Conditioning of linear systems

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Spectral Norm

For a vector $\vec{x} \in \mathbb{R}^n$, we'll exclusively use the Euclidean norm $\|\vec{x}\|_2$

Def Euclidean norm $\|\vec{x}\|_{\underline{2}} = \sqrt{\sum_{i=1}^{n} |x_i|^{\underline{2}}}$

here the notation

aka the "distance formula" as it's called in algebra and calculus classes

There are many other types of norms which hopefully you saw in your linear algebra class

For a matrix, there are also many norms (e.g., think of a $m \times n$ matrix as a length $m \cdot n$ vector)

but we'll focus on the most important norm you never heard of ...

click-bait for mathematicians

the spectral norm.

Def The spectral norm of a matrix A is

$$\|A\|_2 := \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_2}{\|\vec{x}\|_2} = \max_{\|\vec{x}\|_2 = 1} \|A\vec{x}\|_2$$

$$= \text{maximum singular value of } A$$

$$= \sqrt{\text{maximum eigenvalue of } A^T A}$$

we say it's "induced" by the Euclidean norm, hence the reason for writing $\|A\|_2$

Given a matrix A, $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$

so for any particular $x \neq 0$, $\frac{\|Ax\|_2}{\|x\|_2} \leq \max_{x' \neq 0} \frac{\|Ax'\|_2}{\|x'\|_2} = \|A\|_2$

i.e., $\forall x \neq 0$, $\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_2$ i.e., $\forall x$, $\boxed{\underbrace{\|Ax\|_2}_{\text{Euclidean}} \leq \underbrace{\|A\|_2}_{\text{Spectral}} \cdot \underbrace{\|x\|_2}_{\text{Euclidean}}}$

Condition number (i.e. relative condition number)

Recall if we're trying to compute $f(x)$, we perturb $\tilde{x} = x + \Delta x$

input; output is $f(x)$

and the **relative condition number** is

$$K_f(x) = \lim_{\Delta x \to 0} \frac{\frac{|f(x) - f(\tilde{x})|}{|f(x)|}}{\frac{|x - \tilde{x}|}{|x|}}$$

$|x - \tilde{x}| = \Delta x$

For solving linear equations $A\vec{x} = \vec{b}$, our input is $\vec{b}$ (or, $\vec{A}$ and $\vec{b}$, but we'll focus on $\vec{b}$)

and our output is $\vec{x}$, so "x" is now the output not the input!

(we had a similar issue in the root-finding case... this is just a multi-dimensional extension)

So, perturb $\tilde{b} = \vec{b} + \Delta b$, and "$f(\tilde{b})$" is $\tilde{x}$, i.e.,

all vectors

$$A\tilde{x} = \tilde{b}.$$

If we write $\tilde{x} = x + \Delta x$, then

I'm going to stop writing $\vec{x}$. Just remember it's a vector

$$A \cdot (x + \Delta x) = b + \Delta b$$

So

$$K(b) = \lim_{\|\Delta b\|_2 \to 0} \frac{\frac{\|x - \tilde{x}\|_2}{\|x\|_2}}{\frac{\|b - \tilde{b}\|_2}{\|b\|_2}} = \lim_{\|\Delta b\|_2 \to 0} \frac{\frac{\|\Delta x\|_2}{\|x\|_2}}{\frac{\|\Delta b\|_2}{\|b\|_2}}$$

we call it K_2 because we use Euclidean norm.

You can use other norms but K_2 is most common.

Now, $A(x + \Delta x) = b + \Delta b$ and $Ax = b$ so $A \cdot \Delta x = \Delta b$

so $\underline{\Delta x = A^{-1} \cdot \Delta b}$

Think back to $\|\cdot\|_2$ for matrices (spectral norm): $\boxed{\|A^{-1}\Delta b\|_2 \le \|A^{-1}\|_2 \cdot \|\Delta b\|_2}$

Euclidean; Spectral; Euclidean

so $\|\Delta x\|_2 = \|A^{-1}\Delta b\|_2 \le \|A^{-1}\|_2 \cdot \|\Delta b\|_2$

and in fact $\exists\ \Delta b$ s.t. $\|A^{-1}\Delta b\|_2 = \|A^{-1}\|_2 \|\Delta b\|_2$ so it can be tight.

So...

$$K_2(b) = \lim_{\|\Delta b\|_2 \to 0} \frac{\frac{\|\Delta x\|_2}{\|x\|_2}}{\frac{\|\Delta b\|_2}{\|b\|_2}} = \lim_{\|\Delta b\|_2 \to 0} \frac{\|\Delta x\|_2 \cdot \|b\|_2}{\|x\|_2 \cdot \|\Delta b\|_2}$$

$$\le \lim_{\|\Delta b\|_2 \to 0} \frac{\|A^{-1}\|_2 \cdot \|\cancel{\Delta b}\|_2 \cdot \|b\|_2}{\|x\|_2 \cdot \|\cancel{\Delta b}\|_2}$$

and since $b = Ax$

$$\|b\|_2 = \|Ax\|_2 \le \|A\|_2 \cdot \|x\|_2$$

(same trick)

$$\le \frac{\|A^{-1}\|_2 \; \|A\|_2 \cdot \|\cancel{x}\|_2}{\|\cancel{x}\|_2}$$

$$= \|A^{-1}\|_2 \cdot \|A\|_2$$

This was a <u>bound</u> on $\kappa_2(b)$.

In general, we define the <u>condition number</u>, independent of b, to be

$$\boxed{\kappa_2(A) := \|A^{-1}\|_2 \cdot \|A\|_2}$$

spectral norms

(slight modification of relative condition number that we use for numerical linear algebra)

$$\|A^{-1}\|_2 \cdot \|A\|_2 = \frac{\text{largest singular value of } A}{\text{smallest singular value of } A}$$

a measure of dynamic range in a sense.

<u>Facts</u>: $\|A\|_2 = \|A^T\|_2$ so $\kappa_2(A) = \kappa_2(A^T)$

$\|A^TA\|_2 = \|AA^T\|_2 = \|A\|_2^2$ so $\kappa_2(A^TA) = \kappa_2(AA^T) = \kappa_2(A)^2$

In Matlab, use `cond(A)` to find $\kappa_2(A)$

In Python, it's `numpy.linalg.cond(A)`

So... $\kappa_2(A)$ measures the inherent difficulty of solving $A\vec{x} = \vec{b}$

(it's also useful for other problems too)

We can't complain if our algorithm "loses" $\log_{10}(\kappa_2(A))$ digits

but if our algorithm does worse than this, e.g., if we ever form an intermediate quantity with $\kappa_2(A)^2$, then the algorithm is unstable, i.e., it's doing worse than it needs to.

As before, we're distinguishing:

Conditioning $\longleftrightarrow$ math problem, i.e., find x s.t. $Ax = b$

vs.

Stability $\longleftrightarrow$ algorithm / implementation