# Capturing Flexible Price Elasticities in Direct Utility Models*

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**Abstract**

This paper investigates the role of the outside good utility function on admissible substitution patterns in direct utility models of discrete/continuous demand. We first present a set of novel results that characterize the functional form of price effects within this class of models. The results highlight the relative inflexibility of many standard outside good utility functions. We then propose a new outside good utility function that admits more flexible marginal utility curves. Our empirical analysis uses household scanner panel data from the potato chip category, where we find empirical support for non-standard rates of satiation for the outside good. We then show how the restrictive substitution patterns induced by standard utility specifications may distort price elasticities and optimal pricing decisions.

*Keywords*: Outside good, satiation, multiple discreteness, price elasticity, category pricing

# 1 Introduction

Models of consumer demand are often specified with an outside good to account for preferences outside of the focal inside goods being studied. The advantage of an outside good specification is that it allows individuals to do other things with their money if prices were set to be too high, which in turn, allows total category demand to expand and contract depending on prices. The additional richness afforded by an outside good in part depends on the curvature of its utility function. While much attention has been given to functional form choices of inside good utility, relatively little attention has been given to outside good utility. Instead, empirical models of demand are often specified with "default" assumptions that can place strong restrictions on demand elasticities. For example, utility that is linear in the outside good eliminates income or budgetary effects altogether (Allenby et al., 2017; Dubé, 2019).

In this paper, we formally investigate the role of the outside good utility function on the implied substitution patterns for direct utility models of multiple discrete/continuous demand. Our first contribution is to analytically derive the functional form of various price effects induced by this class of models. We assume additively separable logarithmic utility for the inside goods but do not impose any restrictions on the outside good utility function beyond the usual regularity conditions. The results highlight the mechanism through which outside good utility shapes substitution. Specifically, price effects depend on the outside good only through a ratio of second to first derivatives of its utility function. We then use this framework to examine possible limitations of standard outside good utility assumptions.

Our second contribution is to propose a new outside good utility function that induces more flexible satiation of the outside good and therefore allows for more flexible substitution patterns. The utility function is parsimonious and easy to implement. It contains an intercept and slope parameter which can generate a variety of satiation patterns. This additional flexibility of the marginal utility curves also allows price effects to depend on consumption of the outside good, which is ruled out under standard outside good utility functions.

Our empirical application uses household scanner panel data from the potato chip category. We first present model-free evidence that consumers tend to substitute more towards the inside goods as the consumption of the outside good increases. We then fit our model to the data and find empirical support for non-standard rates of outside good satiation. Comparing the estimated marginal utilities of the outside good shows that, on average, consumers have a higher baseline value but satiate faster than what would be predicted by the standard logarithmic utility function.

We highlight the impact of misspecified outside good satiation in two ways. First, we compute the full table of price elasticities and find that relative to the standard logarithmic model, our model generates larger (in magnitude) cross-price elasticities for the inside goods, smaller cross-price elasticities for the outside good, and smaller own-price elasticities. We then show that these differences matter by solving the retailer's category pricing problem. When prices are optimized using the proposed model, total category revenue increases by 3.5% relative to the standard model. These gains can impactful given the low margins and increased challenges to profitability in the grocery industry (Kuijpers et al., 2019).

Our work contributes to a larger literature that investigates functional forms in direct utility models. These models stem from early work by Wales and Woodland (1983) and Hanemann (1978, 1984) who provide unified frameworks for modeling discrete/continuous (i.e., purchase incidence and quantity) decisions. We specifically focus on models of multiple discrete/continuous demand as outlined in Kim et al. (2002) and Bhat (2005, 2008). In these models, stochastic direct utility functions are maximized subject to a budget constraint and the associated demand equations are solved using the Kuhn-Tucker conditions of constrained optimization.

There are at least three key assumptions embedded within the typical implementation of direct utility models that contribute to their flexibility, or lack thereof. The first is an assumption of additively separable utility which requires the marginal rate of substitution between any two goods to be independent of consumption for other goods. This, in turn, implies that all pairs of goods are substitutes, which is likely reasonable for the narrow product categories to which these models are commonly applied. Accommodating more flexible substitution generally poses challenges, as additive separability is needed for getting tractable demand equations (Mehta, 2015).

The second assumption, which has arguably received most attention in the literature, is the choice of functional form for inside good utility. There have been a variety of functional forms proposed and implemented, with the two most common being logarithmic (translated Cobb-Douglas) utility (e.g., Hanemann, 1978; Phaneuf et al., 2000; Satomura et al., 2011; Hasegawa et al., 2012; Lee et al., 2013; Lee and Allenby, 2014) and power (translated CES) utility (e.g., Kim et al., 2002, 2007; Bhat, 2005, 2008; von Haefen et al., 2004; Lin et al., 2013; Luo et al., 2013). The power utility function presented in Bhat (2008) contains two parameters contributing to satiation and therefore provides additional flexibility relative to the logarithmic utility function. However, the benefit of this flexibility is often limited by identification restrictions. For example, Bhat (2008) fixes one of the satiation parameters to zero which results in the limiting case of logarithmic utility.

The third assumption is the utility of the outside good. In most applications, utility is specified to have the same functional form as inside good utility. Two exceptions are Lee et al. (2013) and Lee and Allenby (2014) who specify logarithmic utility for the inside goods but linear utility for the outside good. This is a much more restrictive assumption and is made in order to simplify the evaluation of the likelihood function. There has been relatively little attention given to more flexible outside good utility models. However, a recent paper by Griffith et al. (2018) studies of the effects of nonlinear specifications of outside good utility in the class of logit demand models. In the discrete choice modeling literature, accounting for budgetary effects is usually accomplished through indirect utility that is either linear (e.g., Nevo, 2001) or logarithmic (e.g., Berry et al., 1995; Petrin, 2002) in the outside good. Griffith et al. (2018) find empirical support for a nonlinear utility specification even more flexible than what logarithmic utility provides, and then show how this flexibility is important for measuring the distributional effects of a tax. Their results point to the importance of investigating similar issues in models of discrete/continuous demand.

The remainder of this paper is organized as follows. Section 2 presents formal results on the functional form of price effects and discusses the implications of common assumptions. Section 3 proposes a new outside good utility function that can accommodate more flexible substitution patterns. Section 4 presents the results from our empirical application. Section 5 demonstrates the importance of allowing flexible substitution for category pricing. Section 6 concludes.

# 2 Direct Utility Models

## 2.1 Framework

Direct utility models are defined by the following constrained utility maximization problem.

$$\begin{aligned}&\max_{x,z} U(x,z)\\ &\text{s.t.} \sum_{j=1}^{J} p_j x_j + z \leq M \quad \text{and} \quad (x,z) \geq 0\end{aligned} \tag{1}$$

The direct utility function $U(x,z)$ is assumed to be quasi-concave, increasing, and continuously differentiable in the vector of quantities $(x,z)$. Utility is then maximized subject to two constraints: a budget constraint which says that the total expenditure on inside and outside goods cannot exceed the budgetary allotment $M$, and a non-negativity constraint to ensure valid purchase quantities.

