Stability 1 (definitions, one-step methods)

Thursday, November 12, 2020 10:48 AM

two interrelated chapters in Burden & Faires, 5.10 "stability"
5.11 "Stiff differential equations"

we'll have 3 main concepts (some with variants)

local 1) Consistency (straightforward, intuitive)

global 2) Convergence *our goal* As $h \to 0$ $\underbrace{w_i}_{\text{numerical approx.}} \to \underbrace{y(t_i)}_{\text{"true answer"}}$

3) Stability a bit more complicated, less intuitive

Our main results:
"Equivalence Theorem" (paraphrased for now) For a numerical method,
to get convergence, we only need $\underbrace{\text{consistency}}_{\text{local errors} \to 0}$ and $\underbrace{\text{stability}}_{\text{small errors don't accumulate catastrophically}}$

Chapter 5.10 "Stability" is about the limit $h \to 0$ (h is our stepsize)

Chapter 5.11 "stiff" eq'n is about a different variant of stability,
in particular, we look at the case of a fixed $h > 0$

This is the culmination of ch.5 (building on ch. 3 & 4 and some ch.2)
Trickiest part of the course, so we'll take our time

Outline

1. Define the concepts of consistency, convergence, and stability
2. Stability, for $h \to 0$
   - 2a One-step
   - 2b Multi-step Concepts: root condition, 0-stable, strongly/weakly stable, characteristic polynomial Q
3. stiff equations, $h > 0$ Concepts: Dahlquist test eq'n, Region of absolute stability, A-stable, characteristic polynomial P
4. Examples, eg, "compute region of absolute stability for RK4" etc.

① The concepts of consistency, convergence, and stability

Consistency is about whether we're modeling the ODE

(whereas convergence is about the solution to the ODE)

Recall our IVP is $y' = f(t,y)$, $y(0) = y_0$, for $t \in [0,T]$

(or $y(a) = \alpha$, for $t \in [a,b]$)

and recall for a one-step method $w_{i+1} = w_i + h\,\phi(t_i, w_i)$ Ex: RK

we defined the local truncation error

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\,\phi(t_i, y_i))}{h}$$

$y_i := y(t_i)$

$y(t)$ is true sol'n to IVP

and similarly for the linear multi-step method

$$w_{i+1} = \overbrace{a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}}^{m\text{-terms}}$$

$$+\, h \cdot \big(\underbrace{b_m f_{i+1} + b_{m-1} \cdot f_i + \dots + b_0 f_{i+1-m}}_{m+1 \text{ terms}} \big)$$

Ex: AB AM BD

Recall $f_i := f(t_i, w_i)$

we similarly defined the local truncation error

$$\tau_{i+1}(h) = \frac{1}{h}\left[y_{i+1} - \left(\sum_{j=1}^{m} a_{m-j}\, y_{i+1-j} + h \sum_{j=0}^{m} b_{m-j}\, f(t_{i+1-j}, y_{i+1-j}) \right) \right]$$

Def A numerical method (one-step or multistep*) is consistent (meaning it is "consistent with the ODE") if

$$\lim_{h \to 0} \max_{1 \le i \le n} |\tau_{i+1}(h)| = 0$$

(where $n = \frac{b-a}{h}$)

A local property, not global. It will be a necessary condition for better results (ie., a means-to-an-end, not a goal in itself)

So, if we say a method is $O(h)$ or $O(h^4)$ etc.,

(meaning $\tau_{i+1}(h) = O(h)$ or $O(h^4)$ ...), this implies it is consistent

[If we hadn't put the $\frac{1}{h}$ in the definition of τ, then for any continuous y, for a bad choice of ϕ, like $\phi(t,y)=0$, would have appeared to be consistent.]

* For multistep methods, we need a slight variant, as we must worry about previous values. We assume the starting value $w_0 = \alpha$ is exact, but w_1 we can't apply our multistep formula yet since for $m>1$ we'd need to know w_{-1}. So, we use a one-step method (like RK) to approximate the first few w, specifically $w_1, \ldots, w_{m-1}$. Clearly these approximations need to be good. So, we define a multistep method to be consistent if

$$\lim_{h\to 0} \max_{m \le i \le n} |\tau_i(h)| = 0$$

and

$$\lim_{h\to 0} \max_{1 \le i \le m-1} |w_i - y_i| = 0 \quad \Big\} \text{"warmup period"}$$

Convergence This is what we want in the end!

