FRAMED DIFFEOMORPHISMS OF A TORUS

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Abstract. We identify the topological group of framed diffeomorphisms of the standardly framed torus as a semi-direct product of the torus and a braid group on 3 strands. It follows that the framed mapping class group is the braid group on 3 strands.

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Introduction

Vector addition, as well as the standard vector norm, gives $\mathbb{R}^2$ the structure of a topological abelian group. Consider its closed subgroup $\mathbb{Z}^2 \subset \mathbb{R}^2$. The ***torus*** is the quotient in the short exact sequence of topological abelian groups:

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{\text{inclusion}} \mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2 \longrightarrow 0 .$$

Because $\mathbb{R}^2$ is connected, and because $\mathbb{Z}^2$ acts cocompactly by translations on $\mathbb{R}^2$, the torus $\mathbb{T}^2$ is connected and compact. The quotient map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ endows the torus with the structure of a Lie group, and in particular a smooth manifold. Using that the smooth map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ is a covering space and $\mathbb{T}^2$ is connected, the canonical homomorphism between groups

$$\mathsf{GL}_2(\mathbb{Z}) \ := \ \mathsf{Aut}_{\mathsf{Groups}}(\mathbb{Z}^2) \xleftarrow{\cong} \mathsf{Aut}_{\mathsf{LieGroups}}(\mathbb{Z}^2 \hookrightarrow \mathbb{R}^2) \xrightarrow{\cong} \mathsf{Aut}_{\mathsf{LieGroups}}(\mathbb{T}^2) \ , \tag{1}$$

$$A \mapsto \Big(q \mapsto Aq := \mathsf{quot}(A\widetilde{q})\Big) \qquad (\text{for any } \widetilde{q} \in \mathsf{quot}^{-1}(q)) \ ,$$

is an isomorphism. Note that (1) does not depend on $\widetilde{q} \in \mathsf{quot}^{-1}(q)$ and this homomorphism defines a semi-direct product topological group:

$$\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \ . \tag{2}$$

Consider the space of smooth diffeomorphisms of the torus:

$$\mathsf{Diff}(\mathbb{T}^2) \ \subset \ \mathsf{Map}(\mathbb{T}^2, \mathbb{T}^2) \ ,$$

which is endowed with the subspace topology of the C^∞ topology on the set of smooth self-maps of the torus. Consider the following map.

$$\mathsf{Aff} : \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \longrightarrow \mathsf{Diff}(\mathbb{T}^2) \ , \qquad (p, A) \mapsto \Big(q \mapsto Aq + p\Big) \ . \tag{3}$$

The following lemmas show that Aff is a continuous homomorphism and a homotopy equivalence.

2010 *Mathematics Subject Classification.* Primary ??. Secondary ??.
Key words and phrases. ??.
The author was supported by ...

Lemma 0.1. *The map defined in (3)*

$$\mathsf{Aff} : \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \longrightarrow \mathsf{Diff}(\mathbb{T}^2), \qquad \mathsf{Aff}(p, A) = (q \mapsto Aq + p)$$

is a continuous homomorphism.

Proof. Since the maps $\mathsf{GL}_2(\mathbb{Z}) \xrightarrow{A \mapsto (q \mapsto Aq)} \mathsf{Diff}(\mathbb{T}^2)$ and $\mathbb{T}^2 \xrightarrow{p \mapsto (q \mapsto q+p)} \mathsf{Diff}(\mathbb{T}^2)$ are both continuous, we have that Aff is also continuous. Now, for $(p_1, A_1), (p_2, A_2) \in \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z})$ we have that

$$(p_1, A_1) \cdot (p_2, A_2) = (A_1 p_2 + p_1, A_1 A_2).$$

So the image of this map is

$$\mathsf{Aff}((p_1, A_1) \cdot (p_2, A_2)) = \mathsf{Aff}((A_1 p_2 + p_1, A_1 A_2)) = (q \mapsto A_1 A_2 q + A_1 p_2 + p_1).$$

The product structure on $\mathsf{Diff}(\mathbb{T}^2)$ is composition, so

$$\begin{aligned} \mathsf{Aff}(p_1, A_1) \cdot \mathsf{Aff}(p_2, A_2) = (q \mapsto A_1 q + p_2) \circ (q \mapsto A_2 q + p_2) &= (q \mapsto A_2 q + p_2 \mapsto A_1(A_2 q + p_2) + p_1) \\ &= (q \mapsto A_1 A_2 q + A_1 p_2 + p_1). \end{aligned}$$

Therefore we have that $\mathsf{Aff}((p_1, A_1) \cdot (p_2, A_2)) = \mathsf{Aff}(p_1, A_1) \cdot \mathsf{Aff}(p_2, A_2)$, showing that Aff is indeed a group homomorphism. □

Lemma 0.2. *The continuous homomorphism* $\mathsf{Aff} : \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{(3)} \mathsf{Diff}(\mathbb{T}^2)$ *is a homotopy equivalence.*

Proof. For G a locally-connected topological group, denote by $G_e \subset G$ the path-component containing the identity element in G. This subspace $G_e \subset G$ is a normal subgroup, and the sequence of continuous homomorphisms

$$1 \longrightarrow G_e \xrightarrow{\text{inclusion}} G \xrightarrow{\text{quotient}} \pi_0(G) \longrightarrow 1$$

is a short-exact sequence. This short-exact sequence is evidently functorial in the argument G. In particular, there is a commutative diagram among topological groups

$$\begin{tikzcd}
1 \arrow[r] \arrow[d, "="'] & \mathbb{T}^2 = \big(\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z})\big)_e \arrow[r, "\text{inc}"] \arrow[d, "\mathsf{Aff}_e"'] & \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \arrow[r, "\text{quot}"] \arrow[d, "\mathsf{Aff}"'] & \pi_0\big(\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z})\big) = \mathsf{GL}_2(\mathbb{Z}) \arrow[r] \arrow[d, "\pi_0(\mathsf{Aff})"'] & 1 \arrow[d, "="'] \\
1 \arrow[r] & \mathsf{Diff}(\mathbb{T}^2)_e \arrow[r, "\text{inc}"] & \mathsf{Diff}(\mathbb{T}^2) \arrow[r, "\text{quot}"] & \pi_0\big(\mathsf{Diff}(\mathbb{T}^2)\big) \arrow[r] & 1.
\end{tikzcd}$$