A further assumption commonly employed in the literature is that total utility is additively separable in the inside and outside goods.

$$U(x,z) = \sum_{j=1}^{J} u_x(x_j) + u_z(z) \tag{2}$$

Here $u_x(\cdot)$ denotes the utility of the inside goods and $u_z(\cdot)$ denotes the utility of the outside good. Additive separability implies that there are no additional gains from co-consumption and therefore requires all pairs of goods to be substitutes. While this restriction limits model flexibility, it is important for retaining demand functions that are tractable (Mehta, 2015).

We specify inside good utility $u_x(\cdot)$ to have a logarithmic functional form:[1]

$$u_x(x_j) = \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) \tag{3}$$

where $\psi_j > 0$ is a baseline utility parameter and $\gamma_j > 0$ is a satiation parameter. The specific role of $\psi_j$ is to set the baseline value of marginal utility at the point $x_j = 0$, while the role of $\gamma_j$ is to govern the rate at which marginal utility diminishes. Other inside good utility functions have been

[1]This functional form also arises from a logarithmic transformation of Cobb-Douglas utility with an additional translation term (Pollak and Wales, 1992). It is also often referred to as LES utility, as it is the utility function which gives rise to the linear expenditure system (Deaton and Muellbauer, 1980).

proposed in the literature, the most notable being the power (or translated CES) utility function (Kim et al., 2002; Bhat, 2008). The power utility function contains two parameters $(\gamma_j, \alpha_j)$ that govern the rate of diminishing marginal utility and therefore offer more flexibility over the shape of satiation. However, these two satiation parameters are not separately identified and a common assumption is to set $\alpha_j = 0$ for all offerings $j = 1, \ldots, J$ which reduces to the logarithmic utility function (Bhat, 2008). Thus, the logarithmic functional form remains popular for its ability to capture satiation in a parsimonious way (Hanemann, 1978; Chintagunta and Nair, 2011; Allenby et al., 2017).

Given the assumption of additively separable utility along with the inside good utility function given in (3), the standard Kuhn-Tucker first-order conditions can be summarized as follows.

$$\begin{aligned} &\text{if } x_j > 0 \;\; \text{then } u_x'(x_j) = \frac{\psi_j}{\gamma_j x_j + 1} = u_z'(z) p_j \\ &\text{if } x_j = 0 \;\; \text{then } u_x'(x_j) = \frac{\psi_j}{\gamma_j x_j + 1} < u_z'(z) p_j \end{aligned} \tag{4}$$

If the outside good utility function is linear, then $u_z'(z) = 1$ and the first-order conditions yield simple demand equations: $x_j = (\psi_j - p_j)/(\gamma_j p_j)$. For nonlinear outside good utility functions, however, these first-order conditions give rise to an implicit function for demand. This is one of the challenges in systematically studying the substitution patterns induced by direct utility models.

## 2.2 Substitution Patterns

In this section, we investigate the substitution patterns induced by the class of direct utility models described above. In particular, the following proposition presents a set of results to show how the curvature of the outside good utility function shapes various price effects.

**Proposition 1.** *Consider the vector of demanded quantities $(x, z)$ induced by the constrained utility maximization problem in (1) with additively separable utility, inside good utility given by $u_x(x_j) = \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1)$, and outside good utility that satisfies $u_z'(z) > 0$ and $u_z''(z) < 0$. Also define $h(z) = -u_z''(z)/u_z'(z)^2$ and let $R$ denote the set of chosen inside goods. Then as the price of inside good $k \in R$ increases, the following statements hold:*

*a. (Outside good cross-price effect) The demand of the outside good increases;*

$$\mathcal{E}_{zk}(z) = \frac{\partial z}{\partial p_k} = \frac{1}{\gamma_k}\left(\frac{1}{1 + h(z)\sum_{\ell \in R} \psi_\ell/\gamma_\ell}\right) > 0 \tag{5}$$

*b. (Inside good cross-price effect) The demand of any inside good $j \in R$, $j \neq k$ increases;*

$$\mathcal{E}_{jk}(z) = \frac{\partial x_j}{\partial p_k} = \frac{1}{p_j \gamma_k}\left(\frac{h(z)\psi_j/\gamma_j}{1 + h(z)\sum_{\ell \in R} \psi_\ell/\gamma_\ell}\right) > 0 \tag{6}$$

*c. (Own-price effect) The demand of good $k$ decreases.*

$$\mathcal{E}_{kk}(z) = \frac{\partial x_k}{\partial p_k} = -\frac{1}{p_k \gamma_k}\left(\frac{1 + h(z)\sum_{\ell \in R, \ell \neq k} \psi_\ell/\gamma_\ell}{1 + h(z)\sum_{\ell \in R} \psi_\ell/\gamma_\ell}\right) - \frac{x_k}{p_k} < 0 \tag{7}$$

*Proof.* Once the functional form of a given price effect is derived, then the direction of the effect directly follows from the assumed concavity of $u_z(z)$. However, the functional forms themselves are non-trivial because the Kuhn-Tucker conditions lead to an implicit function for optimal demand. Details of the derivations are provided in Appendix A. □

Proposition 1 highlights two key properties of direct utility models. The first is that all cross-price effects are positive and the inside and outside goods are all pairwise substitutes. This is a direct consequence of the assumed additively separable structure of utility. The second is that the curvature of the outside good utility function is fully captured through the ratio of derivatives $h(z) = -u_z''(z)/u_z'(z)^2$ and directly impacts the magnitude of all price effects. For example, $h(z)$ acts as a coefficient on chosen good preference parameters $(\psi_\ell/\gamma_\ell)$ and therefore influences the weight these parameters have on price effects. If $h(z)$ is constant, then price effects only depend on preferences for the chosen goods and not levels of outside good satiation or consumption.

Further inspection reveals that the own-price effect is decreasing in $h(z)$ and the outside (inside) good cross effect is decreasing (increasing) in $h(z)$. This provides an interpretation for the moderating role of outside good demand on substitution patterns. For example, if $h(z)$ is increasing in $z$ (i.e., $h'(z) > 0$), then as the demand for the outside good increases, the presence of satiation will lead to substitution towards other inside goods rather than the outside good. This should then generate more elastic inside good cross effects, less elastic outside good cross effects, and less elastic own effects.

