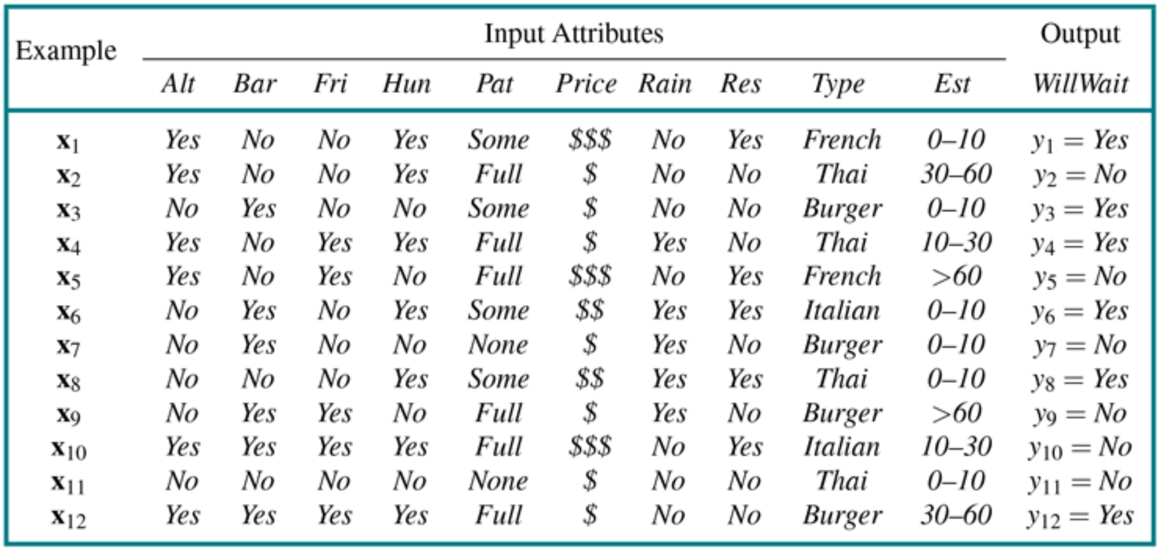
We will describe a sample supervised learning problem in detail: the problem of deciding whether to wait for a table at a restaurant. This problem will be used throughout the chapter to demonstrate different model classes. For this problem the output, y, is a Boolean variable that we will call *WillWait*; it is true for examples where we do wait for a table. The input, x, is a vector of ten attribute values, each of which has discrete values:

1. **ALTERNATE:** whether there is a suitable alternative restaurant nearby.
2. **BAR:** whether the restaurant has a comfortable bar area to wait in.
3. **FRI/SAT:** true on Fridays and Saturdays.
4. **HUNGRY:** whether we are hungry right now.
5. **PATRONS:** how many people are in the restaurant (values are *None*, *Some*, and *Full*).
6. **PRICE:** the restaurant's price range ($, $$, $$$).
7. **RAINING:** whether it is raining outside.
8. **RESERVATION:** whether we made a reservation.
9. **TYPE:** the kind of restaurant (French, Italian, Thai, or burger).
10. **WAITESTIMATE:** host's wait estimate: $0 - 10$, $10 - 30$, $30 - 60$, or >60 minutes.

A set of 12 examples, taken from the experience of one of us (SR), is shown in Figure 19.2. Note how skimpy these data are: there are $2^6 \times 3^2 \times 4^2 = 9,216$ possible combinations of values for the input attributes, but we are given the correct output for only 12 of them; each of the other 9,204 could be either true or false; we don't know. This is the essence of induction: we need to make our best guess at these missing 9,204 output values, given only the evidence of the 12 examples.

