NATURAL SYMMETRIES OF SECONDARY HOCHSCHILD HOMOLOGY

DAVID AYALA AND ADAM HOWARD

ABSTRACT. We identify the group of framed diffeomorphisms of the torus as a semi-direct product of the torus with the braid group on 3 strands; we also identify the topological monoid of framed local-diffeomorphisms of the torus in similar terms. It follows that the framed mapping class group is this braid group. We show that the group of framed diffeomorphisms of the torus acts on twice-iterated Hochschild homology, and explain how this recovers a host of familiar symmetries of such. In the case of a Cartesian symmetric monoidal structure, we show that this action extends to the monoid of framed local-diffeomorphisms of the torus. Based on this, we propose a definition of an unstable secondary cyclotomic structure, and show that iterated Hochschild homology possesses such.

CONTENTS

Yo Adam – do these things to the extent that you're able.

(1) Address all red comments in the body.
(2) On second thought, I think the notation $\widetilde{\mathsf{E}}_2^+(\mathbb{Z})$ isn't good. Perhaps better would be $\widetilde{\mathsf{E}}_2^+(\mathbb{Z})$, or something like that.

INTRODUCTION

This paper has two main results. The first main result identifies the continuous group of **framed diffeomorphisms**, and the continuous monoid of **framed local-diffeomorphisms**, of a framed torus. The second main result constructs actions of these monoids on iterated Hochschild homology. We abstract such action as **unstable secondary cyclotomic structure**. We state these results right away, as Theorem A and Theorem B.1, and direct a reader to the body of the paper for definitions and proofs.

Convention.

- Out of convenience, we work in the ∞-category Spaces of spaces, or ∞-groupoids, an object in which is a **space**. This ∞-category can be presented as the ∞-categorical localization of the ordinary category of compactly-generated Hausdorff topological spaces that are homotopy equivalent with a CW complex, localized on the weak homotopy equivalences. So we present some objects in Spaces by naming a topological space.

- By a pullback square among spaces we mean a pullback square in the ∞-category Spaces. Should the square be presented by a homotopy-commutative square among topological spaces, then the canonical map from the initial term in the square to the homotopy-pullback is a weak homotopy-equivalence.

- By a **continuous group** (resp. **continuous monoid**) we mean a group-object (resp. monoid-object) in Spaces.

- For $X \in \mathfrak{X}$ an object in an ∞-category, and for M a continuous monoid, an **action of M on X**, denoted $M \curvearrowright X$, is an extension $\langle X \rangle \colon * \to \mathfrak{B}M \xrightarrow{\langle M \curvearrowright X \rangle} \mathfrak{X}$. The ∞-category of **M-modules in $\mathfrak{X}$** is
$$\mathsf{Mod}_M(\mathfrak{X}) := \mathsf{Fun}(\mathfrak{B}M, \mathfrak{X}) .$$

- For $G \curvearrowright X$ an action of a continuous group on a space, the space of **coinvariants** is the colimit
$$X_{/G} := \mathsf{colim}\big(\mathsf{B}G \xrightarrow{\langle G \curvearrowright X \rangle} \mathsf{Spaces}\big) \in \mathsf{Spaces} .$$
Should the action $G \curvearrowright X$ be presented by a continuous action of a topological group on a topological space, then this space of coinvariants can be presented by the homotopy-coinvariants.

- We work with ∞-operads, as developed in [?]. As so, they are implicitly symmetric. Some ∞-operads are presented as discrete operads, such as Assoc, while some are presented as topological operads, such as the little 2-disks opera $\mathcal{E}_2$.

0.1. **Moduli and isogeny of framed tori.** Here we state our first result, which identifies the entire symmetries of a framed torus.

The **braid group on 3 strands** can be presented as

$$(1) \qquad\qquad \mathsf{Braid}_3 \cong \Big\langle \tau_1 , \tau_2 \mid \tau_1\tau_2\tau_1 = \tau_2\tau_1\tau_2 \Big\rangle .$$

Through this presentation, there is a standard representation
(2)
$$\Phi\colon \mathsf{Braid}_3 \xrightarrow{\big\langle\, \tau_1 \mapsto U_1 \,,\, \tau_2 \mapsto U_2\, \big\rangle} \mathsf{GL}_2(\mathbb{Z})\,, \qquad \text{where } U_1 := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U_2 := \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}\,.$$

The homomorphism Φ defines an action $\mathsf{Braid}_3 \xrightarrow{\Phi} \mathsf{GL}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group. This action defines a topological group:
$$\mathsf{Braid}_3 \ltimes \mathbb{T}^2\,.$$

The following result, which is essentially due to Milnor, is the starting point of this paper.

Proposition 0.1 ([**?**])**.** *The image of Φ is the subgroup $\mathsf{SL}_2(\mathbb{Z})$; the kernel of Φ is central, and is freely generated by the element $(\tau_1\tau_2)^6 \in \mathsf{Braid}_3$. In other words, Φ fits into a central extension among groups:*

(3)
$$1 \longrightarrow \mathbb{Z} \xrightarrow{\big\langle (\tau_1\tau_2)^6 \big\rangle} \mathsf{Braid}_3 \xrightarrow{\Phi} \mathsf{SL}_2(\mathbb{Z}) \longrightarrow 1\,.$$

Furthermore, this central extension (3) is classified by the element
$$\left[\mathsf{BSL}_2(\mathbb{Z}) \xrightarrow{\text{B standard}} \mathsf{BSL}_2(\mathbb{R}) \simeq \mathsf{B}^2\mathbb{Z} \right] \ \in \ \mathsf{H}^2\big(\mathsf{SL}_2(\mathbb{Z}); \mathbb{Z}\big)\,,$$

which is to say there is a canonical top horizontal homomorphism making a pullback among groups:

$$
\begin{array}{ccc}
\mathsf{Braid}_3 & \dashrightarrow & \widetilde{\mathsf{SL}}_2(\mathbb{R}) \\
{\scriptstyle \Phi}\big\downarrow & & \big\downarrow {\scriptstyle \text{universal cover}} \\
\mathsf{SL}_2(\mathbb{Z}) & \xrightarrow[\text{standard}]{\mathbb{R}\underset{\mathbb{Z}}{\otimes}} & \mathsf{SL}_2(\mathbb{R}).
\end{array}
$$

Consider the subgroup $\mathsf{GL}_2^+(\mathbb{R}) \subset \mathsf{GL}_2(\mathbb{R})$ consisting of those 2×2 matrices with positive determinant – it is the connected component of the identity matrix. Consider the submonoid
$$\mathbb{R}\underset{\mathbb{Z}}{\otimes}\colon \ \mathsf{E}_2^+(\mathbb{Z}) \ \subset \ \mathsf{GL}_2^+(\mathbb{R})$$
consisting of those 2×2 matrices with positive determinant whose entries are integers. Consider the pullback among monoids:

(4)
$$
\begin{array}{ccc}
\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) & \longrightarrow & \widetilde{\mathsf{GL}}_2^+(\mathbb{R}) \\
{\scriptstyle \Psi}\big\downarrow & & \big\downarrow {\scriptstyle \text{universal cover}} \\
\mathsf{E}_2^+(\mathbb{Z}) & \xrightarrow{\mathbb{R}\underset{\mathbb{Z}}{\otimes}} & \mathsf{GL}_2^+(\mathbb{R}).
\end{array}
$$

This morphism Ψ supplies a canonical action $\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \xrightarrow{\Psi} \mathsf{E}_2^+(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group. This action defines a topological monoid
$$\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2\,.$$

For $\varphi\colon \tau_{\mathbb{T}^2} \cong \epsilon_{\mathbb{T}^2}^2$ a framing of the torus, as Definition 1.14, we introduce the continuous group $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$ of **framed diffeomorphisms**, and the continuous monoid $\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$ of **framed local-diffeomorphisms** of the torus.

Theorem A.
   (1) *The set of framed-diffeomorphism-types of framed tori is canonically a torsor for $\mathbb{Z}^2$.*
   (2) *Let $\varphi\colon \epsilon_{\mathbb{T}^2} \cong \tau_{\mathbb{T}^2}$ be a framing of the torus.*
      (a) *There is a canonical identification of the continuous group of framed diffeomorphisms of $(\mathbb{T}^2, \varphi)$ with a semi-direct product of the torus with the braid group on 3 strands:*
$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \ \simeq \ \mathsf{Braid}_3 \ltimes \mathbb{T}^2\,.$$

(b) *There is a canonical identification of the continuous monoid of framed local-diffeomorphisms of $(\mathbb{T}^2, \varphi)$ as a semi-direct product of the torus with the monoid $\widetilde{\mathsf{E}}_2^+(\mathbb{Z})$:*

$$\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \; \simeq \; \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \; .$$

Taking path-components, Theorem A(2a) has the following immediate consequence.

Corollary 0.2. *Let φ be a framing of the torus. There is a canonical identification of the framed mapping class group of $(\mathbb{T}^2, \varphi)$ as the braid group on 3 strands:*

$$\mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \; \cong \; \mathsf{Braid}_3 \; .$$

Remark 0.3. Consider the moduli space $\boldsymbol{\mathcal{M}}_1^{\mathsf{fr}}$ of framed tori. Theorem A(1)&(2a) can be phrased as the assertion that $\boldsymbol{\mathcal{M}}_1^{\mathsf{fr}}$ has $\mathbb{Z}^2$-many path-components, each of which is the space of homotopy coinvariants $(\mathbb{CP}^\infty)^2_{\mathsf{h}\,\mathsf{Braid}_3}$ with respect to the action $\mathsf{Braid}_3 \xrightarrow{\Phi} \mathsf{GL}_2(\mathbb{Z}) \curvearrowright \mathsf{B}^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^{\times 2}$. A neat result of [?] gives an isomorphism between groups:

$$\mathsf{Braid}_3 \; \cong \; \pi_1(\mathbb{S}^3 \smallsetminus \mathsf{Trefoil}) \; .$$

Using that $\mathbb{S}^3 \smallsetminus \mathsf{Trefoil}$ is a path-connected 1-type, this isomorphism reveals that each path-component $(\boldsymbol{\mathcal{M}}_1^{\mathsf{fr}})_{[\varphi]} \subset \boldsymbol{\mathcal{M}}_1^{\mathsf{fr}}$ fits into a fiber sequence:

$$(\mathbb{CP}^\infty)^2 \longrightarrow (\boldsymbol{\mathcal{M}}_1^{\mathsf{fr}})_{[\varphi]} \longrightarrow \left(\mathbb{S}^3 \smallsetminus \mathsf{Trefoil}\right) \; .$$

Remark 0.4. Theorem A(2a) implies the equivalence-type of the continuous group $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$ is independent of the framing φ of the torus.

Theorem ?? of [?] identifies the oriented mapping class group of a punctured torus with parametrized boundary as the braid group on 3 strands, as it is equipped with a homomorphism to the oriented mapping class group of the torus. Together with Corollary 0.2, this results in an identification between mapping class groups. The next result lifts this identification to continuous groups; it is proved in §??.

Corollary 0.5. *Let φ be a framing of the torus. Fix a smooth framed embedding from the closed 2-disk $\mathbb{D}^2 \hookrightarrow \mathbb{T}^2$ extending the inclusion $\{0\} \hookrightarrow \mathbb{T}^2$ of the origin. There are canonical identifications among continuous groups over $\mathsf{Diff}(\mathbb{T}^2)$:*

$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \text{ rel } 0, \varphi) \; \simeq \; \mathsf{Braid}_3 \; \simeq \; \mathsf{Diff}(\mathbb{T}^2 \text{ rel } \mathbb{D}^2) \; .$$

In particular, there are canonical isomorphisms among groups over $\mathsf{MCG}(\mathbb{T}^2)$:

$$\mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \; \cong \; \mathsf{Braid}_3 \; \cong \; \mathsf{MCG}(\mathbb{T}^2 \smallsetminus \mathbb{B}^2 \text{ rel } \partial) \; ,$$

where $\mathbb{B}^2 \subset \mathbb{D}^2$ is the open 2-ball.

0.2. **Natural symmetries of iterated Hochschild homology.** Here we state our second main result, which identifies natural symmetries of ***secondary Hochschild homology***,

$$\mathsf{HH}^{(2)}(A) \; := \; \mathsf{HH}\big(\mathsf{HH}(A)\big) \; ,$$

which is simply twice-iterated Hochschild homology, of a ***2-algebra***

$$A \; \in \; \mathsf{Alg}_2(\mathcal{V}) \; := \; \mathsf{Alg}_{\mathsf{Assoc}}\big(\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V})\big) \; ,^1$$

which is simply an (associative) algebra among (associative) algebras in $\mathcal{V}$.

Notation 0.6. Here, and throughout this Section §0.2, we fix $\mathcal{V}$ to be a $\otimes$-presentable symmetric monoidal ∞-category.

---

[1]Dunn's additivity [?] (see also Theorem ?? of [?]), gives a canonical equivalence between ∞-categories $\mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xrightarrow{\simeq} \mathsf{Alg}_2(\mathcal{V})$ from that of algebras over the operad of little 2-disks, to that of 2-algebras. In particular, there is a canonical forgetful functor from commutative algebras $\mathsf{CAlg}(\mathcal{V}) \to \mathsf{Alg}_2(\mathcal{V})$, so each commutative algebra canonically determines a 2-algebra.

Theorem A(2a) has the following consequence, proved in §2.2 using factorization homology.

Theorem B.1. *Let $A \in \mathsf{Alg}_2(\mathcal{V})$ be a 2-algebra in a $\otimes$-presentable symmetric monoidal ∞-category $\mathcal{V}$. There is a canonical action of the continuous group $\mathsf{Braid}_3 \ltimes \mathbb{T}^2$ on iterated Hochschild homology:*

$$(5) \qquad \mathsf{Braid}_3 \ltimes \mathbb{T}^2 \; \curvearrowright \; \mathsf{HH}^{(2)}(A) \; .$$

We now explain how Theorem B.1 extends familiar, or at least expected, symmetries of $\mathsf{HH}^{(2)}(A)$, and how the action $\mathsf{Braid}_3 \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}^{(2)}(A)$ can be phrased in terms of these expected symmetries.

Let $\mathcal{W}$ be an $\otimes$-presentable symmetric monoidal ∞-category. Let B be an associative algebra in $\mathcal{W}$. Taking Hochschild homology with coefficients defines a functor from the ∞-category of (B,B)-bimodules:

$$\mathsf{BiMod}_{(B,B)} \xrightarrow{\; \mathsf{HH}(B,-) \;} \mathcal{W} \; .$$

Each endomorphism $B \xrightarrow{\sigma} B$ of the associative algebra B determines a (B,B)-bimodule structure B_σ on the underlying object B, which is characterized by $B \xrightarrow{\mathsf{id}} B$ being equivariant with respect to $(B,B) \xrightarrow{(\mathsf{id},\sigma)} (B,B)$. This assignment $\sigma \mapsto B_\sigma$ canonically assembles as a functor from the space of endomorphisms to the ∞-category of (B,B)-bimodules:

$$\mathsf{End}_{\mathsf{Alg}(\mathcal{W})}(B) \longrightarrow \mathsf{BiMod}_{(B,B)} \; , \qquad \sigma \mapsto B_\sigma \; .$$

This results in a composite functor

$$\mathsf{End}_{\mathsf{Alg}(\mathcal{W})}(B) \longrightarrow \mathsf{BiMod}_{(B,B)} \xrightarrow{\; \mathsf{HH}(B,-) \;} \mathcal{W} \; , \qquad \sigma \mapsto \mathsf{HH}(B, B_\sigma) \; .$$

This functor restricts to automorphisms of $\mathsf{id} \mapsto \mathsf{HH}(B, B_{\mathsf{id}}) = \mathsf{HH}(B)$ as a morphism between continuous groups:

$$(6) \qquad \Omega_{\mathsf{id}}\, \mathsf{Aut}_{\mathsf{Alg}(\mathcal{W})}(B) \; = \; \Omega_{\mathsf{id}}\, \mathsf{End}_{\mathsf{Alg}(\mathcal{W})}(B) \longrightarrow \mathsf{Aut}_{\mathcal{W}}\big(\mathsf{HH}(B)\big) \; .$$

Now, take $\mathcal{W} = \mathsf{Alg}(\mathcal{V})$ to be the ∞-category of associative algebras in an $\otimes$-presentable symmetric monoidal ∞-category $\mathcal{V}$, and take $B = \mathsf{HH}(A)$ to be the Hochschild homology of a 2-algebra $A \in \mathsf{Alg}\big(\mathsf{Alg}(\mathcal{V})\big)$. ***Connes' cyclic operator*** is a $\mathbb{T}$-action $\mathbb{T} \simeq \mathsf{B}\mathbb{Z} \curvearrowright \mathsf{HH}(A)$, which is canonically functorial in the argument A. This results in a morphism between continuous groups

$$(7) \qquad \mathbb{T} \xrightarrow{\; \big\langle \mathbb{T} \curvearrowright \mathsf{HH}(A) \big\rangle \;} \mathsf{Aut}_{\mathsf{Alg}(\mathcal{V})}\big(\mathsf{HH}(A)\big) \; .$$

We conclude a symmetry of Hochschild homology, which manifests as the composite morphisms between continuous groups:

$$\mathbb{Z} \; \simeq \; \Omega_1 \mathbb{T} \xrightarrow{\; \Omega(7) \;} \Omega_{\mathsf{id}}\, \mathsf{Aut}_{\mathsf{Alg}(\mathcal{V})}\big(\mathsf{HH}(A)\big) \xrightarrow{\; (6) \;} \mathsf{Aut}_{\mathcal{V}}\big(\mathsf{HH}^{(2)}(A)\big) \; .$$

This symmetry interacts with the two Connes' cyclic operators $\mathbb{T}^2 \curvearrowright \mathsf{HH}^{(2)}(A)$ by assembling as a ***sheer-action***,

$$(8) \qquad \mathsf{Sheer}_1 \colon \mathbb{Z} \underset{U_1}{\ltimes} \mathbb{T}^2 \; \curvearrowright \; \mathsf{HH}^{(2)}(A) \; ,$$

where this semi-direct product is defined by $\mathbb{Z} \xrightarrow{\langle U_1 \rangle} \mathsf{GL}_2(\mathbb{Z}) \simeq \mathsf{Aut}_{\mathsf{Groups}}(\mathbb{T}^2)$ (see (2)). Corollary 2.10 states that swapping the two associative algebra structures on A does not effect Hochschild homology. There results another ***sheer-action***:

$$(9) \qquad \mathsf{Sheer}_2 \colon \mathbb{Z} \underset{U_2}{\ltimes} \mathbb{T}^2 \xrightarrow[\cong]{\; (-1) \ltimes \mathsf{id} \;} \mathbb{Z} \underset{U_2^{-1}}{\ltimes} \mathbb{T}^2 \; \curvearrowright \; \mathsf{HH}^{(2)}(A) \; ,$$

where this semi-direct product is defined by $\mathbb{Z} \xrightarrow{\langle U_2 \rangle} \mathsf{GL}_2(\mathbb{Z}) \simeq \mathsf{Aut}_{\mathsf{Groups}}(\mathbb{T}^2)$ (see (2)).

