Collapsing Notions

1. Allow greater flexibility with conclusions from nested conjunctions. The numbers in the justifications below correspond to the position of the conjunct in the conjunction.

   Example Several conclusions can be made from the same line.

1	$\varphi(x) \,\&\, \varphi(y) \,\&\, \varphi(z)$	Premise
2	$\varphi(x)$	$\&E_1$: 1
3	$\varphi(z)$	$\&E_3$: 1
4	$\varphi(y) \,\&\, \varphi(x)$	$\&E_{2,1}$: 1

   Example Allows immediate combinations to produce more complex conclusions.

1	$u \in A \,\&\, u \in B \,\&\, u \in C$	Premise
2	$\varphi(u)$	Premise
3	$(\exists x \in B)\varphi(x)$	$\exists_\epsilon$I: 1,2

   An alternative is to use $(u \in A, u \in B, u \in C)$ in place of $(u \in A \,\&\, u \in B \,\&\, u \in C)$.

2. Abbreviate conjunctions expressing elementhood of several elements to the same set by:

   $$x_1 \in A \,\&\, \cdots \,\&\, x_n \in A \Leftrightarrow x_1, \dots, x_n \in A$$

   Example Several conslusions can be made from the same line.

1	$x, y, z \in \mathbb{N}$	Premise
2	$x \in \mathbb{N}$	$\in$E: 1
3	$z \in \mathbb{N}$	$\in$E: 1
4	$y, z \in \mathbb{N}$	$\in$E: 1

   Example Allows immediate combinations to produce more complex conclusions.

1	$x_1, x_2, x_3 \in \mathbb{N}$	Premise
2	$\varphi(x_2)$	Premise
3	$(\exists x \in \mathbb{N})\varphi(x)$	$\exists_\epsilon$I: 1,2

3. Abbreviate series of quantifiers.

   $$(\forall x_1)(\forall x_2)\cdots(\forall x_n)\varphi(x_1, x_2, \dots x_n) \Leftrightarrow (\forall x_1, x_2, \dots, x_n)\varphi(x_1, x_2, \dots x_n)$$

Example Multiple instantiations are handled in a single step.

Elimination

1	$(\forall x, y, z)\varphi(x, y, z)$	Premise
2	$\varphi(u, v, w)$	$\forall$E: 1

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1	$\vdots$	
2	$\varphi(u, v, w)$	$\cdots$
3	$(\forall x, y, z)\varphi(x, y, z)$	$\forall$I: 2

Similarly for existential quantifiers.

4. Abbreviate series of bounded quantifiers by combining the previous two notions.

$$(\forall x_1 \in A)(\forall x_2 \in A) \cdots (\forall x_n \in A)\varphi(x_1, x_2, \ldots, x_n) \Leftrightarrow (\forall x_1, x_2, \ldots, x_n \in A)\varphi(x_1, x_2, \ldots, x_n)$$

Example

Elimination

1	$u \in \mathbb{N}$	Premise
2	$v, w \in \mathbb{N}$	Premise
3	$(\forall x, y, z \in \mathbb{N})\varphi(x, y, z)$	Premise
4	$\varphi(u, v, w)$	$\forall_\epsilon$E: 3,1,2

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1	$\vdots$	
2	$u \in \mathbb{N}$	$\cdots$
3	$v \in \mathbb{N}$	$\cdots$
4	$\varphi(u, v)$	$\cdots$
5	$(\forall x, y \in \mathbb{N})\varphi(x, y)$	$\forall_\epsilon$I: 2,3,4

Similarly for existential quantifiers.

5. Automatically generate the appropriate structure for existentially-quantified conjunctions and goal universally-quantified conditionals/biconditionals.

Elimination (Note that this case relates to how conjunctions are used, see 1 and 2).

1	$(\exists x)(\varphi(x) \,\&\, \psi(x))$	Premise
2	$\varphi(u), \psi(u)$	Assume
3	$\vdots$	
4	*Goal*	$\cdots$
5	*Goal*	$\exists_\epsilon(\&)$E: 1,4

1	$(\forall x)(\varphi(x) \rightarrow \psi(x))$	Premise
2	$\varphi(u)$	Premise
3	$\psi(u)$	$\forall(\rightarrow)$E: 1,2

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1	$\varphi(u)$	Premise
2	$\psi(u)$	Premise
3	$(\exists x)(\varphi(x) \,\&\, \psi(x))$	$\exists(\&)$I: 1,2

1	$\quad \varphi(u)$	Assume
2	$\quad \vdots$	
3	$\quad \psi(u)$	$\ldots$
4	$(\forall x)(\varphi(x) \rightarrow \psi(x))$	$\forall(\rightarrow)$I: 3

Similarly for bounded quantifiers, differing only in the additional assumption or goal required by the bound.

Also, if a conjunction is present in the antecedent of a conditional, the conjuncts would be presented as assumptions:

Example

1	$\quad u \in \mathbb{N}, \varphi(u), \psi(u), \beta(u)$	Assume
2	$\quad \vdots$	
3	$\quad \alpha(u)$	$\ldots$
4	$(\forall x \in \mathbb{N})((\varphi(x) \,\&\, \psi(x) \,\&\, \beta(x)) \rightarrow \alpha(x))$	$\forall_\epsilon(\rightarrow)$I: 3

6. Freedom to manipulate the logical structure of a quantified formula within the scope of the quantifier.

   Examples

$$(\forall x)(\varphi(x) \,\&\, \psi(x)) \Leftrightarrow (\forall x)(\psi(x) \,\&\, \varphi(x))$$

$$(\forall x)(\forall y)\neg(\varphi(x) \vee \psi(y)) \Leftrightarrow (\forall x)(\forall y)(\neg\varphi(x) \,\&\, \neg\psi(y))$$

7. Take symmetry of equality as primitive.

   Example

$$u \in \{v\} \Leftrightarrow u = v$$

$$u \in \{v\} \Leftrightarrow v = u$$

8. Allow (1)partial conclusions to follow from definitions, and (2)definitions to follow from partial conclusions.

   Example(1)

$$x \in A \cap B \Rightarrow x \in A$$

   Example(2)

$$x \in A \Rightarrow x \in A \cup B$$

9. Introduce extensionality and subset definition directly.

Example

1	$\quad x \in A$	Assume
2	$\quad \vdots$	
3	$\quad x \in B$	$\cdots$
4	$\quad x \in B$	Assume
5	$\quad \vdots$	
6	$\quad x \in A$	$\cdots$
7	$A = B$	ExtI: 3,6

Example

1	$\quad x \in A$	Assume
2	$\quad \vdots$	
3	$\quad x \in B$	$\cdots$
4	$A \subseteq B$	DefI($\subseteq$)I: 3

10. Maintain a set of immediate contradictions (contradictory notions that are not contingent on the proof at hand and follow from the definition or our intuition about the notion of concern).

    Some examples of these notions include: $x \in \emptyset$, $x \neq x$, $x = s(x)$, $x < \emptyset$, etc.

    Example

1	$\quad x \in A$	Assume
2	$\quad \vdots$	
3	$\quad x \in \emptyset$	$\cdots$
4	$\quad x \in \emptyset$	Assume
5	$\quad x \in A$	Imm($\perp$): 4
6	$A = \emptyset$	ExtI: 3,5