Figure 19.2



Example	Input Attributes										Output
	Alt	*Bar*	*Fri*	*Hun*	*Pat*	*Price*	*Rain*	*Res*	*Type*	*Est*	*WillWait*
$\mathbf{x}_1$	*Yes*	*No*	*No*	*Yes*	*Some*	*$$$*	*No*	*Yes*	*French*	*0–10*	$y_1 = Yes$
$\mathbf{x}_2$	*Yes*	*No*	*No*	*Yes*	*Full*	*$*	*No*	*No*	*Thai*	*30–60*	$y_2 = No$
$\mathbf{x}_3$	*No*	*Yes*	*No*	*No*	*Some*	*$*	*No*	*No*	*Burger*	*0–10*	$y_3 = Yes$
$\mathbf{x}_4$	*Yes*	*No*	*Yes*	*Yes*	*Full*	*$*	*Yes*	*No*	*Thai*	*10–30*	$y_4 = Yes$
$\mathbf{x}_5$	*Yes*	*No*	*Yes*	*No*	*Full*	*$$$*	*No*	*Yes*	*French*	*>60*	$y_5 = No$
$\mathbf{x}_6$	*No*	*Yes*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Italian*	*0–10*	$y_6 = Yes$
$\mathbf{x}_7$	*No*	*Yes*	*No*	*No*	*None*	*$*	*Yes*	*No*	*Burger*	*0–10*	$y_7 = No$
$\mathbf{x}_8$	*No*	*No*	*No*	*Yes*	*Some*	*$$*	*Yes*	*Yes*	*Thai*	*0–10*	$y_8 = Yes$
$\mathbf{x}_9$	*No*	*Yes*	*Yes*	*No*	*Full*	*$*	*Yes*	*No*	*Burger*	*>60*	$y_9 = No$
$\mathbf{x}_{10}$	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$$$*	*No*	*Yes*	*Italian*	*10–30*	$y_{10} = No$
$\mathbf{x}_{11}$	*No*	*No*	*No*	*No*	*None*	*$*	*No*	*No*	*Thai*	*0–10*	$y_{11} = No$
$\mathbf{x}_{12}$	*Yes*	*Yes*	*Yes*	*Yes*	*Full*	*$*	*No*	*No*	*Burger*	*30–60*	$y_{12} = Yes$

Examples for the restaurant domain.

19.3 Learning Decision Trees

A **decision tree** is a representation of a function that maps a vector of attribute values to a single output value—a “decision.” A decision tree reaches its decision by performing a sequence of tests, starting at the root and following the appropriate branch until a leaf is reached. Each internal node in the tree corresponds to a test of the value of one of the input attributes, the branches from the node are labeled with the possible values of the attribute, and the leaf nodes specify what value is to be returned by the function.

Decision tree

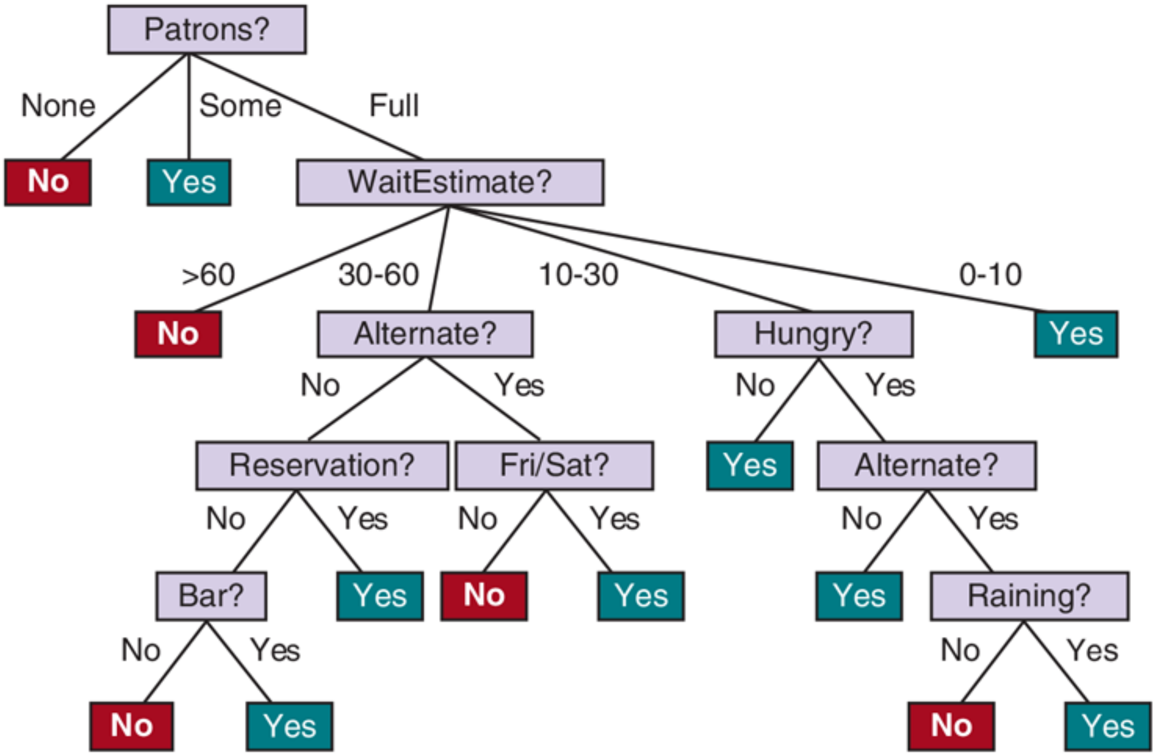
In general, the input and output values can be discrete or continuous, but for now we will consider only inputs consisting of discrete values and outputs that are either *true* (a **positive** example) or *false* (a **negative** example). We call this **Boolean classification**. We will use j to index the examples ($\mathbf{x}_j$ is the input vector for the jth example and y_j is the output), and $x_{j,i}$ for the ith attribute of the jth example.

