APPM 4515-001: Homework #6

Due on October 19, 2020 at 11:59pm

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Problem 1
From figure 1 we can tell that:

$$
\begin{aligned}
y'' + 5y' + 6y = \cos t;\ y(0) = 1;\ y'(0) = 0 \\
\text{Let } y_h \text{ be the solution to } y'' + 5y' + 6y = 0 \\
\text{ansatz } y_h = e^{rt} \text{ where r:} \\
r^2 + 5r + 6 = 0 \\
(r+2)(r+3) = 0 \\
r = -2;\ r = -3 \\
y_h = C_1 e^{-2t} + C_2 e^{-3t} \\
\text{Let } y_p \text{ be the particular solution to the ODE:} \\
\text{ansatz } y_p = B_0 \sin t + A_0 \cos t \\
\implies y_p' = B_0 \cos t - A_0 \sin t \\
\implies y_p'' = -B_0 \sin t - A_0 \cos t \\
\text{Substituting these into the ODE:} \\
-B_0 \sin t - A_0 \cos t + 5B_0 \cos t - 5A_0 \sin t + 6B_0 \sin t + 6A_0 \cos t = \cos t \\
(-A_0 + 5B_0 + 6A_0) \cos t + (-B_0 - 5A_0 + 6B_0) \sin t = \cos t + 0 \sin t \\
\implies -A_0 + 5B_0 + 6A_0 = 1;\ -B_0 - 5A_0 + 6B_0 = 0 \\
\implies A_0 = \frac{1}{10};\ B_0 = \frac{1}{10} \\
\text{Thus } y_p = \frac{1}{10} \sin t + \frac{1}{10} \cos t \\
\text{So the general solution is:} \\
y(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{10} \sin t + \frac{1}{10} \cos t \\
\text{Now implementing our initial conditions:} \\
y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + \frac{1}{10} \cos t - \frac{1}{10} \sin t \\
y(0) = C_1 e^{-2*0} + C_2 e^{-3*0} + \frac{1}{10} \sin 0 + \frac{1}{10} \cos 0 = 1 \\
\implies C_1 + C_2 + \frac{1}{10} = 1 \ \mathbf{(1)} \\
y'(0) = -2C_1 e^{-2*0} - 3C_2 e^{-3*0} + \frac{1}{10} \cos 0 - \frac{1}{10} \sin 0 = 0 \\
\implies 2C_1 + 3C_2 = \frac{1}{10} \ \mathbf{(2)} \\
\text{From } \mathbf{(1)} \text{ and } \mathbf{(2)} \text{ we can conclude:} \\
C_1 = \frac{13}{5};\ C_2 - \frac{17}{10} \\
\text{Therefore our solution is:} \\
y(t) = \frac{13}{5} e^{-2t} - \frac{17}{10} e^{-3t} + \frac{1}{10} \sin t + \frac{1}{10} \cos t
\end{aligned}
$$

So the radius of $S^{n-2}(x_1, r)$ is $\sqrt{1-\epsilon^2}$.

Problem 2
$Vol(B^n(x_1, r)) = \frac{n^{n/2}}{\Gamma(n/2+1)})\sqrt{1-\epsilon^2}^n$

$Vol(B^n(0,1)) = \frac{n^{n/2}}{\Gamma(n/2+1})$
So it follows that $Vol(B^n(x_1,r)) = Vol(B^n(0,1))\sqrt{1-\epsilon^2}^n = Vol(B^n(0,1))(1-\epsilon^2)^{\frac{n}{2}}$

Problem 3
Graphically we can see that for every $\gamma \in C(k)$, $\gamma \in B^n(x_1,r)$ So this tells us that $C(k) \subset B^n(x_1,r)$. Formally we choose a $\gamma \in C(k)$ and show that because the radius of $B^n(x_1,r)$ is greater than or equal to the radius of $C(k)$, $\gamma \in B^n(x_1,r)$.

Problem 4
We know from problem 3 that $Vol(C(k)) \leq Vol(B^n(x_1,r))$ because it is a subset,
(note $1+x \leq e^x \implies (1+x)^a \leq (e^x)^a$), so it follows that:

$$\begin{aligned} y'' + 5y' + 6y = \cos t \\ y'' = \cos t - 5y' - 6y \\ \text{Let } y_1 = y;\ y_2 = y' \\ \text{So our system is:} \\ y_1' = y_2 = u' \\ y_2' = y'' = -6y_1 - 5y_2 + cos(t) = -6u - 5v + \cos t = v' \end{aligned}$$

Problem 5

$$\begin{aligned} \mu(k(x_0,\epsilon)) &\leq Vol(C(k)) \\ &\leq Vol(B^n(0,1))e^{-\frac{n\epsilon^2}{2}} \\ &\leq e^{-\frac{n\epsilon^2}{2}} \end{aligned}$$

Problem 6
This is really close to what we got for Levy's bound, but not quite a good.I believe Levy's bound would have been $\sqrt{\frac{\pi}{8}}e^{\frac{-(n-2)\epsilon^2}{2}}$.