Theorem 2.D.4. of [3] along with the Kirby torus trick gives us that the vertical homomorphism $\pi_0(\mathsf{Aff})$ is an isomorphism. Because both horizontal sequences are exact, if the vertical homomorphism Aff_e is a homotopy equivalence, then the vertical homomorphism Aff will aslo be a homotopy equivalence by the five lemma. We will now show that Aff_e is a homotopy equivalence. With respect to the canonical action of $\mathsf{Diff}(\mathbb{T}^2)_e$ on $\mathbb{T}^2$, the orbit of the identity element $0 \in \mathbb{T}^2$ is the evaluation map

$$\mathsf{ev}_0 \colon \mathsf{Diff}(\mathbb{T}^2)_e \longrightarrow \mathbb{T}^2 \ . \tag{4}$$

Note that the composition,

$$\mathsf{id}_{\mathbb{T}^2} : \mathbb{T}^2 \xrightarrow{(\mathsf{Aff})_e} \mathsf{Diff}(\mathbb{T}^2)_e \xrightarrow{(4)} \mathbb{T}^2 \ ,$$

is the identity map. Therefore, to show that Aff_e is a homotopy equivalence, it is sufficient to show that the stabilizer, $\mathsf{Stab}_0\big(\mathsf{Diff}(\mathbb{T}^2)_e\big) \subset \mathsf{Diff}(\mathbb{T}^2)$ of $0 \in \mathbb{T}^2$ (elements $f \in \mathsf{Diff}(\mathbb{T}^2)_e$ for which $f(0) = 0$) is contractible. Theorem 1b of [1] shows that the space diffeomorphisms of the torus which leave a specified point fixed is contractible, and therefore $\mathsf{Stab}_0\big(\mathsf{Diff}(\mathbb{T}^2)_e\big)$ is contractible. □

The quotient map $\mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2$ endows the smooth manifold $\mathbb{T}^2$ with a standard framing (ie, a trivialization of its tangent bundle): for

$$\mathsf{trans}\colon \mathbb{T}^2 \times \mathbb{T}^2 \xrightarrow{(p,q)\mapsto \mathsf{trans}_p(q):=p+q} \mathbb{T}^2$$

the abelian multiplication rule of the Lie group $\mathbb{T}^2$,

$$\varphi_0\colon \epsilon^2_{\mathbb{T}^2} \xrightarrow{\cong} \tau_{\mathbb{T}^2}\ , \qquad \mathbb{T}^2 \times \mathbb{R}^2 \ni (p,v) \mapsto \big(p, \mathsf{D}_0(\mathsf{trans}_p \circ \mathsf{quot})(v)\big) \in \mathsf{T}\mathbb{T}^2\ . \tag{5}$$

In this way we regard $\mathbb{T}^2$ as a ***framed smooth 2-manifold***. Consider the space of framings of the torus:

$$\mathsf{Fr}(\mathbb{T}^2) \ := \ \mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\epsilon^2_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}\big) \ \subset \ \mathsf{Map}(\mathbb{T}^2 \times \mathbb{R}^2, \mathsf{T}\mathbb{T}^2)\ ,$$

which is endowed with the subspace topology of the C^∞ topology on the set of smooth maps between total spaces. Note the evident continuous conjugation action of $\mathsf{Diff}(\mathbb{T}^2)$ on $\mathsf{Fr}(\mathbb{T}^2)$:

$$\begin{aligned} \mathsf{Act}\colon \mathsf{Diff}(\mathbb{T}^2) \times \mathsf{Fr}(\mathbb{T}^2) \ & \xrightarrow{(f,\varphi)\mapsto(Df,\varphi,(f^{-1},\mathsf{id}_{\mathbb{R}^2}))} \ \mathsf{Aut}_{\mathsf{Bdl}_{\mathbb{T}^2}}(\tau_{\mathbb{T}^2}) \times \mathsf{Iso}_{\mathsf{Bdl}_{\mathbb{T}^2}}\big(\epsilon^2_{\mathbb{T}^2}, \tau_{\mathbb{T}^2}\big) \times \mathsf{Aut}_{\mathsf{Bdl}_{\mathbb{T}^2}}(\epsilon^2_{\mathbb{T}^2})^{\mathsf{op}} \\ & \xrightarrow{(a,\varphi,b)\mapsto a\circ\varphi\circ b} \ \mathsf{Fr}(\mathbb{T}^2)\ , \end{aligned} \tag{6}$$

$$(f,\varphi) \mapsto \Big(\mathbb{T}^2 \times \mathbb{R}^2 \xrightarrow{(p,v)\mapsto \mathsf{D}f\big(\varphi(f^{-1}(p),v)\big)} \mathsf{T}\mathbb{T}^2\Big)\ .$$

Consider the orbit map of φ_0 for this continuous action:

$$\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(\ \mathsf{id}_{\mathsf{Diff}(\mathbb{T}^2)}\ ,\ \mathsf{constant}_{\varphi_0}\ )} \mathsf{Diff}(\mathbb{T}^2) \times \mathsf{Fr}(\mathbb{T}^2) \xrightarrow{(6)} \mathsf{Fr}(\mathbb{T}^2)\ . \tag{7}$$

The fiber of (7) is the stabilizer of φ_0 under the action (6), which consists of those diffeomorphisms that 'strictly' preserve the standard framing φ_0 of $\mathbb{T}^2$. The homotopy fiber of (7) is the homotopy stabilizer of φ_0 under the action of (6), which consists of those diffeomorphisms that 'homotopy coherently' preserve the standard framing φ_0 of $\mathbb{T}^2$.

