Adaptive Multistep Methods & Extrapolation

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ch. 5.7 Variable Stepsize Multistep Methods } Burden + Faires textbook
ch. 5.8 Extrapolation

We saw error control/estimation and adaptive stepsizes for RK.

Conceptually, it's quite similar for multistep methods, but details different.

Embedded Formula

- For RK, need special embedded formula in order to make it efficient
- For multistep, much simpler — any pair will do

Ex from Burden + Faires: pair error estimation w/ predictor corrector

① Predictor: AB4 (4-step Adams-Bashforth, explicit)

$$w_{t+1}^{(P)} = w_t + \frac{h}{24}\left(55 f_i - 59 f_{i-1} + 37 f_{i-2} - 9 f_{i-3}\right)$$

$$f_i := f(t_i, w_i)$$

$y^{(2)}$ means y'', etc.

which has local truncation error

$$\tau_{i+1}^{(P)}(h) := \frac{y_{i+1} - w_{i+1}^{(P)}}{h} = \frac{251}{720} y^{(5)}(\xi_i^{(P)})\, h^4 = O(h^4)$$

② Corrector AM 3 (3-step Adams-Moulton, implicit)

$$w_{t+1} = w_t + \frac{h}{24}\left(9 f_{i+1} + 19 f_i - 5 f_{i-1} + f_{i-2}\right)$$

$$= f(t_{i+1}, w_{t+1}^{(P)})$$

which has local truncation error

$$(*) \qquad \tau_{i+1}(h) := \frac{y_{i+1} - w_{i+1}}{h} = \frac{-19}{720} y^{(5)}(\xi_i)\, h^4 = O(h^4)$$

③ Combining to get error estimate is a bit different than adaptive methods for integration or RK. Now, both methods are the same order, $O(h^4)$

We'll assume $y^{(5)}(\xi_i^{(P)}) \approx y^{(5)}(\xi_i)$

thus with some algebra, $\dfrac{w_{i+1} - w_{i+1}^{(p)}}{h} = \dfrac{h^4}{720}\left(251\, y^{(5)}(\xi_i^{(p)}) + 19\, y^{(5)}(\xi_i)\right)$

$$\approx \tfrac{3}{8} h^4 \cdot y^{(5)}(\xi_i)$$

$$\Rightarrow \underbrace{y^{(5)}(\xi_i)}_{\text{plug into } (*)} \approx \tfrac{8}{3} h^{-5} \left(w_{i+1} - w_{i+1}^{(p)} \right)$$

$$\Rightarrow |\tau_{i+1}(h)| \approx \frac{19}{270\, h} \left| w_{i+1} - w_{i+1}^{(p)} \right|$$

so we can check if this is acceptable

(4) If step wasn't acceptable...

estimate new step $q\,h$ that we predict would be acceptable

See book for details

For target accuracy ε, choose

$$q \approx 1.5 \left(\frac{h\,\varepsilon}{\left| w_{i+1} - w_{i+1}^{(p)} \right|} \right)^{1/4}$$

(5) Implement step size change

less pleasant than for RK

Since this is multi-step, formulas only work if previous values $w_i,\ w_{i-1},\ w_{i-2}, \dots$ were equispaced.

So, we need to either
(1) re-evaluate f at "old" points
(2) interpolate f and evaluate interpolant
(3) switch to RK until we have enough history at the new stepsize

For this reason, we rarely use $q > 1$ to increase a stepsize (only if $q > 10$, say), and mainly use it to decrease stepsize

That's it. See Algo. 5.5 in the book if you want.

It's mostly $\underbrace{\text{details}}$, not math

very important details, but only important to a few hundred people in the world.

Extrapolation

Based off the midpoint method (a non-Adams type of multistep method)

$$w_{i+1} = \underbrace{w_{i-1}}_{\text{not } w_i} + 2h\, f(t_i, w_i)$$

i.e., derived from

$$y(t_{i+1}) = y(t_{i-1}) + \underbrace{\int_{t_{i-1}}^{t_{i+1}} y'(s)\,ds}_{\text{use midpoint quad. rule}}$$

Not going into details

Based off same principles as Richardson extrapolation

+ Romberg integration, but much messier.

Can repeat the extrapolation to get higher and higher order methods

(or until something breaks, like smoothness assumptions)