Big-O Notation

Tuesday, August 25, 2020 8:00 AM

Definition Case 1 $f(x) = O(g(x))$ as $x \to \infty$ if

$\exists\ x_0$ and $\exists\ M < \infty$ s.t. if $x > x_0$ then $|f(x)| \le M \cdot g(x)$

"exists"

Case 2 $f(x) = O(g(x))$ as $x \to a$ (usually $a=0$) if

$\exists\ \delta > 0$ and $\exists\ M < \infty$ s.t. if $|x-a| < \delta$ then $|f(x)| \le M \cdot g(x)$

(Both cases, equivalent def'n)

$f = O(g)$ if $\limsup\limits_{x \to \infty \text{ or } a} \left|\frac{f(x)}{g(x)}\right| < \infty$] valid

or, if $\lim\limits_{x \to \infty \text{ or } a} \left|\frac{f(x)}{g(x)}\right|$ exists and is $< \infty$] sufficient

Interpretations and Examples

∞ "infinite case"

$f = O(g)$ means, eventually, f grows no faster than g (up to constants)

(typically, $g(x) \to \infty$ as $x \to \infty$)

Use case: n = size of input, $f(n)$ = how long it takes to run

$f(n) = n^3 + 3n^2 - 4n + 7$, $f(n) = O(n^3)$

$f(n) = 10n^3 + 3n^2 \ldots$ $f(n) = O(n^3)$, $f(n) \ne O(n^2)$

$f(n) = n^3$, $g(n) = n^2$, is $f = O(g)$? No

$\lim\limits_{n \to \infty} \frac{f(n)}{g(n)} = \lim\limits_{n \to \infty} \frac{n^3}{n^2} = \lim n = \infty$. No

$n^2 = O(n^3)$ ✓

$n^2 = O(n^2)$

$x^2 = O(e^x)$, $e^x \ne O(x^2)$

$a=0$ "infinitesimal case"

typically, $a=0$, write "h" instead of "x", $g(h)\to 0$ as $h\to 0$

discretization

$f=O(g)$ means, eventually, f decays to 0 at least as fast as g (up to a constant)

<u>Ex</u>: $3h^2=O(h^2)$, $\left.\begin{array}{l} h^2=O(h) \\ h^2\neq O(h^3)\end{array}\right\}$ check: $\lim_{h\to 0}\frac{h^2}{h}=\lim_{h\to 0}h=0<\infty$

WARNING $x=O(x^2)$ as $x\to\infty$, $x^2\neq O(x)$ as $x\to\infty$ ∞ case

$h\neq O(h^2)$ as $h\to 0$, $h^2=O(h)$ as $h\to 0$ $a=0$ case

Smaller exponent is "better"

larger exponent is "better" (usually $f(h)$ is error term in a Taylor Series)

<u>Ex</u>: $f(x_0+h)=\underbrace{f(x_0)+f'(x_0)h+\frac{f''(x_0)h^2}{2!}}_{P}+\frac{f'''(\xi)h^3}{3!}$

then we can say
$f(x_0+h)=P(h)+O(h^3)$

if $|f'''(\xi)|$ is bounded for all $\xi\in[x_0,x_0+h]$

Variants

little-o notation:

$f=o(g)$ as $x\to\infty$ means (asymptotically) f grows slower than $c\cdot g(x)$ for all constants c

i.e., $\forall c>0,\ \exists x_0$ s.t. $x>x_0$, $|f(x)|\leq c\cdot g(x)$

more 'precise' than big-O notation

$x^2=O(x^2)$, $x^2\neq o(x^2)$, $x^2=o(x^3)$

big-theta θ

$f=\theta(g)$ means $f=O(g)$ and $g=O(f)$

ex: $5x^3=\theta(10x^3)$ $x^3\neq\theta(x^2)$, $x^2\neq\theta(x^3)$

$f\sim g$ even stronger: means $\lim_{x\to \infty \text{ or } a}\frac{f(x)}{g(x)}=1$

$5x^3\nsim 10x^3$, $5x^3+3x^2\sim 5x^3-x$

$\tilde{O}$ ← ignore log factors

<u>Ex</u>: is $e^x = O(x!)$ or is $x! = O(e^x)$? $(x \to \infty)$

Use Stirling's Formula: $x! \sim \sqrt{2\pi x}\,(x/e)^x$,

ie, $x! = \Theta\left(\sqrt{x}\left(\frac{x}{e}\right)^x\right)$

x^x grows faster than e^x

so $x!$ grows faster than e^x, $e^x = O(x!)$, $x! \neq O(e^x)$

$x! \neq \Theta(e^x)$