Conditioning of the rootfinding problem

Sunday, August 30, 2020 3:37 PM

Before we get into algorithms, let's ask what we can expect

Goal / assumptions:

let $f: \mathbb{R} \to \mathbb{R}$ be continuous. → w/o this assumption, no hope to solve the problem in general. In reality, we only need continuity near the root

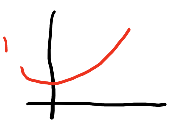
Find a root r, i.e., solve $f(r) = 0$

- Sometimes we know $r \in [a,b]$
- ... sometimes we don't know this.

- We don't expect to find the root w/ ∞ precision

Ex:

- $f(x) = x^2 - 3x + 1$ → polynomials (of low degree) we can handle algebraically ...
  but already, this brings up a good point: do roots always exist? $f(x) = x^2 + 1$



- $e^{-x} - x^2 = 0$
- $\tan(x) - 2x = 0$
- $x \cdot e^x - c = 0$

} don't have algebraic solutions (i.e., most transcendental equations)

Condition number

Assume for now that f' exists (at least near the root)

Let r be a root, so $f(r) = 0$

If we perturb our "problem data" (ie., perturb coefficients of a polynomial, or if $f(x) = \tan(x) - 2x$, what if we can't compute tan exactly)

we get a new function $\tilde{f}$

and let $\boxed{\varepsilon = \tilde{f}(r) - f(r) \quad \text{i.e., } \tilde{f}(r) = f(r) + \varepsilon}$ ε

Let $\tilde{r}$ be the root of $\tilde{f}$, i.e., $\tilde{f}(\tilde{r}) = 0$, and $\boxed{\delta = \tilde{r} - r \quad \text{i.e., } \tilde{r} = r + \delta}$ δ

Q: if ε is small, is δ small too?

Quantify this by condition number.

Before, we discussed a relative condition number

$$\kappa_g^{\text{rel.}}(x) = \left| \frac{x}{g(x)} g'(x) \right|$$

now we'll look at the absolute condition number

⚠ Warning: "g" is not our f.

Ex: find a root of $f(x) = x^2 - c^2$

$K_g^{abs}(x) = |g'(x)|$

"g" is a bit tricky to pin down in general so we won't explicitly use it.

Instead,

$$K^{abs}(r) = \lim_{\varepsilon \to 0} |\delta / \varepsilon|$$

and to derive a nicer expression, use Taylor Series

then we might ask about perturbations in the coefficient c

so $g(c) = \sqrt{c}$

i.e., $g(\text{data}) = \text{root}$

vs. $f(\text{root}) = 0$

$$0 = \tilde{f}(\tilde{r})$$

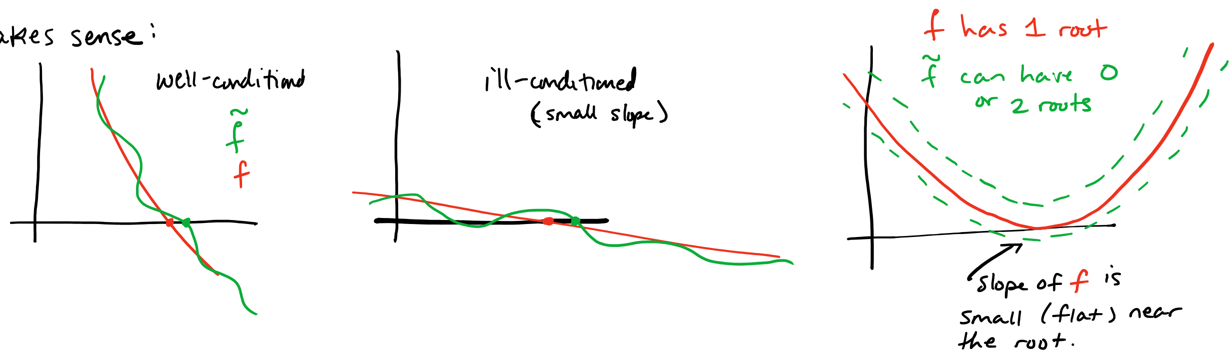
$$= f(r+\delta) + \varepsilon$$

$$= \overbrace{\underbrace{f(r)}_{=0} + f'(r)\cdot\delta + \cancel{O(\delta^2)}}^{\text{Taylor}} + \varepsilon \quad \text{ignore} \qquad \Rightarrow \qquad \delta/\varepsilon = -1/f'(r)$$

i.e. absolute condition number for root-finding is

$$K^{abs}(r) = |f'(r)|^{-1}$$

This makes sense:



Notation:

The error is how far our estimated root is from the true root, $|\tilde{r} - r|$

The residual is $|f(\tilde{r})|$

$f(\tilde{r}) = 0$ (residual $= 0$) $\Rightarrow$ $\tilde{r} = r$ (error is 0) but $|f(\tilde{r})|$ small (but nonzero) does not guarantee error is small.

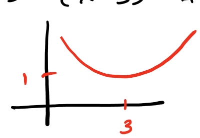
Observe if we define $\tilde{f}(x) = f(x) - \overbrace{f(\tilde{r})}^{\text{constant}}$

then $\tilde{f}(\tilde{r}) = 0$. So $\tilde{r}$ exactly solves the wrong problem

i.e., $f(\tilde{r})$ small means small backward error

One last piece of math...

how do we even know if a root exists? $f(x) = (x-3)^2 + 1$ has no roots



Our main tool is the Intermediate Value Thm.

i.e., if $f(a) < 0$ and $f(b) > 0$

(or $f(a) > 0$, $f(b) < 0$)

and f is continuous on $[a,b]$ (ie. $f \in C([a,b])$)

then $\exists\ r \in (a,b)$ such that

$$f(r) = 0$$