Bisimulation metric

A bit of history:

Bisimulation is an equivalence relation which captures when 2 states of a system "behave the same way."

1970's Robin Milner U. of Edinburgh

1980ish David Park → fixed pt def.

1989 Kim Larsen & Arne Skou : prob. bisim.

1997 Blute, Desharnais, Edalat, P.
→ continuous state spaces

1999 Bisimulation metrics → Desharnais, ..., P

2003 Bisimulation hits the ML community

Givan & Dean → rewards

2004 Bisimulation metrics for MDPs

Ferns, P, D. Precup

2005 " " for continuous state spaces

⋮

2021 NeurIPS: using metrics for representation learning: Tyler Kastner, Pablo Castro, Mark Rowland, P.

Prehistory : lumpability : OR in the 60's

Def A Markov Decision Process (MDP)

$$M = (S, \Sigma, A, P_s^a \in dist(\Sigma), r_s^a \in [0,1])$$

$S \downarrow$ states, $A \downarrow$ actions

If you are in "s" and execute "a" then you end up in a set $B (\in \Sigma)$ with prob $P_s^a(B)$.

You earn a reward $r_s^a \in [0,1]$

Def An equivalence relation $R \subseteq S \times S$ is said to be a bisimulation if whenever sRt then

$\Rightarrow$ (i) $\forall a$, $r_s^a = r_t^a$

(ii) $\forall a$, $\forall C$ R-closed sets

$$P_s^a(C) = P_t^a(C)$$

X is R-closed means that $s \in X$, sRs' then $s' \in X$

Bisimulation is too unstable.

A better concept is a pseudometric.

(X,d) is a <u>metric</u> <u>space</u> if X is a set and $d: X \times X \to \mathbb{R}^{\geq 0}$ such that

(i) $\forall x \in X, d(x,x) = 0$

(ii) $\forall x,y \in X \quad d(x,y) = d(y,x)$

(iii) $\forall x,y,z \in X \quad d(x,y) \leq d(x,z) + d(z,y)$

(iv) $\quad d(x,y) = 0 \Rightarrow x = y$

$\to$ If we drop (iv) we get a pseudometric.

If you keep (i), (iii) & (iv) quasimetric

If you keep (i), (ii), (iv) semi-metric

<u>Basic Fact</u>:

$f:(X,d) \to (X,d)$ we say f is <u>contractive</u> if $\exists\, c \in (0,1)$ s.t $\forall x,y \in X$

$$d(f(x),f(y)) \leq c \cdot d(x,y)$$

<u>Thm.</u> If (X,d) is a complete metric space & $f: X \to X$ is contractive there is a <u>unique</u> point $x_0 \in X$ s.t.

$$f(x_0) = x_0$$

BANACH FIXED POINT THEOREM

This theorem is very useful but does not always apply.

Another fixed-point theorem:

$(S, \leq)$ posets

<u>Def</u> A lattice is a poset in which every <u>pair</u> of elements has a least upper bound (lub, sup) and a greatest lower bound (glb, inf).

<u>Ex</u> Rational numbers with $\leq$

<u>Def</u> A <u>complete</u> <u>lattice</u> is one in which every subset has a lub (sup).

<u>Remark 1</u>: Yes even the empty set: there is a smallest element

<u>Remark 2</u> It follows that every set has a glb (inf). So a complete lattice has a greatest element.

<u>Remark 3</u> Just because a poset has a greatest & a least element does not mean that it is a complete lattice.

<u>Ex</u> Take <u>any</u> set X and look at the power set $\mathcal{P}(X)$ $[2^X]$ ordered by inclusion. This is always complete

<u>Ex</u> Take any set X and look at the collection of all equivalence relations ordered by inclusion. Complete lattice

Def A function $f : (S, \leq) \to (S', \leq')$ is said to be <u>monotone</u> (monotonic) if $\forall x, y \in S$, $x \leq y \Rightarrow f(x) \leq' f(y)$.

Thm If L is a complete lattice and $f : L \to L$ is monotone, then there are a whole bunch of fixed points. [The fixed points themselves form a complete lattice]

KNAESTER-TARSKI theorem.

In particular there is a least fixed point and a <u>greatest</u> fixed point.

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METRICS defined on spaces of probability distributions.

Given a function $f : (X, d) \to (X', d')$ we say it is non-expansive (or 1-Lipschitz) if $\forall x, y$ $\quad d'(f(x), f(y)) \leq d(x, y)$.

special case : $(X, d) \to \mathbb{R}$

$\forall x, x' \in X \quad |f(x) - f(x')| \leq d(x, x')$.