The following result is proved in §**??**.

Corollary 0.7. *The action of Theorem B.1 is a canonical extension of these sheer-actions: for* $i = 1, 2$,

$$\mathsf{Sheer}_i \colon \mathbb{Z} \underset{U_i}{\ltimes} \mathbb{T}^2 \xrightarrow{\langle \tau_i \rangle \ltimes \mathsf{id}} \mathsf{Braid}_3 \ltimes \mathbb{T}^2 \underset{\mathrm{Thm\ B.1}}{\curvearrowright} \mathsf{HH}^{(2)}(A) \ .$$

Remark 0.8. One phrasing of Theorem B.1 is that the two actions (8) and (9) are not independent. For starters, these sheer-actions (8) and (9) define an action

$$(10) \qquad\qquad \mathsf{Sheers} \colon (\mathbb{Z} \amalg \mathbb{Z}) \underset{U_1, U_2}{\ltimes} \mathbb{T}^2 \curvearrowright \mathsf{HH}^{(2)}(A) \ . \ ^{2}$$

Next, the two symmetries $U_1, U_2 \in \mathsf{GL}_2(\mathbb{Z}) \simeq \mathsf{Aut}_{\mathsf{Groups}}(\mathbb{T}^2)$ satisfy the relation $R := U_1 U_2 U_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = U_2 U_1 U_2$. Denoting the generators $\langle \tau_1, \tau_2 \rangle = \mathbb{Z} \amalg \mathbb{Z}$, this results in two natural actions

$$(11) \qquad\qquad \mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2 \xrightarrow[\langle \tau_2 \tau_1 \tau_2 \rangle \ltimes \mathsf{id}]{\langle \tau_1 \tau_2 \tau_1 \rangle \ltimes \mathsf{id}} (\mathbb{Z} \amalg \mathbb{Z}) \underset{U_1, U_2}{\ltimes} \mathbb{T}^2 \underset{(10)}{\curvearrowright} \mathsf{HH}^{(2)}(A) \ .$$

Via the standard presentation (1) of Braid_3, Theorem B.1 can be precisely rephrased as an identification of these two actions (11).

Next, the short exact sequence (3) of Proposition 0.1 implies an identification between moduli spaces:

$$\Big\{ \text{ extensions of } \mathsf{Braid}_3 \curvearrowright \mathsf{HH}^{(2)}(A) \text{ along } \Phi \text{ to an action } \mathsf{SL}_2(\mathbb{Z}) \curvearrowright \mathsf{HH}^{(2)}(A) \Big\}$$

$$\simeq \Big\{ \text{ trivializations of } \mathsf{Ker}(\Phi) \curvearrowright \mathsf{HH}^{(2)}(A) \Big\} \ .$$

LaTexing issue. How to break the lines in the above braces into, like, three, and enlarge the braces, so that the above expression can fit into a single (horizontally aligned) display.

Remark 0.9. In the case that $\mathcal{V} = (\mathsf{Mod}_{\Bbbk}, \underset{\Bbbk}{\otimes})$ for some commutative ring $\Bbbk$, then an extension of $\mathsf{Braid}_3 \curvearrowright \mathsf{HH}^{(2)}(A)$ along Φ to $\mathsf{SL}_2(\mathbb{Z}) \curvearrowright \mathsf{HH}^{(2)}(A)$ is obstructed by the element

$$\big[\mathsf{HH}^{(2)}(\beta_A) \big] \ \in \ \mathsf{Ext}^1 \big(\mathsf{HH}^{(2)}(A), \Bbbk[\epsilon] \big) \ .$$

We are interested in identifying the action $\mathsf{Ker}(\Phi) \curvearrowright \mathsf{HH}^{(2)}(A)$ in familiar, or at least expected, terms. Corollary 0.13 does just this, in terms of the familiar/expected symmetry of secondary Hochschild homology given by ***braiding-conjugation***, as we now explain. A starting point for this symmetry is given from the following result which was essentially due to Dunn. Recall the topological operad $\mathcal{E}_2$ of little 2-disks, which we regard as an ∞-operad in the standard manner.

Theorem 0.10 ([**?**], Theorem ?? of [**?**])**.** *There is a canonical equivalence from the ∞-category of $\mathcal{E}_2$-algebras in $\mathcal{V}$ to that of 2-algebras in $\mathcal{V}$:*

$$\mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \xrightarrow{\ \simeq\ } \mathsf{Alg}_2(\mathcal{V}) \ .$$

After Theorem 0.10, the standard continuous action $\mathsf{O}(2) \curvearrowright \mathcal{E}_2$ on the topological operad immediately implies the following.

Corollary 0.11. *There is a canonical action of the continuous group $\mathsf{O}(2) \curvearrowright \mathsf{Alg}_2(\mathcal{V})$. In particular, for each 2-algebra A in $\mathcal{V}$, the orbit map with respect to the action of Corollary 0.11 lends to a canonical symmetry of A:*

$$\beta_A \colon \mathbb{Z} \ \simeq \ \Omega_{\mathbb{1}} \mathsf{SO}(2) \ = \ \Omega_{\mathbb{1}} \mathsf{O}(2) \xrightarrow{\ \Omega \mathsf{Orbit}_A\ } \mathsf{Aut}_{\mathsf{Alg}_2(\mathcal{V})}(A) \ .$$

---

[2]This pushout appearing here is in the category of groups, where it is often referred to as a *free product*.

Remark 0.12. This symmetry β_A on each 2-algebra A is ***braiding-conjugation***. For instance, this this symmetry β_A is the identity on the underlying object (so, $\beta_A(1) = \mathsf{id}_A$), and for $\mu \in \mathcal{E}_2(2)$ it supplies the commutativity of the diagram in $\mathcal{V}$,

$$
\begin{array}{ccc}
A \otimes A & \xrightarrow{\;\mathsf{id} \otimes \mathsf{id}\;} & A \otimes A \\
{\scriptstyle \mu_A} \downarrow & & \downarrow {\scriptstyle \mu_A} \\
A & \xrightarrow{\quad \mathsf{id} \quad} & A,
\end{array}
\qquad\qquad
\begin{array}{l}
\text{given by the point} \\[2mm]
\beta_A(2)\colon * \xrightarrow{\;\langle 1 \rangle\;} \mathbb{Z} \simeq \Omega_\mu \mathcal{E}_2(2) \to \Omega_{\mu_A} \mathsf{Hom}_{\mathcal{V}}(A \otimes A, A) \ .
\end{array}
$$

The following result is a direct consequence of Observation 1.17, and inspection of the action $\mathsf{Braid}_3 \curvearrowright \mathsf{HH}^{(2)}(A)$ of Theorem B.1, proved in §**??**.

Corollary 0.13. *Let A be a 2-algebra in $\mathcal{V}$. Through the action of Theorem B.1, the kernel of Φ acts $\mathsf{HH}^{(2)}(\beta_A)$: there is a canonically commutative diagram among continuous groups:*

$$
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{\qquad\qquad \beta_A \qquad\qquad} & \mathsf{Aut}_{\mathsf{Alg}_2(\mathcal{V})}(A) \\
{\scriptstyle \left\langle (\tau_1\tau_2)^6 \right\rangle} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \mathsf{HH}^{(2)}} \\
\mathsf{Ker}(\Phi) & \longrightarrow \mathsf{Braid}_3 \xrightarrow{\;\mathrm{Thm\ B.1}\;} & \mathsf{Aut}_{\mathcal{V}}\big(\mathsf{HH}^{(2)}(A)\big).
\end{array}
$$

In particular, there is the following immediate consequence of Proposition 0.1.

Corollary 0.14. *Let A be a 2-algebra in $\mathcal{V}$. An $\mathsf{SO}(2)$-fixed-point-structure on $A \in \mathsf{Alg}_2(\mathcal{V})$ determines a trivialization of the action $\mathsf{Ker}(\Phi) \curvearrowright \mathsf{HH}^{(2)}(A)$, and thereafter an extension along Φ of the actions $\mathsf{Braid}_3 \to \mathsf{Braid}_3 \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}^{(2)}(A)$ to actions*

$$
\mathsf{SL}_2(\mathbb{Z}) \to \mathsf{SL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \ \curvearrowright \ \mathsf{HH}^{(2)}(A) \ .
$$

Remark 0.15. It is not generally true that the action $\mathsf{Braid}_3 \curvearrowright \mathsf{HH}^{(2)}(A)$ extends along Φ as an action $\mathsf{SL}_2(\mathbb{Z}) \curvearrowright \mathsf{HH}^{(2)}(A)$. Examples demonstrating this can be derived from [**?**].

0.3. **Isogenic symmetries of secondary Hochschild homology.** Recall the ∞-category $\mathsf{Orbit}_{\mathbb{T}^2}^{\mathsf{fin}}$ of transitive $\mathbb{T}^2$-spaces with finite isotropy, and $\mathbb{T}^2$-equivariant maps between them. For $\mathcal{X}$ an ∞-category, the ∞-category of ***genuine finite $\mathbb{T}^2$-objects in*** $\mathcal{X}$ is

$$
\mathcal{X}^{\mathsf{g}_{\mathsf{fin}}.\mathbb{T}^2} \ := \ \mathsf{Fun}\big((\mathsf{Orbit}_{\mathbb{T}^2}^{\mathsf{fin}})^{\mathsf{op}}, \mathcal{X}\big) \ .
$$

The action $\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \to \mathsf{E}_2^+(\mathbb{Z}) \curvearrowright \mathbb{T}^2$ as a topological group supplies an action:

(12)
$$
\widetilde{\mathsf{E}}_2^+(\mathbb{Z})^{\mathsf{op}} \ \curvearrowright \ \mathcal{X}^{\mathsf{g}_{\mathsf{fin}}.\mathbb{T}^2} \ .
$$

We propose the following.

Definition 0.16. The ∞-category of ***unstable 2-cyclotomic objects*** in an ∞-category $\mathcal{X}$ is

$$
\mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{X}) \ := \ \big(\mathcal{X}^{\mathsf{g}_{\mathsf{fin}}.\mathbb{T}^2}\big)^{\mathsf{h}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z})} \ .
$$

Theorem A(2a) has the following consequence, which is proved in §**??**.

Corollary 0.17. *Let $\mathcal{X}$ be an ∞-category. Let φ be a framing of the torus. There is a canonical equivalence between the ∞-category of $\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$-modules in $\mathcal{X}$ and the left??-lax invariants with respect to the action (12):*

$$
\mathsf{Mod}_{\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2,\varphi)}(\mathcal{X}) \ \xrightarrow{\ \simeq\ } \ \big(\mathcal{X}^{\mathsf{g}_{\mathsf{fin}}.\mathbb{T}^2}\big)^{\mathsf{l.lax}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z})} \ .
$$

In particular, there is a canonical fully faithful functor from the (strict) invariants:

$$
\mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{X}) \ \hookrightarrow \ \mathsf{Mod}_{\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2,\varphi)}(\mathcal{X}) \ .
$$

Remark 0.18. The underpinning of Corollary 0.17 is that the group Braid_3 has no non-trivial finite subgroups, unlike the group $\mathsf{SL}_2(\mathbb{Z})$ which has the single non-trivial finite subgroup: $\left\langle \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle \subset \mathsf{SL}_2(\mathbb{Z})$.

Theorem B.2. *Let $\mathcal{X}$ be a presentable ∞-category in which finite products distribute over colimits separately in each variable. Regard $\mathcal{X}$ as a symmetric monoidal ∞-category via the Cartesian symmetric monoidal structure. For each 2-algebra $A \in \mathsf{Alg}_2(\mathcal{X})$, the action (5) of Theorem B.1 canonically extends*

$$(13) \qquad\qquad \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \; \curvearrowright \; \mathsf{HH}^{(2)}(A) \; .$$

Furthermore, through the fully faithful functor of Corollary 0.17, the isogeny-action (13) defines an unstable 2-cyclotomic structure on the iterated Hochschild homology of A:

$$\left(\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \; \curvearrowright \; \mathsf{HH}^{(2)}(A) \right) \; \in \; \mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{X}) \; .$$

Remark 0.19. We explain a relationship between an unstable secondary cyclotomic structure and an iterated unstable cyclotomic structure. Notice the commutative diagram among groups,

$$
\begin{array}{ccc}
\mathbb{N}^\times \times \mathbb{N}^\times & \dashrightarrow & \mathbb{R}_{>0}^\times \times \mathbb{R}_{>0}^\times \\
\downarrow & & \downarrow \\
\mathsf{E}_2^+(\mathbb{Z}) & \xrightarrow{\;\; \mathbb{R} \otimes_{\mathbb{Z}} \;\;} & \mathsf{GL}_2^+(\mathbb{R}),
\end{array}
$$

in which the downward homomorphisms are inclusions as diagonal matrices: $(r,s) \mapsto \begin{bmatrix} s & 0 \\ 0 & r \end{bmatrix}$, Contractibility of the continuous group $\mathbb{R}_{>0}^\times$ thusly supplies a lift there is a continuous section

$$(14) \qquad\qquad \mathbb{N}^\times \times \mathbb{N}^\times \longrightarrow \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \; .$$

With respect to this morphism between monoids (14), the product isomorphism $\mathbb{T} \times \mathbb{T} \xrightarrow{\;\times\;} \mathbb{T}^2$ is equivariant. For $\mathcal{X}$ an ∞-category, this results in a forgetful functor from unstable secondary cyclotomic objects to iterated unstable cyclotomic objects:

$$\mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{X}) \longrightarrow \mathsf{Cyc}^{\mathsf{un}}\big(\mathsf{Cyc}^{\mathsf{un}}(\mathcal{X}) \big) \; .$$

This functor is generally not an equivalence.[3]

(This whole subsection might reasonably be removed.)

---

[3](This footnote could be commented out entirely.)
Indeed, first note that it is evidently $\mathbb{T}^2$-equivariant. Next, consider the commutative square among ∞-categories, and restriction functors among them:

$$
\begin{array}{ccc}
\mathsf{Mod}_{\widetilde{\mathsf{GL}}_2^+(\mathbb{Q})}(\mathcal{X}) & \longrightarrow & \mathsf{Cyc}^{(2)}(\mathcal{X})^{\mathbb{T}^2} \\
\downarrow & & \downarrow \\
\mathsf{Mod}_{(\mathbb{Q}_{>0}^\times)^2}(\mathcal{X}) & \longrightarrow & \mathsf{Cyc}\big(\mathsf{Cyc}(\mathcal{X}) \big)^{\mathbb{T}^2}.
\end{array}
$$

Proposition 1.21 implies the horizontal functors are fully faithful. To finish, simply note that the left vertical restriction functor is not generally fully faithful, as demonstrated by the homomorphism between group cohomologies with coefficients in a commutative ring $\Bbbk$,

$$\mathsf{H}^*(\widetilde{\mathsf{GL}}_2^+(\mathbb{Q}); \Bbbk) \longrightarrow \mathsf{H}^*((\mathbb{Q}_{>0}^\times)^2; \Bbbk) \; ,$$
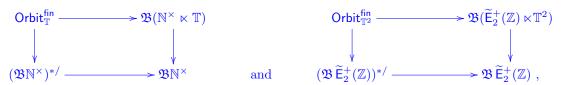
not being injective.

0.4. **Analogy with dimension 1.** Theorems A and B.1, as well as Definition 0.16, are in precise analogy with the established 1-dimensional array of results and definitions. We explain.

(1) There are pullback squares among monoid-objects in the ∞-category Spaces:

$$
\begin{array}{ccc}
\mathbb{N}^{\times} & \longrightarrow & \mathsf{End}^{\det \neq 0}(\mathbb{Z}) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathsf{End}^{\det \neq 0}(\mathbb{R}) \simeq \mathsf{O}(1)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) & \longrightarrow & \mathsf{E}_2(\mathbb{Z}) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \mathsf{End}^{\det \neq 0}(\mathbb{R}^2) \simeq \mathsf{O}(2).
\end{array}
$$

(2) There are pullback squares among ∞-categories:

$$
\begin{array}{ccc}
\mathsf{Orbit}_{\mathbb{T}}^{\mathsf{fin}} & \longrightarrow & \mathfrak{B}(\mathbb{N}^{\times} \ltimes \mathbb{T}) \\
\downarrow & & \downarrow \\
(\mathfrak{B}\mathbb{N}^{\times})^{*/} & \longrightarrow & \mathfrak{B}\mathbb{N}^{\times}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathsf{Orbit}_{\mathbb{T}^2}^{\mathsf{fin}} & \longrightarrow & \mathfrak{B}(\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2) \\
\downarrow & & \downarrow \\
(\mathfrak{B}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z}))^{*/} & \longrightarrow & \mathfrak{B}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z})
\end{array},
$$

which in particular codify actions as ∞-categories:

$$
\mathbb{N}^{\times} \curvearrowright \mathsf{Orbit}_{\mathbb{T}}^{\mathsf{fin}} \qquad \text{and} \qquad \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \curvearrowright \mathsf{Orbit}_{\mathbb{T}^2}^{\mathsf{fin}}.
$$

(3) For each ∞-category $\mathcal{X}$, there are identifications between ∞-categories of modules and of left-lax fixed-points:

$$
\mathsf{Mod}_{\mathbb{N}^{\times} \ltimes \mathbb{T}}(\mathcal{X}) \simeq \left(\mathcal{X}^{\mathsf{g_{fin}}.\mathbb{T}}\right)^{\mathsf{l.lax}\mathbb{N}^{\times}} \qquad \text{and} \qquad \mathsf{Mod}_{\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2}(\mathcal{X}) \simeq \left(\mathcal{X}^{\mathsf{g_{fin}}.\mathbb{T}^2}\right)^{\mathsf{l.lax}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z})}.
$$

These identifications yield fully faithful functors

$$
\tag{15}
\mathsf{Cyc}^{\mathsf{un}}(\mathcal{X}) := \left(\mathcal{X}^{\mathsf{g_{fin}}.\mathbb{T}}\right)^{\mathsf{h}\mathbb{N}^{\times}} \hookrightarrow \mathsf{Mod}_{\mathbb{N}^{\times} \ltimes \mathbb{T}}(\mathcal{X}) \qquad \text{and} \qquad \mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{X}) := \left(\mathcal{X}^{\mathsf{g_{fin}}.\mathbb{T}^2}\right)^{\mathsf{h}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z})} \hookrightarrow \mathsf{Mod}_{\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2}(\mathcal{X}).
$$

(4) There are canonical identifications between monoid-objects in the ∞-category Spaces:

$$
\mathbb{N}^{\times} \ltimes \mathbb{T} \xrightarrow{\simeq} \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}) \qquad \text{and} \qquad \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\simeq} \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2).
$$

(5) For $\mathcal{V}$ a $\otimes$-presentably symmetric monoidal ∞-category, there are canonical actions

$$
\mathbb{T} \curvearrowright \mathsf{HH}(A) \qquad \text{and} \qquad \mathsf{Braid}_3 \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}\big(\mathsf{HH}(A_2)\big).
$$

If the monoidal structure of $\mathcal{V}$ is Cartesian, these actions canonically extend as actions:

$$
\mathbb{N}^{\times} \ltimes \mathbb{T} \curvearrowright \mathsf{HH}(A) \qquad \text{and} \qquad \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}\big(\mathsf{HH}(A_2)\big).
$$

Through the fully faithful functors (15), these extended actions define objects:

$$
\Big(\mathbb{N}^{\times} \ltimes \mathbb{T} \curvearrowright \mathsf{HH}(A)\Big) \in \mathsf{Cyc}^{\mathsf{un}}(\mathcal{V}) \qquad \text{and} \qquad \Big(\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}\big(\mathsf{HH}(A_2)\big)\Big) \in \mathsf{Cyc}^{\mathsf{un}(2)}(\mathcal{V}).
$$

0.5. **Remarks on secondary cyclotomic trace.** We see the role of Corollary 0.17 as informing an approach to secondary cyclotomic traces.

