Introduction to Rates of Change

Introduction

Differential calculus, broadly speaking, is the study of *rates of change*. That is, differential calculus is the branch of mathematics required whenever we want to describe a how one quantity changes with respect to another. This makes it of particular importance in physics, which is largely about studying how objects move through space as time advances, but anything that changes over time (like a chemical reaction, an electrical charge, a population, or a stock price) can be studied more closely using calculus.

One of the most important rates of change in physics, and a good place for us to start, is *velocity*, the rate of change in an object's position with respect to time. From high school physics you should already be familiar with one type of velocity: the *average velocity* over some interval of time. For example, if a car is driving along a highway and it crosses mile marker 74 at 2:16pm and then mile marker 106 at 2:41pm, then its average velocity between 2:16pm and 2:41pm is

$$v_{\text{avg}} = \frac{\text{change in position between 2:16 and 2:41}}{\text{change in time between 2:16 and 2:41}} = \frac{106\text{ mi} - 74\text{ mi}}{41\text{ min} - 16\text{ min}} = \frac{32\text{ mi}}{25\text{ min}} = 1.28\ \frac{\text{mi}}{\text{min}}$$

which is 76.8 mph. Average velocity is perfectly easy to define and compute (it's what's called a *difference quotient*, which is exactly what its name suggests), but it doesn't tell us everything about the car's motion. The car's speed probably wasn't exactly 76.8 mph over the entire time interval; it was probably traveling slightly faster at some times and slightly slower at others. Presumably if we had more information (more positions at more times) we would be able to describe the car's *instantaneous velocity*, or its velocity at a specific moment in time.

Instantaneous velocity seems like a simple concept, and in everyday speech we often talk about how fast something is happening "right now" or at some other specific moment in time. However, mathematics is all about making vague ideas precise enough to actually use in applications, and it turns out that the notion of instantaneous velocity is surprisingly tricky to define in a sensible, usable way. This worksheet will take you through a sequence of numerical experiments and activities to allow you to play around with these ideas for yourself to build up an intuition about how to approach rates of change.

Required Materials

This lab includes computational activities in the provided spreadsheet file `rates_of_change.xlsx`. You will need Microsoft Excel[1] installed in order to access and work with the file. If you cannot access Excel, there are several functionally-equivalent free alternatives that will work just as well (we recommend either Google Sheets[2] or LibreOffice Calc[3]).

It is expected that you have some basic familiarity with how to use Excel to perform basic computations, including how to use cell formulas, how to reference other cells, and how to generate plots. If not, the best way to learn is simply to try using it to solve some problems, asking for help from your classmates or the instructor or looking up help online when you get stuck. Excel is a skill, and like with any skill there is no substitute for practice.

Learning Objectives

After completing this project, you should be able to:

- Explain how the idea of an instantaneous rate of change is related to average rates of change.
- Use a computer to help to answer questions about a function's rate of change, like when its value is increasing/decreasing/constant and when it is changing the fastest.
- Work with a real-world function defined as a table of values rather than as a simple formula.
- Be comfortable using a computer alongside analytical work by hand in order to solve a complicated problem.

[1] `https://www.microsoft.com/en-us/microsoft-365/excel`
[2] `https://www.google.com/sheets/`
[3] `https://www.libreoffice.org/discover/calc/`

Instantaneous Velocity of a Weather Balloon

As noted above, average velocity is easy to compute, but we don't yet have a way to approach computing instantaneous velocities. In this activity you will work with a (lightly edited) set of position data gathered from a weather balloon launched from the Salton Sea weather station on February 28, 2021[4] to try to see how we might approach the problem of how to describe the balloon's instantaneous velocity.

Activity

1. Open the activity spreadsheet `rates_of_change.xlsx` and make sure that you're on the first tab, labeled "Weather Balloon". You should see two very long columns of data: column **A** indicates the time (in seconds since the launch) and column **B** indicates the position (in meters north of the launch site) of the balloon[5]. The data series includes 5409 times in 1-second increments spanning a period of just over 1.5 hours.

2. Shortly we will be computing some velocity measurements for the balloon, but before that we can at least try to come up with some preliminary results using our eyes. Create a scatter plot of the balloon's position versus time, and use the graph to estimate the following:

   - Over what time interval was the balloon moving north? South?
   - Are there any times at which the balloon stopped moving momentarily?
   - At what time was the balloon moving the fastest?