### 2.3 Standard Assumptions and Their Implications

We now use the results of Proposition 1 to comment on the relative flexibility or restrictiveness of standard outside good utility assumptions. The three mostly commonly used specifications are: linear $u_z(z) = \psi_z z$, logarithmic $u_z(z) = \psi_z \log(z+\gamma)$, and power $u_z(z) = \psi_z(z+\gamma)^{\alpha}$. The drawback of linear utility is that the optimal demand for good $k$ will only depend on its own price and not the price of the other goods $j \neq k$ or the budgetary allotment $M$. This, in turn, implies that all cross-price effects and income effects will be zero.[2]

The solution to this problem is to use a nonlinear specification for the outside good (Allenby et al., 2017). By allowing the outside good to satiate, the budgetary allotment and full vector of prices will enter the demand function. One of the most widely used nonlinear specifications is logarithmic utility (Chintagunta and Nair, 2011; Allenby et al., 2017). However, Proposition 1 shows that the magnitude of these cross effects is governed by $h(z)$, which is constant under logarithmic utility.[3] Therefore, the substitution patterns induced by the standard logarithmic model are still restrictive in the sense that price effects only depend on the preferences of the chosen goods and not the demand of the outside good.

The power utility function used by Bhat (2008) offers additional flexibility and so it may not be subject to the same concerns. His specification is a modification of the standard power utility function: $u_z(z) = \alpha^{-1}((z+1)^{\alpha} - 1)$, where $\alpha < 1$ captures the degree of satiation. There is no satiation when $\alpha = 1$, satiation equal to logarithmic satiation when $\alpha = 0$, and immediate satiation when $\alpha = -\infty$. The associated marginal utilities are also flexible enough to yield a non-constant ratio of derivatives: $h(z) = (1-\alpha)(z+1)^{-\alpha}$. However, the moderating effects of $z$ on substitution patterns implied Proposition 1 require $h'(z) > 0$. For this functional form, $h(z)$ is only increasing in $z$ if $\alpha < 0$, in which case satiation is stronger than what is implied by logarithmic utility. This implies that power utility is not both fully flexible and globally valid.

[2] By Proposition 1, if $u_z(z) = \psi_z z$, then $u_z''(z) = 0$ implying that $h(z) = -u_z''(z)/u_z'(z)^2 = 0$ and $\mathcal{E}_{jk}(z) = 0$.

[3] If $u_z(z) = \psi_z \log(z+\gamma)$, then $u'(z) = \psi_z/(z+\gamma)$, $u''(z) = -\psi_z/(z+\gamma)^2$, and $h(z) = 1/\psi_z$.

## 3 Proposed Utility Function

The shortcomings of existing utility functions are tied to their relatively inflexible marginal utility curves. For example, logarithmic and power utility functions induce marginal utility curves which are polynomial functions of $z$. In this section, we propose a new outside good utility function that belongs to the class of hyperbolic functions.

$$u_z(z) = \frac{1}{b} \cdot \frac{1 + \exp(bz + c)}{1 - \exp(bz + c)} \tag{8}$$

The proposed utility function contains a slope parameter $b > 0$ and an intercept parameter $c > 0$. The slope captures the degree of satiation of outside good while the intercept takes the role of a translation parameter allowing for corner solutions (i.e., $z = 0$). One benefit of hyperbolic functions is that their derivatives are also hyperbolic functions that can be expressed as combinations of exponential functions. In this case, marginal utility takes the following form:

$$u_z'(z) = \frac{2\exp(bz + c)}{(1 - \exp(bz + c))^2}. \tag{9}$$

Figure 1 compares the marginal utility curves from the proposed model (with different slopes and intercepts) to those from the standard logarithmic model. In the proposed model, marginal utility become more L-shaped as the slope parameter gets larger, which implies stronger satiation. The slope parameter allows this utility function to freely capture any range of satiation, from no satiation ($b = 0$) to full satiation ($b = \infty$). The intercept parameter shifts the marginal utility curves along the $x$-axis. Specifically, as $c$ gets larger, marginal utility shifts to the left. The intercept also represents a measure of the baseline marginal utility of the outside good since the slope parameter vanishes at the point $z = 0$: $u_z'(0) = 2\exp(c)/(1 - \exp(c))^2$.

We can also use the results in Proposition 1 to better understand the additional flexibility of the proposed model. We find that the proposed model yields a simple yet non-constant ratio of derivatives as a function of $z$.

$$h(z) = -\frac{u_z''(z)}{u_z'(z)^2} = b\sinh(bz + c) \tag{10}$$

**(a)** Different slopes **(b)** Different intercepts

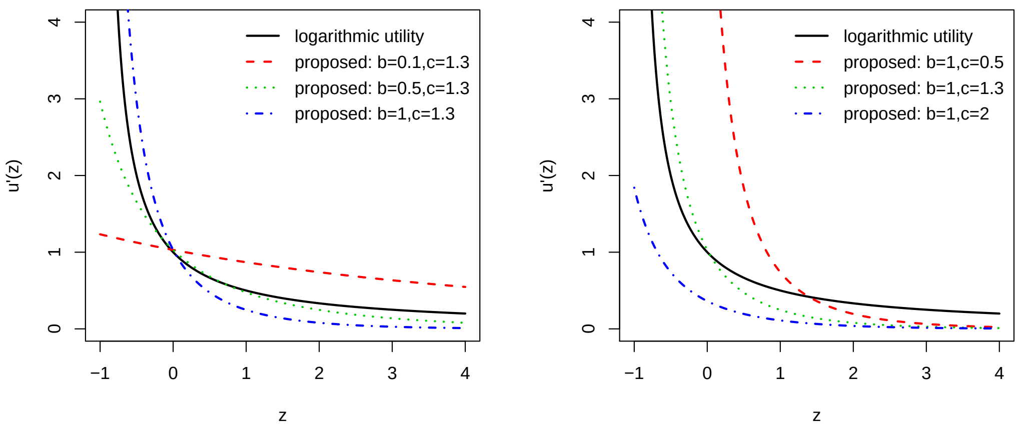


**Figure 1:** Marginal utility of the proposed model

The impact that this can have on admissible cross-price effects is shown in Figure 2. In the standard log model, $h(z)$ is constant so the cross effects associated with an inside good (red dashed line) or an outside good (solid black line) are also constant as a function of $z$. In comparison, the proposed model induces a non-constant function ratio of derivatives so the inside and outside good cross effects are allowed to move with $z$. For example, the patterns displayed in the right panel of Figure 2 show the outside good cross effect decreasing with $z$ and the inside good cross effect increasing with $z$ (note that this is only possible when $h'(z) > 0$). This corresponds to a case where the consumer's satiation on the outside good leads to increased substitution towards other inside goods relative to the outside good.