Let $\Bbbk$ be a commutative ring spectrum. Let $A \in \mathsf{Alg}_2(\mathsf{Mod}_{\Bbbk})$. Recall the $\Bbbk$-linear Dennis trace map: $\mathsf{K}(A) \xrightarrow{\mathsf{tr}} \mathsf{HH}(A)$ (see [**?**]). The cyclic trace map is a canonical factorization of this Dennis trace map through negative-cyclic homology $\mathsf{K}(A) \xrightarrow{\mathsf{tr}^{\mathbb{T}}} \mathsf{HH}^-(A) := \mathsf{HH}(A)^{\mathbb{T}}$ (see [**?**]). Iterating this cyclic trace map results in a map between $\Bbbk$-modules:

$$
\mathsf{K}\big(\mathsf{K}(A)\big) \xrightarrow{\mathsf{tr}^{\mathbb{T}}(\mathsf{tr}^{\mathbb{T}})} \mathsf{HH}^-\big(\mathsf{HH}^-(A)\big).
$$

We expect the work of [**?**] on universal properties of secondary K-theory to yield a solution to the following.

Conjecture 1. *The iterated cyclic trace map factors through the* Braid_3*-fixed points:*
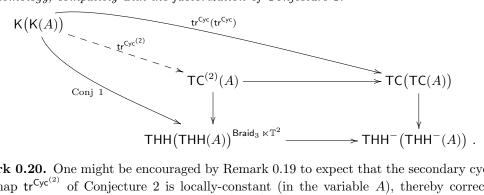
$$
\begin{array}{ccc}
& & \mathsf{HH}\big(\mathsf{HH}(A)\big)^{\mathsf{Braid}_3 \ltimes \mathbb{T}^2} \\
& \nearrow & \downarrow \\
\mathsf{K}\big(\mathsf{K}(A)\big) & \xrightarrow{\;\mathsf{tr}^{\mathbb{T}}(\mathsf{tr}^{\mathbb{T}})\;} & \mathsf{HH}^-\big(\mathsf{HH}^-(A)\big).
\end{array}
$$

For the case in which $\mathbb{k} = \mathbb{S}$ is the sphere spectrum, where standard notation is $\mathsf{HH} = \mathsf{THH}$ and referred to as *topological* Hochschild homology, the cyclic trace map factors further through the topological cyclotomic homology, $\mathsf{K}(A) \xrightarrow{\mathsf{tr}^{\mathsf{Cyc}}} \mathsf{TC}(A) := \mathsf{THH}(A)^{\mathsf{Cyc}}$, which is the *cylotomic* fixed-points with respect to a canonical *cyclotomic structure* on topological Hochschild homology. Iterating this cyclotomic trace map results in a map between spectra:

$$
\mathsf{K}\big(\mathsf{K}(A)\big) \xrightarrow{\;\mathsf{tr}^{\mathsf{Cyc}}(\mathsf{tr}^{\mathsf{Cyc}})\;} \mathsf{TC}\big(\mathsf{TC}(A)\big) \ .
$$

Following the developments in **??**, we further expect Definition 0.16 to lend to a definition of a (stable) secondary cyclotomic object, and that secondary topological Hochschild homology canonically admits a secondary cyclotomic structure. For secondary topological cyclotomic homology to be the fixed-points with respect to this structure, $\mathsf{TC}^{(2)}(A) := \mathsf{THH}\big(\mathsf{THH}(A)\big)^{\mathsf{Cyc}^{(2)}}$, we expect the work [**?**] on secondary K-theory to further lend a secondary cyclotomic trace map, which we state as the following.

Conjecture 2. *The iterated cyclotomic trace map factors through the secondary topological cyclotomic homology, compatibly with the factorization of Conjecture 1:*

$$
\begin{array}{ccccc}
\mathsf{K}\big(\mathsf{K}(A)\big) & & \xrightarrow{\;\mathsf{tr}^{\mathsf{Cyc}}(\mathsf{tr}^{\mathsf{Cyc}})\;} & & \\
& \searrow^{\mathsf{tr}^{\mathsf{Cyc}^{(2)}}} & & & \\
\downarrow_{\mathsf{Conj\ 1}} & & \mathsf{TC}^{(2)}(A) \longrightarrow & & \mathsf{TC}\big(\mathsf{TC}(A)\big) \\
& & \downarrow & & \downarrow \\
& \mathsf{THH}\big(\mathsf{THH}(A)\big)^{\mathsf{Braid}_3 \ltimes \mathbb{T}^2} & \longrightarrow & & \mathsf{THH}^-\big(\mathsf{THH}^-(A)\big) \ .
\end{array}
$$

Remark 0.20. One might be encouraged by Remark 0.19 to expect that the secondary cyclotomic trace map $\mathsf{tr}^{\mathsf{Cyc}^{(2)}}$ of Conjecture 2 is locally-constant (in the variable A), thereby correcting the failure for the iterated cyclotomic trace map $\mathsf{tr}^{\mathsf{Cyc}}(\mathsf{tr}^{\mathsf{Cyc}})$ to be locally-constant. However, we do not expect for this to be so. Namely, the local-constancy of the cyclotomic trace map $\mathsf{K}(A) \xrightarrow{\mathsf{tr}^{\mathsf{Cyc}}} \mathsf{TC}(A)$ relies in an essential way on calculations of [**?**], which identify the fiber of the canonical map $\mathsf{TC}(A \ltimes V) \to \mathsf{TC}(A)$ associated to a square-zero extension of A. These calculations in turn rely on the simple observation that, for each $i \geq 0$, the canonical action $\mathbb{T} \simeq \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}) \curvearrowright \mathsf{Conf}_i(\mathbb{T})_{\Sigma_i}$ on unordered configuration space canonically factors as a $\mathbb{T}_{/\mathsf{C}_i}$-torsor. Because the canonical action $\mathsf{Braid}_3 \ltimes \mathbb{T}^2 \simeq \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \curvearrowright \mathsf{Conf}_i(\mathbb{T}^2)_{\Sigma_i}$ does not apparently have any such property, we do not expect for the secondary cyclotomic trace map of Conjecture 1 to be locally constant.

1. Moduli and isogeny of framed tori

1.1. **Moduli and isogeny of tori.** Vector addition, as well as the standard vector norm, gives $\mathbb{R}^2$ the structure of a topological abelian group. Consider its closed subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$. The **torus** is the quotient in the short exact sequence of topological abelian groups:

$$
0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\;\mathsf{inclusion}\;} \mathbb{R}^2 \xrightarrow{\;\mathsf{quot}\;} \mathbb{T}^2 \longrightarrow 0 \ .
$$

Because $\mathbb{R}^2$ is connected, and because $\mathbb{Z}^2$ acts cocompactly by translations on $\mathbb{R}^2$, the torus $\mathbb{T}^2$ is connected and compact. The quotient map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ endows the torus with the structure of a Lie group, and in particular a smooth manifold. Consider the submonoid

$$\mathsf{E}_2(\mathbb{Z}) \; := \; \left\{ \mathbb{Z}^2 \xrightarrow{A} \mathbb{Z}^2 \mid \det(A) \neq 0 \right\} \; \subset \; \mathsf{End}_{\mathsf{Groups}}(\mathbb{Z}^2) \; ,$$

consisting of the cofinite endomorphisms of the group $\mathbb{Z}^2$. Using that the smooth map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ is a covering space and $\mathbb{T}^2$ is connected, there is a canonical continuous action on the topological group:

(16) $$\mathsf{E}_2(\mathbb{Z}) \curvearrowright \mathbb{T}^2 \; , \qquad A \cdot q := \mathsf{quot}(A\widetilde{q}) \qquad \left(\text{for any } \widetilde{q} \in \mathsf{quot}^{-1}(q) \right) .[4]$$

This homomorphism (16) defines a semi-direct product topological monoid:

$$\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \; .$$

Consider the topological monoid of smooth local-diffeomorphisms of the torus:

$$\mathsf{Imm}(\mathbb{T}^2) \; \subset \; \mathsf{Map}(\mathbb{T}^2, \mathbb{T}^2) \; ,$$

which is endowed with the subspace topology of the C^∞-topology on the set of smooth self-maps of the torus.

Observation 1.1.
    (1) The standard inclusion $\mathsf{GL}_2(\mathbb{Z}) \hookrightarrow \mathsf{E}_2(\mathbb{Z})$ witnesses the maximal subgroup. It follows that the standard inclusion $\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \hookrightarrow \mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2$ witnesses the maximal subgroup, both as topological monoids and as monoid-objects in the ∞-category $\mathcal{S}\mathsf{paces}$.
    (2) The standard monomorphism $\mathsf{Diff}(\mathbb{T}^2) \hookrightarrow \mathsf{Imm}(\mathbb{T}^2)$ witnesses the maximal subgroup, both as topological monoids and as monoid-objects in the ∞-category $\mathcal{S}\mathsf{paces}$.

Consider the morphism between topological monoids:

(17) $$\mathsf{Aff} : \mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \longrightarrow \mathsf{Imm}(\mathbb{T}^2) \; , \qquad (A, p) \mapsto \left(q \mapsto Aq + p \right) .$$

We record the following classical result.

Lemma 1.2. *The restriction of the morphism (17) to maximal subgroups is a homotopy-equivalence:*

$$\mathsf{Aff} : \; \mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\; \simeq \;} \mathsf{Diff}(\mathbb{T}^2) \; , \qquad (A, p) \mapsto \left(q \mapsto Aq + p \right) .$$

Proof. Let G be a locally path-connected topological group, which we regard as a continuous group. Denote by $G_\mathbb{1} \subset G$ the path-component containing the identity element in G. This subspace $G_\mathbb{1} \subset G$ is a normal subgroup, and the sequence of continuous homomorphisms

$$1 \longrightarrow G_\mathbb{1} \xrightarrow{\;\text{inclusion}\;} G \xrightarrow{\;\text{quotient}\;} \pi_0(G) \longrightarrow 1$$

is a fiber-sequence among continuous groups. This fiber sequence is evidently functorial in the argument G. In particular, there is a commutative diagram among topological groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{T}^2 = \left(\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \right)_\mathbb{1} & \xrightarrow{\;\text{inc}\;} & \mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{\;\text{quot}\;} & \pi_0\left(\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \right) = \mathsf{GL}_2(\mathbb{Z}) & \longrightarrow & 1 \\
& = \Big\downarrow & \Big\downarrow{\scriptstyle \mathsf{Aff}_\mathbb{1}} & & \Big\downarrow{\scriptstyle \mathsf{Aff}} & & \Big\downarrow{\scriptstyle \pi_0(\mathsf{Aff})} & & \Big\downarrow{\scriptstyle =} \\
1 & \longrightarrow & \mathsf{Diff}(\mathbb{T}^2)_\mathbb{1} & \xrightarrow{\;\text{inc}\;} & \mathsf{Diff}(\mathbb{T}^2) & \xrightarrow{\;\text{quot}\;} & \pi_0\left(\mathsf{Diff}(\mathbb{T}^2) \right) & \longrightarrow & 1.
\end{array}
$$

Now, by the 5-lemma applied to homotopy groups, if both horizontal sequences are exact, if the vertical homomorphisms $\mathsf{Aff}_\mathbb{1}$ and $\pi_0(\mathsf{Aff})$ are homotopy equivalences, then the vertical homomorphism Aff is a homotopy equivalence as well.

Theorem 2.D.4 of [4], along with the Kirby torus trick Improve citation here, implies $\pi_0(\mathsf{Aff})$ is an isomorphism.

----

[4]Note that (16) indeed does not depend on $\widetilde{q} \in \mathsf{quot}^{-1}(q)$.

It remains to show $\mathsf{Aff}_{\mathbb{1}}$ is a homotopy equivalence. With respect to the canonical continuous action $\mathsf{Diff}(\mathbb{T}^2)_{\mathbb{1}} \curvearrowright \mathbb{T}^2$, the orbit of the identity element $0 \in \mathbb{T}^2$ is the evaluation map

(18) $$\mathsf{ev}_0 \colon \mathsf{Diff}(\mathbb{T}^2)_{\mathbb{1}} \longrightarrow \mathbb{T}^2 \ .$$

Note that the composition,

$$\mathsf{id} \colon \mathbb{T}^2 \xrightarrow{\ \mathsf{Aff}_{\mathbb{1}}\ } \mathsf{Diff}(\mathbb{T}^2)_{\mathbb{1}} \xrightarrow{\ (18)\ } \mathbb{T}^2 \ ,$$

is the identity map. Now, Theorem 1b of [1] gives that the subspace of those diffeomorphisms of the torus that fix $0 \in \mathbb{T}^2$ has contractible path-components. This implies the stabilizer $\mathsf{Stab}_0\big(\mathsf{Diff}(\mathbb{T}^2)_{\mathbb{1}}\big)$ is contractible, which implies $\mathsf{Aff}_{\mathbb{1}}$ is a homotopy equivalence.

$\square$

Remark 1.3. By the classification of compact surfaces, the moduli space $\boldsymbol{\mathcal{M}}_1$ of smooth tori is path-connected, and as so is

$$\boldsymbol{\mathcal{M}}_1 \ \simeq \ \mathsf{BDiff}(\mathbb{T}^2) \underset{\mathrm{Lem}\ 1.2}{\simeq} \mathsf{B}\big(\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \ .$$

In particular, this path-connected moduli space fits into a fiber sequence

$$(\mathbb{CP}^\infty)^2 \longrightarrow \boldsymbol{\mathcal{M}}_1 \longrightarrow \mathsf{BGL}_2(\mathbb{Z}) \ ,$$

which is classified by the standard action $\mathsf{GL}_2(\mathbb{Z}) \curvearrowright \mathsf{B}^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^2$.

Consider the set of **cofinite subgroups** of $\mathbb{Z}^2$:

$$\mathcal{L}(2) \ := \ \Big\{ \Lambda \underset{\mathrm{cofin}}{\subseteq} \mathbb{Z}^2 \Big\} \ .$$

Observation 1.4.

(1) The orbit-stabilizer theorem immediately verifies the composite map $\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\ \mathsf{pr}\ } \mathsf{E}_2(\mathbb{Z}) \xrightarrow{\ \mathsf{Image}\ } \mathcal{L}(2)$ witnesses the quotient:

$$\mathsf{E}_2(\mathbb{Z})_{/\,\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2} \ltimes \mathbb{T}^2 \xrightarrow{\ \cong\ } \mathsf{E}_2(\mathbb{Z})_{/\,\mathsf{GL}_2(\mathbb{Z})} \xrightarrow{\ \cong\ } \mathcal{L}(2) \ .$$

(2) Using that each finite-sheeted cover over $\mathbb{T}^2$ is diffeomorphic with $\mathbb{T}^2$, the classification of covering spaces implies the map given by taking the image of degree-1 homology $\mathsf{Imm}(\mathbb{T}^2) \xrightarrow{\ \mathsf{Image}(\mathsf{H}_1)\ } \mathcal{L}(2)$ witnesses the quotient:

$$\mathsf{Imm}(\mathbb{T}^2)_{/\,\mathsf{Diff}(\mathbb{T}^2)} \xrightarrow{\ \cong\ } \mathcal{L}(2) \ .$$

(3) The diagram

$$
\begin{array}{ccc}
\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{\quad\mathsf{Aff}\quad} & \mathsf{Imm}(\mathbb{T}^2) \\
{\scriptstyle\mathsf{pr}}\Big\downarrow & {\scriptstyle\mathsf{H}_1}\quad\quad & \Big\downarrow{\scriptstyle\mathsf{Image}(\mathsf{H}_1)} \\
\mathsf{E}_2(\mathbb{Z}) & \xrightarrow[\quad\mathsf{Image}\quad]{} & \mathcal{L}(2)
\end{array}
$$

commutes.

Corollary 1.5. *The morphism (17) between topological monoids is a homotopy-equivalence:*

$$\mathsf{Aff} \colon \mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\ \cong\ } \mathsf{Imm}(\mathbb{T}^2) \ .$$

Proof. Consider the morphism between fiber sequences in the ∞-category Spaces:

$$
\begin{array}{ccccc}
\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{\ \mathsf{quotient}\ } & \big(\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big)_{\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2} & \longrightarrow & \mathsf{B}\big(\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \\
{\scriptstyle\mathsf{Aff}}\Big\downarrow & & {\scriptstyle\mathsf{Aff}_{\mathsf{Aff}}}\Big\downarrow & & \Big\downarrow{\scriptstyle\mathsf{B}\,\mathsf{Aff}} \\
\mathsf{Imm}(\mathbb{T}^2) & \xrightarrow{\ \mathsf{quotient}\ } & \mathsf{Imm}(\mathbb{T}^2)_{/\,\mathsf{Diff}(\mathbb{T}^2)} & \longrightarrow & \mathsf{B}\,\mathsf{Diff}(\mathbb{T}^2).
\end{array}
$$

Lemma 1.2 implies the right vertical map is an equivalence. Observation 1.5 implies the middle vertical map is an equivalence. It follows that the left vertical map is an equivalence, as desired.

