MATH235 Course Summary

Public ID: 93

1 Differential Equation Intro

1.1 Wait, what are they?

A differential equation establishes a relationship between value and the way it changes. A function models value from another independently-changing value, and the derivative of the function describes the rate of change of the dependent value with respect to the independent value.

Imagine what would happen if a function were set equal to its own derivative. This process marks the birth of a differential equation—before solving it, be sure to send your congratulations on the parents' new arrival. Now, *what sort of function spits out the same value and slope for any value you plug in?*

$$y(t) = e^t$$

0 also does the trick, but it isn't as exciting. This also denotes a particular solution to this problem of y' = y.

1.2 Wait, particular solution?

Yeah. Notice how when you differentiate something like $2e^t$, you still get $2e^t$? Turns out, you can slap on any real coefficient and the solution won't budge. This means our solution of e^t only gives us a finite picture of infinite solutions. To remedy this, we can add an *arbitrary constant*. It's good practice to call it C, but you can get away with using any Latin-derived character, or even Greek or Cyrillic if you desire spicing up your nomenclature.

$$y(t) = Ce^t$$

Everything's good now, having found the *general solution*. But, unfortunately, this does not exhaust the entire differential equation doctrine. Be sure not to miss my next sermon for a fuller picture.

1.3 But this is written text, everything is already in the document

Same deal with textbooks, why don't you read one of those instead?

1.4 ...Okay, fine, I've left and now I'm back

Welcome. To be able to broaden our scope of ordinary differential equation solving beyond intuition, we need to define a few kinds of differential equations first.

2 Differential Equation Classification

2.1 Order

The rank of "order" of a differential equation is simply the count of successions of differentiating the dependent variable. y', or for the Leibniz proponents, dy/dx, is the first derivative of y. If this is the highest succession of derivatives found in the differential equation, then it is *first order*. This document will really only cover first order and second order differential equations.

2.2 Autonomy

The name is indicative of the behavior. When you have an autonomous differential equation, the dependent variable is self-governing, indifferent to any state of the independent variable. However, when it comes time to solve the differential equation, this independent variable is sometimes necessary to properly model its behavior as a function, like with the first example.

2.3 Linearity

This can be most easily defined by the form of a first order linear differential equation. It must be in or can derive to this form:

$$\mathbf{F}(\vec{y}, t) = \mathbf{A}(t)\vec{y} + \vec{b}(t)$$

There's a lot of emboldening and h*arrow*ing symbols that you may not be familiar with. Don't worry, this is just the most general form which can assume a much-friendlier appearance with the assumption of variable dependence and single-dimensionality. Here's a nicer-looking, more specific case of first order linearity:

$$y' = f(t)y + g(t)$$

Very friendly, but very specific. That tends to be a recurring trend in differential equations; it's just something you have to get used to.

2.4 Homogeneity

We can piggyback off of the definition of linearity to define homogeneous differential equations. For first order cases:

$$\mathbf{F}(\vec{y}, t) = \mathbf{A}(t)\vec{y}$$

Notice how the equation is the same, except for the lack of $\vec{b}(t)$. For the more specific case:

$$y' = f(t)y$$

The $g(t)$ is missing here instead. It's that simple! Now you're ready to solve differential equations.

2.5 Cool, but wait, how can I use this to solve differential equations‽

Sorry, I got ahead of myself there. I need to show you the various *methods* of solving these things. The giants have plenty of room on their shoulder, hop on!

3 Differential Equation Methods

3.1 Separation of Variables

This is a natural starting place to the venture to solving differential equations on your own! While it only pertains for us to a certain flavor of differential equations (first order, separable, and ordinary), it is a rather intuitive method with some background of Calculus. Our first example can serve nicely as yet another example, but with a twist!

$$\frac{dy}{dt} = yt$$

This differential equation is linear, homogeneous, but most importantly, *separable*. Yesiree, this basically means that all of the y terms can stay on one side of the equation, and all of the t terms can stay on the other side. Separated but equal. Here's what the end result looks like:

$$\frac{dy}{y} = tdt$$

If only we knew how to indefinitely add up an arbitrarily small change in a variable... oh yeah, Calculus!

$$\int \frac{dy}{y} = \int tdt$$

This ends us up with:

$$ln|y| = t^2 + C$$

Yes, a constant is necessary when antidifferentiating, but two aren't necessary because this can just simplify to yet another constant. Hence, the C stays where it is most convenient, where the t terms lie.

$$y(t) = e^{t^2+C} = C_1 e^{t^2}$$

And this is the end result for the general solution for the differential equation. Exponent rules can place another C as a coefficient to the exponent, and exponentiating both sides can give us an explicit solution.