**(a)** Logarithmic utility **(b)** Proposed utility

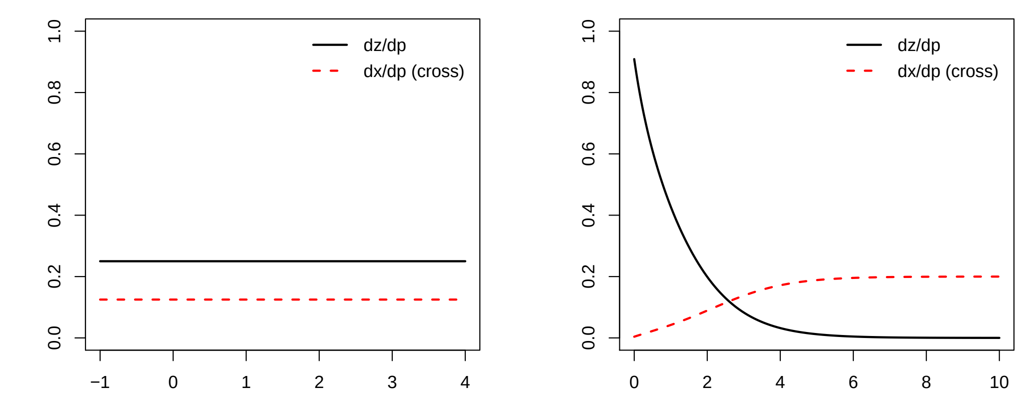


**Figure 2:** Comparison of price effects as a function of outside good demand

# 4 Empirical Application

## 4.1 Data

We apply our model to data from the potato chip product category using the IRI household panel data set (Bronnenberg et al., 2008). We use data from a single store in Eau Claire, Wisconsin covering the four-year period from July 2001 to July 2005. We select the top four brands based on market share: Lays (LA), Wavy Lays (WL), Ruffles (RF), and Old Dutch (OD). UPCs are aggregated to the brand level and a share-weighted average of UPC prices is used to define weekly prices for each brand. We use data from households that have made at least three purchases in the category, which results in a sample of 476 total households. The budget for each household is taken to be the maximum expenditure on the four brands during shopping trips over the four-year period. The last 20% of each household's purchase occasions are used as a hold-out sample for the purpose of model validation.

Table 1 shows the descriptive statistics of our data. We report average prices, total purchase quantity, and total purchase incidence. Purchase incidence is further broken down based on quantity and the frequency with which brands are purchased alone or with other brands. We find that two or more units of the same brand are purchased more than 25% of the time. We also find that many purchases include multiple brands. For example, Lays is purchased with other brands 24% of the time and Wavy Lays is purchased with other brands 38% of the time. These summary statistics point to the need to use models of multiple discrete/continuous demand in order to properly account for simultaneous quantity and incidence decisions.

**Table 1:** Chip Data Summary Statistics

| | Lays (LA) | Wavy Lays (WL) | Ruffles (RF) | Old Dutch (OD) |
|---|---|---|---|---|
| Average Price ($) | 2.21 | 2.38 | 2.67 | 0.87 |
| Total Quantity | 4,598 | 2,447 | 1,072 | 755 |
| Total Incidence | 3,246 | 1,869 | 839 | 591 |
| 1 unit | 66% | 75% | 75% | 76% |
| 2 units | 28% | 21% | 23% | 22% |
| more than 2 units | 6% | 4% | 2% | 2% |
| purchased alone | 76% | 62% | 89% | 92% |
| purchased with other brands | 24% | 38% | 11% | 8% |

## 4.2 Model-Free Evidence

One feature of the proposed utility function is that the induced substitution patterns can depend on outside good consumption. We first look to the raw data for evidence of this dependence. We approximate the effect of a price change of good $k$ on the demand of good $j$ using the ratio $(q_j^* - q_j)/(p_k^* - p_k)$ where $(q_j^*, p_k^*)$ and $(q_j, p_k)$ are observed quantities and prices for $(j, k)$ at two points in time. To do this, we start with the set of price vectors shown to each household and enumerate all possible pairs of price vectors. We then isolate the pairs of observations for which only the price of good $k$ is observed to change and compute the associated difference in prices and quantities. We define the consumption quantities of the outside good at time $t$ to be equal to $M$, the maximum observed spend on inside goods across all weeks, minus the total spend on the inside goods at time $t$.

Figure 3 plots the three different price effects (own, cross, and outside) as a function of outside good demand. The shown price effects are averaged across all goods and households, with 95% point-wise confidence bands showing measures of uncertainty. We find evidence that the price effect of the outside good tends to decrease as the outside good demand increases, while the own and cross-price effects tend to increase. These results suggest that consumers tend to substitute more towards the inside goods as the demand of the outside good increases. These data patterns are also in line with the price effects induced by the proposed utility function shown in Figure 2.

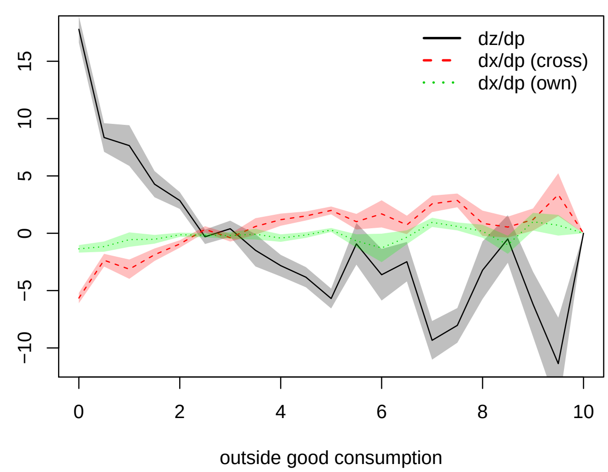


**Figure 3:** Observed price effects as a function of outside good consumption

### 4.3 Likelihood Function, Heterogeneity, and Identification

**Likelihood Function** Our empirical analysis focuses on two models: (1) a benchmark model which uses the standard logarithmic outside good utility function; and (2) a model which uses the proposed utility function for the outside good. In order to take these models to household purchase data, we first introduce household and time specific subscripts to the modeling framework in (1).

$$\begin{aligned} &\max_{x_{it}, z_{it}} \sum_{j=1}^{J} \frac{\psi_{ijt}}{\gamma_{ij}} \log(\gamma_{ij} x_{ijt} + 1) + u_z(z_{it}) \\ &\text{s.t. } \sum_{j=1}^{J} p_{jt} x_{ijt} + z_{it} \leq M_{it} \quad \text{and} \quad (x_{it}, z_{it}) \geq 0 \end{aligned} \tag{11}$$

We then introduce normal error terms through the standard multiplicative transformation of the baseline utility parameters to ensure positive marginal utility.

$$\psi_{ijt} = \exp(\psi^*_{ij} + \varepsilon_{ijt}), \qquad \varepsilon_{ijt} \sim N(0,1) \tag{12}$$

The original Kuhn-Tucker optimality conditions in (4) can be rewritten as:

$$\begin{aligned} &\varepsilon_{ijt} = g_{ijt} \quad \text{if } x_{ijt} > 0 \\ &\varepsilon_{ijt} < g_{ijt} \quad \text{if } x_{ijt} = 0 \end{aligned} \tag{13}$$

where

$$g_{ijt} = -\psi^*_{ij} + \log(\gamma_{ij} x_{ijt} + 1) + \log(p_{jt}) + \log(u'_z(z_{it})). \tag{14}$$