Definition 0.3. The topological space of ***framed diffeomorphisms*** of the framed smooth manifold $(\mathbb{T}^2, \varphi_0)$ is

$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \ := \ \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2, \varphi_0) \ := \ \mathsf{hofib}_{\varphi_0}\Big(\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(7)} \mathsf{Fr}(\mathbb{T}^2)\Big)\ . \tag{8}$$

In other words, there is a homotopy-pullback diagram among topological spaces:

$$\begin{tikzcd} \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \arrow[r] \arrow[d] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(7)"] \\ * \arrow[r, "\langle\varphi_0\rangle"] & \mathsf{Fr}(\mathbb{T}^2). \end{tikzcd} \tag{9}$$

Remark 0.4. The image of f under (7) is the framing

$$\begin{tikzcd} \mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "f^{-1}\times id"] \arrow[d] & \mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "\varphi_0"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[r, "\mathsf{D}f"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[d] \\ \mathbb{T}^2 \arrow[r, "f^{-1}"] & \mathbb{T}^2 \arrow[r, "id"] & \mathbb{T}^2 \arrow[r, "f"] & \mathbb{T}^2, \end{tikzcd} \tag{10}$$

so the strict fiber of (7) over φ_0, will be those diffeomorphisims f such that diagram (10) is precisely φ_0. The only diffeomorphisms of $\mathbb{T}^2$ which satisfy this rigid condition are translations. In considering the homotopy fiber of (7), we get all diffeomorphisms f such that (10) is homotopic to φ_0. This results in $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$ being a much larger class of diffeomorphisms including small perturbations such as multiplication by bump functions in neighborhoods of $\mathbb{T}^2$.

Lemma 0.5. *There is a canonical equivalence*

$$\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \xrightarrow{\simeq} \Omega\big(\mathsf{Fr}(\mathbb{T}^2)_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)}\big)$$

over $\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{\simeq} \Omega\big(*_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)}\big)$. *In particular, the underlying space of* $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$ *can be endowed with the structure of a group-object in the* ∞*-category* $\mathcal{S}\mathsf{paces}$*, with respect to which the canonical map* $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \to \mathsf{Diff}(\mathbb{T}^2)$ *is a morphism between group-objects in* $\mathcal{S}\mathsf{paces}$,.

Proof. The commutative diagram among topological spaces (9) extends as a commutative diagram among topological spaces:

(11)
$$\begin{tikzcd}
\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \arrow[r] \arrow[d] & \mathsf{Diff}(\mathbb{T}^2) \arrow[r, "\text{quotient}"] \arrow[d, "(7)"] & \mathsf{Diff}(\mathbb{T}^2)_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)} \arrow[d, "(7)_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)}"] \\
* \arrow[r, "\langle\varphi_0\rangle"] & \mathsf{Fr}(\mathbb{T}^2) \arrow[r, "\text{quotient}"] & \mathsf{Fr}(\mathbb{T}^2)_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)},
\end{tikzcd}$$

in which the rightmost terms are homotopy coinvariants. Observe the canonical contractibility $\mathsf{Diff}(\mathbb{T}^2)_{\mathsf{h}\,\mathsf{Diff}(\mathbb{T}^2)} \simeq *$. Through this contractibility, notice that the right square is a homotopy pullback square. Because the lefthand square is defined as a homotopy pullback square, it follows that the outer square is a homotopy pullback square, from which the result follows. $\square$

Consider the composable sequence of group homomorphism The ***braid group on 3 strands*** has a standard presentation, generated by adjacent twists:

(12)
$$\mathsf{Braid}_3 \;\cong\; \Big\langle\; \tau_{1,2}\;,\; \tau_{2,3} \;\mid\; \tau_{1,2}\tau_{2,3}\tau_{1,2} \;=\; \tau_{2,3}\tau_{1,2}\tau_{2,3} \;\Big\rangle\;.$$

Lemma 1.6 below verifies that the assignments

(13)
$$\mathsf{Braid}_3 \xrightarrow{\tau_{1,2}\mapsto \begin{bmatrix}1&1\\0&1\end{bmatrix}\;,\;\tau_{2,3}\mapsto\begin{bmatrix}1&0\\-1&1\end{bmatrix}} \mathsf{GL}_2(\mathbb{Z})$$

define a homomorphism between groups.

Remark 0.6. The homomorphism $\mathsf{Braid}_3 \xrightarrow{(13)} \mathsf{GL}_2(\mathbb{Z})$ factors through the subgroup $\mathsf{SL}_2(\mathbb{Z}) \subset \mathsf{GL}_2(\mathbb{Z})$. This factorization is surjective, with central kernel freely generated by the element $(\tau_{1,2}\tau_{2,3})^6 \in \mathsf{Braid}_3$. In other words, there is a central extension among groups:

(14)
$$1 \longrightarrow \mathbb{Z} \xrightarrow{\big\langle(\tau_{1,2}\tau_{2,3})^6\big\rangle} \mathsf{Braid}_3 \xrightarrow{(13)} \mathsf{SL}_2(\mathbb{Z}) \longrightarrow 1\;.$$

The composite homomorphism

(15)
$$\mathsf{Braid}_3 \xrightarrow{(13)} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{(1)} \mathsf{Aut}_{\mathsf{Groups}}(\mathbb{T}^2)$$

defines the semi-direct product topological group:

(16)
$$\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3\;.$$

There results a composite continuous group homomorphism

(17)
$$\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \xrightarrow{\mathsf{id}_{\mathbb{T}^2}\rtimes(13)} \mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{(3)} \mathsf{Diff}(\mathbb{T}^2)\;.$$

We now state our main result; its proof occupies Section 1.

Theorem 0.7. *There is a canonical homotopy-commutative diagram among topological spaces:*

$$
\begin{tikzcd}
\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \arrow[r, "(17)"] \arrow[d] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(7)"] \\
\ast \arrow[r, "\langle \varphi_0 \rangle"] & \mathsf{Fr}(\mathbb{T}^2).
\end{tikzcd}
\tag{18}
$$

From the universal property of homotopy-pullbacks, this induces a continuous map

$$
\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \longrightarrow \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)
\tag{19}
$$

Even more, the continuous map (19) is an equivalence between group-objects in $\mathcal{S}\mathsf{paces}$.

