MATH 462

Classification

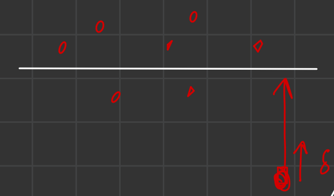
Sept 17

Friday

Lect 6

Classification loss design

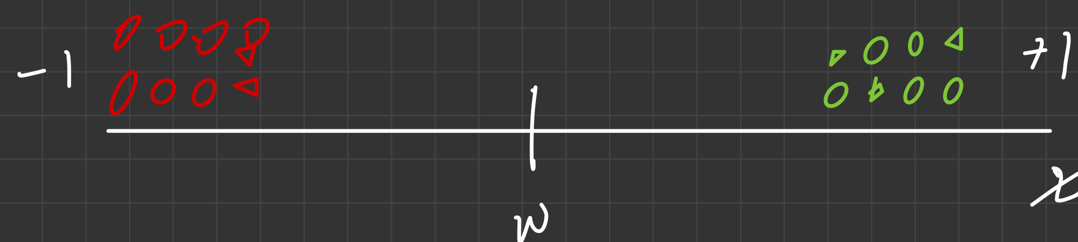
Recall Regression



Binary CAT / DOG

pos / Neg

$$h(x,w) = \begin{cases} +1 = \circ & \text{if } x \geq w \\ -1 = \circ & \text{if } x < w \end{cases}$$



$$S_m = \{(x_1, y_1), \dots, (x_m, y_m)\}$$

$$x \in \mathbb{R}^d \qquad y \in \mathcal{Y}_k = \{1, 2, \dots, k\}$$

special case binary $\mathcal{Y}_2 = \{-1, +1\}$

$x \in \mathbb{R}^d$ "scores"

$d = 1$ x scores

threshold model $h_w(x) = x - w \in \mathbb{R}$

classifier $c(h) = \text{sign}(h) = \begin{cases} +1 & h \geq 0 \\ -1 & h < 0 \end{cases}$

simplest loss $\ell_{01} : \mathcal{Y} \times \mathcal{Y} \to \{0, 1\}$

$$\ell(h, y) = \begin{cases} 0 & h = y \quad \text{correct} \\ 1 & h \neq y \quad \text{wrong} \end{cases}$$

