Homework 9

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MATH 4650

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1)

$$y'' + 5y' + 6y = \cos t;\ y(0) = 1;\ y'(0) = 0$$
$$\text{Let } y_h \text{ be the solution to } y'' + 5y' + 6y = 0$$
$$\text{ansatz } y_h = e^{rt} \text{ where r:}$$
$$r^2 + 5r + 6 = 0$$
$$(r+2)(r+3) = 0$$
$$r = -2;\ r = -3$$
$$y_h = C_1 e^{-2t} + C_2 e^{-3t}$$
$$\text{Let } y_p \text{ be the particular solution to the ODE:}$$
$$\text{ansatz } y_p = B_0 \sin t + A_0 \cos t$$
$$\implies y_p' = B_0 \cos t - A_0 \sin t$$
$$\implies y_p'' = -B_0 \sin t - A_0 \cos t$$
$$\text{Substituting these into the ODE:}$$
$$-B_0 \sin t - A_0 \cos t + 5B_0 \cos t - 5A_0 \sin t + 6B_0 \sin t + 6A_0 \cos t = \cos t$$
$$\left(-A_0 + 5B_0 + 6A_0\right)\cos t + \left(-B_0 - 5A_0 + 6B_0\right)\sin t = \cos t + 0 \sin t$$
$$\implies -A_0 + 5B_0 + 6A_0 = 1;\ -B_0 - 5A_0 + 6B_0 = 0$$
$$\implies A_0 = \frac{1}{10};\ B_0 = \frac{1}{10}$$
$$\text{Thus } y_p = \frac{1}{10}\sin t + \frac{1}{10}\cos t$$
$$\text{So the general solution is:}$$
$$y(t) = C_1 e^{-2t} + C_2 e^{-3t} + \frac{1}{10}\sin t + \frac{1}{10}\cos t$$
$$\text{Now implementing our initial conditions:}$$
$$y'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + \frac{1}{10}\cos t - \frac{1}{10}\sin t$$
$$y(0) = C_1 e^{-2*0} + C_2 e^{-3*0} + \frac{1}{10}\sin 0 + \frac{1}{10}\cos 0 = 1$$
$$\implies C_1 + C_2 + \frac{1}{10} = 1 \ \textbf{(1)}$$
$$y'(0) = -2C_1 e^{-2*0} - 3C_2 e^{-3*0} + \frac{1}{10}\cos 0 - \frac{1}{10}\sin 0 = 0$$
$$\implies 2C_1 + 3C_2 = \frac{1}{10} \ \textbf{(2)}$$
$$\text{From } \textbf{(1)} \text{ and } \textbf{(2)} \text{ we can conclude:}$$
$$\frac{13}{\ } \qquad \frac{17}{\ }$$

$$C_1 = \frac{5}{} ; C_2 - \frac{10}{}$$

Therefore our solution is:

$$y(t) = \frac{13}{5} e^{-2t} - \frac{17}{10} e^{-3t} + \frac{1}{10} \sin t + \frac{1}{10} \cos t$$

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2)

In [72]:

```
import numpy as np
import pandas as pd
from numpy import sin, cos, exp
import matplotlib.pyplot as plt
import matplotlib as mpl
import scipy.interpolate # use for scipy.interpolate.CubicHermiteSpline
from scipy.integrate import solve_ivp
mpl.rcParams["figure.figsize"] = [8,6] # or 7, 4 or 10,8
mpl.rcParams["lines.linewidth"] = 2
mpl.rcParams["lines.markersize"] = 4
mpl.rcParams.update({'font.size': 20})
mpl.rcParams['mathtext.fontset'] = 'cm'
```

a)

$$y'' + 5y' + 6y = \cos t$$

$$y'' = \cos t - 5y' - 6y$$

Let $y_1 = y; y_2 = y'$

So our system is:

$$y_1' = y_2 = u'$$

$$y_2' = y'' = -6y_1 - 5y_2 + cos(t) = -6u - 5v + \cos t = v'$$

b)