Corollary 0.8. *The* framed mapping class group of $\mathbb{T}^2$ *is canonically isomorphic with the braid group on 3 strands:*

$$
\mathsf{Braid}_3 \xrightarrow{\cong} \mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \ := \ \pi_0\Big(\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)\Big) \ .
$$

Proof. We explain the sequence of homomorphisms among groups:

$$
\begin{aligned}
\mathsf{Braid}_3 \ &\xrightarrow{\cong} \ \pi_0(\mathsf{Braid}_3) \\
&\xleftarrow{\cong} \ \pi_0(\mathbb{T}^2) \underset{\pi_0(15)}{\rtimes} \pi_0(\mathsf{Braid}_3) \\
&\xleftarrow{\cong} \ \pi_0(\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3) \\
&\xrightarrow{\cong} \ \pi_0\big(\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)\big) \ =: \ \mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \ .
\end{aligned}
$$

The first isomorphism follows from the topological space Braid_3 being endowed with the discrete topology. The second isomorphism follows from $\pi_0(\mathbb{T}^2)$ being a singleton, which is so because $\mathbb{T}^2$ is connected. The third isomorphism is $[(p,\omega)] \mapsto ([p],[\omega])$. The fourth map is the induced map on homotopy of (19), and as (19) is a homotopy equivalence its induced map on homotopy groups is an isomorphism. ◻

Remark 0.9. Corollary 0.8 is compatible with the standard isomorphism $\mathsf{SL}_2(\mathbb{Z}) \cong \mathsf{MCG}^{\mathsf{or}}(\mathbb{T}^2) =: \pi_0\Big(\mathsf{Diff}^{\mathsf{or}}(\mathbb{T}^2)\Big)$, in the sense that the diagram among groups

$$
\begin{tikzcd}
\mathsf{Braid}_3 \arrow[r, "(13)"] \arrow[d, "\cong"'] & \mathsf{SL}_2(\mathbb{Z}) \arrow[d, "\cong"] \\
\mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \arrow[r, "\text{forget structure}"] & \mathsf{MCG}^{\mathsf{or}}(\mathbb{T}^2)
\end{tikzcd}
$$

commutes. Remark 0.6 thusly grants a central extension among groups:

$$
1 \longrightarrow \mathbb{Z} \longrightarrow \mathsf{MCG}^{\mathsf{fr}}(\mathbb{T}^2) \longrightarrow \mathsf{MCG}^{\mathsf{or}}(\mathbb{T}^2) \longrightarrow 1 \ .
\tag{20}
$$

Remark 0.10. Let $(\mathcal{V},\otimes)$ be a presentably symmetric monoidal ∞-category.[1] Let $A \in \mathsf{Alg}\Big(\mathsf{Alg}(\mathcal{V})\Big)$ an associative algebra among associative algebras in $\mathcal{V}$. Consider its iterated Hochschild homology

$$
\mathsf{HH}\big(\mathsf{HH}(A)\big) \ \in \ \mathcal{V} \ .
$$

[1] For example, for $\Bbbk$ a commutative ring, take $(\mathcal{V},\otimes) = \big(\mathsf{Ch}_{\Bbbk}[\{\text{quasi-isos}\}^{-1}], \underset{\Bbbk}{\otimes}^{\mathbb{L}}\big)$ is the ∞-categorical localization of chain complexes over $\Bbbk$ on quasi-isomorphisms, with derived tensor product over $\Bbbk$. More generally, for R a commutative ring spectrum, take $(\mathcal{V},\otimes) := (\mathsf{Mod}_R, \underset{R}{\wedge})$ to be R-module spectra and smash product over R.

Through Dunn's additivity ([**?**]), this object is canonically identified as factorization homology (in the sense of [**?**]) over the framed torus:

$$\int_{\mathbb{T}^2} A \ \simeq\ \mathsf{HH}\big(\mathsf{HH}(A)\big)\ \in\ \mathcal{V}\ .$$

There results an action of the topological group $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$ on $\mathsf{HH}\big(\mathsf{HH}(A)\big)$. Through Theorem 0.7 is identical with a morphism between group-objects in spaces:

$$\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \longrightarrow \mathsf{Aut}_{\mathcal{V}}\Big(\mathsf{HH}\big(\mathsf{HH}(A)\big)\Big)\ .$$

Remark 0.11. We follow up on Remark 0.10. Let $A \in \mathsf{CAlg}(\mathcal{S}\mathsf{pectra})$ be a commutative ring spectrum, which forgets as an algebra among ring spectra. It is known ([**?**]) that the iterated cyclotomic trace map

$$\mathsf{K}\big(\mathsf{K}(A)\big) \longrightarrow \mathsf{TC}\big(\mathsf{TC}(A)\big)$$

is not locally constant (in the argument A). Well, there is a canonical factorization of the iterated trace, from iterated K-theory:

$$\begin{tikzcd}
\mathsf{Obj}\Big(\mathsf{Perf}\big(\mathsf{K}(A)\big)\Big) \arrow[r] \arrow[dd] & \mathsf{HH}\Big(\mathsf{HH}\big(\mathsf{Perf}\big(\mathsf{Perf}(A)\big)\big)\Big)^{\mathsf{h}(\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3)} \arrow[d] & \mathsf{HH}\big(\mathsf{HH}(A)\big)^{\mathsf{h}(\mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3)} \arrow[l, "\simeq"'] \arrow[d] \\
 & \mathsf{HH}\Big(\mathsf{HH}\big(\mathsf{Perf}\big(\mathsf{Perf}(A)\big)^{\mathsf{h}\mathbb{T}}\big)\Big)^{\mathsf{h}\mathbb{T}} \arrow[d] & \mathsf{HH}\big(\mathsf{HH}(A)^{\mathsf{h}\mathbb{T}}\big)^{\mathsf{h}\mathbb{T}} \arrow[l, "\simeq"'] \arrow[d] \\
\mathsf{K}\big(\mathsf{K}(A)\big) \arrow[r, "\mathsf{tr}(\mathsf{tr})"] \arrow[uur, dashed] \arrow[ur, dashed, "\mathsf{tr}^-(\mathsf{tr}^-)"] & \mathsf{HH}\Big(\mathsf{HH}\big(\mathsf{Perf}\big(\mathsf{Perf}(A)\big)\big)\Big) & \mathsf{HH}\big(\mathsf{HH}(A)\big). \arrow[l, "\text{Morita invariance}", "\simeq"']
\end{tikzcd}$$