$\square$

1.2. Framings.
A **framing** of the torus is a trivialization of its tangent bundle: $\varphi \colon \tau_{\mathbb{T}^2} \cong \epsilon_{\mathbb{T}^2}^2$. Consider the topological **space of framings** of the torus:

$$\mathsf{Fr}(\mathbb{T}^2) \; := \; \mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\tau_{\mathbb{T}^2}, \epsilon_{\mathbb{T}^2}^2\big) \; \subset \; \mathsf{Map}(\mathsf{T}\mathbb{T}^2, \mathbb{T}^2 \times \mathbb{R}^2) \; ,$$

which is endowed with the subspace topology of the C^∞-topology on the set of smooth maps between total spaces. The quotient map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ endows the smooth manifold $\mathbb{T}^2$ with a **standard framing** φ_0: for

$$\mathsf{trans} \colon \mathbb{T}^2 \times \mathbb{T}^2 \xrightarrow{\;(p,q) \mapsto \mathsf{trans}_p(q) := p+q\;} \mathbb{T}^2$$

the abelian multiplication rule of the Lie group $\mathbb{T}^2$,

$$(\varphi_0)^{-1} \colon \epsilon_{\mathbb{T}^2}^2 \xrightarrow{\;\cong\;} \tau_{\mathbb{T}^2} \; , \qquad \mathbb{T}^2 \times \mathbb{R}^2 \ni (p,v) \mapsto \big(p, \mathsf{D}_0(\mathsf{trans}_p \circ \mathsf{quot})(v)\big) \in \mathsf{T}\mathbb{T}^2 \; .$$

The next sequence of observations culminates as an identification of this space of framings.

Observation 1.6.
  (1) Postcomposition gives the topological space $\mathsf{Fr}(\mathbb{T}^2)$ the structure a left-torsor for the topological group $\mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\epsilon_{\mathbb{T}^2}^2, \epsilon_{\mathbb{T}^2}^2\big)$. In particular, the orbit map of a of framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ is a homeomorphism:

$$(19) \qquad\qquad \mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\epsilon_{\mathbb{T}^2}^2, \epsilon_{\mathbb{T}^2}^2\big) \xrightarrow{\;\cong\;} \mathsf{Fr}(\mathbb{T}^2) \; , \qquad \alpha \mapsto \alpha \circ \varphi \; .$$

  (2) Consider the topological space $\mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big)$ of smooth maps from the torus to the standard smooth structure on $\mathsf{GL}_2(\mathbb{R})$, which is endowed with the C^∞-topology. The map

$$(20) \qquad \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \xrightarrow{\;\cong\;} \mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\epsilon_{\mathbb{T}^2}^2, \epsilon_{\mathbb{T}^2}^2\big) \; , \qquad a \mapsto \Big(\mathbb{T}^2 \times \mathbb{R}^2 \xrightarrow{\;(p,v) \mapsto \big(p, a_p(v)\big)\;} \mathbb{T}^2 \times \mathbb{R}^2\Big) \; ,$$

  is a homeomorphism.
  (3) The map to the product with based maps,

$$(21) \; \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \xrightarrow{\;\cong\;} \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R}) \; , \qquad a \mapsto \big(\, a(0)^{-1}a \,,\, a(0) \,\big) \; ,$$

  is a homeomorphism.
  (4) Because both of the spaces $\mathbb{T}^2$ and $\mathsf{GL}_2(\mathbb{R})$ are 1-types with the former path-connected, the map,

$$\pi_1 \colon \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \xrightarrow{\;\simeq\;} \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \; ,$$

  is a homotopy-equivalence.
  (5) Evaluation on the standard basis for $\pi_1(0 \in \mathbb{T}^2) \xrightarrow{\;\cong\;} \pi_1(0 \in \mathbb{T})^2 \cong \mathbb{Z}^2$ defines a homeomorphism:

$$(22) \qquad\qquad \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \xrightarrow{\;\cong\;} \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})^2\big) \; \cong \; \mathbb{Z}^2 \; .$$

Observation 1.6, together with the Gram-Schmidt homotopy-equivalence $\mathsf{O}(2) \xrightarrow{\;\simeq\;} \mathsf{GL}_2(\mathbb{R})$, yields the following.

Corollary 1.7. *A choice of framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ determines a composite homotopy-equivalence:*

$$\mathsf{Fr}(\mathbb{T}^2) \quad \xleftarrow[\simeq]{(20)\circ(19)} \quad \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big)$$

$$\xrightarrow[\simeq]{(21)} \quad \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R})$$

$$\xrightarrow[\simeq]{\pi_1 \times \mathsf{id}} \quad \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R})$$

$$\xrightarrow[\simeq]{(22)\times\mathsf{id}} \quad \mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R})$$

$$\xleftarrow[\simeq]{\mathsf{id}\times\mathsf{GS}} \quad \mathbb{Z}^2 \times \mathsf{O}(2) \ .$$

1.3. **Moduli of framed tori.** Consider the map:

$$\mathsf{Act}\colon \mathsf{Fr}(\mathbb{T}^2) \times \mathsf{Imm}(\mathbb{T}^2) \longrightarrow \mathsf{Fr}(\mathbb{T}^2) \ ,$$

$$(\varphi, f) \mapsto \left(\tau_{\mathbb{T}^2} \xrightarrow{\mathsf{D}f}_{\cong} f^*\tau_{\mathbb{T}^2} \xrightarrow{f^*\varphi}_{\cong} f^*\epsilon^2_{\mathbb{T}^2} = \epsilon^2_{\mathbb{T}^2} \right) \ .$$

Lemma 1.8. *The map Act is a continuous right action of the topological monoid $\mathsf{Imm}(\mathbb{T}^2)$ on the topological space $\mathsf{Fr}(\mathbb{T}^2)$. In particular, there is a continuous right action of the topological group $\mathsf{Diff}(\mathbb{T}^2)$ on the topological space $\mathsf{Fr}(\mathbb{T}^2)$.*

Proof. Consider the topological subspace of the topological space of smooth maps between total spaces of tangent bundles, which is endowed with the C^∞-topology,

$$\mathsf{Bdl}^{\mathsf{fw.iso}}(\tau_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}) \ \subset \ \mathsf{Map}\big(\mathsf{T}\mathbb{T}^2, \mathsf{T}\mathbb{T}^2\big) \ ,$$

consisting of the smooth maps between tangent bundles that are fiberwise isomorphisms. Notice the factorization

$$\mathsf{Act}\colon \mathsf{Fr}(\mathbb{T}^2) \times \mathsf{Imm}(\mathbb{T}^2) \xrightarrow{\mathsf{id}\times\mathsf{D}} \mathsf{Fr}(\mathbb{T}^2) \times \mathsf{Bdl}^{\mathsf{fw.iso}}(\tau_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}) \xrightarrow{\circ} \mathsf{Fr}(\mathbb{T}^2)$$

as first taking the derivative, followed by composition of bundle morphisms. The definition of the C^∞-topology is so that the first map in this factorization is continuous. The second map in this factorization is continuous because composition is continuous with respect to C^∞-topologies. We conclude that Act is continuous.

We now show that Act is an action. Clearly, for each $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$, there is an equality $\mathsf{Act}(\varphi, \mathsf{id}) = \varphi$. Next, let $g, f \in \mathsf{Imm}(\mathbb{T}^2)$, and let $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$. The chain rule, together with universal properties for pullbacks, gives that the diagram among smooth vector bundles

commutes. Inspecting the definition of Act, the commutativity of this diagram implies the euqlity $\mathsf{Act}\big(\mathsf{Act}(\varphi, g), f\big) = \mathsf{Act}(\varphi, g \circ f)$, as desired.

$\square$

Remark 1.9. The space of homotopy-coinvariants with respect to this conjugation action Act can be interpreted as the ***moduli space of framed tori***:

$$\mathcal{M}_1^{\mathsf{fr}} \ := \ \mathsf{Fr}(\mathbb{T}^2)_{/\mathsf{Diff}(\mathbb{T}^2)} \ .$$

Observation 1.10. Through Corollary 1.7, the action Act is compatible with familiar actions. Specifically, a choice of framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ fits Act into a commutative diagram among topological spaces:

$$
\begin{array}{ccc}
\mathsf{Fr}(\mathbb{T}^2) \times \mathsf{Imm}(\mathbb{T}^2) & \xrightarrow{\quad\quad\quad\quad\quad\quad\quad \mathsf{Act} \quad\quad\quad\quad\quad\quad\quad} & \mathsf{Fr}(\mathbb{T}^2) \\
{\scriptstyle \text{Cor } 1.7 \times \mathsf{Aff}} \Big\uparrow {\scriptstyle \simeq} & & {\scriptstyle \cong}\Big\uparrow {\scriptstyle \text{Cor } 1.7} \\
\mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \times \big(\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \xrightarrow{\mathsf{id} \times \mathsf{pr}} \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \times \mathsf{E}_2(\mathbb{Z}) & \xrightarrow[\text{multiply}]{\text{value-wise}} & \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \\
{\scriptstyle \text{Cor } 1.7 \times \mathsf{id}}\Big\downarrow {\scriptstyle \simeq} & {\scriptstyle \simeq}\Big\downarrow {\scriptstyle \mathsf{id} \times \text{Cor } 1.7} & {\scriptstyle \simeq}\Big\downarrow {\scriptstyle \text{Cor } 1.7} \\
\big(\mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R})\big) \times \big(\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2\big) \xrightarrow{\mathsf{id} \times \mathsf{pr}} \big(\mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R})\big) \times \mathsf{E}_2(\mathbb{Z}) & \xrightarrow{(a,b,A;B) \mapsto (a,b,AB)} & \mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R}).
\end{array}
$$

Corollary 1.11. *A choice of framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ canonically determines an identification in the ∞-category $\mathbb{S}\mathrm{paces}$ of the homotopy-coinvariants for Act,*

$$
\mathsf{Fr}(\mathbb{T}^2)_{/\mathsf{Diff}(\mathbb{T}^2)} \xrightarrow{\simeq} \mathbb{Z}^2 \times \left((\mathbb{CP}^\infty)^2{}_{\Omega\left(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\right)} \right) ,
$$

through which φ selects the $(0,0)$-path-component. Each path-component in this expression fits into the fiber sequence,

$$
(\mathbb{CP}^\infty)^2 \longrightarrow (\mathbb{CP}^\infty)^2{}_{/\Omega\left(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\right)} \longrightarrow \big(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\big) ,
$$

which is classified by the action

$$
\Omega\big(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\big) \xrightarrow{\pi_0} \pi_1\big(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\big) \longrightarrow \mathsf{GL}_2(\mathbb{Z}) \curvearrowright \mathsf{B}^2\mathbb{Z}^2 \simeq (\mathbb{CP}^\infty)^2 .
$$

Furthermore, the resulting projection is given by

$$
\mathsf{Fr}(\mathbb{T}^2)_{/\mathsf{Diff}(\mathbb{T}^2)} \longrightarrow \mathbb{Z}^2 , \qquad [\varphi'] \mapsto \left[\mathbb{T} \vee \mathbb{T} = \mathsf{sk}_1(\mathbb{T}^2) \xrightarrow{\varphi^{-1}\varphi'_{|\mathsf{sk}_1(\mathbb{T}^2)}} \mathsf{GL}_2(\mathbb{R}) \right] \in \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)^2 \cong \mathbb{Z}^2 .
$$

Proof. We explain the following sequences of identifications in the ∞-category $\mathbb{S}\mathrm{paces}$.

$$
\begin{array}{cccc}
\mathsf{Fr}(\mathbb{T}^2)_{/\mathsf{Diff}(\mathbb{T}^2)} & \underset{\text{Obs } 1.10}{\overset{\simeq}{}} & \left(\mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R})\right)_{\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2} \\[2mm]
& \underset{\text{triv on } \mathbb{Z}^2}{\overset{\simeq}{}} & \mathbb{Z}^2 \times \big(\mathsf{GL}_2(\mathbb{R})\big)_{\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2} \\[2mm]
& \underset{\text{iterate coinvariants}}{\overset{\simeq}{}} & \mathbb{Z}^2 \times \left(\big(\mathsf{GL}_2(\mathbb{R})\big)_{\mathbb{T}^2}\right)_{\mathsf{GL}_2(\mathbb{Z})} \\[2mm]
& \underset{\mathbb{T}^2-\text{action trivial}}{\overset{\simeq}{}} & \mathbb{Z}^2 \times \left((\mathbb{CP}^\infty)^2 \times \mathsf{GL}_2(\mathbb{R})\right)_{\mathsf{GL}_2(\mathbb{Z})} \\[2mm]
& \underset{\Omega-\text{Puppe}}{\overset{\simeq}{}} & \mathbb{Z}^2 \times \left((\mathbb{CP}^\infty)^2\right)_{\Omega\left(\mathsf{GL}_2(\mathbb{R})_{/\mathsf{GL}_2(\mathbb{Z})}\right)} .
\end{array}
$$

The first identification follows from Observation 1.10. The bottom horizontal map in Observation 1.10 reveals that the action $\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \curvearrowright \mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R})$ is the diagonal action of the trivial action on $\mathbb{Z}^2$ and the action

$$
\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\mathsf{pr}} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow[\text{left mult}]{\text{include}} \curvearrowright \mathsf{GL}_2(\mathbb{R}) .
$$

This supplies the second identification. The third identification is iteration of coinvariants. The fourth identification follows because the restricted $\mathbb{T}^2$-action is trivial. The final identification is an instance of the following general fact.

Let $H \to G$ be a morphism between continuous groups that is π_0-surjective. Let $H \curvearrowright X$ be an action on a space. The Ω-Puppe sequence supplies a canonical morphism $\Omega(G_{/H}) \to H$ between

continuous groups. Specifically, this morphism witnesses the stabilizer of $* \xrightarrow{\text{unit}} G$ with respect to the action $H \to G \underset{\text{left trans}}{\curvearrowright} G$:

$$
\begin{array}{ccc}
\Omega(G_{/H}) & \longrightarrow & H \\
\downarrow & & \downarrow \\
* & \xrightarrow{\quad \text{unit} \quad} & G.
\end{array}
$$

In particular, there is a canonical $\Omega(G_{/H})$-equivariant map

$$ X \simeq X \times * \xrightarrow{\text{id} \times \text{unit}} X \times G . $$

Taking coinvariants lends a canonically commutative diagram among spaces:

$$
\begin{array}{ccccc}
X_{\Omega(G_{/H})} & \longrightarrow & (X \times G)_{/H} & \longrightarrow & X_{/H} \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{B}\Omega(G_{/H}) & \longrightarrow & G_{/H} & \longrightarrow & \mathsf{B}H.
\end{array}
$$

Because groupoid-objects are effective in the ∞-category $\mathcal{S}\text{paces}$, the coinvariants functor,

$$ \mathsf{Fun}(\mathsf{B}H, \mathcal{S}\text{paces}) \longrightarrow \mathcal{S}\text{paces}_{/\mathsf{B}H} , \qquad (H \curvearrowright X) \mapsto (X_{/H} \to \mathsf{B}H) , $$

is an equivalence between ∞-categories. In particular, it preserves products. It follows that the right square witnesses a pullback. By definition of coinvariants of the restricted action $\Omega(G_{/H}) \to H \curvearrowright X$, the outer square is a pullback. The connectivity assumption on the morphism $H \to G$ implies the left bottom horizontal map is an equivalence. We conclude that the left top horizontal map is also an equivalence, as desired.

$\square$

For $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ a framing of the torus, consider the orbit map of φ for this continuous action of Lemma 1.8:

$$ \mathsf{Orbit}_\varphi \colon \mathsf{Imm}(\mathbb{T}^2) \xrightarrow{(\ \text{constant}_\varphi \ , \ \text{id}\ )} \mathsf{Fr}(\mathbb{T}^2) \times \mathsf{Imm}(\mathbb{T}^2) \xrightarrow{\ \mathsf{Act}\ } \mathsf{Fr}(\mathbb{T}^2) , \qquad f \mapsto \mathsf{Act}(\varphi, f) . $$

Observation 1.12. After Observation 1.10, for each framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$, the orbit map for φ fits into a commutative diagram among topological spaces:

The existence and commutativity of the fillers follows form Observation 1.5.

Remark 1.13. The point-set fiber of Orbit_φ over φ, which is the point-set stabilizer of the action $\mathsf{Fr}(\mathbb{T}^2) \curvearrowright \mathsf{Imm}(\mathbb{T}^2)$ of Lemma 1.8, consists of those local-diffeomorphisims f for which the diagram among vector bundles,

$$
\begin{array}{ccc}
\tau_{\mathbb{T}^2} & \xrightarrow{\ \varphi\ } & \epsilon^2_{\mathbb{T}^2} \\
{\scriptstyle \mathsf{D}f}\downarrow & & \uparrow{\scriptstyle =} \\
f^*\tau_{\mathbb{T}^2} & \xrightarrow{\ f^*\varphi\ } & f^*\epsilon^2_{\mathbb{T}^2},
\end{array}
$$

16

commutes. In the case of the standard framing φ_0, a local-diffeomorphism f satisfies this rigid condition if and only if $f = \mathsf{trans}_{f(0)} \circ \mathsf{quot}_C$ is translation in the group $\mathbb{T}^2$ after a group-theoretic quotient $\mathbb{T}^2 \xrightarrow{\text{quotient}} \mathbb{T}^2$. In particular, the point-set fiber of $\left(\mathsf{Orbit}_{\varphi_0}\right)_{|\,\mathsf{Diff}(|TT^2)}$ over φ_0 is $\mathbb{T}^2$, and the homomorphism $\mathbb{T}^2 \hookrightarrow \mathsf{Diff}(\mathbb{T}^2)$ witnesses the inclusion of those diffeomorphisms that *strictly* fix φ_0.

On the other hand, the *homotopy*-fiber of $\mathsf{Orbit}_{\varphi_0}$ over φ_0 is more flexible: it consists of pairs (f, γ) in which f is a local-diffeomorphism and γ is a homotopy

$$\varphi_0 \underset{\gamma}{\sim} \mathsf{Act}(\varphi_0, f) \ .$$

As we will see, every orientation-preserving local-diffeomorphism f admits a lift to this homotopy-fiber. In particular, small perturbations of such f, such as multiplication by bump functions in neighborhoods of $\mathbb{T}^2$, can be lifted to this homotopy-fiber.

Definition 1.14. Let $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ be a framing of the torus. The space of ***framed local-diffeomorphisms***, and the space of ***framed diffeomorphisms***, of the framed smooth manifold $(\mathbb{T}^2, \varphi)$ are respectively the pullbacks in the ∞-category $\mathbb{S}\mathsf{paces}$:

$$
\begin{array}{ccc}
\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) & \longrightarrow & \mathsf{Imm}(\mathbb{T}^2) \\
\downarrow & & \downarrow {\scriptstyle \mathsf{Orbit}_\varphi} \\
* & \xrightarrow{\langle\varphi\rangle} & \mathsf{Fr}(\mathbb{T}^2)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) & \longrightarrow & \mathsf{Diff}(\mathbb{T}^2) \\
\downarrow & & \downarrow {\scriptstyle \mathsf{Orbit}_\varphi} \\
* & \xrightarrow{\langle\varphi\rangle} & \mathsf{Fr}(\mathbb{T}^2)
\end{array} \ .
$$

In the case that the framing $\varphi = \varphi_0$ is the standard framing, we simply denote

$$\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2) \ := \ \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi_0) \qquad \text{and} \qquad \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \ := \ \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi_0) \ .$$

Lemma 1.15. *Let $G \curvearrowright X$ be an action of a continuous on a space X. Let $* \xrightarrow{\langle x \rangle} X$ be a point in X. Consider stabilizer of x, which is the fiber of the orbit map of x:*

$$
\text{(23)} \qquad
\begin{array}{ccc}
\mathsf{Stab}_G(x) & \longrightarrow & * \\
\downarrow & & \downarrow {\scriptstyle \langle x \rangle} \\
G \simeq G \times * \xrightarrow{\mathsf{id} \times \langle x \rangle} G \times X & \xrightarrow{\mathsf{act}} & X.
\end{array}
$$

with Orbit_x the composite from $G \simeq G \times *$ to X.

