A. Problems to Submit

1. BE Exercises: 9.21, 9.23, 9.26, 9.31, 9.35

2. Suppose that $X_1, \ldots, X_n$ is a random sample from GAM$(\theta, 2)$.

   (a) Show that the MLE of θ is $\hat{\theta} = \sum_{i=1}^{n} X_i/2n$.

   (b) Is the MLE of θ simply consistent?

3. Consider again the MME and MLE of θ in BE Exercise 9.26. Suppose you wanted to compare their performance for estimating θ.

   (a) Is it appropriate to compare their performance by computing their relative efficiency? If so, what is it and which estimator has better performance? If not, explain why not.

   (b) Is it appropriate to compare their performance by computing their asymptotic relative efficiency? If so, what is it and which estimator has better performance? If not, explain why not.

   (c) Could either of the metrics described in part (a) or (b) be used to compare the finite-sample performance of these estimators? If so, which one? If not, propose another metric that could be used.

   (d) Could either of the metrics described in part (a) or (b) be used to compare the large-sample performance of these estimators? If so, which one? If not, propose another metric that could be used.

4. Consider again the scenario given in BE Exercise 9.21. It can be shown that $\bar{X}_n$ is the MLE of p.

   (a) What is the MLE of $p(1-p)$?

   (b) Is the MLE found in part (a) a UMVUE for $p(1-p)$? Explain your reasoning.

   (c) Does the result found in part (b) contradict the following result given on page 316 of the course text?

   $$\tau(\hat{\theta})_{\text{MLE}} \overset{d}{\sim} N\Big(\tau(\theta), \text{CRLB}\big(\tau(\theta)\big)\Big)$$

   Explain your reasoning.

   <u>Note</u>: The result above extends the result given on page 316 from θ to $\tau(\theta)$; this extension can be established using Cramer's Theorem.

B. Additional Practice Problems

1. BE Exercises: 9.17, 9.22, 9.33, 9.36
2. Suppose that $X_1, \ldots, X_n$ is a random sample from Exponential(θ). Consider two estimators for θ: $\hat{\theta}_1 = \bar{X}_n$ and $\hat{\theta}_2 = nX_{1:n}$.
   (a) Show that $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased for θ.
   (b) Find the variance of $\hat{\theta}_1$ and of $\hat{\theta}_2$.
   (c) If appropriate, find the relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_2$. Which is more efficient?
   (d) Is either estimator efficient? (Hint: Derive the CRLB for θ).

C. Advanced Problems

1. BE Exercises: 9.10, 9.13, 9.37
2. $X_1, \ldots, X_n$ is an iid sample from distribution that admits a density function $f[x;\theta]$ where $\theta \in \Omega \subset \mathbb{R}$. Denote the Maximum Likelihood Estimator (MLE) by $\hat{\theta}_n$.

   Show the following results (with proper details)

   (a) $I[\theta] = \mathbb{E}[(\frac{\partial}{\partial\theta}\ln f[X_1, \theta])^2] = -\mathbb{E}[\frac{\partial^2}{\partial\theta^2}\ln f[X_1, \theta]] = \mathbb{V}[\frac{\partial}{\partial\theta}\ln f[X_1, \theta]]$.
   (b) $\hat{\theta}_n$ is asymptotically normal. To solve this problem, let $\ell_n[\theta] = \sum_{i=1}^n \ln f[X_i;\theta]$. Let $\dot{\ell}_n[\theta] = \frac{\partial}{\partial\theta}\ell_n[\theta]$ (the first derivative of $\ell_n[\theta] = \sum_{i=1}^n \ln f[X_i;\theta]$ with respect to θ). Similarly define the second and third derivatives as $\ddot{\ell}_n[\theta]$ and $\dddot{\ell}_n[\theta]$ respectively. Proceed by using a first-order Taylor expansion of $\dot{\ell}_n[\hat{\theta}_n]$ (the first derivative of $\ell_n[\theta]$ evaluated at $\theta = \hat{\theta}_n$) and use the fact that $\dot{\ell}_n[\hat{\theta}] = 0$. The Taylor expansion should be around $\theta = \theta_0$ where θ_0 denotes the *true* but *unknown* value of the parameter. Be sure to carefully and rigorously deal with the remainder term R_n and to outline the necessary conditions.
3. Prove that the MLE is consistent. Outline the necessary conditions.
4. Prove the invariance theorem for the MLE.