With independent and identically distributed error terms, the likelihood reduces to a product of probability density (PDF) and mass (CDF) components.

$$L(x_{it}, z_{it}) = |J_{it}| \prod_{j \in R_{it}} \phi(g_{ijt}) \prod_{k \in R^*_{it}} \Phi(g_{ijt}) \tag{15}$$

Here $\phi(\cdot)$ and $\Phi(\cdot)$ are the normal PDF and CDF, respectively, $R_{it} = \{k : x_{ikt} > 0\}$ is the set of chosen goods, $R^*_{it} = \{k : x_{ikt} = 0\}$ is the set of non-chosen goods, and $J_{it}$ is the Jacobian from the transformation of errors to the observed data. The determinant of the Jacobian takes the form:

$$|J_{it}| = \left( \prod_{j \in R_{it}} \frac{\gamma_{ij}}{\gamma_{ij} x_{ijt} + 1} \right) \left( 1 - \frac{u_z''(z_{it})}{u_z'(z_{it})} \sum_{j \in R_{it}} \frac{\gamma_{ij} x_{ijt} + 1}{\gamma_{ij}} p_{jt} \right). \tag{16}$$

**Identification** In the proposed model, the complete set of model parameters includes the baseline utility parameters $\psi$, satiation parameters $\gamma$, and outside good utility parameters $(b, c)$. The baseline utility parameters are identified by variation in purchase shares for each brand. The inside good satiation parameters are identified by contemporaneous variation in purchase quantities. The slope parameter $b$ governs the curvature of the outside good utility function, and is therefore identified by the variation in the magnitude of outside good demand. For example, $b$ would have to be large in order to rationalize low consumption of the outside good. The intercept parameter $c$ acts as a baseline utility parameter for the outside good. Baseline utility parameters for the outside good are often fixed for identification (e.g., $\psi_z = 1$). To allow for complete flexibility, we estimate $c$ and instead fix the baseline utility of the first good ($\psi_1 = 1$). This also allows us to estimate the baseline utility parameter in the standard log model. Both parameters $(\psi_z, c)$ are identified by the overall share of consumption of the outside good relative to inside goods.

**Heterogeneity** The full set of model parameters can be split into two groups: preference parameters for the inside goods $\beta_i = (\{\psi_{ij}^*\}, \{\gamma_{ij}^*\})$ and preference parameters for the outside good $\theta_i = (b_i^*, c_i^*)$. Note that we model the log-transformed parameters in order to impose the required sign constraints. We allow for heterogeneity across households through the following priors.

$$\beta_i \sim N(\bar{\beta}, V_\beta) \tag{17}$$

$$\theta_i \sim N(\bar{\theta}, V_\theta) \tag{18}$$

Diffuse conjugate priors are used for the population level parameters $(\bar{\beta}, V_\beta)$ and $(\bar{\theta}, V_\theta)$.

### 4.4 Results

The statistical fit of the benchmark and proposed model is reported in Table 2. We measure in-sample fit using the log marginal density (LMD) and sum of squared errors (SSE) and measure out-of-sample fit using the predictive SSE. We find that the proposed model outperforms the benchmark model both in-sample and out-of-sample, providing evidence that supports the non-standard rates of satiation offered by the proposed model.

**Table 2:** Model Fit Statistics

| Model | In-Sample LMD | In-Sample SSE | Predictive SSE |
|---|---|---|---|
| Benchmark | -11,583 | 15,325 | 17,362 |
| Proposed | -11,015 | 7,237 | 15,062 |

Next we compare the estimates of model parameters. Table 3 reports the posterior means and standard deviations of the population-level parameters. We also report the posterior mean of the variance parameters in the distribution of heterogeneity (labeled $V$). The ranking of baseline utility parameters is the same in both models and also matches the ranking of observed market shares. We do not find significant differences between the estimates of inside good utility parameters.

**Table 3:** Parameter Estimates

| | Benchmark | | | Proposed | | |
|---|---|---|---|---|---|---|
| | Mean | SD | $V$ | Mean | SD | $V$ |
| $\psi_1^*$ | 0 | | | 0 | | |
| $\psi_2^*$ | -0.65 | 0.09 | 2.45 | -0.56 | 0.07 | 1.54 |
| $\psi_3^*$ | -1.47 | 0.09 | 2.24 | -1.36 | 0.08 | 1.28 |
| $\psi_4^*$ | -3.10 | 0.12 | 4.00 | -3.00 | 0.10 | 2.64 |
| $\gamma_1^*$ | -0.71 | 0.07 | 1.04 | -1.10 | 0.06 | 0.46 |
| $\gamma_2^*$ | -1.48 | 0.11 | 1.11 | -1.71 | 0.14 | 1.26 |
| $\gamma_3^*$ | -4.05 | 0.28 | 2.58 | -4.46 | 0.46 | 2.20 |
| $\gamma_4^*$ | -1.50 | 0.15 | 1.11 | -1.66 | 0.13 | 1.18 |
| $\psi_z^*$ | 0.53 | 0.05 | 0.90 | | | |
| $b^*$ | | | | -1.21 | 0.04 | 0.50 |
| $c^*$ | | | | -0.12 | 0.02 | 0.11 |

The parameter estimates associated with the outside good utility function allow us to compare the rates of satiation implied by each model. Figure 4 plots the outside good marginal utility curves for each model. In both cases, we can clearly reject the case of linear utility with zero satiation. However, we also find that the proposed model generates different satiation patterns from the standard logarithmic utility function. In particular, we find higher baseline utility (i.e., higher values at the point $z = 0$) as well as faster rates of satiation. Together, the differences of the marginal utility curves imply that substitution towards the inside goods will be understated when

$z$ is relatively small and overstated when $z$ is relatively large.

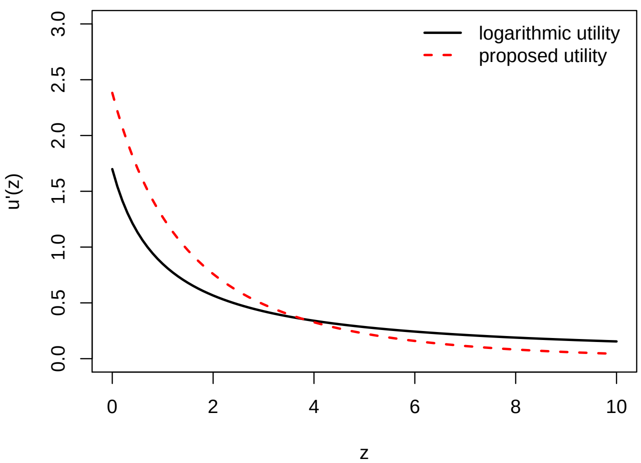


**Figure 4:** Estimated marginal utility of the outside good

To formally show that more flexible outside good utility leads to more flexible substitution patterns, we report the full cross-price elasticity matrix in Table 4. We find systematic differences between substitution patterns predicted by each model: the proposed model induces smaller (more inelastic) own-price elasticities, larger (more elastic) cross-price elasticities for the inside goods, and smaller (more inelastic) cross-price elasticities for the outside good. The overestimated outside good cross effect in the benchmark model is a direct consequence of the limited flexibility of logarithmic marginal utility function. The observed shares of the outside good must be rationalized by low satiation, implying that price increases for the inside goods leads to overestimated substitution towards the outside good.

