Richardson Extrapolation

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used in Romberg integration, and similar in spirit to Aitken Acceleration

Suppose we have a method $N_1(h)$, a function of h (or $n = 1/h$ sometimes) which is used to approximate a number M,

i.e., $\lim_{h\to 0} N_1(h) = M$

Ex: $M = f'(x)$, $N_1(h) = \dfrac{f(x+h) - f(x-h)}{2h}$

We need to assume $M = N_1(h) + c\,h^\alpha + o(h^\alpha)$ "little o"

(or more generally, $M = N_1(h) + c_1 h^{\alpha_1} + c_2 h^{\alpha_2} + c_3 h^{\alpha_3} + \ldots$

where we know α

but we don't need to know c

Note: Since handwritten big O's "O" and little o's "o" are hard to distinguish, all my little o's will be in purple

Ex again

N_1 is the 3-pt. centered diff. formula, we know this is $O(h^2)$

Or, more precisely

$$f(x+h) = \cancel{f(x)} + f'(x)h + \cancel{f''(x)\tfrac{h^2}{2}} + f'''(x)\tfrac{h^3}{3!} + \cancel{f''''(x)\tfrac{h^4}{4!}} \ldots$$

$$-\; f(x-h) = -\left[\cancel{f(x)} - f'(x)h + \cancel{f''(x)\tfrac{h^2}{2}} - f'''(x)\tfrac{h^3}{3!} + \cancel{f''''(x)\tfrac{h^4}{4!}} \ldots\right]$$

$$\div 2h \quad = \frac{2f'(x)h + \frac{2}{3!}f'''(x)h^3 + \frac{2}{5!}f^{(5)}(x)h^5}{2h}$$

$$= \underbrace{f'(x)}_{M} + \tfrac{1}{6}f'''(x)h^2 + \tfrac{1}{5!}f^{(5)}(x)h^4 + O(h^6)$$

let $\alpha = 1$ for now

So, $-1\times\left(M = N_1(h) + c\,h^1 + o(h^1) \right)$ (*)

then note

$2\times\left(M = N_1(h/2) + c(h/2)^1 + o(h^1) \right)$ (**)

so $** + (** - *) = M + (M - M) = M$ [LHS]

$$= N_1(h/2) + N_1(h/2) - N_1(h)$$
$$+ \underbrace{c(h/2) + c(h/2) - c\,h}_{=0} + o(h^1) \quad \text{[RHS]}$$

So $N_2(h) := N_1\left(\tfrac{h}{2}\right) + \left(N_1\left(\tfrac{h}{2}\right) - N_1(h)\right) = M + o(h)$

instead of $N_1(h) = M + O(h)$

little-o is better than big-O

If $\alpha \neq 1$,

it looks similar but we pick different coefficients in order to make the cancellation happen

Ex again

$$N_1(h) = \frac{f(x+h) - f(x-h)}{2h}$$

$$N_1(h) = \underbrace{f'(x)}_{M} + c \cdot h^2 + \tilde{c} \cdot h^4 + O(h^6)$$

so

different coeff. than earlier since here $\alpha = 2$

$$N_2(h) = \frac{1}{3}\left[4N_1(h/2) - N_1(h) \right]$$

$$= \frac{1}{3}\Big[4f'(x) + 4 \cdot c \cdot (h/2)^2 + 4\tilde{c}(h/2)^4 - f'(x) - c\,h^2 - \tilde{c}\,h^4 + O(h^6) \Big]$$

$$= f'(x) + \frac{1}{3}\tilde{c}\left(\frac{4}{2^4} - 1\right)h^4 + O(h^6)$$

So... $N_1(h)$ is $O(h^2)$

$N_2(h)$ is $O(h^4)$

Conclusion

If we know α (but don't need to know c), we can make N_2 which converges to 0 (as $h \to 0$) faster than N_1

Note

If $N_1(h) = M + c\,h^2 + O(h^4)$

and $N_2(h) = M + \tilde{c}\,h^4 + O(h^6)$,

we can apply extrapolation to N_2 ! Call this N_3

$N_3(h) = M + \tilde{\tilde{c}}\,h^6 + O(h^8)$, and so-on

$O(h^2)$	$O(h^4)$	$O(h^6)$	$O(h^8)$
$N_1(h)$			
$N_1(h/2)$	$N_2(h)$		
$N_1(h/4)$	$N_2(h/2)$	$N_3(h)$	
$N_1(h/8)$	$N_2(h/4)$	$N_3(h/2)$	$N_4(h)$

shows dependencies

$$N_j(h) = N_{j-1}(h/2) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}$$

is $O(h^{2j})$

this is <u>only</u> for approximations that only have <u>even</u> powers of h

i.e., centered diff. approximations

If you have all powers, formula must be modified and chart is like

$$
\begin{array}{ccccc}
O(h^1) & O(h^2) & O(h^3) & O(h^4) & O(h^5) \; \cdots \\
\hline
N_1(h) & & & & \\
N_1(h/2) & N_2(h) & & & \\
 & \cdots & & &
\end{array}
$$

same dependence, different formula

General Formula

If $N_1(h) = M + c h^{\alpha_1} + O(h^{\alpha_2})$, $\alpha_2 > \alpha_1$

then

$$N_2(q h) := N_1(h) + \frac{N_1(h) - N_1(q h)}{q^{\alpha_1} - 1}$$

Satisfies

$$N_2(q h) = M + O(h^{\alpha_2})$$

better!

q is up to us but usually $q = 2$ is most convenient

Actually, for notation consistent with our book, we would write

$$N_2(h) := N_1(h/q) + \frac{N_1(h/q) - N_1(h)}{q^{\alpha_1} - 1}$$

Comparison to Aitken Acceleration

Aitken: $x_n \to 0$ as $n \to \infty$ linearly (or nearly so), i.e., $x_n = .9^n$

Richardson $N(h) \to 0$ as $h \to 0$ sublinearly, $N(h) = h^\alpha$

different forms

or, $n = 1/h$

$N(n) \to 0$ as $n \to \infty$