This suggests that the group Braid_3, as it acts on $\mathbb{T}^2$, could play a fundamental role in improving the iterated cyclotomic trace map

$$\mathsf{K}\big(\mathsf{K}(A)\big) \longrightarrow \mathsf{TC}\big(\mathsf{TC}(A)\big)$$

toward being closer to locally constant (in the argument A).

1. Proof of Theorem 0.7

Here we prove Theorem 0.7. First consider the following maps:

$$\mathsf{Fr}(\mathbb{T}^2) \to \mathsf{Map}(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R}), \qquad \psi \mapsto (p \mapsto (\phi_0^{-1} \circ \psi)_p). \tag{21}$$

$$\mathsf{Map}(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R}) \to \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R}), \tag{22}$$

$$(\mathbb{T}^2 \xrightarrow{f} GL_2(\mathbb{R})) \mapsto ((\mathbb{T}^2, 0) \xrightarrow{(f(0))^{-1}f} (GL_2(\mathbb{R}), Id), f(\mathbf{0})).$$

Lemma 1.1. *Each of the continuous maps*

$$
\begin{aligned}
\text{(23)} \qquad \mathsf{Fr}(\mathbb{T}^2) &\xrightarrow{(21)} \mathsf{Map}\big(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})\big) \\
&\xrightarrow{(22)} \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R}) \\
&\xrightarrow{\pi_1 \times \mathsf{id}_{\mathsf{GL}_2(\mathbb{R})}} \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R}) \\
&\xrightarrow{(25) \times \mathsf{id}_{\mathsf{GL}_2(\mathbb{R})}} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})
\end{aligned}
$$

is a homotopy equivalence. In particular, the composite continuous map is a homotopy equivalence.

Proof. The map (21) is a homeomorphism. It's a continuous map with a continuous inverse sending $(\mathbb{T}^2 \xrightarrow{f} GL_2(\mathbb{R}))$ to the framing

$$
\begin{tikzcd}
\mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "F"] \arrow[d] & \mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "\phi_0"] \arrow[d] & T\mathbb{T}^2 \arrow[d] \\
\mathbb{T}^2 \arrow[r, "id"] & \mathbb{T}^2 \arrow[r, "id"] & \mathbb{T}^2
\end{tikzcd}
$$

where the map F is defined by $F(p,v) = (p, f(p)v)$. The map (22) is also a homeomorphism. It's a continuous map with continuous inverse

$$
\mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \times \mathsf{GL}_2(\mathbb{R}) \to \mathsf{Map}(\mathbb{T}^2, \mathsf{GL}_2(\mathbb{R})), \qquad (f, M) \mapsto Mf.
$$

In general, for two topological groups G and H, passing to fundamental groups induces a bijection between $[K(G,1), K(H,1)] \xrightarrow{\pi_1} \mathsf{homo}(G,H)$ where $[-,-]$ indicates homotopy classes of based maps. This is the homotopy inverse to the classifying space functor. Therefore

$$
\text{(24)} \qquad \mathsf{Map}_*\Big(K(G,1), K(H,1)\Big) \xrightarrow{\pi_1} \mathsf{homo}(G,H)
$$

is a homotopy equivalence. Because $\mathbb{S}^1 = SO(2) \hookrightarrow GL_2(\mathbb{R})$ is a homotopy equivalence onto the connected component containing $\mathbb{1}$, the map

$$
\mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \xrightarrow{\pi_1} \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big)
$$

factors through

$$
\mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \to \mathsf{Map}_*\Big(\big(0 \in \mathbb{T}^2\big), \big(\mathbb{1} \in SO_2(\mathbb{R})\big)\Big) \xrightarrow{\pi_1} \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big),
$$

where the first map post composes with the Gram-Schmidt process and the second map is the homotopy equivalence (24) for $G = \mathbb{Z}^2$ and $H = \mathbb{Z}$.

Finally, the last map to consider is the identification

$$
\text{(25)} \qquad \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big) \longrightarrow \mathbb{Z} \times \mathbb{Z}
$$

$$
h \mapsto (h([\mathbb{T}^1_1]), h([\mathbb{T}^1_2]))
$$

for generators $[\mathbb{T}^1_i]$ of $\pi_1\big(0 \in \mathbb{T}^2\big)$. This is a bijection with inverse

$$
\mathbb{Z} \times \mathbb{Z} \to \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big), \qquad (a,b) \mapsto (\mathbb{Z}^2 \xrightarrow{h_{(a,b)}} \mathbb{Z})
$$

where $h_{(a,b)}(p,q) = ap + bq$. So we have that all maps are homotopy equivalences and therefore the composite will be a homotopy equivalence as well. □

The diagram among topological spaces

$$
\begin{tikzcd}
\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \arrow[r, "(3)"] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(7)"] & \\
& \mathsf{Fr}(\mathbb{T}^2) \arrow[r, "(23)"] & \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}) .
\end{tikzcd}
\tag{26}
$$

determines a span among homotopy fibers:

$$
\begin{tikzcd}
\mathsf{hofib}_{(23(\varphi_0)}\Big(\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{(26)} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})\Big) \arrow[d] \\
\mathsf{hofib}_{(23(\varphi_0)}\Big(\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(23)\,\circ\,(7)} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})\Big) \\
\mathsf{hofib}_{\varphi_0}\Big(\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(7)} \mathsf{Fr}(\mathbb{T}^2)\Big) \ =: \ \mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2) \arrow[u]
\end{tikzcd}
\tag{27}
$$

Observation 1.2. Because the horizontal maps in (26) is an equivalence, by Lemma (1.7) below, each of the maps in the span among topological spaces (27) is a homotopy equivalence. Note that $(23)\varphi_0) = (0, 0, \mathbb{1}) \in \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})$. We will compute

$$
\mathsf{hofib}_{(0,0,\mathbb{1})}\Big(\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(23)\,\circ\,(7)} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})\Big)
$$

below, allowing us to determine the homotopy type of $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$.