Positive

Negative

The tree representing the decision function that SR uses for the restaurant problem is shown in Figure 19.3. Following the branches, we see that an example with $Patrons = Full$ and $WaitEstimate =$ 0–10 will be classified as positive (i.e., yes, we will wait for a table).

Figure 19.3



A decision tree for deciding whether to wait for a table.

19.3.1 Expressiveness of decision trees

A Boolean decision tree is equivalent to a logical statement of the form:

$$Output \quad \Leftrightarrow \quad (Path_1 \vee Path_2 \vee \cdots),$$

where each $Path_i$ is a conjunction of the form $(A_m = v_x \wedge A_n = v_y \wedge \cdots)$ of attribute-value tests corresponding to a path from the root to a *true* leaf. Thus, the whole expression is in disjunctive normal form, which means that any function in propositional logic can be expressed as a decision tree.

For many problems, the decision tree format yields a nice, concise, understandable result. Indeed, many "How To" manuals (e.g., for car repair) are written as decision trees. But some functions cannot be represented concisely. For example, the majority function, which returns true if and only if more than half of the inputs are true, requires an exponentially large decision tree, as does the parity function, which returns true if and only if an even number of input attributes are true. With real-valued attributes, the function $y > A_1 + A_2$ is hard to represent with a decision tree because the decision boundary is a diagonal line, and all decision tree tests divide the space up into rectangular, axis-aligned boxes. We would have to stack a lot of boxes to closely approximate the diagonal line. In other words, decision trees are good for some kinds of functions and bad for others.

Is there *any* kind of representation that is efficient for *all* kinds of functions? Unfortunately, the answer is no—there are just too many functions to be able to represent them all with a small number of bits. Even just considering Boolean functions with n Boolean attributes, the truth table will have 2^n rows, and each row can output *true* or *false*, so there are 2^{2^n} different functions. With 20 attributes there are $2^{1,048,576} \approx 10^{300,000}$ functions, so if we limit ourselves to a million-bit representation, we can't represent all these functions.

19.3.2 Learning decision trees from examples

We want to find a tree that is consistent with the examples in Figure 19.2 and is as small as possible. Unfortunately, it is intractable to find a guaranteed smallest consistent tree. But with some simple heuristics, we can efficiently find one that is close to the smallest. The Learn-Decision-Tree algorithm adopts a greedy divide-and-conquer strategy: always test the most important attribute first, then recursively solve the smaller subproblems that are defined by the possible results of the test. By "most important attribute," we mean the one that makes the most difference to the classification of an example. That way, we hope to get to the correct classification with a small number of tests, meaning that all paths in the tree will be short and the tree as a whole will be shallow.