There is a canonical identification in $\mathbb{S}\mathsf{paces}$ between this stabilizer and the based-loops at $[x] \colon * \xrightarrow{\langle x \rangle} X \xrightarrow{\text{quotient}} X_{/G}$ of the G-coinvariants,

$$\mathsf{Stab}_G(x) \ \simeq \ \Omega_{[x]} X_{/G} \ ,$$

through which the resulting composite morphism $\Omega_{[x]} X_{/G} \simeq \mathsf{Stab}_G(x) \to G$ canonically lifts to one between continuous groups.

Proof. By definition of a G-action, the orbit map $G \xrightarrow{\mathsf{Orbit}_x} X$ is canonically G-equivariant. Taking G-coinvariants supplies an extension of the commutative diagram (23) in $\mathbb{S}\mathsf{paces}$:

$$
\begin{array}{ccccc}
\mathsf{Stab}_G(x) & \longrightarrow & G & \xrightarrow{\text{quotient}} & G_{/G} \simeq * \\
\downarrow & & \downarrow {\scriptstyle \mathsf{Orbit}_x} & & \downarrow {\scriptstyle (\mathsf{Orbit}_x)_G} \\
* & \xrightarrow{\langle x \rangle} & X & \xrightarrow{\text{quotient}} & X_{/G}.
\end{array}
$$

Through the identification $G_{/G} \simeq *$, the right vertical map is identified as $* \xrightarrow{\langle [x] \rangle} X_{/G}$. Using that groupoids in $\mathbb{S}\mathsf{paces}$ are effective, the right square is a pullback. Because the lefthand square is

17

defined as a pullback, it follows that the outer square is a pullback. The identification $\mathsf{Stab}_G(x) \simeq \Omega_{[x]}X_{/G}$ follows. In particular, the space $\mathsf{Stab}_G(x)$ has the canonical structure of a continuous groups.

Now, this continuous group $\mathsf{Stab}_G(x)$ is evidently functorial in the argument $G \curvearrowright X \ni x$. In particular, the unique G-equivariant morphism $X \xrightarrow{!} *$ determines a morphism between continuous groups

$$\mathsf{Stab}_x(X) \longrightarrow \mathsf{Stab}_*(*) \simeq G .$$

<div align="right">□</div>

Corollary 1.16. *Let $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ be a framing. There space $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$ is canonically endowed with the structure of a continuous group over $\mathsf{Diff}(\mathbb{T}^2)$. With respect to this structure, there is a canonical identification,*

$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \simeq \Omega_{[\varphi]}\big(\mathsf{Fr}(\mathbb{T}^2)_{/\,\mathsf{Diff}(\mathbb{T}^2)}\big) .$$

Observation 1.17. Let $\varphi \in \mathsf{Fr}(\mathbb{T}^2)$ be a framing. The kernel of Φ acts by rotating framing, which is to say there is a canonically commutative diagram among continuous groups:

$$
\begin{array}{ccccc}
\mathbb{Z} & \xrightarrow{\ \simeq\ } & \Omega_{\mathbb{1}}\,\mathsf{GL}_2(\mathbb{R}) & \xrightarrow{\Omega\big(A \mapsto A\cdot\varphi\big)} & \Omega_\varphi\,\mathsf{Fr}(\mathbb{T}^2) \\
{\scriptstyle\big\langle(\tau_1\tau_2)^6\big\rangle}\Big\downarrow{\scriptstyle\cong} & & & & \Big\downarrow \\
\mathsf{Ker}(\Phi) & \xrightarrow{\hspace{2cm}} & \mathsf{Braid}_3 & \xrightarrow{\ \mathsf{Aff}^{\mathsf{fr}}\ } & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi).
\end{array}
$$

Indeed, there is a canonically commutative diagram among spaces, in which each row is an Ω-Puppe sequence:

$$
\begin{array}{ccccccc}
\mathsf{Ker}(\Phi) & \xrightarrow{\hspace{1.5cm}} & \mathsf{Braid}_3 & \xrightarrow{\ \Phi\ } & \mathsf{GL}_2(\mathbb{Z}) & \xrightarrow{\ \mathbb{R}\otimes_{\mathbb{Z}}\ } & \mathsf{GL}_2(\mathbb{R}) \\
\Big\downarrow & & {\scriptstyle\mathsf{Aff}^{\mathsf{fr}}}\Big\downarrow & & {\scriptstyle\mathsf{Aff}}\Big\downarrow & & \Big\downarrow{\scriptstyle\text{Rotate the framing }\varphi} \\
\Omega_\varphi\,\mathsf{Fr}(\mathbb{T}^2) & \xrightarrow{\hspace{1.5cm}} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) & \xrightarrow{\hspace{1.5cm}} & \mathsf{Diff}(\mathbb{T}^2) & \xrightarrow{\ \mathsf{Orbit}_\varphi\ } & \mathsf{Fr}(\mathbb{T}^2).
\end{array}
$$

By definition of $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$, both of the horizontal sequences are Ω-Puppe sequences. The result follows.

1.4. **Braid group and braid monoid.** Here we collect some facts about the braid group on 3 strands, and the braid monoid on 3 strands.

Remark 1.18. We give an explicit description of $\widetilde{\mathsf{E}}_2^+(\mathbb{Z})$. In [3], the author gives an explicit description for the universal cover of $\mathsf{SP}_2(\mathbb{R}) = \mathsf{SL}_2(\mathbb{R})$ (and goes on to establish the pullback square of Proposition 0.1). Following those methods, consider function

$$\phi\colon \mathsf{GL}_2(\mathbb{R}) \longrightarrow \mathbb{S}^1 , \qquad A \mapsto \frac{(a+d)+i(b-c)}{|(a+d)+i(b-c)|} ,$$

and the argument map

$$\eta\colon (\mathsf{GL}_2(\mathbb{R}))^{\times 2} \longrightarrow \mathbb{R} , \qquad (A, B) \mapsto \arg\Big(1 - \alpha_A \overline{\alpha_{B^{-1}}}\Big)$$

where

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \alpha_A = \frac{a^2 + c^2 - b^2 - d^2 - 2i(ad + bc)}{(a+d)^2 + (b-c)^2}.$$

Then the universal cover can be expressed as the subset

$$\widetilde{\mathsf{GL}}_2^+(\mathbb{R}) := \big\{(A, c) \mid \phi(A) = e^{ic}\big\} \subset \mathsf{GL}_2(\mathbb{R}) \times \mathbb{R} , \text{ with group-law } (A, c)\cdot(B, d) := \big(AB, c + d + \eta(A, B)\big) .$$

So the monoid

$$\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) := \big\{(A, c) \mid \phi(A) = e^{ic}\big\} \subset \mathsf{E}_2^+(\mathbb{Z}) \times \mathbb{R} , \text{ with monoid-law } (A, c)\cdot(B, d) := \big(AB, c + d + \eta(A, B)\big) .$$

Observation 1.19. The inclusion $\mathsf{SL}_2(\mathbb{Z}) \subset \mathsf{E}_2^+(\mathbb{Z})$ between submonoids of $\mathsf{GL}_2^+(\mathbb{R})$ determines an inclusion between topological monoids:

$$(24) \qquad \mathsf{Braid}_3 \ltimes \mathbb{T}^2 \longrightarrow \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 \ .$$

After Observation 1.1, this inclusion (24) witnesses the maximal subgroup, both as topological monoids and as monoid-objects in the ∞-category $\mathcal{S}\mathrm{paces}$.

Observation 1.20. In Braid_3, there is an identity of the generator of $\mathsf{Ker}(\Phi)$:

$$(\tau_1\tau_2\tau_1)^4 = (\tau_1\tau_2)^6 = (\tau_2\tau_1\tau_2)^4 \in \mathsf{Ker}(\Phi) \ .$$

For that matter, since the matrix

$$(25) \qquad R \ := \ U_1 U_2 U_3 \ = \ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \ = \ U_2 U_1 U_2 \ \in \ \mathsf{GL}_2(\mathbb{Z})$$

implements rotation by $-\frac{\pi}{2}$, then $R^4 = \mathbb{1}$ in $\mathsf{GL}_2(\mathbb{Z})$.

For $\mathbb{R}\underset{\mathbb{Q}}{\otimes}\colon \mathsf{GL}_2^+(\mathbb{Q}) \subset \mathsf{GL}_2^+(\mathbb{R})$ the subgroup with rational coefficients, consider the pullback among groups:

$$\widetilde{\mathsf{GL}}_2^+(\mathbb{Q}) \ := \ \mathsf{GL}_2^+(\mathbb{Q}) \underset{\mathsf{GL}_2^+(\mathbb{R})}{\times} \widetilde{\mathsf{GL}}_2(\mathbb{R}) \ .$$

Proposition 1.21. *Each of the standard inclusions*

$$\mathbb{Q}\underset{\mathbb{Z}}{\otimes}\colon \ \mathsf{E}_2(\mathbb{Z}) \longrightarrow \mathsf{GL}_2(\mathbb{Q}) \qquad and \qquad \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \longrightarrow \widetilde{\mathsf{GL}}_2^+(\mathbb{Q})$$

witnesses a group-completion between (a priori continuous) monoids. In other words, the induced maps between pointed spaces are equivalences:

$$\mathsf{BE}_2(\mathbb{Z}) \xrightarrow{\ \simeq\ } \mathsf{BGL}_2(\mathbb{Q}) \qquad and \qquad \mathsf{B}\,\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \xrightarrow{\ \simeq\ } \mathsf{B}\widetilde{\mathsf{GL}}_2^+(\mathbb{Q}) \ .$$

Proof. First note that the (a priori continuous) monoid $\mathbb{N}^\times \cong \underset{p \text{ prime}}{\bigoplus} (\mathbb{N}, +)$ is the free (a priori continuous) monoid on the set of prime natural numbers. Next note the monomorphism between monoids $\mathbb{N}^\times \subset \mathsf{E}_2(\mathbb{Z})$ as scalar matrices, the image which is central. Recall the elementary formula for the inverse of a 2x2 matrix:

$$\text{for } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{E}_2(\mathbb{Z}) \qquad \text{then} \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{GL}_2(\mathbb{Q}) \ .$$

Together, these facts imply the canonical inclusion $\mathbb{Q}\underset{\mathbb{Z}}{\otimes}\colon \mathsf{E}_2(\mathbb{Z}) \to \mathsf{GL}_2(\mathbb{Q})$ witnesses the localization on $\mathbb{N}^\times \subset \mathsf{E}_2(\mathbb{Z})$:

$$\mathsf{E}_2(\mathbb{Z})[(\mathbb{N}^\times)^{-1}] \ \cong \ \mathsf{GL}_2(\mathbb{Q}) \ .$$

In particular, this localization is a (a priori continuous) group. The left identification follows from the universal property of group-completion. This immediately implies $\mathbb{Q}\underset{\mathbb{Z}}{\otimes}\colon \mathsf{E}_2^+(\mathbb{Z}) \to \mathsf{GL}_2^+(\mathbb{Q})$ also witnesses a group-completion. The right result follows via base-change along the central extension among continuous groups

$$1 \longrightarrow \mathbb{Z} \longrightarrow \widetilde{\mathsf{GL}}_2^+(\mathbb{R}) \longrightarrow \mathsf{GL}_2^+(\mathbb{R}) \longrightarrow 1 \ .$$

$\square$

The following result is an immediate consequence of the Definition **??**, using that the continuous group $\mathsf{GL}_2^+(\mathbb{R})$ is a path-connected 1-type.

Corollary 1.22. *There are pullbacks among continuous monoids:*

$$
\begin{array}{ccccc}
\mathsf{Braid}_3 & \longrightarrow & \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) & \longrightarrow & * \\
\Phi \downarrow & & \Psi \downarrow & & \downarrow \langle \mathbb{1} \rangle \\
\mathsf{GL}_2(\mathbb{Z}) & \longrightarrow & \mathsf{E}_2(\mathbb{Z}) & \xrightarrow{\ \mathbb{R} \otimes_{\mathbb{Z}}\ } & \mathsf{GL}_2(\mathbb{R}).
\end{array}
$$

In particular, there is a canonical identification between continuous groups over $\mathsf{GL}_2(\mathbb{Z})$*:*

$$
\mathsf{Braid}_3 \ \simeq \ \Omega\big(\mathsf{GL}_2(\mathbb{R})_{/\,\mathsf{GL}_2(\mathbb{Z})}\big) \qquad \big(\ \text{over}\ \ \mathsf{GL}_2(\mathbb{Z})\ \big)\ .
$$

1.5. **Proof of Theorem A and Corollaries 0.5.** Theorem A consists of three statements. Theorem A(1) is a consequence of Corollary 1.11. Theorem A(2) follows immediately from the following.

Lemma 1.23. *There are canonical equivalences in the diagrams among continuous monoids:*

$$
(26) \quad
\begin{array}{ccc}
\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 & \dashrightarrow[\ \mathsf{Aff}^{\mathsf{fr}}\ ]{\ \simeq\ } & \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2) \\
\Psi \ltimes \mathsf{id} \downarrow & & \downarrow \mathsf{forget} \\
\mathbb{T}^2 \ltimes \mathsf{E}_2(\mathbb{Z}) & \xrightarrow[\ \mathsf{Aff}\ ]{\ \simeq\ } & \mathsf{Imm}(\mathbb{T}^2)
\end{array}
\qquad and \qquad
\begin{array}{ccc}
\mathsf{Braid}_3 \ltimes \mathbb{T}^2 & \dashrightarrow[\ \mathsf{Aff}^{\mathsf{fr}}\ ]{\ \simeq\ } & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \\
\Phi \ltimes \mathsf{id} \downarrow & & \downarrow \mathsf{forget} \\
\mathsf{GL}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow[\ \mathsf{Aff}\ ]{\ \simeq\ } & \mathsf{Diff}(\mathbb{T}^2).
\end{array}
$$

Proof. Using Observation 1.1, the canonical equivalence in the commutative diagram on the right follows from that on the left.

Consider the diagram in the ∞-category $\mathcal{S}$paces:

$$
\begin{array}{ccccccc}
\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{\ \mathsf{pr}\ } & \widetilde{\mathsf{E}}_2^+(\mathbb{Z}) & \xrightarrow{\ !\ } & * & \xrightarrow{\ !\ } & * \\
\Psi \ltimes \mathsf{id} \downarrow & & \Psi \downarrow & & \downarrow \langle \mathbb{1} \rangle & & \downarrow \langle (0,0,\mathbb{1}) \rangle \\
\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{\ \mathsf{pr}\ } & \mathsf{E}_2(\mathbb{Z}) & \xrightarrow[\text{standard}]{\ \mathbb{R} \otimes_{\mathbb{Z}}\ } & \mathsf{GL}_2(\mathbb{R}) & \xrightarrow{\langle (0,0) \rangle \times \mathsf{id}} & \mathbb{Z}^2 \times \mathsf{GL}_2(\mathbb{R}) \\
\mathsf{Aff} \downarrow \simeq & & & & & & \uparrow \text{Cor 1.7} \\
\mathsf{Imm}(\mathbb{T}^2) & & & \xrightarrow{\ \ \mathsf{Orbit}_\varphi\ \ } & & & \mathsf{Fr}(\mathbb{T}^2).
\end{array}
$$

Observation 1.12 implies that the bottom rectangle canonically commutes. Lemma 1.2 and Corollary 1.7 together imply this bottom rectangle witnesses a pullback. The top left square is clearly a pullback. Because $\mathbb{Z}^2$ is discrete, the top right square is a pullback. Corollary 1.22 states that the top middle square is a pullback. We conclude that the outer square witnesses a pullback. The result follows by Definition 1.14 of $\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)$.

$\square$

By applying the product-preserving functor $\mathcal{S}$paces $\xrightarrow{\pi_0}$ Sets, Lemma 1.23 implies the following.

Corollary 1.24. *There is a canonical isomorphism in the diagram of groups:*

$$
\begin{array}{ccc}
\mathsf{Braid}_3 & \dashrightarrow[]{\ \cong\ } & \mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \\
\Phi \downarrow & & \downarrow \mathsf{forget} \\
\mathsf{GL}_2(\mathbb{Z}) & \xrightarrow{\ \cong\ } & \mathsf{MCG}(\mathbb{T}^2).
\end{array}
$$

Remark 1.25. Proposition 0.1 and Corollary 1.24 grant a central extension among groups:

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \longrightarrow \mathsf{MCG}^{\mathsf{or}}(\mathbb{T}^2) \longrightarrow 1\ .
$$

Proof of Corollary 0.5. By construction, the diagram among spaces,

$$
\begin{array}{ccc}
\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow[\text{Cor 1.5}]{\simeq} & \mathsf{Imm}(\mathbb{T}^2) \\
& \searrow_{\mathsf{pr}} \quad \swarrow_{\mathsf{ev}_0} & \\
& \mathbb{T}^2 &
\end{array}
\quad,
$$

canonically commutes, in which the left vertical map is projection, and the right vertical map evaluates at the origin $0 \in \mathbb{T}^2$. Therefore, upon taking fibers over $0 \in \mathbb{T}^2$, the (left) commutative diagram (26) among continuous monoids determines the commutative diagram among commutative monoids:

$$
\begin{array}{ccc}
\widetilde{\mathsf{E}}_2^+(\mathbb{Z}) & \xrightarrow{\simeq} & \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2 \ \mathsf{rel}\ 0) \\
\downarrow & & \downarrow \\
\mathsf{E}_2(\mathbb{Z}) & \xrightarrow[\text{Cor 1.5}]{\simeq} & \mathsf{Imm}(\mathbb{T}^2 \ \mathsf{rel}\ 0) \\
& \searrow_{\mathbb{R}\underset{\mathbb{Z}}{\otimes}} \quad \swarrow_{\mathsf{D}_0} & \\
& \mathsf{GL}_2(\mathbb{R}) &
\end{array}
\quad,
$$

in which the map $\mathbb{R}\underset{\mathbb{Z}}{\otimes}$ is the standard inclusion, and D_0 takes the derivative at the origin $0 \in \mathbb{T}^2$. To finish, Corollary 1.22 supplies the left pullback square in the following diagram among continuous groups, while the right pullback square is definitional:

$$
\begin{array}{ccccc}
\mathsf{Braid}_3 & \longrightarrow & * & \longleftarrow & \mathsf{Diff}(\mathbb{T}^2 \smallsetminus \mathbb{B}^2 \ \mathsf{rel}\ \partial) \\
\downarrow & & \downarrow & & \downarrow \\
\mathsf{GL}_2(\mathbb{Z}) & \xrightarrow[\mathbb{R}\underset{\mathbb{Z}}{\otimes}]{} & \mathsf{GL}_2(\mathbb{R}) & \xleftarrow[\mathsf{D}_0]{} & \mathsf{Diff}(\mathbb{T}^2 \ \mathsf{rel}\ 0).
\end{array}
$$

The result follows.

