 7.18

Problem 2

BE Exercise 7.2

BE Exercise 7.19

BE Exercise 7.3

Problem 3

Consider Example 7.2.2 in the textbook, on pages 233. Let $X1, X2, ..., X_n$ be a random sample from an exponential distribution, Xi EXP( ) and let $Y_n = X_{1:n}$ be the smallest order statistic.

a) Using the definition of a CDF (Theorem 2.2.3), show that $G_n(Y)$ is a proper CDF. That is, show that the conditions (2.2.8–2.2.11) hold for $G_n(y)$.

b) Sketch the graph of the limiting function $G(Y)$.

c) Explain why the authors state that G(y) being discontinuous (and not even right-continuous) at y = 0 "is not a problem". How does this connect with our explanations in the lecture notes on pages 1–2?

Problem 4

(Example 7.2.6 in the textbook, on page 235)

Consider $X1, ..., X_n$ a random sample where $X_i \sim EXP(\theta)$. Define the random sequence $Y_n = (1/\theta)X_{n:n} - \ln(n)$. The purpose of this exercise is to demonstrate that Y_n converges in distribution

a) Prove that the CDF of Y_n is

$$
G_n(Y) = \begin{cases} \left[1 - \frac{1}{n}e^{-y}\right]^n, & \text{when } y > \ln(n) \\ 0, & \text{otherwise} \end{cases}
$$

b) Show that $\lim_{n\to\infty} G_n(Y) = G(Y)$, with $e^{-e^{-y}}$, where $-\infty < y < \infty$.

c) Show that $G(Y)$ is a CDF.

Problem 5

BE Exercise 7.5