Figure 19.4(a) shows that *Type* is a poor attribute, because it leaves us with four possible outcomes, each of which has the same number of positive as negative examples. On the other hand, in (b) we see that *Patrons* is a fairly important attribute, because if the value is *None* or *Some*, then we are left with example sets for which we can answer definitively (*No* and *Yes*, respectively). If the value is *Full*, we are left with a mixed set of examples. There are four cases to consider for these recursive subproblems:

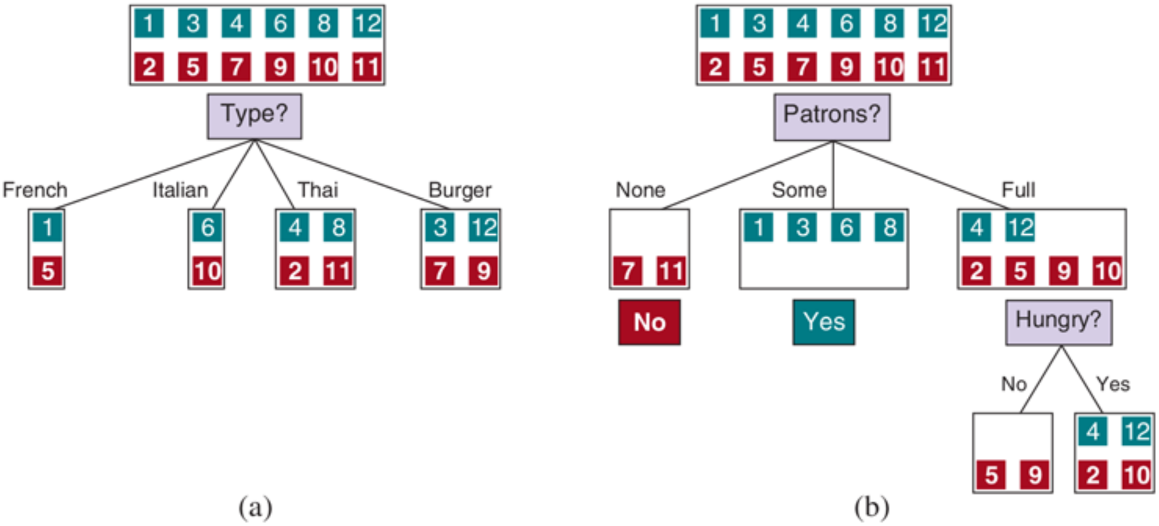
1. If the remaining examples are all positive (or all negative), then we are done: we can answer *Yes* or *No*. Figure 19.4(b) shows examples of this happening in the *None* and *Some* branches.
2. If there are some positive and some negative examples, then choose the best attribute to split them. Figure 19.4(b) shows *Hungry* being used to split the

remaining examples.

3. If there are no examples left, it means that no example has been observed for this combination of attribute values, and we return the most common output value from the set of examples that were used in constructing the node's parent.
4. If there are no attributes left, but both positive and negative examples, it means that these examples have exactly the same description, but different classifications. This can happen because there is an error or **noise** in the data; because the domain is nondeterministic; or because we can't observe an attribute that would distinguish the examples. The best we can do is return the most common output value of the remaining examples.

