APPM 4650 — Extra problems Exam 2

The problems provided here are on the material from the week before the exam. **You should not think of this document as a comprehensive review of the material on the exam.**

1. Show that $f(x) = \sqrt{x^2 + 3}$ is Lipschitz in $\mathbb{R}$.

   Soln: It is enough to show that $|f'(x)|$ is bounded for all $x \in \mathbb{R}$. (We could use the definition and the Mean Value Theorem as an alternative.)

$$|f'(x)| = \left| \frac{x}{(x^2+3)^{1/2}} \right| \leq 1$$

2. (a) Show that $f(x) = e^x$ is not Lipschitz on $\mathbb{R}$

Soln: Let x_1 and x_2 be real numbers.

$$|f(x_1) - f(x_2)| = |e^{x_1} - e^{x_2}| \leq |e^{\mu}||x_1 - x_2|$$

for some $\mu \in (x_1, x_2)$ by the Mean Value Theorem. In $\mathbb{R}$, it is not possible to bound e^{μ} so $f(x)$ is not Lipschitz.

(b) Show that $f(x) = e^x$ is Lipschitz on $[-10, 10]$.

Soln: Let x_1 and x_2 be real numbers in $[-10, 10]$.

$$|f(x_1) - f(x_2)| = |e^{x_1} - e^{x_2}| \leq |e^{\mu}||x_1 - x_2|$$

for some $\mu \in (x_1, x_2)$ by the Mean Value Theorem. In $[-10, 10]$ $|e^{\mu}| \leq e^{10} = L$. So $f(x)$ is Lipshitz in the bounded interval.

3. Show that $y^3 t + yt = 2$ implicitly defines a solution to the following initial value problem.

$$\begin{cases} y' &= -\frac{y^3+y}{(3y^2+1)t} \quad 1 \le t \le 2 \\ y(1) &= 1 \end{cases}$$

Soln: Using implicit differential on the solution, we get

$$y^3 + 3y^2 t y' + y + t y' = 0.$$

Solving for y', we get the ODE.

Plugging $t = 1$ into the solution, we get

$$y^3 + y - 2 = 0.$$

The only real root of this polynomial is $y = 1$.

4. Let $f(t,y) = \frac{1+y}{1+t}$.

   (a) Does f satisfy a Lipschitz condition on $D = \{(t,y) : 0 \le t \le 1, y \in \mathbb{R}\}$?

   Soln: It is enough to show that $|\frac{\partial f}{\partial y}|$ is bounded.

   $$\left|\frac{\partial f}{\partial y}\right| = \left|\frac{1}{1+t}\right| \le 1$$

   Thus f is Lipschitz in D.

   (b) Is the problem

   $$\begin{cases} y' &= f(t,y) \quad 0 \le t \le 1 \\ y(0) &= 1 \end{cases}$$

   well-posed (using the theorems from this class)? Justify your answer.

Soln: f is continuous and Lipschitz in D (by part a). So by Thm 5.6, the problem is well-posed.