**Table 4:** Price Elasticity Estimates

| | Benchmark | | | | | Proposed | | | |
|---|---|---|---|---|---|---|---|---|---|
| | LA | WL | RF | OD | | LA | WL | RF | OD |
| LA | -1.27 | 0.30 | 0.10 | 0.02 | LA | -1.17 | 0.46 | 0.16 | 0.03 |
| WL | 0.61 | -1.63 | 0.14 | 0.02 | WL | 0.96 | -1.61 | 0.20 | 0.03 |
| RF | 0.48 | 0.41 | -2.01 | 0.00 | RF | 0.99 | 0.52 | -1.91 | 0.03 |
| OD | 0.26 | 0.18 | 0.06 | -1.53 | OD | 0.35 | 0.26 | 0.83 | -1.43 |
| $z$ | 0.37 | 0.21 | 0.09 | 0.02 | $z$ | 0.29 | 0.15 | 0.06 | 0.02 |

We further explore the dependence of price effects on outside good consumption levels and use the results of Proposition (1) to trace out the price effect curves for the proposed model as a function of $z$. Figure 5 plots the average own and cross-price effect curves using averages of the individual-level estimates. We find evidence of an interaction between price effects and outside good consumption levels, which is in line with the data patterns reported in Figure 3. Specifically, the outside good price effect decreases with $z$ whereas the inside good own and cross effects both increase with $z$. This suggests that higher levels of outside good consumption tend to be associated with more substitution towards other inside goods as opposed to more consumption of the outside good. The somewhat surprising finding is that these interactive effects are not admissible under logarithmic utility (as shown in Figure 2), even though it allows for satiation of the outside good.

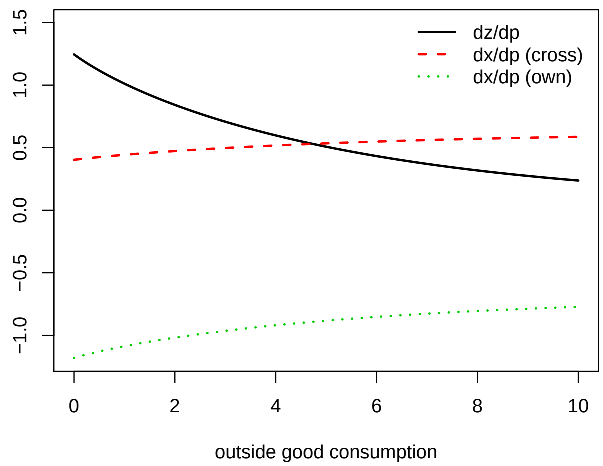


**Figure 5:** Price effects generated from the proposed model as a function of outside good demand

Because the demand of inside and outside goods is solved jointly, it is worth briefly discussing the interpretation of these interactive effects. Many policy questions centered around the measurement of consumer substitution patterns are answered through a counterfactual analysis. That is, if we observe an optimal demand vector $(x(p), z(p))$ at some set of initial prices $p$, we want to know how demand will change when prices move from $p$ to $p^*$. The interactions shown in Figure 5 refer to changes between $z(p^*)$ and $z(p)$. For example, the direction of these effects suggests that if $p^* < p$, then holding all else constant, $x(p^*) - x(p)$ will be larger when $z(p)$ is large relative to the case when $z(p)$ is small. In other words, the satiation of $u_z(\cdot)$ penalizes the outside good when it is consumed at a high level, which encourages more substitution towards other inside goods.

# 5 Implications for Category Pricing

Pricing remains one of the central arms of retail category management. In this context, managers are responsible for setting prices among products in an assortment in order to maximize category profit. This presents a nontrivial task when demand is related across goods, as is often the case. For example, for substitutable goods like Lays and Ruffles potato chips, a price increase on Lays should lead to an increase in the demand for Ruffles, holding all else constant. The magnitude of the demand responds is captured through the various own and cross-price effects, which are induced by primitives of the underlying demand model. Setting the right prices then crucially depends on the accuracy of the price effects, which in turn, depends on the flexibility of the demand model itself.

We therefore investigate how the substitution patterns induced by the proposed model affect the profitability of category pricing. We assume that the retailer sets prices to maximize total category profit:

$$\max_{p_1,\dots,p_J} \Pi = \sum_{j=1}^{J} (p_j - mc_j) x_j(p_j, p_{-j}) \tag{19}$$

where $mc_j$ is the marginal cost of $j$ and $p_{-j}$ denotes the vector of prices for all brands other than $j$. The associated first-order conditions are given by:

$$\frac{\partial \Pi}{\partial p_j} = x_j(p_j, p_{-j}) + \sum_{k=1}^{J} (p_k - mc_k) \frac{\partial x_k(p_k, p_{-k})}{\partial p_j} = 0. \tag{20}$$

This expression shows how the optimality of prices explicitly depends on the set of own and cross-price effects, which appear in the rightmost term above.

For simplicity, we assume away marginal costs and instead focus on maximizing category revenue. For both the benchmark and proposed models, we jointly solve for the vector of optimal prices and calculate the corresponding revenue for the four-good assortment. The results are summarized in Table 5. The first four columns report the difference between the optimal price and the average price observed in the data. The last column reports total revenue calculated over the entire sample period.

We find that the models agree on the direction of price changes in all but one case (Lays). When

**Table 5:** Optimal Price Changes and Revenue ($)

| | $\Delta P_j$ | | | | |
|---|---|---|---|---|---|
| Model | LA | WL | RF | OD | Total Revenue |
| Benchmark | -0.005 | 0.018 | 0.019 | -0.118 | $12,829 |
| Proposed | 0.039 | 0.041 | 0.031 | -0.064 | $13,269 |

the optimal prices are larger than the original prices and $\Delta P_j > 0$ (which is the case for Wavy Lays and Ruffles), the magnitudes of the price changes associated with the benchmark model are roughly half of those in the proposed model. When the optimal prices are smaller than the original prices (which is the case for Old Dutch), the magnitudes of the price changes of the benchmark model are almost two times larger than those in the proposed model. Together, these results imply that the benchmark model tends to underestimate price increases and overestimate the required depth of price discounts. We also find that these differences lead to a meaningful impact on total revenue, which is estimated to be about 3.5% higher using the prices from the proposed model.