3.2 Method of Integrating Factors

Like my grandmother always used to say: "You're out of luck on integrating factors if you can't get a differential equation into an explicitly linear form." And how. We can use the specific definition of a linear differential equation as a starting point for this method:

$$y' = f(t)y + g(t)$$

Now close your eyes, I'm going to change it a bit, but it will still stay ultimately the same. Promise me you're gonna do it.

Okay, you can open them now. Notice anything different?

$$y' + f(t)y = g(t)$$

That's right. There has been a shift in the expression, but it's still the same ol' generic linear differential equation we know and love. Without getting into the proof of why this method actually works, I'm going to throw in the result of the proof that makes differential equations in this form trivial to solve in certain linear cases:

$$\mu = e^{\int f(t)dt}$$

And that is what I mean. Sure, if you have a constant or inverse y in that f(t), it'll be smooth sailing, but things might get hairy with trig or high-order polynomials, as you'll be needing to antidifferentiate the result again afterwards!

After having solved that, all that is left to do is derive y from this equation:

$$\mu y = \int \mu g(t)dt$$

Simple enough. The worst you may have to do is integration by parts—and otherwise—it either isn't possible with your typical antidifferentiation methods or not worth the hassle.

3.3 Method of Undetermined Coefficients

Like any superhero, this method does not stand alone. In fact, it is more analogous to a sidekick—cleaning up the mess after a job well done. This *mess* is none more than a nonhomogeneous component of a differential equation. Oftentimes, this takes longer to figure out than the homogeneous part! Our silent, studious knight has kept in the shadows for long enough; in this section, our underdog will get the spotlight that it finally deserves. How does it get the task done? Guessing.

$$y' - y = e^{2t}$$

Hey look, a nonhomogeneous differential equation! How convenient for it to drop by in a trying time like this. To solve this using the Method of Undetermined Coefficients, we have to first *not* use it. Let's get rid of the nonhomogeneous part, e^{2t}. Just change it to 0. Make it disappear. Now it is homogeneous. Now solve!

$$y_h(t) = Ce^t$$

That's a job for another method, though. After having found the *homogeneous* solution, we now need to use this method for the stuff we set to 0, e^{2t}. Remember the particular solution vs. general solution stuff I talked about a while ago? This is a good time to apply it in the real world of mathematics. We need to make the e^{2t} general to find a certain particular solution! That means an arbitrary constant must be tacked onto it. Let's make it A.

$$y_p(t) = Ae^{2t}$$

This is our best attempt right now at a "particular" solution. To find out if it works, we need to see if it's possible to find the values of the arbitrary constants so that it *is* particular. We take our guess and treat it as the y in the differential equation, and set it equal to the nonhomogeneous piece.

$$2Ae^{2t} - Ae^{2t} = e^{2t}$$

With some algebra, we can deduce that A is equal to 1. The solution to this problem is defined by this method as the sum of the particular and general solutions found:

$$y(t) = Ce^t + e^{2t}$$

And Gotham is yet again saved from prompt and utter destruction. Not necessarily because of the Method of Undetermined Coefficients, but are you prepared to assume responsibility for the loss of millions of lives if that actually turned out to be the case?

3.4 Characteristic Equation

This method is exclusively for second order linear differential equations. Keep away if you have anything else! You're gonna need to have it in this form:

$$Ay'' + By' + Cy = f(t)$$

And a-one, and a-two, and alakazam!

$$Ar^2 + Br + C = 0$$

You're now looking at your friendly neighborhood characteristic equation. Finding the value(s) of r will set you well on your way to finding the homogeneous solution, where the Method of Undetermined Coefficients can take care of the rest if need be. But what do you *do* with the r's once you have them? It depends on their value, of course. The following expressions will *express* which solution form you'll need:

$$\text{Real: } C_1e^{r_1t} + C_2e^{r_2t}$$
$$\text{Repeated: } C_1e^{rt} + C_2te^{rt}$$
$$\text{Complex: } C_1e^{\alpha t}\cos(\beta t) + C_2e^{\alpha t}\sin(\beta t)$$

In the complex case, the α and β refer to the real and imaginary components of the conjugate root pair, respectively. Don't worry that there are two roots, either—it's only a sign change in the imaginary, which is equivalent to a sign change in the expression's coefficient. The C, being arbitrary, is indifferent to this. Don't fret, big C can handle it!