Lemma 1.3. *There is a commutative diagram among topological spaces*

$$
\begin{tikzcd}
\mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \arrow[r, "(3)"] \arrow[d, "\mathsf{pr}"'] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(7)"] \\
\mathsf{GL}_2(\mathbb{Z}) \arrow[d, "\text{inclusion}"'] & \mathsf{Fr}(\mathbb{T}^2) \arrow[d, "(23)"] \\
\mathsf{GL}_2(\mathbb{R}) \arrow[r, "A \mapsto (0,0,A)"] & \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}).
\end{tikzcd}
\tag{28}
$$

Proof. We have that $(p, A) \in \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z})$ maps to the linear diffeomorphism $f := (q \mapsto Aq + p)$ under (3). Then under the orbit map (7) this linear diffeomorphism is mapped to the framing

$$
\begin{tikzcd}
\mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "f^{-1} \times id"] \arrow[d] & \mathbb{T}^2 \times \mathbb{R}^2 \arrow[r, "\varphi_0"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[r, "\mathsf{D}f"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[d] \\
\mathbb{T}^2 \arrow[r, "f^{-1}"] & \mathbb{T}^2 \arrow[r, "id"] & \mathbb{T}^2 \arrow[r, "f"] & \mathbb{T}^2.
\end{tikzcd}
$$

Now under the map (23) we see that this framing is first sent to the map from $\mathbb{T}^2$ to $GL_2(\mathbb{Z})$:

$$
q \mapsto \Big(\{q\} \times \mathbb{R}^2 \xrightarrow{f^{-1} \times \mathsf{id}} \{f^{-1}(q)\} \times \mathbb{R}^2 \xrightarrow{\varphi_0} \mathsf{T}_{f^{-1}(q)}\mathbb{T}^2 \xrightarrow{\mathsf{D}f} \mathsf{T}_q\mathbb{T}^2 \xrightarrow{\varphi_0^{-1}} \{q\} \times \mathbb{R}^2\Big).
\tag{29}
$$

Recall though that the linear diffeomorphism f is actually the composite

$$
\mathbb{T}^2 \xrightarrow{\mathsf{quot}^{-1}} \mathbb{R}^2 \xrightarrow{A} \mathbb{R}^2 \xrightarrow{\mathsf{quot}} \mathbb{T}^2 \xrightarrow{\mathsf{trans}_p} \mathbb{T}^2
$$

where we operate in a small neighborhood so quot^{-1} is well defined. Then by the chain rule $\mathsf{D}f$ is the composite map

$$\begin{tikzcd}
\mathsf{T}\mathbb{T}^2 \arrow[r, "\mathsf{D}\,\mathsf{quot}^{-1}"] \arrow[d] & \mathsf{T}\mathbb{R}^2 \arrow[r, "\mathsf{D}A"] \arrow[d] & \mathsf{T}\mathbb{R}^2 \arrow[r, "\mathsf{D}\,\mathsf{quot}"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[r, "\mathsf{D}\,\mathsf{trans}_p"] \arrow[d] & \mathsf{T}\mathbb{T}^2 \arrow[d] \\
\mathbb{T}^2 \arrow[r, "\mathsf{quot}^{-1}"] & \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \arrow[r, "\mathsf{quot}"] & \mathbb{T}^2 \arrow[r, "\mathsf{trans}_p"] & \mathbb{T}^2.
\end{tikzcd}$$

The map on the fibers of $\mathsf{D}A$ is precisely A, and the map on the fibers of $\mathsf{D}\,\mathsf{trans}_p$ is the identity. So looking at (29) more closely, we see that for $(q,v) \in \{q\} \times \mathbb{R}^2$ the leftmost map is the identity on v, and then is followed by:

$$v \xrightarrow{\varphi_0|_{f^{-1}(q)}} \mathsf{D}_0(\mathsf{trans}_{f^{-1}(q)} \circ \mathsf{quot})(v) = (\mathsf{D}_0\,\mathsf{trans}_{f^{-1}(q)} \circ \mathsf{D}_0\,\mathsf{quot})(v) \xrightarrow{\mathsf{D}f|_{f^{-1}(q)}}$$

$$\Big(\mathsf{D}_{\mathsf{quot}(A(\widetilde{f^{-1}(q)}))}\,\mathsf{trans}_p \circ \mathsf{D}_{A\widetilde{f^{-1}(q)}}\,\mathsf{quot} \circ \mathsf{D}_{\widetilde{f^{-1}(q)}} A \circ \mathsf{D}_{f^{-1}(q)}\,\mathsf{quot}^{-1} \circ \mathsf{D}_0\,\mathsf{trans}_{f^{-1}(q)} \circ \mathsf{D}_0\,\mathsf{quot}\Big)(v)$$

$$= \Big(\mathsf{D}_{\mathsf{quot}(A(\widetilde{f^{-1}(q)}))}\,\mathsf{trans}_p \circ \mathsf{D}_{A\widetilde{f^{-1}(q)}}\,\mathsf{quot} \circ \mathsf{D}_{\widetilde{f^{-1}(q)}} A\Big)(v) \xrightarrow{\varphi_0^{-1}|_q}$$

$$\Big(\mathsf{D}_0\,\mathsf{quot}^{-1} \circ \mathsf{D}_0\mathsf{trans}_q^{-1} \circ \mathsf{D}_{\mathsf{quot}(A(\widetilde{f^{-1}(q)}))}\,\mathsf{trans}_p \circ \mathsf{D}_{A\widetilde{f^{-1}(q)}}\,\mathsf{quot} \circ \mathsf{D}_{\widetilde{f^{-1}(q)}} A\Big)(v) = (\mathsf{D}_{\widetilde{f^{-1}(q)}} A)(v) = Av.$$

