Condition Number of a Problem

Sunday, August 23, 2020 2:18 PM

Evaluate $f(x)$. Ex: $f(x) = x+1$

Conditioning = sensitivity to perturbations in the input

Evaluate f at $\tilde{x} = fl(x)$, $\tilde{x} = x\cdot(1+\varepsilon)$, $|\varepsilon| < \varepsilon_{mach}/2$

look at the rel. error $\dfrac{|f(\tilde{x}) - f(x)|}{|f(x)|}$ } relative error of output

Ex: $f(x) = x+1$, $\left|\dfrac{\tilde{x}+1 - (x+1)}{|x+1|}\right| = \left|\dfrac{x(1+\varepsilon)+1 - (x+1)}{|x+1|}\right|$

$= \left|\dfrac{\varepsilon\cdot x}{x+1}\right|$, if $|x+1|$ is small, lot of rel. error

$x = 1 - \varepsilon_m/2$

Sensitivity: do small changes in input lead to small in output? well-conditioned

large in output? ill-conditioned

Conditioning is a property of f

NOT the implementation

best we could do

$\tilde{x} = x(1+\varepsilon)$

$$\frac{|f(x) - f(\tilde{x})|}{|f(x)|} \Big/ \left(\frac{|x-\tilde{x}|}{|x|} = |\varepsilon|\right)$$

Def The relative condition number of f at x is:

$$K_f(x) = \lim_{\varepsilon\to 0} \frac{|f(x) - f(x+\varepsilon x)|}{|\varepsilon\cdot f(x)|} = \lim_{\varepsilon\to 0}\left|\frac{f(x+\varepsilon x) - f(x)}{\varepsilon x}\cdot\frac{x}{f(x)}\right|$$

$$= \left|\frac{x}{f(x)}\cdot f'(x)\right| \quad *$$

Rules: $h(x) = f(g(x))$

then $K_h(x) = K_f(g(x))\cdot K_g(x)$

⚠ The relative condition number is not the same as relative error (though they are related)

$$\kappa_f(x) = \lim_{\tilde{x} \to x} \left| \frac{f(x) - f(\tilde{x})}{f(x)} \right| \Big/ \left| \frac{x - \tilde{x}}{x} \right|$$

$\left| \frac{f(x) - f(\tilde{x})}{f(x)} \right| \to$ relative error (of output)

$\left| \frac{x - \tilde{x}}{x} \right| \to$ relative error (of input)

We can also define an absolute condition number based on absolute error

$$\kappa_f^{\text{absolute}}(x) = \lim_{\tilde{x} \to x} \frac{|f(x) - f(\tilde{x})|}{|x - \tilde{x}|}$$

$|f(x) - f(\tilde{x})| \to$ absolute error (of output)

$|x - \tilde{x}| \to$ absolute error (of input)

(i.e., absolute condition number is just the slope!

$$\kappa_f^{abs}(x) = |f'(x)| \qquad)$$

Rule-of-thumb interpretation of relative condition number

If ε is small, $\left| \frac{f(\tilde{x}) - f(x)}{f(x)} \right| \approx \kappa_f(x) \cdot \varepsilon$

$\kappa_f(x) \to 10^4$, $\quad \varepsilon \to 10^{-16}$ --- 10^{-12}

12 digits

$*$ $\boxed{\log_{10}(\kappa_f(x)) \text{ is \# of digits we'll likely lose}}$ (no matter how good the algorithm is)

Students ask... is there a precise definition of well-conditioned vs ill-conditioned?

Answer: No. $\quad \kappa = 10$ is definitely "well-conditioned"

$\kappa = 10^{10}$ is definitely "ill-conditioned"

$\kappa = 10^5$ is less clear, depends on context,

or say "somewhat ill-conditioned"