   Explain the reasoning behind each of your guesses.

3. Over the next few parts we will attempt to answer one fundamental question: What was the *instantaneous* velocity of the balloon at time 60 seconds? We will work our way up to that answer by starting with a few smaller, simpler numerical experiments.

   It's not immediately obvious how to compute an instantaneous velocity, but we know exactly how to compute an *average* velocity, so that might be a good place to start. At least intuitively we might expect the instantaneous velocity at 60 seconds to be similar to the average velocities over time intervals that are "close to" 60.

   Compute the average velocity of the balloon over the time interval from 60 to 70 seconds. Then try it from 60 to 65 seconds, then from 60 to 61 seconds. What are the units of the values you've just computed? If you had to choose one of them as your "best guess" for the instantaneous velocity at time 60, which would you pick? Why do you think your choice is more reasonable than the others?

4. We can use Excel to quickly automate this process to compute many different average velocities over many different time intervals. Use column **C** to compute the average velocity

[4] F. M. Ralph, A. M. Wilson, R. Demirdjian, D. Alden, C. Hecht, C. J. Ellis, B. Kawzenuk, F. Cannon, A. Cooper, and K. Paulsson. Radiosonde Data Collected During California Storms. UC San Diego Library Digital Collections, Dataset, 2021. doi:10.6075/J09P31S0

[5] Only the balloon's position in the longitudinal direction is included here. The original data set included latitudes and altitudes, but for this activity we're going to pretend that all of its motion occurs in a single direction. Describing movement through 3D space is a significantly more complicated topic covered in Calculus III (Multivariate Calculus).

between *every* time and time 60. For example, the entry in the row for time 16 should compute the average velocity between 60 and 16 seconds, the entry in the row for time 763 should compute the average velocity between 60 and 763, and so on[6].

5. Create a scatter plot of these average velocities versus time. You should see what appears to be a relatively normal, well-behaved, continuous-looking function.

   It's hard to see what's happening around time 60 in this graph due to the scale. Try plotting the average velocities again, this time only including times between 50 and 70 seconds. What happens to the graph immediately around 60 seconds? What happens to it exactly at 60 seconds?

6. We started looking at average velocities based on the idea that an average velocity over an interval "close to" 60 should give us some idea of the instantaneous velocity there. Look up the average velocity that your Excel formula actually computed at time 60. Why has it resulted in an error?

7. Based on the previous experiments, apparently it's not possible for us to compute an instantaneous velocity at a time by just computing an average velocity over a time interval of zero width. Since the weather balloon's position is given to us as a set of finitely-many discrete values there is a smallest nonzero time interval that we can consider (1 second). In class we will see how all of this changes for a *continuous* position function, but for now let's continue to work with the idea that we can at least *approximate* an instantaneous velocity at a time by using an average velocity over a small time span around that time.

   Apply this idea to compute an instantaneous velocity estimate for *every* time in column **D**. For example, the entry in the row for time 16 should compute an approximation for the instantaneous velocity at time 16, the entry in the row for time 763 should compute an approximation for the instantaneous velocity at time 763, and so on[7].

8. Create a scatter plot of these (approximate) instantaneous velocities versus time. You should see a somewhat jagged but still relatively well-behaved and continuous-looking graph. Use the velocity graph to estimate the following:

   - Over what time interval was the balloon moving north? South?
   - Are there any times at which the balloon stopped moving momentarily?
   - At what time was the balloon moving the fastest?
   - Now that you actually have a velocity graph you can answer a more precise follow-up question: What was the balloon's maximum velocity?

   Explain the reasoning behind each of your guesses.

9. Finally, look at your velocity graph alongside your original position graph. What is happening to the shape of the position graph where the velocity graph crosses the x-axis? When it's positive? When it's negative?

[6]Hint: In Excel, a dollar sign ($) can be used to keep part of a cell reference constant within a formula as it is copied into other cells. In this case each average velocity computation needs to reference cells in two different rows, one of which should remain constant (the reference for time 60) and one of which should not (the reference for the other time's row).

[7]Hint: Most of the cells can be handled with a single formula copied into every row, but the first and last rows may need to be handled separately, since any average velocity computed for the first time cannot include data from an earlier time, and any average velocity computed for the last time cannot include data from a later time.