3.5 Laplace Transforms

When all else fails, or if you really enjoy solving for partial fractions, Laplace can get the job done. There are two components to using Laplace Transforms: finding the transforms and using them to solve for differential equations. Let's focus on the latter, as people will usually have a transform table at hand to get your differential equation in Laplace land. Just ask!

Laplace transformation basically changes a function of time into a function of frequency. This is an important tool for finding solutions to a differential equation with piecewise behavior, as typical methods do not play nicely with this. Play us off, Dirac!

$$y' = \delta_1, y(0) = 2$$

And he does so with an instantaneous value of 1 at $t = 1$, and 0 everywhere else. In our Laplace land of frequency, this is equivalent to e^{-s}. Very quaint, but what can we do with this to solve the differential equation? By putting everything else into Laplace land:

$$sY(s) - y(0) = sY(s) - 2 = e^{-s}$$

Solving for $Y(s)$ will get us to a point where taking us out of Laplace land will turn $Y(s)$ into $y(t)$. With the other side possible to transform back as well, the differential equation is solved!

$$Y(s) = \frac{e^{-s}}{s} + \frac{2}{s} \to y(t) = u_1(t) + 2$$

What is that strange $u_1(t)$ function? It's a step function, my dear Watson. Instead of a different *value* for only an infinitesimal time, there's instead an infinitesimal *change in value*, at $t = 1$ again as noted in the subscript. Before the t, the value is 0, and after the t, it is 1. Solving differential equations with these two type of functions will almost always give you solutions with them still in it. After all, how else will it describe the behavior?

3.6 Eigenvalues and Eigenvectors

If you thought modeling behavior with successive derivatives wasn't enough to wrap your head around, let's now try it with more than one dimension! A matrix will let us do this, which is just a table which lets us do some cool math if we abstract it to a list of equation coefficients. This is also where our general definition of single order linearity can help us out.

$$\mathbf{F}(\vec{y}, t) = \mathbf{A}(t)\vec{y} + \vec{b}(t)$$

But again, let's cut out some of the fluff that does not concern us in very simple cases:

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}y$$

To keep our discussion within the scope of differential equations, instead of linear algebra, I'll just state how to find the values that are necessary to solve for one of these, in flow chart form! It can be found in the back of the document.

3.7 In summary

How about a Venn diagram, showing what methods you should use under a variety of differential equations? All expenses are paid; take it! You can also find it in the back of the document.

Each method under its collection of categories aren't necessarily guaranteed to work, like separation of variables if non-separable, but it gives a pretty good idea of the methods to try. Matrix stuff is also not included as it's easier to treat it as mutually exclusive to single-dimensional differential equations.