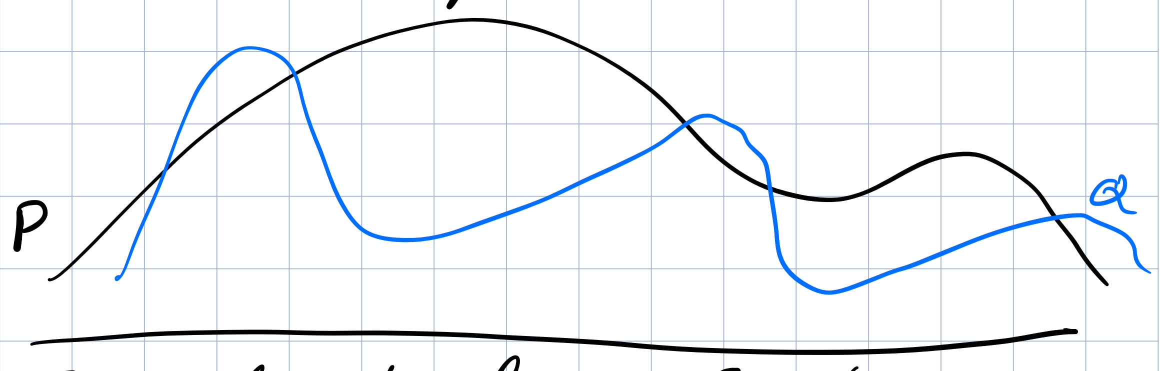
Given a metric space we consider probability distributions defined on the Borel sets.

P, Q : how different are they?
To compare them we compare expectation values

$$K(P,Q) := \sup_{f \in Lip_1} \left| \mathbb{E}_P[f] - \mathbb{E}_Q[f] \right|$$

$$= \sup_{f \in Lip_1} \left| \int f \, dP - \int f \, dQ \right|$$

KANTOROVICH METRIC
There is another way of thinking about the difference between P and Q.
P, Q as "piles of sand"



In order to change P into Q you have to move "sand" around.

A *plan* to move the sand is given by a *joint* distribution γ on

$X \times X$ such that the marginals

$$\left.\begin{aligned} \gamma_1(A) &= \gamma(A \times X) = P(A) \\ \gamma_2(B) &= \gamma(X \times B) = Q(B) \end{aligned}\right\} \text{match } P, Q$$

Such a γ is called a *coupling*

of P, Q : $\mathcal{C}(P, Q) \ni \gamma$.

How much does a plan *cost*?

$$\text{COST of } \gamma = \int_{X \times X} d(x,y)\, \underbrace{d\gamma}_{\text{prob measure on } X \times X}$$

$$W_1(P, Q) = \inf_{\gamma \in \mathcal{C}(P,Q)} \int d(x,y)\, d\gamma$$

$$W_p(P, Q) = \inf_{\gamma \in \mathcal{C}(P,Q)} \left[\int d^p(x,y)\, d\gamma \right]^{1/p}$$

KANTOROVICH - RUBINSTEIN DUALITY:

$$W_1 = \mathcal{K} \qquad (1958)$$

PROP δ_x to be the Dirac dist at x.

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$W_1(\delta_x, \delta_y) = d(x,y)$$

So the original space is isometrically embedded in the space of prob. distributions

Given an MDP $M = (S, \cdot\cdot, A, P, r)$

$\mathcal{M}(S)$: 1-bounded pseudometrics on S ordered by

$m_1 \leq m_2$ if $\forall x, y \in S$ $m_1(x,y) \leq m_2(x,y)$

$\bot(x,y) = 0 \quad \forall x, y$

$$\top(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

$\mathcal{M}$ is a complete lattice

1 bounded means $\forall x, y$
$m(x,y) \leq 1$.

$c \in (0,1)$ $\mathcal{F}^c : \mathcal{M} \longrightarrow \mathcal{M}$

$$\mathcal{F}^c(m)(s,s') = \max_{a \in A} |r_s^a - r_{s'}^a| +$$

$$c\, W_1(m)(P_s^a, P_{s'}^a)$$

$\mathcal{F}_m^c$ is monotone [Easy to see.]

Hence there is a least fixed point

we call this d_{fix}^c : bisimulation (pseudo) metric.

It follows that $d_{fix}(s,s') = 0$ then s, s' are bisimilar.

1999 original def

2001 fixed pt def van Breugel & Worrell

$$\mathcal{F}^c(d_{fix}^c) = d_{fix}^c$$

<u>Ferns' Thm</u> $\gamma \leq c$ then V^* is 1-Lipschitz and

$\longrightarrow |V^*(s) - V^*(s')| \leq d_{fix}^c(s,s').$