$\square$

1.6. **Comparison with sheering.** We use Theorem A(2a) to show that the $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$ is generated by sheering. We quickly tour through some notions and results, which are routine after the above material.

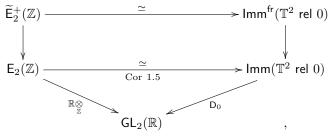**Notation 1.26.** It will be convenient to define the projection $\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}$ to be projection *off* of the i−th coordinate. So for $\mathbb{T}^2 \ni p = (x_p, y_p)$, we have $\mathsf{pr}_1(p) = y_p$ and $\mathsf{pr}_2(p) = x_p$.

Let $i \in \{1, 2\}$. Consider the topological subgroup and topological submonoid,

$$
\mathsf{Diff}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \ \subset\ \mathsf{Diff}(\mathbb{T}^2) \qquad \text{and} \qquad \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \ \subset\ \mathsf{Imm}(\mathbb{T}^2) \ ,
$$

consisting of those (local-)diffeomorphisms $\mathbb{T}^2 \xrightarrow{f} \mathbb{T}^2$ that lie over some (local-)diffeomorphism $\mathbb{T} \xrightarrow{\overline{f}} \mathbb{T}$:

$$
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\ f\ } & \mathbb{T}^2 \\
{\scriptstyle \mathsf{pr}_i}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_i} \\
\mathbb{T} & \xrightarrow[\ \overline{f}\ ]{} & \mathbb{T} \ .
\end{array}
$$

Because pr_i is surjective, for a given f, there is a unique $\overline{f}$ if any. Better, $f \mapsto \overline{f}$ defines a forgetful morphism between topological monoids, whose kernel is the topological monoid of smooth maps from $\mathbb{T}$ to $\mathsf{Imm}(\mathbb{T})$ with value-wise monoid-structure, which is to say there is a short exact sequence of topological monoids:

$$
(27) \qquad 1 \longrightarrow \mathsf{Map}\big(\mathbb{T}, \mathsf{Imm}(\mathbb{T})\big) \longrightarrow \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \underset{\text{forget}}{\overset{}{\longleftarrow\!\!\!\longrightarrow}} \mathsf{Imm}(\mathbb{T}) \longrightarrow 1.
$$

The isotopy-extension theorem implies this short exact sequence among topological groups forgets as a short exact sequence among continuous groups. Furthermore, this short exact sequence of continuous monoids (27) splits, as indicated, by way of the precomposition action $\mathsf{Map}\big(\mathbb{T}, \mathsf{Imm}(\mathbb{T})\big) \curvearrowright \mathsf{Imm}(\mathbb{T})$: there is a canonical equivalence between continuous monoids:

$$\mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \;\simeq\; \mathsf{Imm}(\mathbb{T}) \ltimes \mathsf{Map}\big(\mathbb{T}, \mathsf{Imm}(\mathbb{T})\big)\;.$$

Now, the topological space of **framings** of $\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}$ is the subspace

$$\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \;\subset\; \mathsf{Fr}(\mathbb{T}^2)$$

consisting of those framings $\tau_{\mathbb{T}^2} \xrightarrow{\varphi} \epsilon_{\mathbb{T}^2}^2$ that lie over a framing $\tau_{\mathbb{T}} \xrightarrow{\overline{\varphi}} \epsilon_{\mathbb{T}}^1$:

(28)
$$\begin{array}{ccc} \tau_{\mathbb{T}^2} & \xrightarrow[\;\cong\;]{\varphi} & \epsilon_{\mathbb{T}^2}^2 \\[2pt] {\scriptstyle \mathsf{D}\,\mathsf{pr}_i}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathsf{pr}_i \times \mathsf{pr}_i} \\[2pt] \tau_{\mathbb{T}} & \xrightarrow[\;\cong\;]{\overline{\varphi}} & \epsilon_{\mathbb{T}}^1. \end{array}$$

The continuous right action Act of Lemma 1.8 evidently restricts as a continuous right action

$$\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \;\curvearrowleft\; \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})\;.$$

Now, because pr_i is surjective, for a given φ, there is a unique $\overline{\varphi}$ as in (28) if any. Better, $\varphi \mapsto \overline{\varphi}$ defines a continuous map, which is evidently equivariant with respect to the morphism between topological monoids $\mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \xrightarrow{\text{forget}} \mathsf{Imm}(\mathbb{T})$:

$$\Big(\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \;\curvearrowleft\; \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \Big) \xrightarrow{\text{forget}} \Big(\mathsf{Fr}(\mathbb{T}) \;\curvearrowleft\; \mathsf{Imm}(\mathbb{T}) \Big)\;, \qquad \varphi \mapsto \overline{\varphi}\;.$$

Now let $\varphi \in \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$ be a framing of the projection. The orbit of φ by this action is the map

$$\mathsf{Orbit}_\varphi \colon \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \longrightarrow \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})\;, \qquad f \mapsto \mathsf{Act}(\varphi, f)\;.$$

The space of **framed local-diffeomorphisms**, and the space of **framed diffeomorphisms**, of $(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}, \varphi)$ are respectively the homtopy-pullbacks among spaces:

$$\begin{array}{ccc} \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}, \varphi) & \longrightarrow & \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \\ \downarrow & & \downarrow{\scriptstyle \mathsf{Orbit}_\varphi} \\ * & \xrightarrow{\langle \varphi \rangle} & \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}, \varphi) & \longrightarrow & \mathsf{Diff}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \\ \downarrow & & \downarrow{\scriptstyle \mathsf{Orbit}_\varphi} \\ * & \xrightarrow{\langle \varphi \rangle} & \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})\;. \end{array}$$

The topological space $\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$ is a torsor for the topological group $\mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_{\{i\}\subset 2}(\mathbb{R})\big)$ of smooth maps from $\mathbb{T}^2$ to the subgroup

$$\mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \;:=\; \Big\{ A \mid Ae_i \in \mathsf{Span}\{e_i\} \Big\} \;\subset\; \mathsf{GL}_2(\mathbb{R})$$

consisting of those 2×2 matrices that carry the ith-coordinate line to itself.

Lemma 1.27. *The homotopy equivalence of lemma (1.7) restricts to the following homotopy equivalence of spaces:*

(29)
$$\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \;\simeq\; \mathsf{O}(1) \times \mathsf{O}(1).$$

Proof. Upon a choice of framing $\varphi \in \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$, we have the homeomorphism

$$\mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \approx \mathsf{Map}\Big((\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}), (\mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \to \mathsf{GL}_1(\mathbb{R}))\Big)$$

where the latter space consists of those maps which lie over a map making the diagram below commute

$$\mathsf{Map}\Big((\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}),(\mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \to \mathsf{GL}_1(\mathbb{R}))\Big) := \left\{ \begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \\ {\scriptstyle \mathsf{pr}_i}\downarrow & & \downarrow{\scriptstyle e_i^T(-)e_k} \\ \mathbb{T} & \dashrightarrow & \mathsf{GL}_1(\mathbb{R}). \end{array} \right\}$$

where $k \in \{1,2\} - i$. This homeomorphism is (19) $\circ$ (20) restricted to the appropriate subspaces. Now, $\mathsf{GL}_1(\mathbb{R}) \simeq \mathbb{R}^\times \simeq \mathsf{O}(1)$ and $\mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \simeq \mathbb{R}^\times \times \mathbb{R}^\times \simeq \mathsf{O}(1) \times \mathsf{O}(1)$ as we can take a straight line homotopy to diagonal matrices. Then note we have the following inclusion which is a homotopy equivalence:

$$\left\{ \begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathsf{GL}_{\{i\}\subset 2}(\mathbb{R}) \\ {\scriptstyle \mathsf{pr}_i}\downarrow & & \downarrow \\ \mathbb{T} & \dashrightarrow & \mathsf{GL}_1(\mathbb{R}). \end{array} \right\} \hookleftarrow \left\{ \begin{array}{ccc} \mathbb{T}^2 & \longrightarrow & \mathsf{O}(1) \times \mathsf{O}(1) \\ {\scriptstyle \mathsf{pr}_i}\downarrow & & \downarrow{\scriptstyle \mathsf{pr}_i} \\ \mathbb{T} & \dashrightarrow & \mathsf{O}(1). \end{array} \right\} \simeq \mathsf{Map}(\mathbb{T}^2, \mathsf{O}(1) \times \mathsf{O}(1))$$

where the last equivalence is due to the diagram being fixed after choosing such a map. As $\mathbb{T}^2$ is connected we have that $\mathsf{Map}(\mathbb{T}^2, \mathsf{O}(1) \times \mathsf{O}(1)) \simeq \mathsf{O}(1) \times \mathsf{O}(1)$. □

Lemma 1.28. *There exist the following factorizations between topological groups:*

$$\mathsf{E}_{\{1\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \dashrightarrow^{\mathsf{Aff}_1} \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) \qquad \mathsf{E}_{\{2\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \dashrightarrow^{\mathsf{Aff}_2} \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T})$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$\mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\mathsf{Aff}} \mathsf{Imm}(\mathbb{T}^2), \qquad\qquad \mathsf{E}_2(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\mathsf{Aff}} \mathsf{Imm}(\mathbb{T}^2)$$

where $\mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})$ consists of the upper triangular matrices in $\mathsf{E}_2(\mathbb{Z})$ and $\mathsf{E}_{\{2\}\subset 2}(\mathbb{Z})$ consists of the lower triangular matrices in $\mathsf{E}_2(\mathbb{Z})$.

Proof. We need to show that the image of Aff_1 lies in $\mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T})$. In other words we need to show that for (A,p) in the domain, the map

$$\mathsf{Aff}_1(A,p) : \mathbb{T}^2 \to \mathbb{T}^2; \qquad q \mapsto Aq + p$$

lies over an immersion of $\mathbb{T}$. Note the following commutative square: for $p = (x_p, y_p) \in \mathbb{T}^2$ and

$$A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z}), \qquad \mathbb{T}^2 \ni q = (x_q, y_q) \xrightarrow{\mathsf{Aff}_1(A,p)} (ax_q + by_q + x_p, dy_q + y_p) \in \mathbb{T}^2$$

$$\qquad\qquad\qquad\qquad {\scriptstyle \mathsf{pr}_1}\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow{\scriptstyle \mathsf{pr}_1}$$

$$\qquad\qquad\qquad\qquad \mathbb{T} \ni y_q \xrightarrow{\quad \mathsf{trans}_{y_p} \circ\, \mathsf{scale}_d \quad} dy_q + y_p \in \mathbb{T} .$$

Both a,d are nonzero integers and so we see that the bottom map is indeed an immersion of $\mathbb{T}$. A similar computation shows that $\mathsf{Aff}_2(A,p)$ lies over an immersion of $\mathbb{T}$. □

Proposition 1.29. *For $i = 1,2$ the map Aff_i is a homotopy equivalence of spaces.*

Proof. Consider the following commutative diagram of topological spaces:

$$(30) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})_{ker} \ltimes \mathbb{T} & \xrightarrow{inc_x} & \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 & \xrightarrow{d \ltimes \mathsf{pr}_1} & \mathsf{E}_1(\mathbb{Z}) \ltimes \mathbb{T} & \longrightarrow & 1 \\ & & {\scriptstyle =}\downarrow & & {\scriptstyle F}\downarrow & & {\scriptstyle \mathsf{Aff}_i}\downarrow & & \downarrow{\scriptstyle =} \\ 1 & \longrightarrow & \mathsf{Map}(\mathbb{T}, \mathsf{Imm}(\mathbb{T})) & \longrightarrow & \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) & \longrightarrow & \mathsf{Imm}(\mathbb{T}) & \longrightarrow & 1 \end{array}$$

where $\mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})_{ker}$ consists of matrices of the form

$$\left\{ \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix} : a,b \in \mathbb{Z} \text{ and } a \neq 0 \right\}.$$

and $inc_x : \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})_{ker} \ltimes \mathbb{T} \longrightarrow \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2$ is given by the inclusion $(A,\theta) \mapsto (A,(\theta,0))$. This makes the top sequence exact. The map $\mathsf{Aff}_{\mathbb{T}} : \mathsf{E}_1(\mathbb{Z}) \ltimes \mathbb{T} \to \mathsf{Imm}(\mathbb{T})$ is defined by $(c,\theta) \mapsto (\gamma \mapsto c\gamma + \theta)$, which is well known to be a homotopy equivalence. Then the map F is defined by

$$\mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})_{ker} \ltimes \mathbb{T} \ni \left(A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}, \theta \right) \mapsto (\psi \mapsto f_{a,b,\theta,\psi}) \in \mathsf{Map}(\mathbb{T}, \mathsf{Imm}(\mathbb{T}))$$

where $f_{a,b,\theta,\psi}(\gamma) = a\gamma + b\psi + \theta$. We will show that F is also a homotopy equivalence. Consider the commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \xrightarrow{\ UR\ } & \mathsf{E}_{\{1\}\subset 2}(\mathbb{Z})_{ker} \ltimes \mathbb{T} & \xrightarrow{a\ltimes\mathsf{id}} & \mathsf{E}_1(\mathbb{Z}) \ltimes \mathbb{T} & \longrightarrow & 1 \\
& \ \Big\downarrow{\scriptstyle =} & \Big\downarrow{\scriptstyle G} & & \Big\downarrow{\scriptstyle F} & & \Big\downarrow{\scriptstyle \mathsf{Aff}_{\mathbb{T}}} & & \ \Big\downarrow{\scriptstyle =} \\
1 & \longrightarrow & \mathsf{Map}_*(0 \in \mathbb{T}, \mathsf{id}_{\mathbb{T}} \in \mathsf{Imm}(\mathbb{T})) & \xrightarrow{\ \mathsf{inc}\ } & \mathsf{Map}(\mathbb{T}, \mathsf{Imm}(\mathbb{T})) & \xrightarrow{\ ev_0\ } & \mathsf{Imm}(\mathbb{T}) & \longrightarrow & 1
\end{array}
$$

where

$$UR(b) = \left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}, 0 \right) \quad \text{and} \quad G(b) = (\psi \mapsto (g_{b,\psi} : \mathbb{T} \to \mathbb{T})); \ \ g_{b,\psi}(\gamma) = \gamma + b\psi.$$

Again, $\mathsf{Aff}_{\mathbb{T}}$ is a homotopy equivalence and so

$$\mathsf{Map}_*(0 \in \mathbb{T}, \mathsf{id}_{\mathbb{T}} \in \mathsf{Imm}(\mathbb{T})) \simeq \mathsf{Map}_*(\mathbb{T}, \mathsf{E}_1(\mathbb{Z}) \ltimes \mathbb{T}) = \mathsf{Map}_*(\mathbb{T},\mathbb{T}).$$

Now consider the map $\widetilde{G} : \mathbb{Z} \to \mathsf{Map}_*(\mathbb{T},\mathbb{T})$ which is just G followed by the homotopy equivalence above. Then $\widetilde{G}(b)$ is just the based map $\psi \mapsto b\psi$ which is the homotopy inverse of $\pi_0 : \mathsf{Map}_*(\mathbb{T},\mathbb{T}) \to \mathbb{Z}$. Therefore $\widetilde{G}$ is a homotopy equivalence which implies that G must be a homotopy equivalence. Finally by the 5-lemma we have that the map F is a homotopy equivalence.

Now as the rows of the diagram (30) are exact and the outer columns, F and $\mathsf{Aff}_{\mathbb{T}}$, are homomtopy equivalences the 5-lemma yields that Aff_1 is also a homotopy equivalence. A similar argument shows that Aff_2 is a homotopy equivalence as well. $\qquad\square$

Proposition 1.30. *The following diagrams commute:*

$$
\begin{array}{ccc}
\mathsf{E}_{\{1\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\ \mathsf{Aff}_1\ } \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) & \qquad & \mathsf{E}_{\{2\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\ \mathsf{Aff}_2\ } \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}) \\
\Big\downarrow{\scriptstyle \left(\frac{a}{|a|},\frac{d}{|d|}\right)} \qquad \Big\downarrow{\scriptstyle \mathsf{Orbit}_\varphi} & & \Big\downarrow{\scriptstyle \left(\frac{a}{|a|},\frac{d}{|d|}\right)} \qquad \Big\downarrow{\scriptstyle \mathsf{Orbit}_\varphi} \\
\mathsf{O}(1) \times \mathsf{O}(1) \xleftarrow[\simeq]{} \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}), & & \mathsf{O}(1) \times \mathsf{O}(1) \xleftarrow[\simeq]{} \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}).
\end{array}
$$

Proof. Given some element (A,p) of $\mathsf{E}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2$ we want to identify the associated element of $\mathsf{Orbit}_\varphi \circ \mathsf{Aff}_i(A,p)$ in $\mathsf{O}(1) \times \mathsf{O}(1)$. We know that the differential $\mathsf{DAff}_i(A,p) = A$ and therefore

$$\mathsf{Orbit}_\varphi \circ \mathsf{Aff}_i(A,p) = \tau_{\mathbb{T}^2} \xrightarrow[\cong]{\ A\ } \mathsf{Aff}_i(A,p)^*\tau_{\mathbb{T}^2} \xrightarrow[\cong]{\mathsf{Aff}_i(A,p)^*\varphi} \mathsf{Aff}_i(A,p)^*\epsilon^2_{\mathbb{T}^2} = \epsilon^2_{\mathbb{T}^2}.$$

and then through the homotopy equivalence above in lemma 1.27 we have that the associated element of $\mathsf{O}(1) \times \mathsf{O}(1)$ corresponds to reading off the signs of the diagonal of A. $\qquad\square$

Corollary 1.31. *There exists a homotopy equivalence of spaces*

$$\mathsf{E}^{++}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \simeq \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$$

where $\mathsf{E}^{++}_{\{i\}\subset 2}(\mathbb{Z})$ are those elements of $\mathsf{E}_{\{i\}\subset 2}(\mathbb{Z})$ whose diagonal elements are positive.