## 6 Conclusion

This paper studies the role of the outside good utility function on substitution patterns induced by direct utility models of demand. The accurate measurement of price effects has been the subject of a long literature in marketing and economics and remains a practical challenge for many marketing analysts. We provide evidence, both analytically and empirically, that the outside good utility function plays an important role in this process. In particular, we show that standard specifications such as logarithmic utility are restrictive in the way that induced price effects are allowed to depend on levels of outside good satiation and consumption.

We believe that there is a natural explanation for the interaction between price effects and outside good consumption. When the price of an inside good decreases, the allocation of expenditure towards the outside good is likely to be decreasing in the consumption level of the outside good under the original vector of pre-discounted prices, all else held constant. That is, if consumption of the outside good is high under the original prices, then satiation will penalize any gains of additional consumption and consumers will be more likely to allocate more expenditure towards the inside goods. The somewhat surprising finding is that standard outside good utility functions, which already allow for satiation, cannot admit these interactive effects.

To overcome these limitations, we propose a new outside good utility function that yields more flexible marginal utility curves, but is still analytically tractable and easy to implement. Using household scanner panel data from the potato chip category, we find empirical support for the non-standard rates of satiation induced by the proposed model. Differences of the outside good satiation patterns are then shown to have consequences on substitution patterns. Relative to a model with logarithmic outside good utility, our model produces more elastic inside good cross effects and more inelastic own effects and outside good cross effects. In the context of category pricing, these effects lead to optimal prices that yield a 3.5% gain in total category revenue, which is non-trivial given the low margins in the grocery industry.

There are a few limitations and possible extensions of the current work. First, we have assumed throughout that utility is additively separable. This is a very dogmatic assumption in regards to its implications for admissible substitution patterns. However, it is also often viewed as a necessary assumption in order to retain tractability of the demand system. Further work that attempts to relax additivity would help broaden the class of products and consumption occasions to which direct utility models can be applied.

Second, although our proposed outside good utility function offers additional flexibility relative to existing alternatives, there may be other functional forms which offer similar benefits. Our work also raises a related question of how or why functional forms of utility should differ between the inside and outside goods. Many functional forms are often chosen for simplicity or convenience, but we hope that our results encourage future work that calls into question the behavioral implications of these assumptions.

Finally, the definition, interpretation, and operationalization of the outside good itself varies in the literature. What the outside good represents in a given model is tied to the definition of the budget. For example, in our empirical analysis, we assume that the budget represents the budget for potato chips which we operationalize as the maximum observed expenditure on inside goods during the sample period. Although the properties of outside good utility hold regardless of empirical context, additional research is required to understand the sensitivity of an analysis to the definition of the outside good. We leave these questions for future work.

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# APPENDIX

## A Derivation of Price Effects

Assume the outside good is always chosen and consider the case where $\ell \in R$ is chosen and $m \in R^*$ is not chosen. Then the Kuhn-Tucker conditions imply the following:

$$u_z'(z) = \frac{\psi_\ell}{\gamma_\ell x_\ell + 1}\frac{1}{p_\ell} \quad \text{for } \ell \in R; \tag{A.1}$$

$$u_z'(z) \geq \frac{\psi_m}{\gamma_m x_m + 1}\frac{1}{p_m} \quad \text{for } m \in R^*. \tag{A.2}$$

Rearranging (A.1), we get an expression for demand $x_\ell$.

$$\frac{1}{u_z'(z)} = \frac{p_\ell}{\psi_\ell}(\gamma_\ell x_\ell + 1) \implies x_\ell = \frac{1}{\gamma_\ell}\left(\frac{1}{u_z'(z)}\frac{\psi_\ell}{p_\ell} - 1\right) \tag{A.3}$$

Then plugging (A.3) into the budget constraint yields:

$$\begin{aligned} z &= M - \sum_{\ell \in R} p_\ell x_\ell \qquad (A.4) \\ &= M - \sum_{\ell \in R} p_\ell \left[\frac{1}{\gamma_\ell}\left(\frac{1}{u_z'(z)}\frac{\psi_\ell}{p_\ell} - 1\right)\right] \\ &= M - \sum_{\ell \in R}\left(\frac{\psi_\ell}{\gamma_\ell}\frac{1}{u_z'(z)} - \frac{p_\ell}{\gamma_\ell}\right) \\ &= M - \frac{1}{u_z'(z)}\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell} + \sum_{\ell \in R}\frac{p_\ell}{\gamma_\ell} \end{aligned}$$

and rearranging gives us the following equality:

$$z + \frac{1}{u_z'(z)}\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell} = M + \sum_{\ell \in R}\frac{p_\ell}{\gamma_\ell}. \tag{A.5}$$

We are now ready to derive the results of Proposition 1. We start with the outside good cross-price effect which is the change in the demand of the outside good in response to the price change of good $k \in R$. If we differentiate both sides of (B.6) with respect to $p_k$, we get:

$$\frac{\partial z}{\partial p_k} - \frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k}\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell} = \frac{1}{\gamma_k} \implies \frac{\partial z}{\partial p_k} = \frac{1}{\gamma_k}\left(\frac{1}{1 - \frac{u_z''(z)}{u_z'(z)^2}\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell}}\right) \tag{A.6}$$
$$\implies \frac{\partial z}{\partial p_k} = \frac{1}{\gamma_k}\left(\frac{1}{1 + h(z)\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell}}\right)$$

as desired.

Next, we derive the inside good cross-price effect which is the change in the demand of good $j \in R$ in response to the price change of good $k \in R,\ k \neq j$. If we differentiate both sides of (A.3) with respect to $p_k$, we get:

$$-\frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k} = \frac{p_j\gamma_j}{\psi_j}\frac{\partial x_j}{\partial p_k} \implies \frac{\partial x_j}{\partial p_k} = -\frac{\psi_j}{p_j\gamma_j}\frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k} \tag{A.7}$$

Plugging in the result of (A.6) yields

$$\begin{aligned}\frac{\partial x_j}{\partial p_k} &= -\frac{\psi_j}{p_j\gamma_j}\frac{u_z''(z)}{u_z'(z)^2}\frac{1}{\gamma_k}\left(\frac{1}{1 + h(z)\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell}}\right) \\ &= \frac{1}{p_j\gamma_k}\left(\frac{-\frac{u_z''(z)}{u_z'(z)^2}\frac{\psi_j}{\gamma_j}}{1 + h(z)\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell}}\right) \\ &= \frac{1}{p_j\gamma_k}\left(\frac{h(z)\frac{\psi_j}{\gamma_j}}{1 + h(z)\sum_{\ell \in R}\frac{\psi_\ell}{\gamma_\ell}}\right)\end{aligned} \tag{A.8}$$

as desired.

