

MATH 462 ASSIGNMENT 1
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ADAM M. OBERMAN

1. EXERCISES

1.1. **Example calculations.**

- (1) (One variable quadratic regression) Consider (EL) with $X = [1, 2, 3]^T$, $y = [2, 4, 5]^T$, linear model $h_w(x) = w * x$, and quadratic loss, (EL1d). Solve the problem by finding a critical point, $\hat{L}'(w) = 0$.
- (2) (One variable ℓ_1 regression) Same setup as in the previous problem: consider (EL) with $X = \{1, 2\}$ $Y = \{1, 3\}$. Use linear model $h = w * x$. Solve the problem with the ℓ_1 loss. Hint: plot the function $\hat{L}(w)$, which is piecewise linear, and find the minimum value (by finding the intersection of two lines).
- (3) (ℓ_1 central value) Plot (by hand, or otherwise), $\hat{L}(w) = \frac{1}{m} \sum_i |w - y_i|$ in the case $y = [2, 4, 5]^T$, and in the case $y = [3, 6]$. Find all the minimizers in both cases.
- (4) (Polynomial regression, coding). Suppose our data points are $x = [0, .1, \dots, .9, 1]$. Let $f(x) = (1, x)$ (affine linear regression). Set $y = \sin(2\pi x) + .1e$, where e is uniformly random data on $[0, 1]$. (i) Set up the data matrix, F . What are the sizes of F , $F^T F$, and w ? Plot the error and solution. Is the fit good? (ii) Redo the problem with $f(x) = (1, x, x^2, x^3)$.
- (5) (Two variable quadratic regression) Set $X = \{(3, 0), (0, 2), (1, 1)\}$ $Y = \{6, 2, 5\}$ Setup the quadratic regression problem (i) by minimizing (EL) directly (i.e. take derivatives with respect to w_1 and w_2 and solve. (ii) by setting up the matrix equation (Lin) and solving it.

- (6) Consider https://en.wikipedia.org/wiki/Winsorized_mean Give an example with 10 numbers where the 10% Winsorized mean is the same as the minimizer of the Huber loss (with, say $\delta = 1$). Explain the main difference between the Winsorized mean and the minimizer of the Huber loss? (Hint: the Huber loss has a scale δ which determines the outliers, but the Winsorized mean has a fraction of values).
- (7) (Compare the regression loss functions) Consider y_1, \dots, y_{10} consisting of nine 0 and one value y . What is the solution (as a function of y) of (CVIL) for each of the three main regression losses (take $\delta = 1$ in the Huber).

1.2. Theory exercises.

- (1) Consider (Lin) when $d = 1$. Solve the equation, for w , and give the solution using vector notation.
- (2) Give a different derivation of (Lin) by citing the matrix theory fact that $\min_w \|Xw - y\|^2$ is given by (Lin).
- (3) (Huber loss) Show that the Huber loss is continuous, and differentiable. Find the second derivative of the function.
- (4) *Problem removed*
- (5) Verify that the three main regression losses are all convex.

1.3. Loss design exercises.

- (1) (Loss designs for grading scheme) Consider a grading scheme where there are five assignments. Suppose we want a grading scheme that is less sensitive to outliers, e.g. with a score of .9, .9, .9, 0 (one missed assignment) we don't want the hard penalty given by the average. At the same time, we want every grade to have a small effect (to encourage performance when possible). (i) propose a simple scheme to do this. (ii) Suppose we want a missed assignment to have an effect of no more than $\delta = .2$ on the average. Show that solving (CVIL) using the Huber loss with $\delta = .2$ accomplishes this (at least in the example above).
- (2) (Loss design: flipped huber) Design a 'flipped' Huber loss function, which is quadratic for $|t| \geq \delta$ and equals $|t|$ for otherwise. Set up the quadratic so that the loss is continuously differentiable.
- (3) *Problem removed*

2. REFERENCED FORMULAS

$$(\text{EL}) \quad \hat{L}(w) = \frac{1}{m} \sum_{i=1}^m \ell(h_w(x_i), y_i)$$

$$(\text{EL1d}) \quad \hat{L}(w) = \frac{1}{m} \sum_{i=1}^m q(wx_i - y_i).$$

$$(\text{Lin}) \quad X^T X w = X^T y$$

$$(\text{CVIL}) \quad \min_w \frac{1}{m} \sum_{i=1}^m \ell(w, y_i)$$