Conditioning of linear systems

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Spectral Norm

For a vector $\vec{x} \in \mathbb{R}^n$, we'll exclusively use the Euclidean norm $||\vec{x}||_2$

Def Enclidean norm $\|\vec{x}\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$ hence the notation

aka the "distance formula" as it's called in algebra and calculus classes

There are many other types of norms which hopefully you saw in your linear algebra class

For a matrix, there are also many norms (e.g., think of a mxn matrix as a length min vector)

but we'll focus on the most important norm you never heard of ... click-bait for mothematicians

the spectral norm.

Def The spectral norm of a matrix A is

$$\|A\|_{2} := \max_{\vec{x} \neq 0} \frac{\|A\vec{x}\|_{2}}{\|\vec{x}\|_{2}} = \max_{\|\vec{x}\|_{2}=1} \|A\vec{x}\|_{2}$$

= maximum singular value of A

we say it's "induced" by the Euclidean norm, hence the reason for writing 11 Allz

Given a matrix A, $\|A\|_2 = \max_{x \neq 0} \frac{\|A \times \|_2}{\|x\|_2}$

So for any particular $x \neq 0$, $\left\| \frac{A \times 11_z}{11 \times 11_z} \le \max_{x' \neq 0} \frac{\left\| \frac{A \times '1}{11 \times '11_z} = \left\| A \right\|_2$

i.e.,
$$\forall \times \neq 0$$
, $\|A \times \|_2 \le \|A\|_2$ i.e., $\forall \times$, $\|A \times \|_2 \le \|A\|_2 \cdot \|X\|_2$

Condition number (ie. relative condition number) - input output is f(x) Recall if we're trying to compute f(x), we perturb $\tilde{x} = x + \Delta x$ and the relative condition number is $K_f(x) = \lim_{\Delta x \to 0} \frac{|f(x) - f(x)|}{|f(x)|} = \Delta x$ For solving linear equations $A\vec{x} = \vec{b}$, our input is \vec{b} (or, \vec{A} and \vec{b} , but we'll focus on \vec{b}) and our output is x', so "x" is now the output not the input? (we had a similar issue in the root-finding case ... this is just a multi-dimensional extension) So, perturb $\stackrel{\sim}{b} = \stackrel{\sim}{b} + \Delta b$, and " $f(\stackrel{\sim}{b})$ " is $\stackrel{\sim}{\kappa}$, i.e., $A\tilde{x} = \tilde{b}$. If we write $\tilde{x} = x + \Delta x$, then I'm going to stop $A \cdot (x + \Delta x) = b + \Delta b$ writing x Just remember it's a rector So $K(b) = \lim_{\|\Delta b\|_{2} \to 0} \frac{\|\underline{x} - \widetilde{x}\|_{2}}{\|\Delta b\|_{2}} = \lim_{\|\Delta b\|_{2} \to 0} \frac{\|\Delta x\|_{2}}{\|\Delta b\|_{2}}$ we call it K2 because we use Enclidean norm. You can use other norms but K2 is most common. Now, $A(x+\Delta x) = b+\Delta b$ and Ax = b so $A \cdot \Delta x = \Delta b$ so Δx = A-1 · Δb Think back to $\|\cdot\|_2$ for matrices (spectral norm): $\|A^{-1}\Delta b\|_2 \leq \|A^{-1}\|_2 \cdot \|\Delta b\|_2$ So $\|\Delta x\|^2 \|A^{-1}\Delta b\|_2 \leq \|A^{-1}\|_2 \cdot \|\Delta b\|_2$ Evaluation Spectral Excluden and in fact 3 do s.t. $\|A^{-1}\Delta b\|_2 = \|A^{-1}\|_2 \|\Delta b\|_2$ so it can be tight. 50 ...

 $K_{2}(b) = \lim_{\|\Delta b\|_{2} \to 0} \frac{\|\Delta \times \|_{2}}{\|\Delta b\|_{2}} = \lim_{\|\Delta b\|_{2} \to 0} \frac{\|\Delta \times \|_{2} \cdot \|b\|_{2}}{\|\Delta b\|_{2}}$

and since
$$b = A \times$$

$$||b||_2 = ||A \times ||_2 \le ||A||_2 \cdot ||\times||_2$$
(same trick)

$$||A^{-1}||_{2} ||Ab||_{2} \cdot ||b||_{2}$$

$$||Ab||_{2} \rightarrow 0$$

$$||A||_{2} \cdot ||Ab||_{2} \cdot ||b||_{2}$$

$$||A||_{2} \cdot ||Ab||_{2} \cdot ||b||_{2}$$

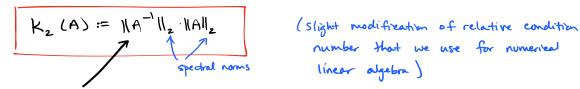
$$||A||_{2} \cdot ||A||_{2} \cdot ||A||_{2} \cdot ||A||_{2}$$

$$||A||_{2} \cdot ||A||_{2} \cdot ||A||$$

= 11 A-11/2 -11 A11/2

This was a bound on K2 (b).

In general, we define the condition number, independent of by to be



 $\|A^{-1}\|_2 - \|A\|_2 =$ largest singular value of A smallest singular value of A

a measure of dynamic range in a sense.

Facts : | Allz = | AT | so K2(A) = K2 (AT) $\|A^{T}A\|_{2} = \|AA^{T}\|_{1} = \|A\|_{2}^{2}$ so $K_{2}(A^{T}A) = K_{2}(AA^{T}) = K_{2}(A)^{2}$

In Matlab, use cond (A) to find K2(A) In Python, it's rumpy liholy, cond (A)

So... K2 (A) measures the inherend difficulty of solving Ax = 6 (it's also useful for other problems too)

We can't complain if our algorithm "loses" (og (K_2(A)) digits

but if our algorithm does worse than this, e.g., if we ever form an intermediate quantity with $k_2(A)^2$, then the algorithm is unstable, i.e., it's doing worse than it needs to.

As before, we're distinguishing?

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rath problem, i.e., find x s.t. Ax = 6
vs.
stability conditioning
co