APPM 4650 – Final Exam Review Sheet

1. Find a way to accurately evaluate $e^x - e^1$ as $x \to 1$.

2. Find the rate of convergence of
$$\lim_{h\to 0} \frac{\sin h}{h} = 1$$
as $h \to 0$.

3. (a) Create a function $f(x)$ so that you can approximate $\sqrt[3]{25}$ with the bissection method.
   (b) Find an upper bound on the number of steps needed to approximate $\sqrt[3]{25}$ within 10^{-4}.

4. Use algebraic manipulation to show that $g(x) = \left(\frac{x+3-x^4}{2}\right)^{\frac{1}{2}}$ has a fixed point p when $f(p) = 0$ for $f(x) = x^4 + 2x^2 - x - 3$.

5. Determine an interval $[a, b]$ on which the fixed point iteration will converge to $x = \frac{2-e^x+x^2}{3}$. Estimate the number of iterations necessary to obtain an approximation that is accurate within 10^{-5}.

6. Derive the Secant method.

7. (a) Use an appropriate Lagrange interpolating polynomial to approximate $f(0.43)$ with the following data.
$$f(0) = 1, \quad f(0.25) = 1.64872, \quad f(0.5) = 2.71828, \quad f(0.75) = 4.48169$$
   (b) The function generating this data is $f(x) = e^{2x}$. Find an upper bound on the error in your approximation in part (a).

8. Derive the truncation for the following approximation of $f'(x_0)$.
$$f'(x_0) \sim \frac{1}{2h}\left(-3f(x_0) + 4f(x_0+h) - f(x_0+2h)\right)$$

9. Suppose that $N(h)$ is an approximation to M for every $h > 0$ and that
$$M = N(h) + k_1 h + k_2 h^2 + k_3 h^3 + \cdots$$
for some constants $\{k_j\}_{j=1}^{\infty}$. Use the values $N(h)$, $N(h/3)$, and $N(h/9)$ to produce an $O(h^3)$ approximation of M.

10. Find the degree of precision of the quadrature formula
$$\int_{-1}^{1} f(x)dx = f\left(\frac{-\sqrt{3}}{3}\right) + f\left(\frac{\sqrt{3}}{3}\right)$$

11. Determine the values of n and h required to approximated

$$\int_0^2 \frac{1}{x+4}dx$$

to within 10^{-5}. Use

(a) Composite Trapezoidal Rule

(b) Composited Simpson's Rule

(c) Composite Midpoint Rule

12. What is the error term for Gaussian quadrature?

13. Use Composite Simpson's rule and the given values of n to approximate the following improper integral.

$$\int_0^1 \frac{e^{-x}}{\sqrt{1-x}}dx$$

with $n = 6$.

14. (a) Use Euler's method to approximate the solution to

$$\begin{aligned} y' &= te^{3t} - 2y \quad \text{for } 0 \le t \le 1 \\ y(0) &= 0 \end{aligned}$$

with $h = 0.5$.

- The exact solution to this problem is

$$y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}.$$

Determine the theoretical error bound. How does this compare with the actual error?

15. (a) Derive a third order Taylor method for approximating solutions to initial value problems of the form

$$\begin{aligned} y' &= f(t,y) \quad t \in [a,b] \\ y(a) &= \alpha. \end{aligned}$$

(b) Derive a formula for the truncation error.

(c) Create an error bound when $f(t,y) = \frac{2-2ty}{t^2+1}$ and $t \in [0,1]$.

16. Show that the difference method

$$\begin{aligned} w_0 &= \alpha \\ w_{i+1} &= w_i + a_1 f(t_i, w_i) + a_2 f(t_i + \alpha_2, w_1 + \delta_2 f(t_i, w_i)), \end{aligned}$$

for each $i = 0, 1, \ldots, N-1$, cannot have a local truncation error $O(h^3)$ for any choice of constants a_1, a_2, α_2, and δ_2.

17. Derive the Adams-Bashforth Two-Step method by using the Lagrange form of the interpolating polynomial.

18. Consider the difference equation

$$w_{i+1} = w_i + h\phi(t_i, w_i, h)$$

where ϕ is continuous and Lipschitz with respect to w with Lipschitz constant L on the set

$$D = \{(t, w, h) \mid t \in [a, b],\ w \in \mathbb{R},\ 0 \le h \le h_0\}$$

. Show that there exist a constant $K > 0$ such that

$$|u_i - v_i| \le K|u_0 - v_0|$$

for each $1 \le i \le N$, whenever $\{u_i\}_{i=1}^N$, and $\{v_i\}_{i=1}^N$ are from satisfy the difference equation.

19. Given the multistep method

$$w_{i+1} = -\frac{3}{2}w_i + 3w_{i-1} - \frac{1}{2}w_{i-2} + 3f(t_i, w_i), \quad \text{for } i = 2, \dots, N-1$$

with starting values w_0, w_1, and w_2.

(a) Find the local truncation error.

(b) Analyze this method for consistency, stability, and convergence.

20. Determine the stability, consistency and convergence for the implicit Trapezoidal method

$$w_{i+1} = w_i + \frac{h}{2}\left(f(t_{i+1}, w_{i+1}) + f(t_i, w_i)\right)$$

for $i = 0, \dots, N-1$.