Noise

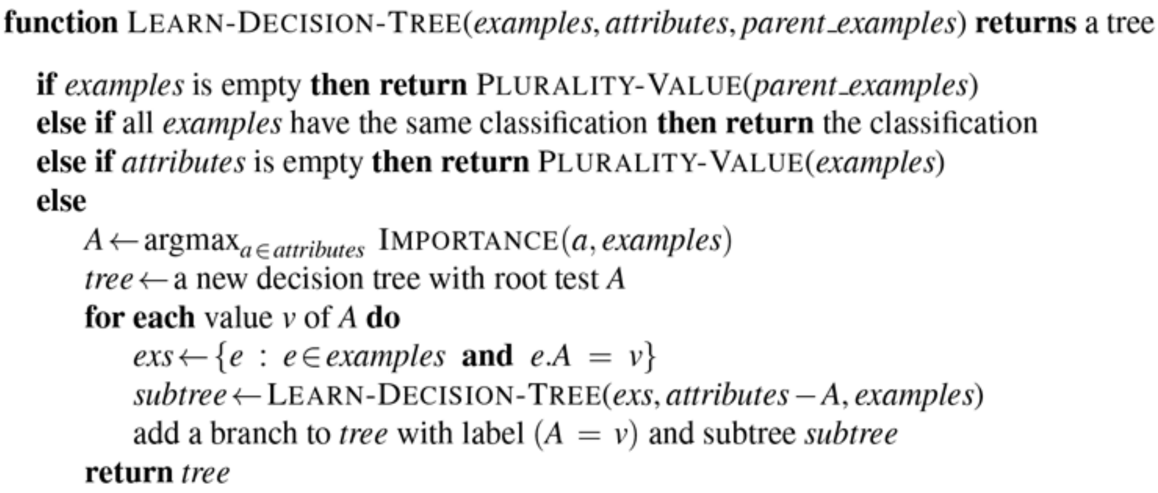
Figure 19.4



Splitting the examples by testing on attributes. At each node we show the positive (light boxes) and negative (dark boxes) examples remaining. (a) Splitting on *Type* brings us no nearer to distinguishing between positive and negative examples. (b) Splitting on *Patrons* does a good job of separating positive and negative examples. After splitting on *Patrons, Hungry* is a fairly good second test.

The Learn-Decision-Tree algorithm is shown in Figure 19.5. Note that the set of examples is an input to the algorithm, but nowhere do the examples appear in the tree returned by the algorithm. A tree consists of tests on attributes in the interior nodes, values of attributes on the branches, and output values on the leaf nodes. The details of the Importance function are given in Section 19.3.3. The output of the learning algorithm on our sample training set is shown in Figure 19.6. The tree is clearly different from the original tree shown in Figure 19.3. One might conclude that the learning algorithm is not doing a very good job of learning the correct function. This would be the wrong conclusion to draw, however. The learning algorithm looks at the *examples*, not at the correct function, and in fact, its hypothesis (see Figure 19.6) not only is consistent with all the examples, but is considerably simpler than the original tree! With slightly different examples the tree might be very different, but the function it represents would be similar.

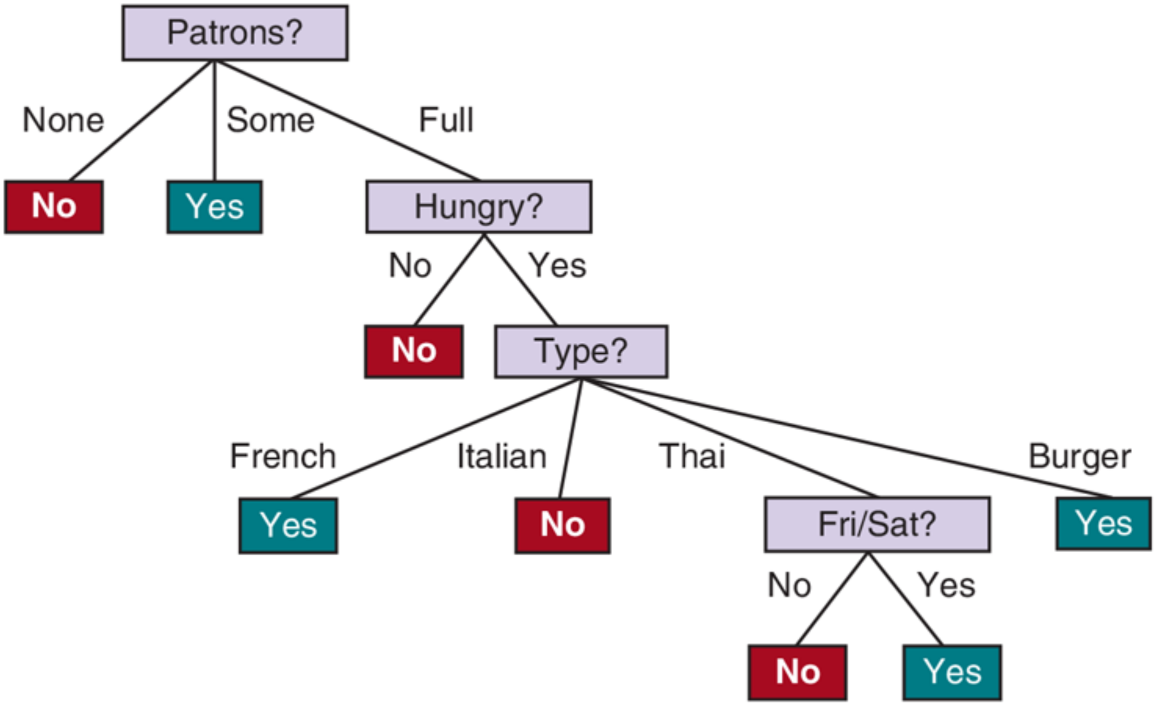
Figure 19.5



```
function LEARN-DECISION-TREE(examples, attributes, parent_examples) returns a tree
  if examples is empty then return PLURALITY-VALUE(parent_examples)
  else if all examples have the same classification then return the classification
  else if attributes is empty then return PLURALITY-VALUE(examples)
  else
    $A \leftarrow \mathrm{argmax}_{a \in attributes}$ IMPORTANCE(a, examples)
    tree ← a new decision tree with root test A
    for each value v of A do
      $exs \leftarrow \{e : e \in examples$ and $e.A = v\}$
      subtree ← LEARN-DECISION-TREE(exs, attributes − A, examples)
      add a branch to tree with label (A = v) and subtree subtree
    return tree
```