4 Conclusion

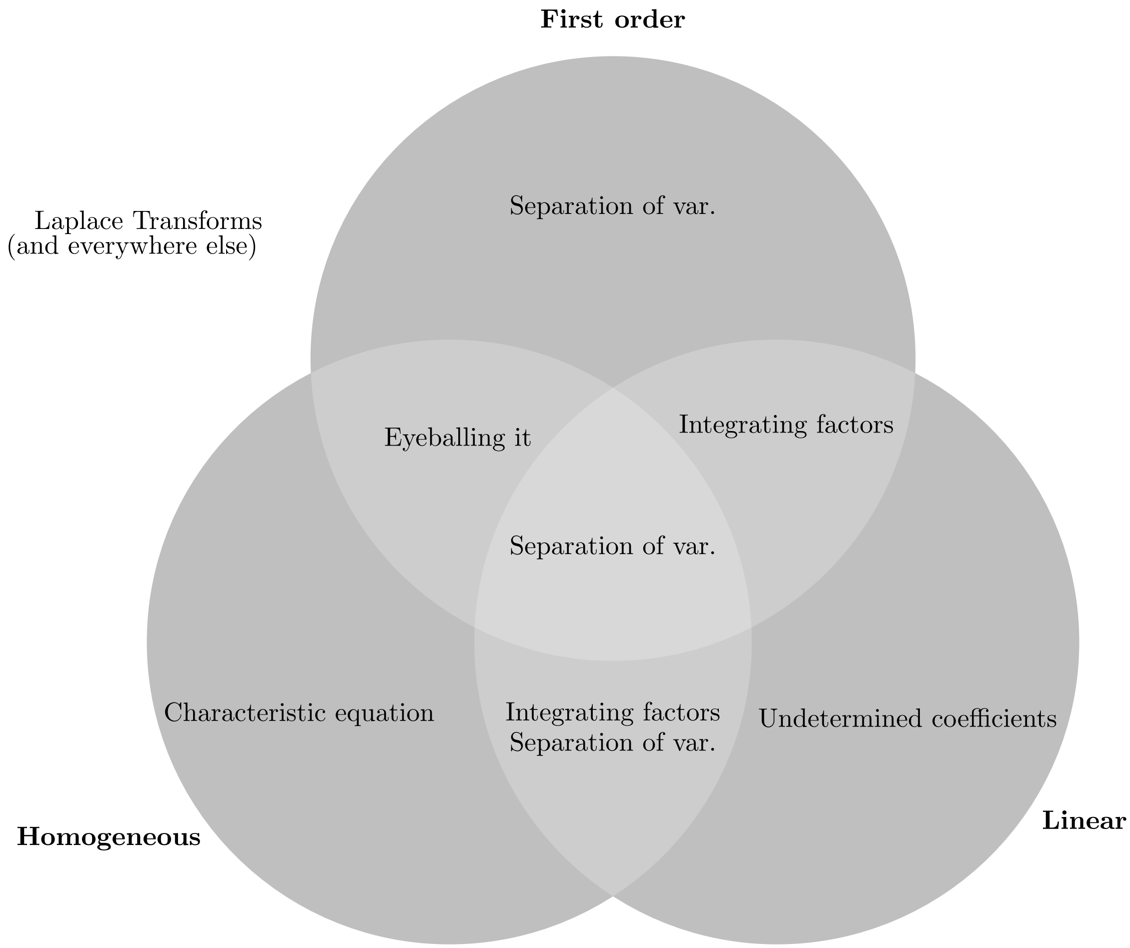
It now seems like you're ready to tackle on the world's problems. Equip your Ticonderoga and set sail!

4.1 It's all in a day's work

It's a far cry from easing geopolitical tensions, but all research finds its application somewhere. Differential equations can give us models, which behave not entirely unlike our chaotic existence. There are an arbitrarily large amount of variables that can go into "modeling" our real world. Can we pinpoint them all? Maybe, but it would be awfully inconvenient to have to simulate the entire observable universe in order to retrieve the exact behavior of a mass-spring system. Or maybe it eventually won't; that would be exciting!

4.2 But I digress

Differential equations are an invaluable tool for modeling behavior based on initial conditions, the behavior of an independent variable, and its own behavior. Where would MATH235 be without it?



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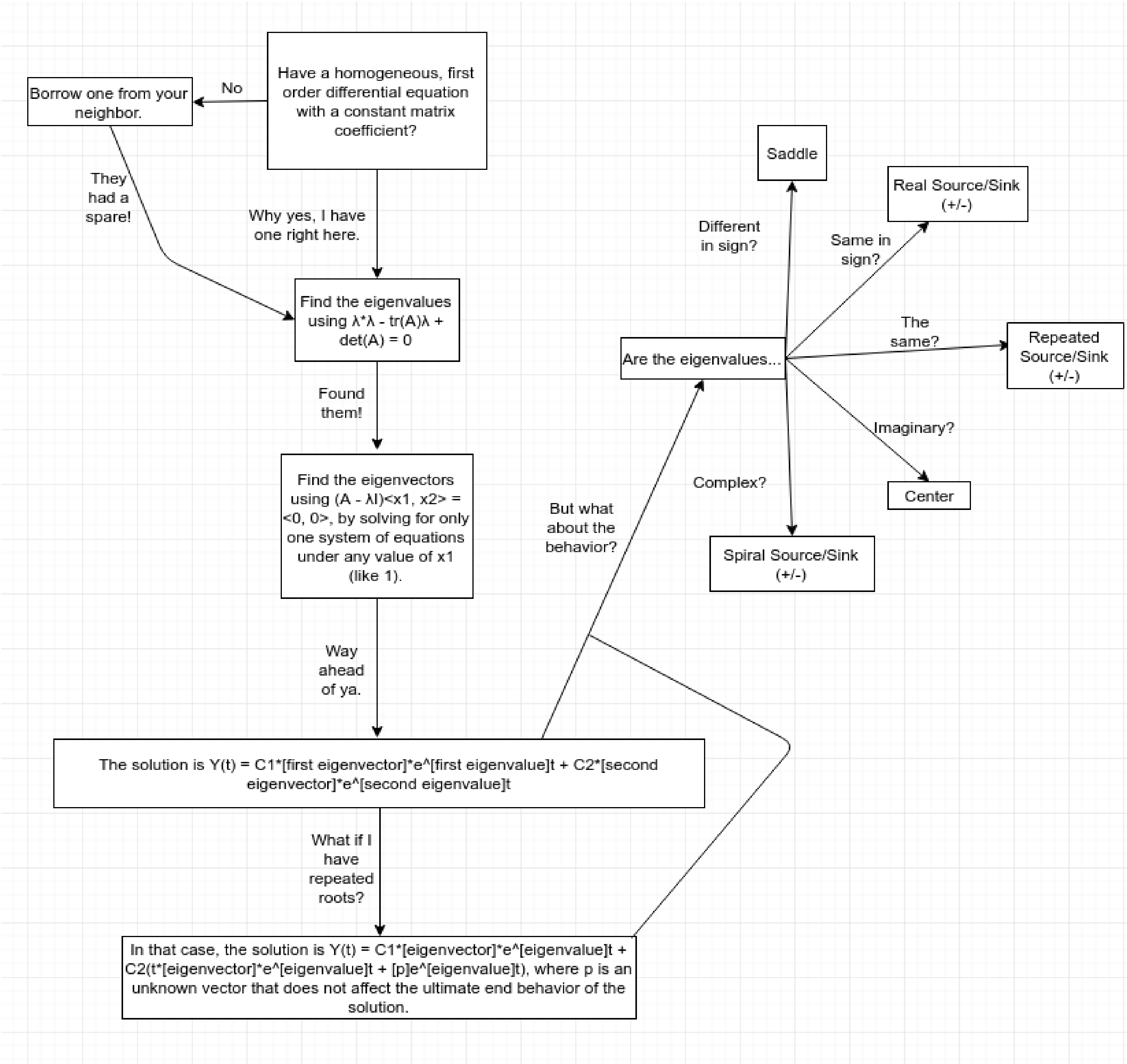