In [107]:

```
def RKMethod(tspan, h, ic):
    n = round((tspan[1]-tspan[0])/h)
    t_step = np.linspace(tspan[0],tspan[1],n+1)
    h = t_step[1] - t_step[0]
    u = np.zeros(len(t_step))
    v = np.zeros(len(t_step))
    f = lambda t,u,v: [v,cos(t)-5*v-6*u]
    u[0] = ic[0]
    v[0] = ic[1]
    for i in range(0, len(t_step)-1):
        t_i = t_step[i]

        k1 = h*pd.Series(f(t_i,u[i],v[i]))
        k2 = h*pd.Series(f(t_i+(h/3), u[i]+k1[0]/3, v[i]-k1[1]/3))
        k3 = h*pd.Series(f(t_i+2*(h/3), u[i]-k1[0]/3+k2[0], v[i]-k1[1]/3+k2[1]))
        k4 = h*pd.Series(f(t_i+h,u[i]+k1[0]-k2[0]+k3[0], v[i]+k1[1]-k2[1]+k3[1]))

        u[i+1] = u[i]+(1/8)*k1[0]+(3/8)*k2[0]+(3/8)*k3[0]+(1/8)*k4[0]
        v[i+1] = v[i]+(1/8)*k1[1]+(3/8)*k2[1]+(3/8)*k3[1]+(1/8)*k4[1]
    return t_step, u
```

In [109]:

```
t_span = [0,20]
i_c = [1,0]
```

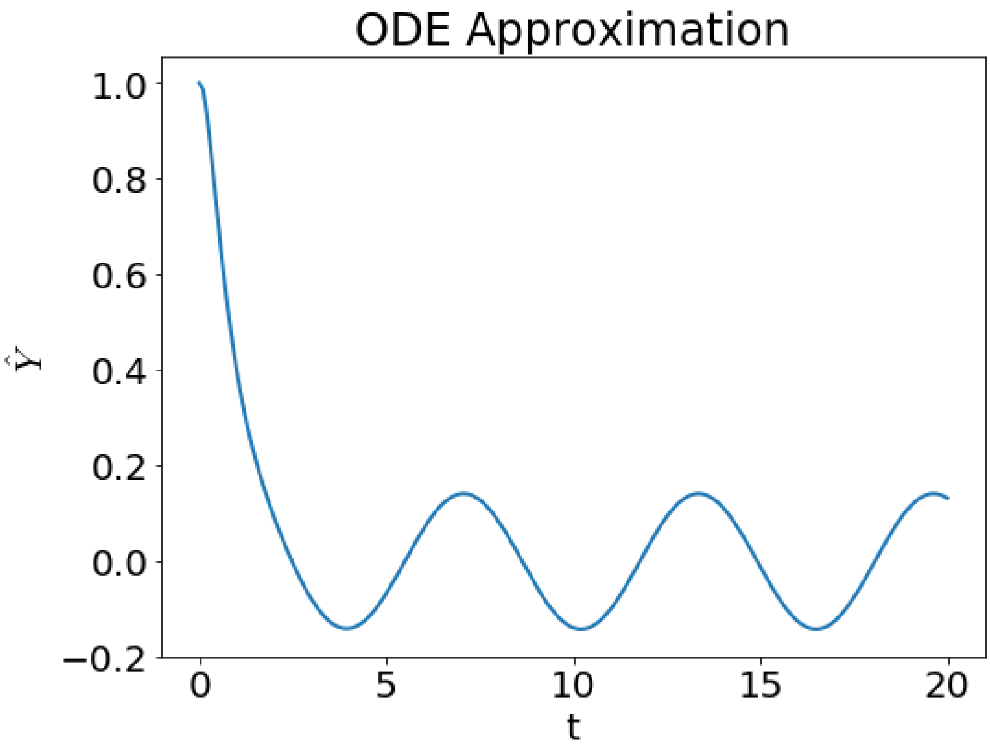
```
y_true = lambda t: (13/5)*np.exp(-2*t)-(17/10)*np.exp(-3*t)+(1/10)*sin(t)+(1/10)*cos(t)


tHist, WHist = RKMethod(t_span, 0.1, i_c)
y = y_true(tHist)
```

In [112]:

```
plt.plot(tHist,WHist)
plt.xlabel("t"); plt.ylabel("$\hat{Y}$")
plt.title("ODE Approximation")
```

Out[112]:

```
Text(0.5,1,'ODE Approximation')
```



c)

From the graph below it looks like we are somewhere in between $O(h^2)$ and $O(h^4)$ but much closer to $O(h^2)$. I think this makes sense because I believe that a 4th step RK Method is $O(h^3)$ so I believe we are doing pretty well!