24

Proof. We have that $\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$ is defined to be the homotopy fiber

$$\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) := \mathsf{hofib}_\varphi(\mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \xrightarrow{\mathsf{Orbit}_\varphi} \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})).$$

Then as Aff_i and (29) are homotopy equivalences of spaces, there exists a homotopy equivalence

$$\mathsf{hofib}_\varphi\big(\mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \xrightarrow{\mathsf{Orbit}_\varphi} \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})\big)$$

$$\simeq \; \mathsf{hofib}_{(1,1)}\big(\mathsf{E}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\mathsf{Aff}_i} \mathsf{Imm}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \xrightarrow{\mathsf{Orbit}_\varphi} \mathsf{Fr}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}) \xrightarrow{(29)} \mathsf{O}(1) \times \mathsf{O}(1)\big)$$

$$\overset{\text{proposition } 1.30}{=} \mathsf{hofib}_{(1,1)}\big(\mathsf{E}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{\left(\frac{a}{|a|}, \frac{d}{|d|}\right)} \mathsf{O}(1) \times \mathsf{O}(1)\big).$$

But as the last map is projection onto a discrete space, the homotopy fiber over $(1,1)$ is equivalent to the standard fiber over $(1,1)$ which is $\mathsf{E}^{++}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2$. $\qquad\square$

Observation 1.32. Note that in restricting the maps of proposition 1.30 to the maximal subgroup of $\mathsf{E}_{\{i\}\subset 2}(\mathbb{Z})$ we have that

$$\mathsf{GL}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \xrightarrow{(29)\circ\mathsf{Orbit}_\varphi\circ\mathsf{Aff}_i} \mathsf{O}(1) \times \mathsf{O}(1)$$

is equal to projection off of the diagonal elements of $\mathsf{GL}_{\{i\}\subset 2}(\mathbb{Z})$. Therefore the homotopy fiber of $(1,1)$ of this map is just the fiber over $(1,1)$ which we will denote as $\mathsf{GL}^{++}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2$, where $\mathsf{GL}^{++}_{\{i\}\subset 2}(\mathbb{Z})$ consists of those matrices in $\mathsf{GL}_{\{i\}\subset 2}(\mathbb{Z})$ with 1's on the diagonal. Then observe the isomorphisms $\mathbb{Z} \xrightarrow{\langle U_i \rangle} \mathsf{GL}^{++}_{\{i\}\subset 2}(\mathbb{Z})$ where $b \mapsto U_i^b$. Therefore we have that

$$\mathbb{Z} \underset{U_i}{\ltimes} \mathbb{T}^2 \xrightarrow{\cong} \mathsf{GL}^{++}_{\{i\}\subset 2}(\mathbb{Z}) \ltimes \mathbb{T}^2 \simeq \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}).$$

where we have $\mathbb{Z} \ni k \curvearrowright p \in \mathbb{T}^2$ by $U_i^k p$.

Observation 1.33. Through observation 1.32, there are canonically commutative diagrams among continuous groups:

$$
\begin{array}{ccc}
\mathbb{Z} \underset{U_1}{\ltimes} \mathbb{T}^2 & \xrightarrow[\text{Obs } 1.32]{\simeq} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) \\
{\scriptstyle \langle \tau_1 \rangle \ltimes \mathsf{id}} \downarrow & & \downarrow {\scriptstyle \text{forget}} \\
\mathsf{Braid}_3 \ltimes \mathbb{T}^2 & \xrightarrow[\text{Lem } 1.23]{\simeq} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{Z} \underset{U_2}{\ltimes} \mathbb{T}^2 & \xrightarrow[\text{Obs } 1.32]{\simeq} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}) \\
{\scriptstyle \langle \tau_2 \rangle \ltimes \mathsf{id}} \downarrow & & \downarrow {\scriptstyle \text{forget}} \\
\mathsf{Braid}_3 \ltimes \mathbb{T}^2 & \xrightarrow[\text{Lem } 1.23]{\simeq} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2).
\end{array}
$$

and

We now explain how the presentation (1) of Braid_3 gives a presentation of the continuous group $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$. Observe the canonically commutative diagram among continuous groups:

$$
\begin{array}{ccc}
\mathbb{T}^2 & \xrightarrow{\hspace{3cm}} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) \\
\downarrow & & \downarrow \\
\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}) & \xrightarrow{\hspace{3cm}} & \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2).
\end{array}
$$

It results in a morphism from the pushout in continuous groups:

$$(31) \qquad \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) \coprod_{\mathbb{T}^2} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}) \longrightarrow \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) .$$

Recall the element $R \in \mathsf{GL}_2(\mathbb{Z})$ from (25). The two homomorphisms

$$\mathbb{Z} \xrightarrow[\langle \tau_2 \tau_1 \tau_2 \rangle]{\langle \tau_1 \tau_2 \tau_1 \rangle} \mathbb{Z} \amalg \mathbb{Z}$$

determine two morphisms among continuous groups under $\mathbb{T}^2$:

25

(32)

$$\mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2 \xrightarrow[\langle \tau_2 \tau_1 \tau_2 \rangle \ltimes \mathsf{id}]{\langle \tau_1 \tau_2 \tau_1 \rangle \ltimes \mathsf{id}} (\mathbb{Z} \amalg \mathbb{Z}) \underset{U_1, U_2}{\ltimes} \mathbb{T}^2 \xrightarrow[\mathrm{Obs\ 1.33}]{\simeq} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_1} \mathbb{T}) \underset{\mathbb{T}^2}{\amalg} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_2} \mathbb{T}) \xrightarrow{(31)} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$$

Corollary 1.34. *The diagram among continuous groups under* $\mathbb{T}^2$, *(32), witnesses a coequalizer. In particular, for each ∞-category $\mathfrak{X}$, there is a pullback diagram among ∞-categories of modules in $\mathfrak{X}$:*

$$
\begin{array}{ccc}
\mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)}(\mathfrak{X}) & \longrightarrow & \mathsf{Mod}_{\mathbb{Z} \underset{U_1}{\ltimes} \mathbb{T}^2}(\mathfrak{X}) \underset{\mathsf{Mod}_{\mathbb{T}^2}(\mathfrak{X})}{\times} \mathsf{Mod}_{\mathbb{Z} \underset{U_2}{\ltimes} \mathbb{T}^2}(\mathfrak{X}) \\
\downarrow & & \downarrow {\scriptstyle \left(\langle \tau_1 \tau_2 \tau_1 \rangle \ltimes \mathsf{id}\right)^* \times \left(\langle \tau_2 \tau_1 \tau_2 \rangle \ltimes \mathsf{id}\right)^*} \\
\mathsf{Mod}_{\mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2}(\mathfrak{X}) & \xrightarrow{\mathrm{diagonal}} & \mathsf{Mod}_{\mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2}(\mathfrak{X}) \underset{\mathsf{Mod}_{\mathbb{T}^2}(\mathfrak{X})}{\times} \mathsf{Mod}_{\mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2}(\mathfrak{X}).
\end{array}
$$

In particular, for $X \in \mathfrak{X}$ an object, an action $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \curvearrowright X$ is

(1) *an action* $\mathbb{T}^2 \curvearrowright X$,
(2) *extensions of this action to actions* $\mathbb{Z} \underset{U_1}{\ltimes} \mathbb{T}^2 \curvearrowright X$ *and* $\mathbb{Z} \underset{U_2}{\ltimes} \mathbb{T}^2 \curvearrowright X$,
(3) *an identification under* $\mathbb{T}^2 \curvearrowright X$ *of the two restricted actions*

$$\mathbb{Z} \underset{R}{\ltimes} \mathbb{T}^2 \xrightarrow[\langle \tau_2 \tau_1 \tau_2 \rangle \ltimes \mathsf{id}]{\langle \tau_1 \tau_2 \tau_1 \rangle \ltimes \mathsf{id}} (\mathbb{Z} \amalg \mathbb{Z}) \underset{U_1, U_2}{\ltimes} \mathbb{T}^2 \curvearrowright X$$

1.7. The finite orbit category of $\mathbb{T}^2$.

(33) $$\mathsf{Orbit}^{\mathsf{fin}}_{\mathbb{T}^2} \longrightarrow \mathsf{Ar}\big(\mathfrak{B}\,\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2)\big) , \qquad \left(\mathbb{T}^2{}_{/C} \mapsto \mathbb{T}^2 \to \mathbb{T}^2{}_{/C}\right) .$$

Remark 1.35. $\big(\mathfrak{B}\,\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)\big)^{*/}$ *is a poset! Say what it is.*

2. Natural symmetries of secondary Hochschild homology

Let A be an associative algebra in $\mathcal{V}$. Consider the ***Hochschild homology*** of A in $\mathcal{V}$:

$$\mathsf{HH}(A) := \mathsf{HH}_{\mathcal{V}}(A) := A \underset{A^{\mathsf{op}} \otimes A}{\otimes} A \simeq \big|\mathsf{Bar}^{\mathsf{cyc}}_{\bullet}(A)\big| \in \mathcal{V} .$$

In [?], Connes constructs a canonical $\mathbb{T}$-action on $\mathsf{HH}(A)$, which is functorial in the algebra A. When working over the sphere spectrum (which is to say $\mathcal{V} = (\mathsf{Spectra}, \wedge)$ so that $\mathsf{HH}_{\mathsf{Spectra}}(A) = \mathsf{THH}(A)$ is ***topological Hochschild homology***, in [?] Bokstedt-Hsaing-Madsen extend this $\mathbb{T}$-action as a ***cyclotomic structure*** on $\mathsf{THH}(A)$. In [?] it is demonstrated how this cyclotomic structure on $\mathsf{THH}(A)$ is derived from an action of the continuous monoid $\mathbb{N}^{\times} \ltimes \mathbb{T}$ on the unstable version $\mathsf{HH}_{\mathsf{Spaces}}(A)$. Here, we prove Theorem **??**, which constructs a canonical $\mathsf{Braid}_3 \ltimes \mathbb{T}^2$-action on $\mathsf{HH}^{(2)}(A)$, which is functorial in the 2-algebra A. We then prove Theorem **??**, which, in the case that $\mathcal{V} = (\mathsf{Spaces}, \times)$, extends this action to one by the continuous monoid $\widetilde{\mathsf{E}}^+_2(\mathbb{Z}) \ltimes \mathbb{T}^2$.

Convention. We fix a symmetric monoidal ∞-category $\mathcal{V}$, and assume it is $\otimes$-presentable (meaning the underlying ∞-category $\mathcal{V}$ is presentable, and $\otimes$ distributes over colimits separately in each variable).

Example 2.1. For $\Bbbk$ a commutative ring, take $(\mathcal{V}, \otimes) = \big(\mathsf{Ch}_{\Bbbk}[\{\mathsf{quasi\text{-}isos}\}^{-1}], \otimes^{\mathbb{L}}_{\Bbbk}\big)$ to be the ∞-categorical localization of chain complexes over $\Bbbk$ on quasi-isomorphisms, with derived tensor product over $\Bbbk$ presenting the symmetric monoidal structure. More generally, for R a commutative ring spectrum, take $(\mathcal{V}, \otimes) := (\mathsf{Mod}_R, \underset{R}{\wedge})$ to be the ∞-category of R-module spectra and smash product over R as the symmetric monoidal structure.

2.1. **Secondary Hochschild homology of 2-algebras.** In order for the Hochschild homology construction to be twice-iterated, we endow the entity $A \in \mathcal{V}$ with an algebra structure among algebras.

Definition 2.2. The ∞-category of **2-algebras (in $\mathcal{V}$)** is

$$\mathsf{Alg}_2(\mathcal{V}) := \mathsf{Alg}_{\mathsf{Assoc}}\big(\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V})\big) .$$

Example 2.3. A commutative algebra $A = (A, \mu)$ in $\mathcal{V}$, determines the 2-algebra (A, μ, μ) in $\mathcal{V}$. This association assembles as a functor

$$\mathsf{CAlg}(\mathcal{V}) \longrightarrow \mathsf{Alg}_2(\mathcal{V}) ,$$

thusly supplying a host of examples of 2-algebras.

Observation 2.4. Using that the tensor product of operads is defined by a "hom-tensor" adjunction, there is a canonical equivalence between ∞-categories:

$$\mathsf{Alg}_{\mathsf{Assoc} \otimes \mathsf{Assoc}}(\mathcal{V}) \simeq \mathsf{Alg}_2(\mathcal{V}) .$$

In particular, swapping the two tensor-factors supplies an involution

$$\Sigma_2 \curvearrowright \mathsf{Alg}_2(\mathcal{V}) .$$

Remark 2.5. After Observation 2.4, a 2-algebra in $\mathcal{V}$ is an object $A \in \mathcal{V}$ together with two associative algebra structures μ_1 and μ_2 on A, and compatibility between them which can be stated as either of the two equivalent structures:

- A lift of the morphism $A \otimes A \xrightarrow{\mu_2} A$ in $\mathcal{V}$ to a morphism $(A, \mu_1) \otimes (A, \mu_1) \xrightarrow{\mu_2} (A, \mu_1)$ in $\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V})$.
- A lift of the morphism $A \otimes A \xrightarrow{\mu_1} A$ in $\mathcal{V}$ to a morphism $(A, \mu_2) \otimes (A, \mu_2) \xrightarrow{\mu_1} (A, \mu_2)$ in $\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V})$.

Example 2.6. Consider the operad $\mathcal{E}_2$ of little 2-disks. There is a standard morphism between operads $\mathsf{Assoc} \otimes \mathsf{Assoc} \to \mathcal{E}_2$. Through Observation 2.4, restriction along this morphism defines a functor between ∞-categories

(34) $$\mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \longrightarrow \mathsf{Alg}_2(\mathcal{V}) ,$$

thusly supplying some rich examples of 2-algebras (in fact, Dunn's additivity (Theorem 2.7) states that the functor (34) is an equivalence). For instance, for $\Bbbk$ a commutative ring, a braided-monoidal $\Bbbk$-linear category R is a 2-algebra in the $(2, 1)$-category of $\Bbbk$-linear categories. For instance, for G a semi-simple Lie group, and for k a level, ...??

Theorem 2.7 (Dunn's additivity [**?**] (see also Theorem ?? of [**?**])). *The functor (34) is an equivalence between ∞-categories.*

Remark 2.8. The action $\mathsf{O}(2) \curvearrowright \mathsf{Alg}_2(\mathcal{V})$ of Corollary ??, afforded by Theorem 2.7, extending the evident $\Sigma_2 \wr \mathsf{O}(1)$-action which swaps the two associative algebra structures (as the Σ_2-factor) and takes opposites of the two associative algebra structures (as the two $\mathsf{O}(1)$-factors).

Definition 2.9. *Secondary Hochschild homology* is the composite functor, given by twice-iterating Hochschild homology:

$$\mathsf{HH}^{(2)} \colon \mathsf{Alg}_2(\mathcal{V}) := \mathsf{Alg}_{\mathsf{Assoc}}\big(\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V})\big) \xrightarrow{\mathsf{HH}} \mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V}) \xrightarrow{\mathsf{HH}} \mathcal{V} ,$$

$$(A, \mu_1, \mu_1) \mapsto \big(\mathsf{HH}(A, \mu_1), \mathsf{HH}(\mu_2)\big) \mapsto \mathsf{HH}\big(\mathsf{HH}(A, \mu_1), \mathsf{HH}(\mu_2)\big) .$$

The following assertion is proved in §2.2.

Corollary 2.10. *For $A = (A, \mu_1, \mu_2)$ a 2-algebra in $\mathcal{V}$, swapping the roles of algebra structures does not change secondary Hochschild homology:*

$$\mathsf{HH}\big(\mathsf{HH}(A, \mu_1), \mathsf{HH}(\mu_2)\big) \;\simeq\; \mathsf{HH}\big(\mathsf{HH}(A, \mu_2), \mathsf{HH}(\mu_1)\big) \; .$$

More precisely, with respect to the involution $\Sigma_2 \curvearrowright \mathsf{Alg}_2(\mathcal{V})$ of Observation 2.4, secondary Hochschild homology is Σ_2-invariant:

$$\mathsf{HH}^{(2)} \colon \; \mathsf{Alg}_2(\mathcal{V})_{\Sigma_2} \longrightarrow \mathcal{V} \; .$$

2.2. **Proof of Theorem B.1 and Corollaries 0.7, 0.13, 0.11.** This subsection proves Theorem B.1 and Corollary 0.7, then Corollary 0.13, and then Corollary 0.11. The key idea is to use **factorization homology**, as established in [**?**].

Recall from [**?**] the symmetric monoidal ∞-category $\mathsf{Mfld}_2^{\mathsf{fr}}$ whose objects are framed 2-manifolds, whose spaces of morphisms are spaces of framed embeddings between them, and whose symmetric monoidal structure is given by disjoint union. Consider the full ∞-subcategories,

$$(35) \qquad\qquad \mathsf{Disk}_2^{\mathsf{fr}} \;\hookrightarrow\; \mathsf{Mfld}_2^{\mathsf{fr}} \;\hookleftarrow\; \mathsf{BDiff}^{\mathsf{fr}}(\mathbb{T}^2) \; ,$$

respectively spanned by those framed 2-manifolds each of whose connected components is equivalent with $\mathbb{R}^2$, and by those framed 2-manifolds that are equivalent with $\mathbb{T}^2$. The left full ∞-subcategory is closed with respect to the symmetric monoidal structure. Restriction along these full ∞-subcategories determines the solid diagram among ∞-categories:

(36)

$$\mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Disk}_2^{\mathsf{fr}}, \mathcal{V}) \xleftarrow{\;\;\text{restrict}\;\;} \mathsf{Fun}^{\otimes}(\mathsf{Mfld}_2^{\mathsf{fr}}, \mathcal{V}) \xrightarrow{\;\;\text{restrict}\;\;} \mathsf{Fun}\big(\mathsf{BDiff}^{\mathsf{fr}}(\mathbb{T}^2), \mathcal{V}\big) \simeq \mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)}(\mathcal{V}) \; .$$

Factorization homology is defined as the left adjoint to the leftward restriction functor; factorization homology over the torus, as it is endowed with a canonical $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$-action, is the rightward composite functor:

$$(37) \qquad\qquad \int_{\mathbb{T}^2} \colon \; \mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \longrightarrow \mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)}(\mathcal{V}) \; .$$

Next, we explain the following diagram among ∞-categories:

(38)

- The functors labeled by "forget" are restriction along evident morphisms between continuous groups. The lower triangles canonically commute.
- The functor $\int_{\mathbb{T}^2}$ is (37).
- For $i = 1, 2$, the functor $\int_{\mathsf{pr}_i}$ is factorization homology over the circle $\mathbb{T}$ of the **pushforward** along the projection $\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T}$ onto the ith-coordinate, as it is endowed with its canonical $\mathsf{Diff}^{\mathsf{fr}}(\mathsf{pr}_i)$-action. (Everywhere, change the notation $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2 \xrightarrow{\mathsf{pr}_i} \mathbb{T})$ simply to $\mathsf{Diff}^{\mathsf{fr}}(\mathsf{pr}_i)$.)
  The **pushforward formula** (Prop pushforward formula?? of [**?**]), $\int_{\mathsf{pr}_i} \simeq \int_{\mathbb{T}} \int_{\mathbb{T}}$, supplies commutativity of the upper triangles.

