Deepening the Automated Search for Gödel's Proofs

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Goal:
Formally verify the representability conditions that allow the metatheoretic notions to be represented inside the object-theory.

$$R1: \quad \mathrm{P}(X_1,\dots,X_n) \longrightarrow ZF \vdash p(\lfloor X_1 \rfloor,\dots,\lfloor X_n \rfloor)$$
$$R2: \mathrm{NotP}(X_1,\dots,X_n) \longrightarrow ZF \vdash \neg p(\lfloor X_1 \rfloor,\dots,\lfloor X_n \rfloor)$$

Metatheory

- Formal theory in the language of binary trees.
- Trees built up from the empty tree S.
- Inductively defined metatheoretic notions.
  - Formulae
  - Proofs
  - Theorems
- Each defined notion satisfies an induction principle.

Object-theory

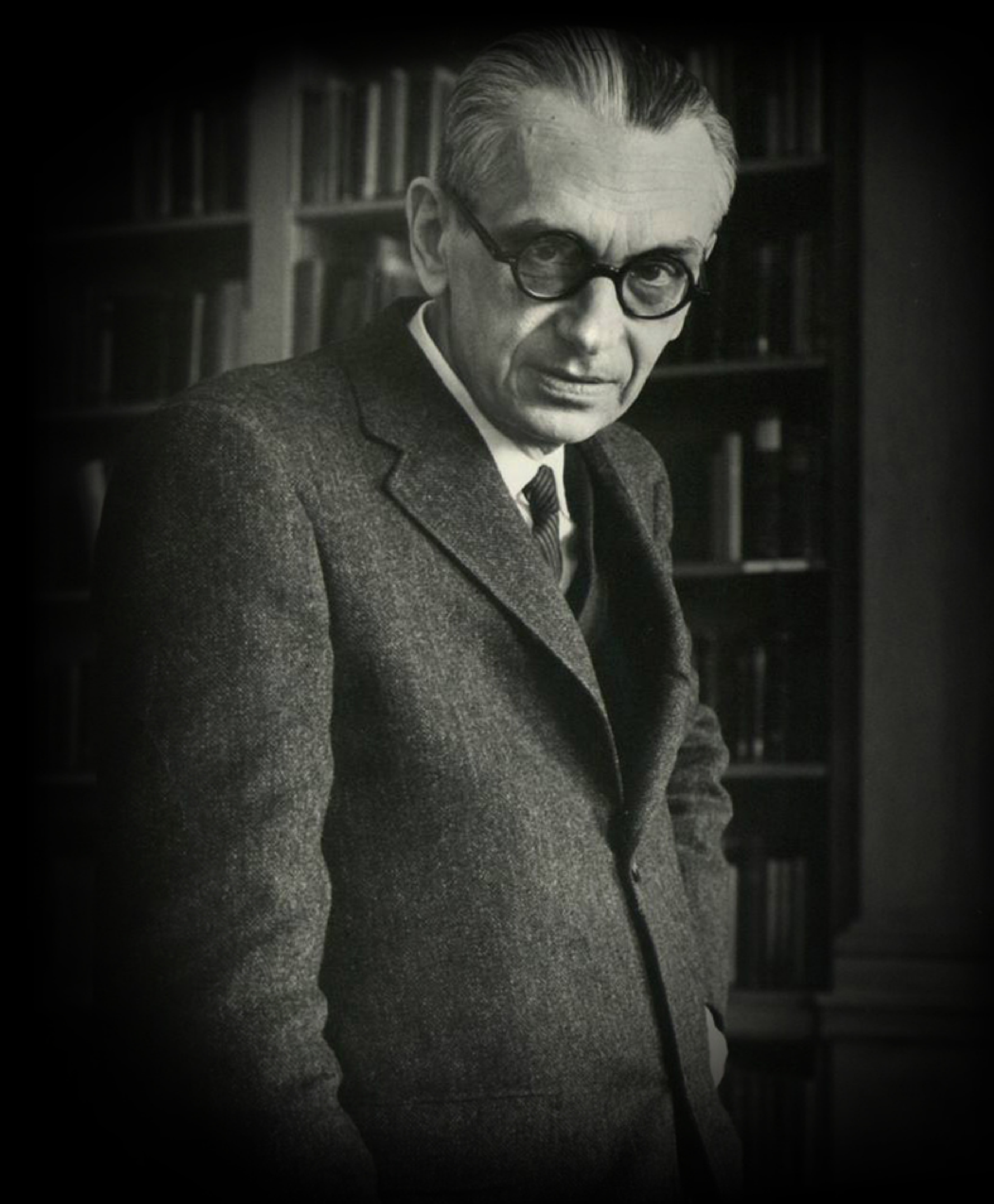
- Zermelo-Fraenkel Set Theory
- Modified Axiom of Infinity:

$$(\exists y)(\emptyset \in y \;\&\; (\forall x_1, x_2 \in y)\langle x_1, x_2\rangle \in y)$$