Def A numerical method (one-step or multistep) for solving an IVP (which has unique solution $y(t)$) is "convergent" if

$$\lim_{h\to 0} \max_{1 \le i \le n} |w_i - y_i| = 0$$

$y_i = y(t_i)$

w_i via the numerical method

A global property.

Note: we only ask about the limit $h \to 0$, so it doesn't say anything about how fast the convergence is.

We've already seen a convergence result: recall Thm 5.9

Theorem 5.9 Global error bound for Forward Euler

Suppose f is continuous and has Lipschitz constant L with respect to y, uniformly in t on $D = \{(t,y) : a \le t \le b, -\infty \le y \le \infty\}$, and $\exists\, M < \infty$ s.t. $|y''(t)| \le M \quad \forall\, t \in [a,b]$ where $y(t)$ is the unique solution to the IVP. Let $(w_i)_{i=0}^{n}$ be via Forward Euler. Then $(\forall\, i = 0, 1, \dots, n)$

$$|y(t_i) - w_i| \le \frac{h \cdot M}{2L}\left(e^{L(t_i - a)} - 1\right).$$

Thm 5.9 was specialized for Euler's method. Is there a more general way to get convergence results? Yes, via stability

Stability

⚠ Stability is such an important concept in many disciplines, so there are many variants of what "stability" means (like how "smooth" functions are vague or context-dependent). They all get at the same concept, but often w/ slight yet important differences.

Broadly speaking, a stable method isn't too sensitive to perturbations. Or, as the book states it, a method is stable if the results depend continuously on the data.

Much like a well-posed problem.

Recall: usually we say a math problem (evaluate $f(x)$, find a root, solve the IVP, simulate the weather)

is well-posed vs. ill-posed (= not well-posed),

for ODEs, this means $\exists!$ solution to the IVP and the solution depends continuously on the input data

or well-conditioned vs. ill-conditioned

(Implementation independent)

whereas we say a numerical method (Newton's method, RK4, Simpson's rule, centered differences)

is stable vs. unstable.

(Implementation dependent)

Same concepts we talked about in ch. 1, just different contexts

(2a) Stability of one-step methods

To cut to the chase, if the underlying function f is nice, then most reasonable one-step methods are stable.

Setup: IVP $y' = f(t,y)$, $a \le t \le b$, $y(a) = \alpha$

One-step method $w_{i+1} = w_i + h\phi(t_i, w_i)$, with $w_0 = \alpha$

main result of (2a)

Thm 5.20 Suppose $\exists\ h_0 > 0$ such that on the set

$$D = \{ (t, w, h) : a \le t \le b,\ -\infty < w < \infty,\ 0 \le h \le h_0 \}$$

the function $\phi(t, w, h)$ is continuous and uniformly Lipschitz continuous in w with Lipschitz constant L, then

1) the method is stable, meaning if we run the method with $w_0 = \alpha$ to generate $\{w_i\}$ for $1 \le i \le n$, and also with $\tilde{w}_0 = \tilde{\alpha}$ to generate $\{\tilde{w}_i\}$ for $1 \le i \le n$, then

$\exists$ constant K such that $|w_i - \tilde{w}_i| \le K \cdot |\alpha - \tilde{\alpha}|$.