- The functor $\mathsf{Alg}_{\mathcal{E}_2} \xrightarrow{\mathsf{fgt}_1} \mathsf{Alg}_2(\mathcal{V})$ is restriction along the standard morphism between operads $\mathsf{Assoc} \otimes \mathsf{Assoc} \xrightarrow{\mathrm{standard}} \mathcal{E}_2$. The functor $\mathsf{Alg}_{\mathcal{E}_2} \xrightarrow{\mathsf{fgt}_2} \mathsf{Alg}_2(\mathcal{V})$ is restriction along the morphism between operads $\mathsf{Assoc} \otimes \mathsf{Assoc} \xrightarrow{\mathrm{swap}} \mathsf{Assoc} \otimes \mathsf{Assoc} \xrightarrow{\mathrm{standard}} \mathcal{E}_2$. Theorem **??** implies that each of these functors are equivalences.
- For $i = 1, 2$, the outer vertical functors are $\mathsf{HH}^{(2)}$, as it is endowed with its canonical action $\mathsf{Sheer}_i \colon \underset{U_i}{\mathbb{Z} \ltimes \mathbb{T}^2} \underset{(9)}{\curvearrowright} \mathsf{HH}^{(2)}(A)$ from §**??**, which is evidently functorial in $A \in \mathsf{Alg}_2(\mathcal{V})$.
- For each $i = 1, 2$, the functor $(\mathsf{Aff}_i^{\mathsf{fr}})^*$ is restriction along the functor $\mathsf{Aff}_i^{\mathsf{fr}}$ of (**??**). Theorem **??** implies each of these functors is an equivalence.
- Theorem **??** of [**?**] implies a canonical identification $\mathsf{HH} \simeq \int_{\mathbb{T}}$ between functors $\mathsf{Alg}_{\mathsf{Assoc}}(\mathcal{V}) \to \mathsf{Mod}_{\mathbb{T}}(\mathcal{V})$. Commutativity of the upper tilted squares follows.

In particular, for each 2-algebra $A \in \mathsf{Alg}_2(\mathcal{V})$, there is a canonical action $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \curvearrowright \mathsf{HH}^{(2)}(A)$. Through Theorem A(2a), this is an action $\mathsf{Braid}_3 \ltimes \mathbb{T}^2 \curvearrowright \mathsf{HH}^{(2)}(A)$, which establishes the statement of Theorem B.1. Furthermore, commutativity of the diagram (38) implies Corollary 0.7. Corollary 0.13 follows from this commutative diagram, upon noticing that the top horizontal composite functor $\mathsf{Alg}_2(\mathcal{V}) \to \mathsf{Alg}_2(\mathcal{V})$ is the involution of Observation 2.4.

Next, observe that both of the full ∞-subcategories (38) are closed with respect to the $\mathsf{O}(2) \simeq \mathsf{GL}_2(\mathbb{R})$-action on $\mathsf{Mfld}_2^{\mathsf{fr}}$ given by change-of-framing. It follows that the left adjoint, $\int$, is $\mathsf{O}(2)$-equivariant. This implies the composite functor

$$\int_{\mathbb{T}^2} \colon \ \mathsf{Alg}_{\mathcal{E}_2}(\mathcal{V}) \longrightarrow \mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)}(\mathcal{V})$$

is canonically $\mathsf{O}(2)$-equivariant. Given this, Corollary 0.11 follows from Observation 1.17.

2.3. **Proof of Theorem B.2.** Here, we fix a presentable ∞-category $\mathfrak{X}$ in which products distribute over colimits separately in each variable. We regard $\mathfrak{X}$ as a $\otimes$-presentable symmetric monoidal ∞-category, via its Cartesian monoidal structure.

We prove Theorem B.2 by extending factorization homology, via the developments of [**?**]. Namely, recall from [**?**] the ∞-category $\mathfrak{Mfd}_2^{\mathsf{sfr}}$ of ***solidly 2-framed stratified spaces***. Consider the full ∞-subcategory $\mathcal{M}_{=2}^{\mathsf{sfr}} \subset \mathfrak{Mfd}_2^{\mathsf{sfr}}$ consisting of those solidly 2-framed stratified spaces each of whose strata is 2-dimensional.

Observation 2.11. Inspection of the definition of $\mathfrak{Mfd}_2^{\mathsf{sfr}}$ reveals the following.

(1) The the moduli space of objects

$$\mathsf{Obj}(\mathcal{M}_{=2}^{\mathsf{sfr}}) \ \simeq \ \coprod_{[\Sigma, \varphi]} \mathsf{BDiff}^{\mathsf{fr}}(\Sigma, \varphi)$$

is that of a framed 2-manifolds. In other words, there is a canonical bijection between framed-diffeomorphism-types of framed 2-manifolds and equivalence-classes of objects in $\mathcal{M}_{=2}^{\mathsf{sfr}}$, and for (Σ, φ) a framed 2-manifold, there is a canonical identification between continuous groups:

$$\mathsf{Diff}^{\mathsf{fr}}(\Sigma, \varphi) \ \simeq \ \mathsf{Aut}_{\mathcal{M}_{=2}^{\mathsf{sfr}}}(\Sigma, \varphi) \ .$$

(2) Let (Σ, φ) and (Σ', φ') be framed 2-manifolds. The space of morphisms from (Σ, φ) to (Σ', φ') in $\mathcal{M}_{=2}^{\mathsf{sfr}}$,

$$\mathsf{Hom}_{\mathcal{M}_{=2}^{\mathsf{sfr}}}\big((\Sigma, \varphi), (\Sigma', \varphi')\big) \ \simeq \ \coprod_{[\widetilde{\Sigma} \xrightarrow{\pi} \Sigma]} \mathsf{Emb}^{\mathsf{fr}}\big((\widetilde{\Sigma}, \pi^*\varphi), (\Sigma', \varphi')\big)_{/\mathsf{Diff}_{/\Sigma}(\widetilde{\Sigma})} \ ,$$

is the moduli space of finite-sheeted covers over Σ together with a framed-embedding from its total space to (Σ', φ').

(3) Composition in $\mathbf{M}^{\mathsf{sfr}}_{=2}$ is given by base-change of framed embeddings along finite-sheeted covers, followed by composition of framed-embeddings:

$$\mathsf{Hom}_{\mathbf{M}^{\mathsf{sfr}}_{=2}}\big((\Sigma,\varphi),(\Sigma',\varphi')\big) \times \mathsf{Hom}_{\mathbf{M}^{\mathsf{sfr}}_{=2}}\big((\Sigma',\varphi'),(\Sigma'',\varphi'')\big) \longrightarrow \mathsf{Hom}_{\mathbf{M}^{\mathsf{sfr}}_{=2}}\big((\Sigma,\varphi),(\Sigma'',\varphi'')\big) \; ,$$

$$\Big(\; (\Sigma,\varphi) \xleftarrow{\pi} (\widetilde{\Sigma}, \pi^*\varphi) \xrightarrow{f} (\Sigma',\varphi') \; , \; (\Sigma',\varphi') \xleftarrow{\pi'} (\widetilde{\Sigma}', \pi'^*\varphi') \xrightarrow{g} (\Sigma'',\varphi'') \; \Big)$$

$$\longmapsto \Big((\Sigma,\varphi) \xleftarrow{\pi \circ \mathsf{pr}_1} (\widetilde{\Sigma}_\pi \underset{\Sigma'\, f}{\times} \widetilde{\Sigma}', (\mathsf{pr}_1 \circ \pi)^*\varphi) \xrightarrow{f \circ \mathsf{pr}_2} (\Sigma',\varphi') \Big) \; .$$

(4) Evidently, framed embeddings form the left factor in a factorization system on $\mathbf{M}^{\mathsf{sfr}}_{=2}$, whose right factor is (the opposite of) framed finite-sheeted covers.

(5) Finite products exist in $\mathbf{M}^{\mathsf{sfr}}_{=2}$, and are implemented by disjoin union of framed 2-manifolds.

(6) For each framing φ of the 2-torus $\mathbb{T}^2$, there is a canonical identification between continuous monoids:

$$\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)^{\mathsf{op}} \simeq \mathsf{End}_{\mathbf{M}^{\mathsf{sfr}}_{=2}}(\mathbb{T}^2, \varphi) \; .$$

Denote the full ∞-subcategory

$$\iota \colon \mathbf{D}^{\mathsf{sfr}}_{=2} \subset \mathbf{M}^{\mathsf{sfr}}_{=2}$$

consisting of those framed 2-manifolds each of that are equivalent with a finite disjoint union of framed Euclidean spaces. Regard both $\mathbf{D}^{\mathsf{sfr}}_{=2}$ and $\mathbf{M}^{\mathsf{sfr}}_{=2}$ as symmetric monoidal ∞-categories, via their Cartesian monoidal structures.[5] Notice the evident monomorphisms be symmetric monoidal ∞-categories,

$$\rho \colon \mathsf{Disk}^{\mathsf{fr}}_2 \hookrightarrow \mathbf{D}^{\mathsf{sfr}}_{=2} \qquad \text{and} \qquad \rho \colon \mathsf{Mfld}^{\mathsf{fr}}_2 \hookrightarrow \mathbf{M}^{\mathsf{sfr}}_{=2} \; ,$$

each of whose images consists of all objects yet only those morphisms $\big((\Sigma,\varphi) \xleftarrow{\pi} (\widetilde{\Sigma}, \pi^*\varphi) \xrightarrow{f} (\Sigma',\varphi')\big)$ in which π is a diffeomorphism.[6]

Consider the full ∞-subcategory

$$\mathsf{Fun}^\times\big((\mathbf{D}^{\mathsf{sfr}}_2, \mathfrak{X})\big) \subset \mathsf{Fun}\big((\mathbf{D}^{\mathsf{sfr}}_2, \mathfrak{X})\big)$$

consisting of those functors that preserve finite products.

Proposition 2.12. *Restriction along ρ defines an equivalence between ∞-categories:*

$$\rho^* \colon \mathsf{Fun}^\times\big((\mathbf{D}^{\mathsf{sfr}}_2, \mathfrak{X})\big) \longrightarrow \mathsf{Fun}^\otimes\big(\mathsf{Disk}^{\mathsf{fr}}_2, \mathfrak{X}\big) \simeq \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) \; .$$

Proof. Let $A \in \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) \simeq \mathsf{Fun}^\otimes(\mathsf{Disk}^{\mathsf{fr}}_2, \mathfrak{X})$. We seek to construct an extension:

(39)

in which $\widetilde{A}$ preserves finite products.

DO IT

The extension $\widetilde{A}$ is evidently functorial in $A \in \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X})$. $\qquad\square$

The inverse of restriction along ρ followed by left Kan extension along ι defines a composite functor

$$\widetilde{\int} \colon \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) \simeq \mathsf{Fun}^\otimes(\mathsf{Disk}^{\mathsf{fr}}_2, \mathfrak{X}) \xrightarrow{\rho^{*-1}} \mathsf{Fun}^\times(\mathbf{D}^{\mathsf{sfr}}_{=2}, \mathfrak{X}) \xrightarrow{\iota_!} \mathsf{Fun}^\times(\mathbf{M}^{\mathsf{sfr}}_{=2}, \mathfrak{X}) \; .$$

---

[5]It is straight-forward to observe that the full ∞-subcategory $\mathbf{D}^{\mathsf{sfr}}_{=2} \subset \mathbf{M}^{\mathsf{sfr}}_{=2}$ is closed under finite products.

[6]In other words, ρ is the inclusion of the left factor in the factorization system of Observation 2.11(4).

Proposition 2.13. *For each framed 2-manifold (Σ, φ), the diagram among ∞-categories canonically commutes:*

$$
\begin{array}{ccc}
\mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) & \xrightarrow{\quad \widetilde{\int} \quad} \mathsf{Fun}^{\times}(\boldsymbol{\mathcal{M}}^{\mathsf{sfr}}_{=2}, \mathfrak{X}) \xrightarrow{\quad \text{restriction} \quad} \mathsf{Fun}\big(\mathsf{BAut}_{\boldsymbol{\mathcal{M}}^{\mathsf{sfr}}_{=2}}(\Sigma, \varphi), \mathfrak{X}\big) \\
{\scriptstyle \int}\big\downarrow & \qquad\qquad\qquad\qquad \simeq \big\downarrow {\scriptstyle \text{Obs } 2.11(1)} \\
\mathsf{Fun}^{\otimes}(\mathsf{Mfld}^{\mathsf{fr}}_2, \mathfrak{X}) \xrightarrow{\text{restriction}} \mathsf{Fun}\big(\mathsf{BAut}_{\mathsf{Mfld}^{\mathsf{fr}}_2}(\Sigma, \varphi), \mathfrak{X}\big) & \xrightarrow{\quad \simeq \quad} \mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\Sigma, \varphi)}(\mathfrak{X}).
\end{array}
$$

Proof. Let $A \in \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) \simeq \mathsf{Fun}^{\otimes}(\mathsf{Disk}^{\mathsf{fr}}_2, \mathfrak{X})$. Let (Σ, φ) be a framed 2-manifold. Using Proposition **??**, the monomorphism ρ determines a canonical morphism between colimits in $\mathfrak{X}$,

$$
\begin{aligned}
(40) \qquad \int_{\mathbb{T}^2} A \quad &\simeq \quad \mathsf{colim}\Big(\mathsf{Disk}^{\mathsf{fr}}_{2/(\mathbb{T}^2, \varphi)} := \mathsf{Disk}^{\mathsf{fr}}_2 \underset{\mathsf{Mfld}^{\mathsf{fr}}_2}{\times} \mathsf{Mfld}^{\mathsf{fr}}_{2/(\mathbb{T}^2, \varphi)} \xrightarrow{\mathsf{pr}} \mathsf{Disk}^{\mathsf{fr}}_2 \xrightarrow{A} \mathfrak{X}\Big) \\
&\xrightarrow{\ \rho\ } \quad \mathsf{colim}\Big(\boldsymbol{\mathcal{D}}^{\mathsf{sfr}}_{=2/(\mathbb{T}^2, \varphi)} := \boldsymbol{\mathcal{D}}^{\mathsf{sfr}}_{=2} \underset{\boldsymbol{\mathcal{M}}^{\mathsf{sfr}}_{=2}}{\times} \boldsymbol{\mathcal{M}}^{\mathsf{sfr}}_{=2/(\mathbb{T}^2, \varphi)} \xrightarrow{\mathsf{pr}} \boldsymbol{\mathcal{D}}^{\mathsf{sfr}}_{=2} \xrightarrow{\rho^{*-1}(A)} \mathfrak{X}\Big) \\
&\simeq \quad \widetilde{\int}_{\mathbb{T}^2} A \ .
\end{aligned}
$$

This morphism is manifestly $\mathsf{Diff}^{\mathsf{fr}}(\Sigma, \varphi)$-equivariant and functorial in $A \in \mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X})$ as so. So the proposition is proved upon showing this morphism (40) is an equivalence. The morphism (40) is an equivalence provided the canonical functor

$$
(41) \qquad\qquad \mathsf{Disk}^{\mathsf{fr}}_{2/(\mathbb{T}^2, \varphi)} \longrightarrow \boldsymbol{\mathcal{D}}^{\mathsf{sfr}}_{=2/(\mathbb{T}^2, \varphi)}
$$

is final. But the factorization system of Observation 2.11(4) reveals that this functor (41) is a right adjoint: its left adjoint given by

$$
\boldsymbol{\mathcal{D}}^{\mathsf{sfr}}_{=2/(\mathbb{T}^2, \varphi)} \longrightarrow \mathsf{Disk}^{\mathsf{fr}}_{2/(\mathbb{T}^2, \varphi)} \ , \qquad \big(D \xleftarrow{\pi} \widetilde{D} \xrightarrow{f} (\mathbb{T}^2, \varphi)\big) \mapsto \big(\widetilde{D} \xrightarrow{f} (\mathbb{T}^2, \varphi)\big) \ .
$$

Finality of the functor (41) follows, which completes this proof.

$\square$

Now let φ be a framing of the 2-torus $\mathbb{T}^2$. Proposition 2.13, together with Observation 2.11(6), immediately supply a filler in the commutative diagram among ∞-categories:

$$
(42)
$$

$$
\begin{array}{ccc}
\Big\langle \mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)^{\mathsf{op}} \curvearrowright \widetilde{\int}_{\mathbb{T}^2} \Big\rangle \ \dashrightarrow & \mathsf{Fun}\big(\mathfrak{B}\,\mathsf{End}_{\boldsymbol{\mathcal{M}}^{\mathsf{sfr}}_{=2}}(\mathbb{T}^2, \varphi), \mathfrak{X}\big) \xrightarrow[\text{Obs } 2.11(6)]{\simeq} \mathsf{Mod}_{\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)^{\mathsf{op}}}(\mathfrak{X}) \xrightarrow[\text{Thm A(2b)}]{\simeq} \mathsf{Mod}_{\Box \ltimes \mathbb{T}^2}(\mathfrak{X}) \\
\nearrow & \big\downarrow {\scriptstyle \text{forget}} \qquad\qquad\qquad\qquad \big\downarrow {\scriptstyle \text{forget}} \\
\mathsf{Alg}_{\mathcal{E}_2}(\mathfrak{X}) \xrightarrow[\qquad\qquad (36) \qquad\qquad]{\big\langle \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \curvearrowright \int_{\mathbb{T}^2} \big\rangle} & \mathsf{Mod}_{\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi)}(\mathfrak{X}) \xrightarrow[\text{Thm A(2a)}]{\simeq} \mathsf{Mod}_{\mathsf{Braid}_3 \ltimes \mathbb{T}^2}(\mathfrak{X}).
\end{array}
$$

The first statement of Theorem B.2 follows from this commutative diagram (42) after the commutative diagram (38).

We now prove the second statement of Theorem B.2, that this action $\mathsf{Imm}^{\mathsf{fr}}(\mathbb{T}^2, \varphi) \curvearrowright \mathsf{HH}^{(2)}(A)$ defines a secondary cyclotomic structure on $\mathsf{HH}^{(2)}(A)$.

References

[1] Earle, C. J.; Eells, J. The diffeomorphism group of a compact Riemann surface. Bull. Amer. Math. Soc. 73 (1967), no. 4, 557–559. https://projecteuclid.org/euclid.bams/1183528956

[2] J. Milnor, Introduction to Algebraic K-theory, Annals of Math. Study 72, Princeton Univ. Press, 1971.

[3] Rawnsley, J. On the Universal Covering Group of the Real Symplectic Group. Journal of Geometry and Physics. 2012

[4] Rolfsen, D.; Knots and Links. Berkeley, Publish or Perish 1976

Department of Mathematics, Montana State University, Bozeman, MT 59717
E-mail address: `david.ayala@montana.edu`

Department of Mathematics, Montana State University, Bozeman, MT 59717
E-mail address: `adam.howard1@montana.edu`