So we see under (29) the framing is sent to the constant map

$$\mathbb{T}^2 \to \mathsf{GL}_2(\mathbb{R}), \qquad q \mapsto A.$$

This constant map is then sent to $(\mathsf{constant}_{\mathbb{1}}, A) \in \mathsf{Map}_*\Big((0 \in \mathbb{T}^2), (\mathbb{1} \in \mathsf{GL}_2(\mathbb{R}))\Big) \times \mathsf{GL}_2(\mathbb{R})$ by (22). As $\mathsf{constant}_{\mathbb{1}}$ is null homotopic, its induced map on fundamental groups is the trivial map $\mathbb{Z}^2 \xrightarrow{\mathsf{constant}_0} \mathbb{Z}$ which is $(0,0)$ as an element of $\mathbb{Z}^2 \simeq \mathsf{Homo}\Big(\pi_1\big(0 \in \mathbb{T}^2\big), \pi_1\big(\mathbb{1} \in \mathsf{GL}_2(\mathbb{R})\big)\Big)$.

So we see that $(p,A) \in \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z})$ is mapped to $(0,0,A)$ by (3) followed by (7) followed by (23) and therefore the diagram commutes. $\square$

Proposition 1.4. *There is a canonical null-homotopy of the composite homomorphism* $\mathsf{Braid}_3 \xrightarrow{(13)} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{\text{inclusion}} \mathsf{GL}_2(\mathbb{R})$. *The resulting homotopy-commutative diagram*

(30)
$$\begin{tikzcd}
\mathsf{Braid}_3 \arrow[r, "(13)"] \arrow[d] & \mathsf{GL}_2(\mathbb{Z}) \arrow[d, "\text{inclusion}"] \\
* \arrow[r, "\langle \mathbb{1} \rangle"] & \mathsf{GL}_2(\mathbb{R})
\end{tikzcd}$$

witnesses a homotopy-pullback.

Proof. We will first show that the following diagram commutes up to homotopy

$$\begin{tikzcd}
\pi_0(\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)) & \Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big) \arrow[l] \arrow[r] \arrow[d] & \mathsf{SL}_2(\mathbb{Z}) \arrow[r] \arrow[d] & \mathsf{GL}_2(\mathbb{Z}) \arrow[d] \\
& * \arrow[r, "\langle \mathbb{1} \rangle"] & \mathsf{SL}_2(\mathbb{R}) \arrow[r] & \mathsf{GL}_2(\mathbb{R}).
\end{tikzcd}$$

The right square is a homotopy pullback because $\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z}) \cong \mathsf{GL}_2(\mathbb{R})/\mathsf{GL}_2(\mathbb{Z})$. The left square is a homotopy fiber because $\mathsf{SL}_2(\mathbb{Z}) \subset \mathsf{SL}_2(\mathbb{R})$, and for any closed subgroup $H \subset G$, the homotopy fiber $\mathsf{hofib}_e(H \hookrightarrow G) \simeq \Omega(G/H)$.

Now since $\pi_n(\mathsf{SL}_2(\mathbb{Z})) = \pi_n(\mathsf{SL}_2(\mathbb{R})) = 0$ for $n > 1$, the long exact sequence of homotopy groups associated to the fibration

$$\begin{tikzcd} \Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big) \arrow[r] & \mathsf{SL}_2(\mathbb{Z}) \arrow[d] \\ & \mathsf{SL}_2(\mathbb{R}), \end{tikzcd}$$

$$\ldots \to \pi_{n+1}(\mathsf{SL}_2(\mathbb{R})) \to \pi_n(\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)) \to \pi_n(\mathsf{SL}_2(\mathbb{Z})) \to \pi_n(\mathsf{SL}_2(\mathbb{R})) \to \ldots,$$

implies that $\pi_n(\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)) = 0$ for all $n > 1$. Therefore, each connected component of $\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)$ is contractible, and this contraction gives the homotopy equivalence

$$\pi_0(\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)) \longleftarrow \Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big).$$

It is shown in [2] that $\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})$ is homeomorphic to the compliment of the trefoil, and it follows that $\pi_1(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})) := \pi_0(\Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big)) \cong \mathsf{Braid}_3$. Therefore we have that

$$\mathsf{hofib}_{\mathbb{1}}(\mathsf{GL}_2(\mathbb{Z}) \to \mathsf{GL}_2(\mathbb{R})) \simeq \Omega\big(\mathsf{SL}_2(\mathbb{R})/\mathsf{SL}_2(\mathbb{Z})\big) \simeq \mathsf{Braid}_3$$

and the diagram (30) is indeed a homotopy-pullback. □

Lemma 1.5. *There is a homotopy-commutative diagram among topological spaces, in which each square is a homotpy-pullback:*

(31)
$$\begin{tikzcd} \mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \arrow[r, "\mathsf{id}_{\mathbb{T}^2} \rtimes (13)"] \arrow[d, "\mathsf{pr}"] & \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \arrow[d, "\mathsf{pr}"] \\ \mathsf{Braid}_3 \arrow[r, "(13)"] \arrow[d] & \mathsf{GL}_2(\mathbb{Z}) \arrow[d, "\text{inclusion}"] \\ * \arrow[r, "\langle \mathbb{1} \rangle"] \arrow[d] & \mathsf{GL}_2(\mathbb{R}) \arrow[d, "A \mapsto (0,0,A)"] \\ * \arrow[r, "\langle (0,0,\mathbb{1}) \rangle"] & \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}). \end{tikzcd}$$

Proof. The lower square is a homotopy-pullback because the bottom right vertical map is the inclusion of the path-component containing the image of the bottom horizontal map. Proposition 1.4 gives that the middle square is a homotopy pullback. The upper square is a homotopy pullback because the induced map between fibers is non-canonically homeomorphic with the identity map $\mathbb{T}^2 \xrightarrow{\mathsf{id}_{\mathbb{T}^2}} \mathbb{T}^2$, which is in particular a homotopy equivalence. □