2) the method is convergent iff it is consistent
"if and only if"

and being consistent is equivalent to $\phi(t, y, \overset{\text{ie. } h=0}{0}) = f(t,y)$ $\forall\ a \le t \le b$

since $\lim_{h \to 0} \phi(t,y,h) = \phi(t,y,0)$ by our assumption that ϕ is continuous

3) we can be more quantitative about the convergence:

we have the following convergence rate, $y_i := y(t_i)$, $n := \frac{b-a}{h}$

The book has a slightly stronger statement

$$(\forall\ 1 \le i \le n) \quad |y_i - w_i| \le \frac{1}{L} \max_{1 \le i \le n} \tau_i(h) \cdot e^{L(t_i - a)}$$

We already saw a very similar result in the "Higher Order Taylor Methods" notes:

Thm 6.2.1 Driscoll and Braun: global error of one-step methods

Consider a one-step method defined by $\phi(t, y, h)$ and suppose the local truncation error satisfies $\tau_{i+1}(h) \le c \cdot h^p \quad (\forall i)$ for $p > 0$, and that ϕ is uniformly Lipschitz in y w/ constant L, and assume there is a unique solution $y(t)$ to the IVP.

Then the global error is bounded

$$|y(t_i) - w_i| \le \frac{c\, h^p}{L}\left(e^{L(t_i - a)} - 1\right) = O(h^p)$$

We've talked about the IVP having certain Lipschitz properties

i.e., $f(t,y)$ uniformly Lipschitz in y $\Rightarrow$

1) existence of sol'n
2) uniqueness of sol'n
3) continuous dependence on data

i.e., Lipschitz f $\Rightarrow$ "well-posed" or "well-conditioned" IVP

but this Theorem asks for $\phi(t, y, h)$ to be Lipschitz in y.

Well, that's related!

Ex: Euler $\phi(t, y) = f(t, y)$.

So $\underbrace{f \text{ uniformly Lipschitz in } y}_{\text{very common, since guarantees a well-posed}} \Rightarrow \underbrace{\phi \text{ uniformly Lipschitz in } y}_{\Rightarrow \text{ convergent}}$

Ex: Modified Euler (both a 2-stage RK and a predictor-corrector)

$$w_{i+1} = w_i + h \underbrace{\frac{1}{2}\left(f(t_i, w_i) + f(\overset{= t_i + h}{t_{i+1}}, w_i + h f(t_i, w_i))\right)}_{\phi(t_i, w_i, h)}$$

Suppose f is uniformly $\underbrace{\text{Lipschitz in } y}_{\text{w/ constant } L}$, then check if ϕ is too!

i.e., want to show there exists L' such that

$\forall t,\ \forall w, \tilde{w},\ \forall\ 0 \le h \le h_0,$ then

$$\underbrace{|\phi(t,w,h) - \phi(t,\tilde{w},h)|}\ \le\ L'|w-\tilde{w}|$$

$$= \Big|\ \tfrac{1}{2}f(t,w) + \tfrac{1}{2}f(t+h, w+hf(t,w)) - \tfrac{1}{2}f(t,\tilde{w}) - \tfrac{1}{2}f(t+h, \tilde{w}+hf(t,\tilde{w}))\ \Big|$$

triangle inequality

$$\le \tfrac{1}{2}|f(t,w)-f(t,\tilde{w})| + \tfrac{1}{2}|f(t+h, w+hf(t,w)) - f(t+h, \tilde{w}+hf(t,\tilde{w}))|$$

Lipschitz of f

$$\le \tfrac{1}{2}L|w-\tilde{w}| + \tfrac{1}{2}L|w+hf(t,w) - \tilde{w} - hf(t,\tilde{w})|$$

triangle ineq.

$$\le \tfrac{1}{2}L\underbrace{|w-\tilde{w}| + \tfrac{1}{2}L|w-\tilde{w}|} + \tfrac{1}{2}Lh|f(t,w)-f(t,\tilde{w})|$$

Lipschitz of f

$$\le L|w-\tilde{w}| + \tfrac{1}{2}L^2h|w-\tilde{w}|$$

$$= \left(L + \tfrac{1}{2}hL^2\right)\cdot|w-\tilde{w}|$$

$$\le \underbrace{\left(L + \tfrac{1}{2}h_0L^2\right)}_{L'}|w-\tilde{w}| \quad \text{since } h \le h_0$$

$\star$

All our RK methods will be like this (exact value of L' may change depending on which RK method)

For very high-order methods, L' will be very large unless h_0 is very small.

(We also need to check ϕ is continuous, meaning jointly continuous in t, w, h. For RK methods, as long as $f(t,y)$ is continuous, then so is ϕ)