Finally, we derive own price effect which is change in the demand of good $k \in R$ in response to its own price change. If we differentiate both sides of (A.3) with respect to $p_k$, we get:

$$-\frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k} = \frac{\gamma_k}{\psi_k}\left(\frac{\partial x_k}{\partial p_k}p_k + x_k\right) + \frac{1}{\psi_k} \implies \frac{\partial x_k}{\partial p_k} = -\frac{1}{p_k\gamma_k}\left(\psi_k\frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k} + 1\right) - \frac{x_k}{p_k}. \tag{A.9}$$

Plugging in the result of (A.6) yields

$$
\begin{aligned}
\frac{\partial x_k}{\partial p_k} &= -\frac{1}{p_k\gamma_k}\left(\psi_k \frac{u_z''(z)}{u_z'(z)^2}\frac{\partial z}{\partial p_k} + 1\right) - \frac{x_k}{p_k} \\
&= -\frac{1}{p_k\gamma_k}\left[\psi_k \frac{u_z''(z)}{u_z'(z)^2}\frac{1}{\gamma_k}\left(\frac{1}{1+h(z)\sum_{\ell\in R}\frac{\psi_\ell}{\gamma_\ell}}\right) + 1\right] - \frac{x_k}{p_k} \\
&= -\frac{1}{p_k\gamma_k}\left[\left(\frac{-h(z)\frac{\psi_k}{\gamma_k}}{1+h(z)\sum_{\ell\in R}\frac{\psi_\ell}{\gamma_\ell}}\right) + 1\right] - \frac{x_k}{p_k} \\
&= -\frac{1}{p_k\gamma_k}\left(\frac{-h(z)\frac{\psi_k}{\gamma_k} + 1 + h(z)\sum_{\ell\in R}\frac{\psi_\ell}{\gamma_\ell}}{1+h(z)\sum_{\ell\in R}\frac{\psi_\ell}{\gamma_\ell}}\right) - \frac{x_k}{p_k} \\
&= -\frac{1}{p_k\gamma_k}\left(\frac{1 + h(z)\sum_{\ell\in R,\ell\neq k}\frac{\psi_\ell}{\gamma_\ell}}{1+h(z)\sum_{\ell\in R}\frac{\psi_\ell}{\gamma_\ell}}\right) - \frac{x_k}{p_k}
\end{aligned}
\tag{A.10}
$$

as desired.

# B Derivation of Price Effects with Quadratic Utility

$$U(x) = \sum_{j=1}^{J} \frac{\psi_j}{\gamma_j} \log(\gamma_j x_j + 1) + \sum_{j=1}^{J} \sum_{k \neq j} \frac{\theta_{jk}}{\gamma_j \gamma_k} \log(\gamma_j x_j + 1) \log(\gamma_k x_k + 1) + u_z(z) \tag{B.1}$$

$$\frac{\partial}{\partial x_j} U(x) = \frac{\psi_j}{\gamma_j x_j + 1} \left( 1 + \frac{1}{\psi_j} \sum_{k \neq j} \frac{\theta_{jk}}{\gamma_k} \log(\gamma_k x_k + 1) \right) \tag{B.2}$$

Define $\Omega_j = \left(1 + \frac{1}{\psi_j} \sum_{k \neq j} \frac{\theta_{jk}}{\gamma_k} \log(\gamma_k x_k + 1)\right)$.

Kuhn-Tucker conditions imply:

$$u_z'(z) = \frac{u_\ell'(x)}{p_\ell} \implies u_z'(z) = \frac{\psi_\ell}{\gamma_\ell x_\ell + 1} \cdot \Omega_\ell \cdot \frac{1}{p_\ell} \implies u_z'(z) = \frac{\psi_\ell}{\gamma_\ell x_\ell + 1} \cdot \frac{1}{p_\ell^*} \tag{B.3}$$

where $p_\ell^* = \Omega_\ell^{-1} p_\ell$. Solving for $x_\ell$:

$$x_\ell = \frac{1}{\gamma_\ell} \left( \frac{1}{u_z'(z)} \frac{\psi_\ell}{p_\ell^*} - 1 \right) \tag{B.4}$$

Then plugging into the budget constraint and rearranging:

$$\begin{aligned} z &= M - \sum_{\ell \in R} p_\ell x_\ell \\ &= M - \sum_{\ell \in R} p_\ell \left[ \frac{1}{\gamma_\ell} \left( \frac{1}{u_z'(z)} \frac{\psi_\ell}{p_\ell^*} - 1 \right) \right] \\ &= M - \sum_{\ell \in R} p_\ell \left[ \frac{1}{\gamma_\ell} \left( \frac{1}{u_z'(z)} \frac{\psi_\ell}{p_\ell \Omega_\ell^{-1}} - 1 \right) \right] \\ &= M - \sum_{\ell \in R} \left( \frac{1}{u_z'(z)} \frac{\psi_\ell}{\gamma_\ell} \Omega_\ell - \frac{p_\ell}{\gamma_\ell} \right) \end{aligned} \tag{B.5}$$

and rearranging gives us the following equality:

$$z + \frac{1}{u_z'(z)} \sum_{\ell \in R} \frac{\psi_\ell}{\gamma_\ell} \Omega_\ell = M + \sum_{\ell \in R} \frac{p_\ell}{\gamma_\ell}. \tag{B.6}$$

After differentiating w.r.t. $p_k$ we have:

$$\frac{\partial z}{\partial p_k} = \left( \frac{\frac{1}{\gamma_k} - \frac{1}{u'(z)} \sum_{\ell \in R} \frac{\psi_\ell}{\gamma_\ell} \frac{\partial \Omega_\ell}{\partial p_k}}{1 + h(z) \sum_{\ell \in R} \frac{\psi_\ell}{\gamma_\ell} \Omega_\ell} \right) \tag{B.7}$$

$$\frac{d\Omega_j}{dp_k} = \frac{1}{\psi_j} \sum_{m \neq j} \frac{\theta_{jm}}{\gamma_m} \frac{\partial}{\partial x_m} \log(\gamma_m x_m + 1) \frac{\partial x_m}{\partial p_k} = \frac{1}{\psi_j} \sum_{m \neq j} \frac{\theta_{jm}}{\gamma_m x_m + 1} \frac{\partial x_m}{\partial p_k}$$

$$\frac{\partial z}{\partial p_k} = \left( \frac{\frac{1}{\gamma_k} - \frac{1}{u'(z)} \sum_{\ell \in R} \frac{1}{\gamma_\ell} \sum_{m \neq \ell} \frac{\theta_{\ell m}}{\gamma_m x_m + 1} \frac{\partial x_m}{\partial p_k}}{1 + h(z) \sum_{\ell \in R} \frac{\psi_\ell}{\gamma_\ell} \Omega_\ell} \right) \tag{B.8}$$