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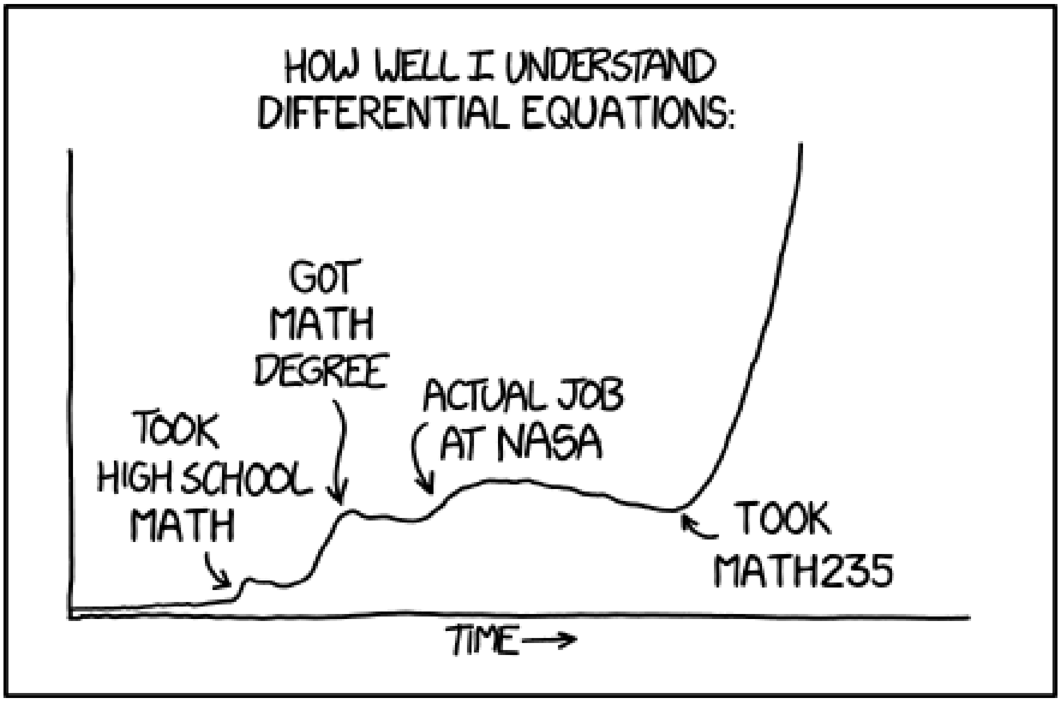

Courtesy of https://xkcd.com/1356/