The decision tree learning algorithm. The function Importance is described in Section 19.3.3. The function Plurality-Value selects the most common output value among a set of examples, breaking ties randomly.

Figure 19.6

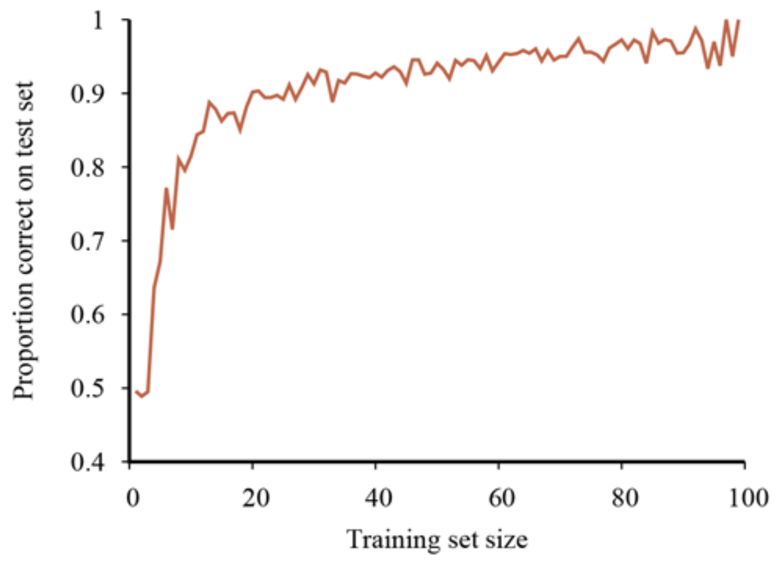


The decision tree induced from the 12-example training set.

The learning algorithm has no reason to include tests for *Raining* and *Reservation*, because it can classify all the examples without them. It has also detected an interesting and previously unsuspected pattern: SR will wait for Thai food on weekends. It is also bound to make some mistakes for cases where it has seen no examples. For example, it has never seen a case where the wait is 0–10 minutes but the restaurant is full. In that case it says not to wait when *Hungry* is false, but SR would certainly wait. With more training examples the learning program could correct this mistake.

We can evaluate the performance of a learning algorithm with a **learning curve**, as shown in Figure 19.7. For this figure we have 100 examples at our disposal, which we split randomly into a training set and a test set. We learn a hypothesis h with the training set and measure its accuracy with the test set. We can do this starting with a training set of size 1 and increasing one at a time up to size 99. For each size, we actually repeat the process of randomly splitting into training and test sets 20 times, and average the results of the 20 trials. The curve shows that as the training set size grows, the accuracy increases. (For this reason, learning curves are also called **happy graphs**.) In this graph we reach 95% accuracy, and it looks as if the curve might continue to increase if we had more data.

Figure 19.7



A learning curve for the decision tree learning algorithm on 100 randomly generated examples in the restaurant domain. Each data point is the average of 20 trials.

Learning curve

Happy graphs

19.3.3 Choosing attribute tests

The decision tree learning algorithm chooses the attribute with the highest IMPORTANCE. We will now show how to measure importance, using the notion of information gain, which is defined in terms of **entropy**, which is the fundamental quantity in information theory (Shannon and Weaver, 1949).