Coding:
Assign to each binary tree of the metatheory a canonical name in the language of set theory.

$$S \longmapsto \emptyset$$
$$[X, Y] \longmapsto \langle \lfloor X \rfloor, \lfloor Y \rfloor \rangle$$

Procedure:
1. Inductively define metatheoretic notion P.
2. Define a set in ZF to represent the metatheoretic notion that has a subset of basic elements and is closed under finitely many generating conditions.
3. Obtain the smallest such set p by taking the intersection of all sets satisfying those conditions.
4. Prove by induction that for every object X satisfying the metatheoretic definition, ZF $\vdash \lfloor X \rfloor \in p$
5. Show also for every object X satisfying the definition of NotP (also inductively defined) that ZF $\vdash \lfloor X \rfloor \notin p$



Kurt Gödel. 1906-1978

1931: *On formally undecidable propositions of Principia Mathematica and related systems I.*

Kurt Gödel established that mathematics cannot be fully formalized and that any formal system containing a modicum of number theory cannot prove its own consistency.

Example:
Formulae are representable in ZF.

Metatheoretic Definition

$$\begin{aligned}(\forall X)((&\mathrm{Atom}(X)\\ &\vee [X = [\neg, (X)_2] \;\&\; \mathrm{Form}((X)_2)]\\ &\vee [X = [\square, [(X)_{21}, (X)_{22}]] \;\&\; \mathrm{Form}((X)_{2i})]\\ &\vee [X = [[Q, (X)_{12}], (X)_2] \;\&\; \mathrm{Var}((X)_{12}) \;\&\; \mathrm{Form}((X)_2)])\\ &\longrightarrow \mathrm{Form}(X))\end{aligned}$$

Object-theoretic Definition

$$\begin{aligned}\mathbb{F} = \bigcap\{z \in \wp(\mathbb{B}) \mid \mathbb{A} \subseteq z\\ &\;\&\; (\forall x \in z)\langle \lfloor \neg \rfloor, x\rangle \in z\\ &\;\&\; (\forall x_1, x_2 \in z)\langle \lfloor \square \rfloor, \langle x_1, x_2\rangle\rangle \in z\\ &\;\&\; (\forall x_1 \in z)(\forall x_2 \in \mathbb{V})\langle\langle \lfloor Q \rfloor, x_2\rangle, x_1\rangle \in z\}\end{aligned}$$

Example Proof:
Inductive case for negation.

Induction Hypothesis: Assume for arbitrary X that

$$\mathrm{Form}((X)_2) \longrightarrow ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{F}$$

1		$X = [\neg, (X)_2]$	Premise
2		$\mathrm{Form}((X)_2)$	Premise
3		$ZF \vdash \lfloor (X)_2 \rfloor \in \mathbb{F}$	IH: 2
4	$*$	$\lfloor (X)_2 \rfloor \in \mathbb{F}$	ProvE: 3
5	$*$	$z \in f$	Assume
6	$*$	$(\forall x \in z)\langle \lfloor \neg \rfloor, x\rangle \in z$	DefE(f): 5
7	$*$	$\lfloor (X)_2 \rfloor \in z$	Lemma: 4
8	$*$	$\langle \lfloor \neg \rfloor, \lfloor (X)_2 \rfloor\rangle \in z$	$\forall_\in$E: 6,7
9	$*$	$\lfloor [\neg, (X)_2] \rfloor \in z$	DefI(Code): 8
10	$*$	$\lfloor X \rfloor \in z$	RepE(=): 9,1
11	$*$	$(\forall z \in f)\lfloor X \rfloor \in z$	$\forall_\in$I: 10
12	$*$	$\lfloor X \rfloor \in \bigcap f$	DefI($\cap$): 11
13	$*$	$\lfloor X \rfloor \in \mathbb{F}$	DefI($\mathbb{F}$): 12
14		$ZF \vdash \lfloor X \rfloor \in \mathbb{F}$	ProvI: 13