Binary name for errors

False Neg

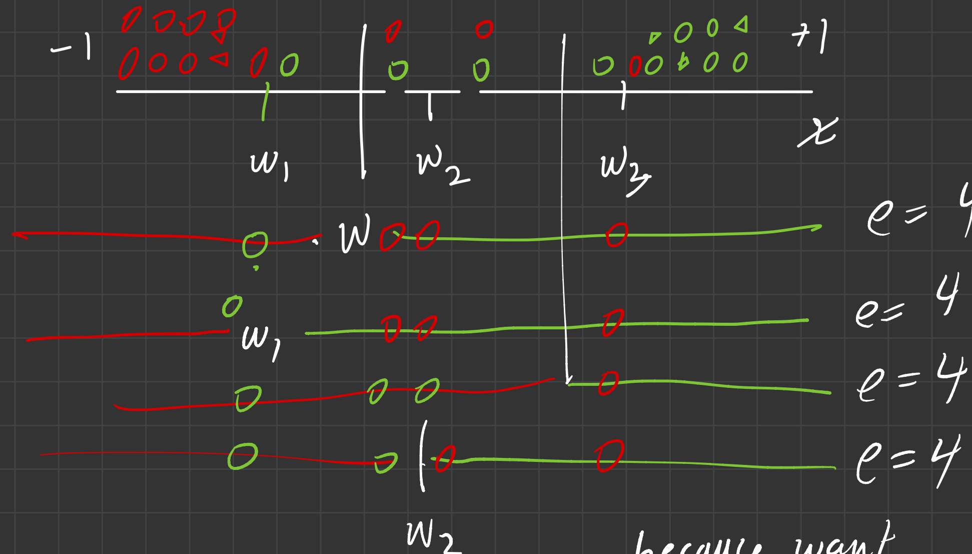
$h = -1$

$y = +1$

False Pos

$h = +1$ $y = -1$

Claim 0-1 loss has no pref between w_1, w_2, w_3



WANT: loss choose w_2 because want balanced error types

Later can choose ratio of FP to FN

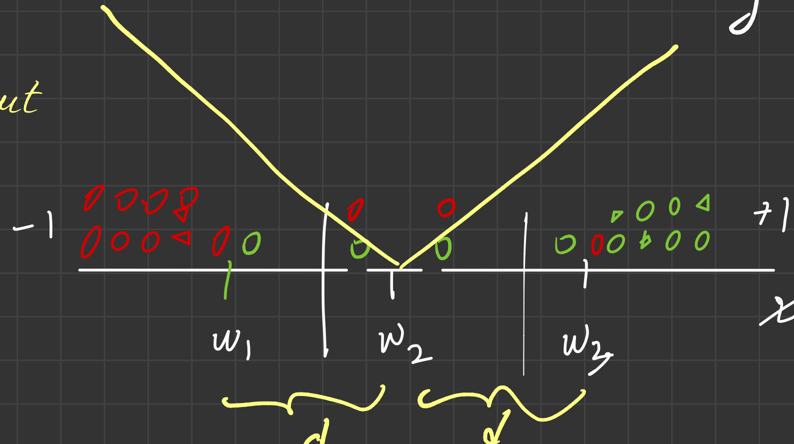
ANOTHER POINT

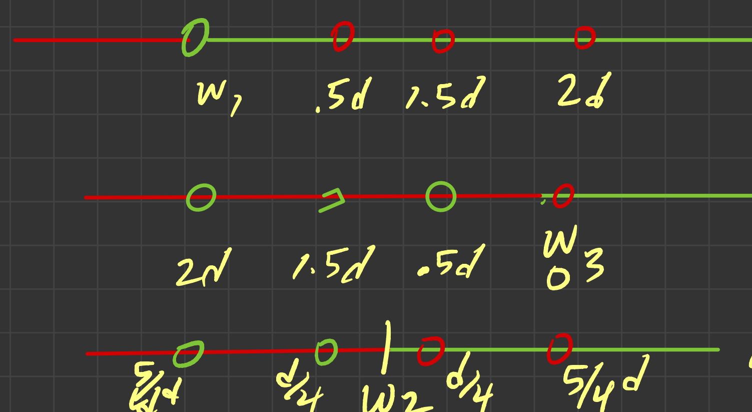
0-1 Loss only tells pos/neg

numerical loss: how close

Another loss with distance to class boundary

$$\text{loss} = \begin{cases} 0 & \text{correct} \\ \text{dist to } w & \text{incorrect} \end{cases}$$





$$\hat{L}(w_1) = 4d$$

$$\hat{L}(w_3) = 4d$$

$$\hat{L}(w_2) = 3d$$

threshold model $\quad h_w(x) = x - w \in \mathbb{R}$

classifier $\quad c(h) = \text{sign}(h) = \begin{cases} +1 & h \geq 0 \\ -1 & h < 0 \end{cases}$

$$\ell(h, y) = \begin{cases} 0 & \text{if } \text{sign}(h) = y \\ |\underbrace{x - w}_{h}| & \text{if not} \end{cases}$$

$h(x, w) = x - w \quad \frac{\partial}{\partial w} h = -1$

$$\ell(h, y) = \max(-hy, 0)$$
$$= (-hy)^+$$
$$= ((w - x)y)^+$$

Note $\quad \frac{\partial}{\partial h} \ell(h, y) = \begin{cases} 0 & \ell = 0 \\ -y & \ell > 0 \end{cases}$

$$= -y \, \mathbb{1}_{\{hy < 0\}}$$

$$\hat{L}(w) = \frac{1}{m} \sum_{i=1}^{m} \ell(h, y)$$

$$\frac{\partial}{\partial w} \hat{L}(w) = 0$$

$$0 = \frac{1}{m} \sum_{i=1}^{m} \frac{\partial}{\partial h} \ell(h, y) \underbrace{\frac{\partial h}{\partial w}}_{-1}$$

$$\sum_{i=1}^{m} y \, \mathbb{1}_{\{hy < 0\}} = 0$$

$$\sum_{i=1}^{m} y\,\mathbb{1}\{hy<0\} = 0$$

$$\underbrace{\sum_{\{y_i=1\}} \mathbb{1}_{h<0}}_{\#FN} + \underset{\#FP}{\sum_{\{y_i=-1\}} -\mathbb{1}\{h>0\}}$$

$$\#FN = \#FP$$

$$\ell(h,y) = \begin{cases} 0 & \text{if } \operatorname{sign}(h) = y \\ C_{FP} |h| y & \text{if } h>0 \quad y<0 \\ C_{FN} \cdot |h| y & \text{if } h<0 \quad y>0 \end{cases}$$

$$\frac{\partial \ell}{\partial h} = \begin{cases} 0 & \text{correct} \\ -C_{FP} & \text{if } FP \\ C_{FN} & \text{if } FN \end{cases}$$

$$\frac{\partial}{\partial w} \hat{L}(h) = \ldots$$

$$C_{FP}(\#FP) = C_{FN}(\#FN)$$

<u>HW</u>

which is what we wanted

By choosing C_{FP} C_{FN}

$$\#FP/\#FN = r = \frac{C_{FN}}{C_{FP}}$$

Multiclass.

scores for class

$$\mathcal{Y}_K = \{1, 2, \dots, K\}$$

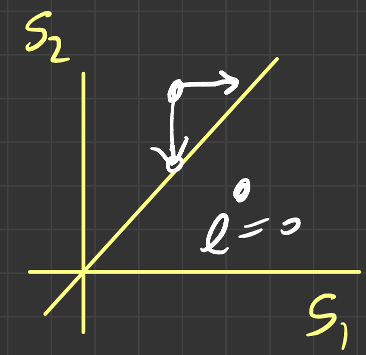
$$s(x) = (s_1(x), \dots, s_K(x))$$

$$c(s(x)) = \arg\max\ s_i(x)$$

choose class with highest score

$$\ell_{0\text{-}1}(\underbrace{c(s(x))}_{\in \mathcal{Y}_K}, \underbrace{y}_{\in \mathcal{Y}_K}) = \begin{cases} 0 & y = c \\ 1 & \text{o.w.} \end{cases}$$

case $y=1$



__Want__ : loss ~ "distance" to class boundary if incorrect

$$\ell_m(s(x), y) = \begin{cases} 0 & s_y > \max_{y \neq j} s_j \\ \max_{j \neq y} s_j - s_y & \text{o.w.} \end{cases}$$

how far need to increase s to reach c.

$$= \left(\max_{j \neq y} s_j - s_y\right)^+$$

suppose

1 each x_j is a score for class j

but not comparable

$x_j > x_i$ means x_j more likely than x_i

learning problem

$$h_i(x_i) = w_i x_i + b_i$$

$$\ell(h, y) = \ell_m(h, y)$$

(last page)

$2k$ vars