Entropy

Entropy is a measure of the uncertainty of a random variable; the more information, the less entropy. A random variable with only one possible value—a coin that always comes up heads—has no uncertainty and thus its entropy is defined as zero. A fair coin is equally likely to come up heads or tails when flipped, and we will soon show that this counts as "1 bit" of entropy. The roll of a fair *four*-sided die has 2 bits of entropy, because there are 2^2 equally probable choices. Now consider an unfair coin that comes up heads 99% of the time. Intuitively, this coin has less uncertainty than the fair coin—if we guess heads we'll be wrong only 1% of the time—so we would like it to have an entropy measure that is close to zero, but positive. In general, the entropy of a random variable V with values v_k having probability $P(v_k)$ is defined as

$$\text{Entropy:} \quad H(V) = \sum_k P(v_k) \log_2 \frac{1}{P(v_k)} = -\sum_k P(v_k) \log_2 P(v_k).$$

We can check that the entropy of a fair coin flip is indeed 1 bit:

$$H(Fair) = -(0.5 \log_2 0.5 + 0.5 \log_2 0.5) = 1\ .$$

And of a four-sided die is 2 bits:

$$H(Die4) = -(0.25 \log_2 0.25 + 0.25 \log_2 0.25 + 0.25 \log_2 0.25 + 0.25 \log_2 0.25) = 2$$

For the loaded coin with 99% heads, we get

$$H(Loaded) = -(0.99 \log_2 0.99 + 0.01 \log_2 0.01) \approx 0.08 \text{ bits}.$$

It will help to define $B(q)$ as the entropy of a Boolean random variable that is true with probability q:

$$B(q) = -(q \log_2 q + (1-q) \log_2 (1-q)).$$

Thus, $H(Loaded) = B(0.99) \approx 0.08$. Now let's get back to decision tree learning. If a training set contains p positive examples and n negative examples, then the entropy of the output variable on the whole set is

$$H(Output) = B\left(\frac{p}{p+n}\right).$$

The restaurant training set in Figure 19.2 has $p = n = 6$, so the corresponding entropy is $B(0.5)$ or exactly 1 bit. The result of a test on an attribute A will give us some information, thus reducing the overall entropy by some amount. We can measure this reduction by looking at the entropy remaining after the attribute test.

An attribute A with d distinct values divides the training set E into subsets $E_1, \ldots, E_d$. Each subset E_k has p_k positive examples and n_k negative examples, so if we go along that branch, we will need an additional $B(p_k/\,(p_k + n_k))$ bits of information to answer the question. A randomly chosen example from the training set has the kth value for the attribute (i.e., is in E_k with probability $(p_k + n_k)/(p + n)$), so the expected entropy remaining after testing attribute A is

$$Remainder(A) = \sum_{k=1}^{d} \frac{p_k + n_k}{p + n} B\left(\frac{p_k}{p_k + n_k}\right).$$

The **information gain** from the attribute test on A is the expected reduction in entropy:

$$Gain(A) = B\left(\frac{p}{p + n}\right) - Remainder(A).$$

Information gain

In fact $Gain(A)$ is just what we need to implement the Importance function. Returning to the attributes considered in Figure 19.4, we have

$$\begin{aligned} Gain(Patrons) &= 1 - \left[\tfrac{2}{12}B\left(\tfrac{0}{2}\right) + \tfrac{4}{12}B\left(\tfrac{4}{4}\right) + \tfrac{6}{12}B\left(\tfrac{2}{6}\right)\right] \approx 0.541 \text{ bits},\\ Gain(Type) &= 1 - \left[\tfrac{2}{12}B\left(\tfrac{1}{2}\right) + \tfrac{2}{12}B\left(\tfrac{1}{2}\right) + \tfrac{4}{12}B\left(\tfrac{2}{4}\right) + \tfrac{4}{12}B\left(\tfrac{2}{4}\right)\right] = 0 \text{ bits}, \end{aligned}$$

confirming our intuition that *Patrons* is a better attribute to split on first. In fact, *Patrons* has the maximum information gain of any of the attributes and thus would be chosen by the decision tree learning algorithm as the root.

19.3.4 Generalization and overfitting