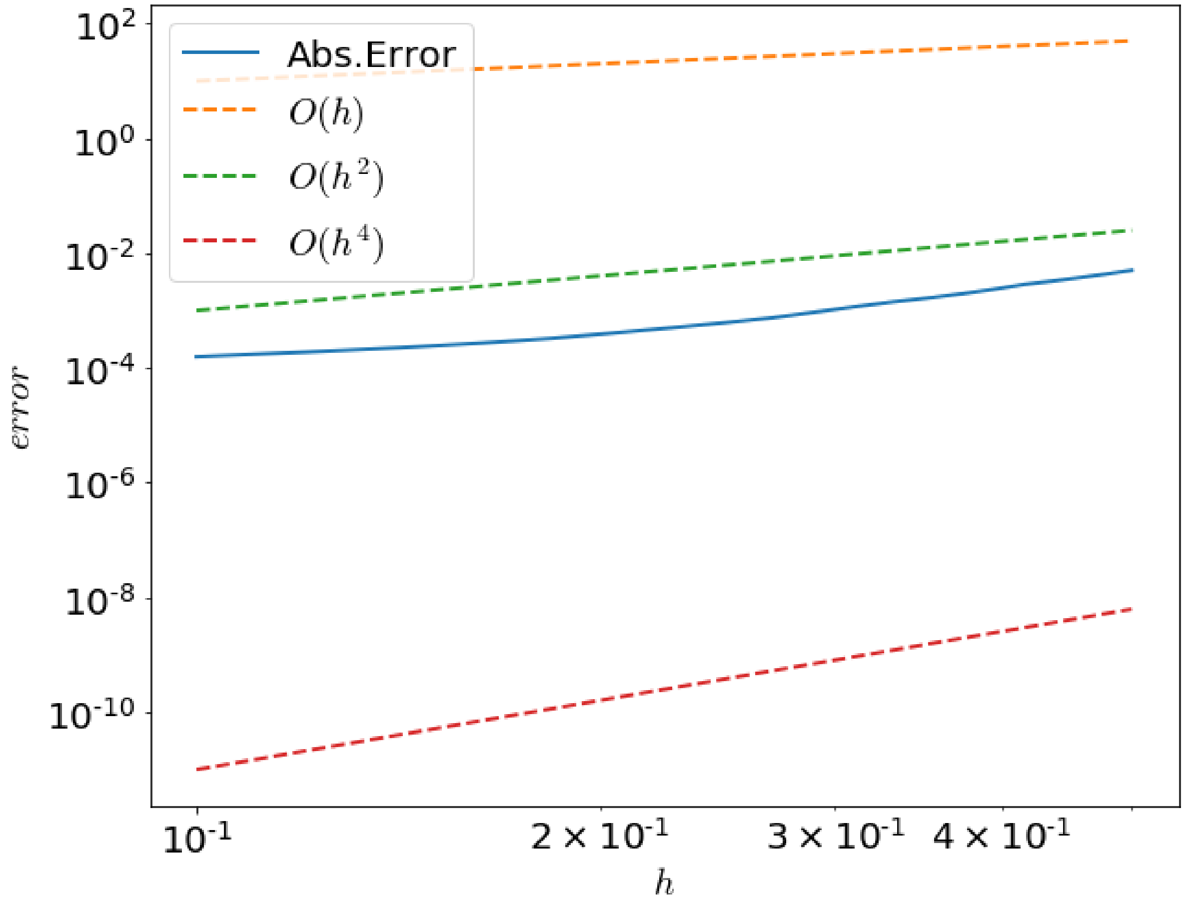
In [104]:

```
hList = np.linspace(0.1,.5,20)
t_span = [0,20]
i_c = [1,0]
err = []

for i in hList:
    tHist, wHist = RKMethod(t_span, i, i_c)
    a = wHist[-1]
    err.append(abs(y_true(20)-a))
    #err.append(a)

plt.figure(figsize=(10,8))
plt.loglog(hList, err,label = 'Abs. Error')
plt.loglog(hList,hList/(10**(-2)),'--',label = '$O(h)$')
plt.loglog(hList,(hList**2)/10,'--',label = '$O(h^2)$')
plt.loglog(hList,(hList**4)/(10**7),'--',label = '$O(h^4)$')
plt.xlabel("$h$"); plt.ylabel("$error$");
plt.legend(['Abs.Error','$O(h)$','$O(h^2)$','$O(h^4)$'])
```

```
/anaconda3/lib/python3.6/site-packages/ipykernel_launcher.py:3: DeprecationWarning: objec
t of type <class 'numpy.float64'> cannot be safely interpreted as an integer.
  This is separate from the ipykernel package so we can avoid doing imports until
```

Out[104]:

```
<matplotlib.legend.Legend at 0x181b5117f0>
```



b)

See code above