Proof of Theorem 0.7. For the composite map

(32)
$$\mathbb{T}^2 \rtimes \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{\mathsf{pr}} \mathsf{GL}_2(\mathbb{Z}) \xrightarrow{\text{inclusion}} \mathsf{GL}_2(\mathbb{R}) \xrightarrow{A \mapsto (0,0,A)} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}),$$

we have the diagram

$$\begin{tikzcd} \mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \arrow[r, "\mathsf{id}_{\mathbb{T}^2} \rtimes (13)"] \arrow[d] & \mathbb{T}^2 \underset{(1)}{\rtimes} \mathsf{GL}_2(\mathbb{Z}) \arrow[r, "(3)"] \arrow[d, "(32)"] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(7)"] \\ * \arrow[r, "(0,0,\mathbb{1})"] & \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}) & \mathsf{Fr}(\mathbb{T}^2) \arrow[l, "(23)"'] \end{tikzcd}$$

where the left square is the homotopy pullback from Lemma (1.5) and the right square is the commutative diagram from Lemma (1.3). As both (3) and (23) are homotopy equivalences, we have that the following diagram is a homotopy pullback:

$$\begin{tikzcd} \mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3 \arrow[r, "(17)"] \arrow[d] & \mathsf{Diff}(\mathbb{T}^2) \arrow[d, "(23)\circ(7)"] \\ * \arrow[r, "(0,0,\mathbb{1})"] & \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R}). \end{tikzcd}$$

In particular we have that $\mathsf{hofib}_{(0,0,\mathbb{1})}(\mathsf{Diff}(\mathbb{T}^2) \xrightarrow{(23)\circ(7)} \mathbb{Z} \times \mathbb{Z} \times \mathsf{GL}_2(\mathbb{R})) \simeq \mathbb{T}^2 \underset{(15)}{\rtimes} \mathsf{Braid}_3$, and then Observation (1.2) gives us an induced homotopy equivalence from $\mathsf{hofib}_{(0,0,\mathbb{1})}(\mathsf{Diff}(\mathbb{T}^2)$ to $\mathsf{Diff}^{\mathsf{fr}}(\mathbb{T}^2)$. □

Miscellaneous Lemmas

Lemma 1.6. *The assignments*

$$\tau_{1,2} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \qquad \tau_{2,3} \mapsto \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

uniquely extend as a group homomorphism

$$\mathsf{Braid}_3 \xrightarrow{(13)} \mathsf{GL}_2(\mathbb{Z}) \ .$$

Proof. The given assignments uniquely define a homomorphism from the free group $\Phi\colon \langle \tau_{1,2}, \tau_{2,3} \rangle \to \mathsf{GL}_2(\mathbb{Z})$. Given the presentation (12) of the braid group, the existence and uniqueness of such a sought extension follows upon verifying an equality in $\mathsf{GL}_2(\mathbb{Z})$:

$$\Phi(\tau_{1,2}\tau_{2,3}\tau_{1,2}) := \Phi(\tau_{1,2})\Phi(\tau_{2,3})\Phi(\tau_{1,2}) = \Phi(\tau_{2,3})\Phi(\tau_{1,2})\Phi(\tau_{2,3}) =: \Phi(\tau_{2,3}\tau_{1,2}\tau_{2,3}).$$

Indeed

$$\Phi(\tau_{1,2})\Phi(\tau_{2,3})\Phi(\tau_{1,2}) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and

$$\Phi(\tau_{2,3})\Phi(\tau_{1,2})\Phi(\tau_{2,3}) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

□

Lemma 1.7. *The diagram among topological spaces*

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

induces a map

$$\mathsf{hofib}_{c_0}(A \xrightarrow{g\circ f} C) \longrightarrow \mathsf{hofib}_{c_0}(B \xrightarrow{g} C).$$

Furthermore, if f is a homotopy equivalence then the induced map between homotopy fibers is a homotopy equivalence. Similarly, the diagram among topological spaces

$$\begin{tikzcd} B \arrow[d, "g"] & \\ C \arrow[r, "h"] & D \end{tikzcd}$$

where $h(c_0) = d_0$ induces a map

$$\mathsf{hofib}_{c_0}(B \xrightarrow{g} C) \longrightarrow \mathsf{hofib}_{d_0}(B \xrightarrow{h \circ g} D).$$

Furthermore, if h is a homotopy equivalence then the induced map between homotopy fibers is a homotopy equivalence.

Proof. The natural map,

$$A \times C^I \xrightarrow{f \times \mathsf{id}} B \times C^I,$$

respects the homotopy fiber conditions. So the induced map

$$A \times C^I \supset \mathsf{hofib}_{c_0}(A \xrightarrow{g \circ f} C) \ni (a, \gamma) \mapsto (f(a), \gamma) \in \mathsf{hofib}_{c_0}(B \xrightarrow{g} C) \subset B \times C^I$$

is well defined. As id is a homotopy equivalence, if f is a homotopy equivalence we have that the induced map is a homotopy equivalence. Similarly, we may consider the natural map

$$B \times C^I \xrightarrow{\mathsf{id} \times h} B \times D^I.$$

Then for $h(c_0) = d_0$ we have that the induced map

$$\mathsf{hofib}_{c_0}(B \xrightarrow{g} C) \ni (b, \gamma) \mapsto (b, h \circ \gamma) \in \mathsf{hofib}_{d_0}(B \xrightarrow{h \circ g} D)$$

is well defined since for any $\gamma \in C^I$ where $\gamma(0) = c_0$ and $\gamma(1) = g(b)$, the path $h \circ \gamma$ will have $(h \circ \gamma)(0) = h(\gamma(0)) = d_0$ and $(h \circ \gamma)(1) = h(\gamma(1)) = h(g(b))$. Again, as we have a product map with id and h, if h is a homotopy equivalence then the induced maps between homotopy fibers will be a homotopy equivalence as well. $\square$

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