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Final Exam

1)
a) You can my code form me over Zoom since I could not connect to a kernel.

I got $X_2 = 6.910062$

for $|x_2 - 2\pi|$ I got $1.594619e^{-07}$

for $|x_3 - 2\pi|$ I got 0.0

Code:
```
def Newton(f, fprime, X0)
        X1 = X0 - f(X0)/fprime(X0)
        X2 = X1 - f(X1)/fprime(X1)
        X3 = X2 - f(X2)/fprime(X2)
        return X1, X2, X3
    f = np.sin(x)
    fprime = np.cos(x)
    X0 = 6

X1, X2, X3 = Newton(f, fprime, X0)
```

b) It would be better to use $g_+(x)$ since

Problem 2

a) $A f(x-h) + B f(x+2h) + C f(x+3h)$; $A = \frac{-5}{12}$ $B = \frac{2}{3}$

Taylor expansion:

$$\frac{A}{h}\left(f(x) - hf'(x) - \frac{h^2}{2} f''(x) + O(h^3)\right)$$

$$+\frac{B}{h}\left(f(x) + 2hf'(x) + \frac{6h^2}{2} f''(x) + O(h^3)\right)$$

$$+\frac{C}{h}\left(f(x) + 3hf'(x) + \frac{9h^2}{2} f''(x) + O(h^3)\right)$$

$$= h^{-1}\left((A+B+C) f(x) + (-A+2B+3C) h f'(x) + (+A+9B+9C)\frac{h^2}{2} f''(x) + O(h^3)\right)$$

$$\Rightarrow \left[\begin{array}{ccc|c} A & B & C & 0 \\ -A & 2B & 3C & 1 \\ -A & 9B & 9C & 0 \end{array}\right] = \left[\begin{array}{ccc|c} -5/12 & 2/3 & C & 0 \\ 5/12 & 4/3 & 3C & 1 \\ 5/12 & 6 & 9C & 0 \end{array}\right]$$

from (1)

$$\Rightarrow -5/12 + 2/3 = C \Rightarrow C = \frac{1}{4}$$

But $C = 1/4$ does not make row 2 true.

So there is no value of C for which the difference formula is an $O(h^2)$ approx.

2 b)

lets interpolate.

Assume w/o loss of generality that $x_0 = 0$

so our nodes are $\{x_0 = 0,\ x_1 = 2h\}$

$$p(x) = f(0)\,l_0(x) + f(2h)\,l_1(x)$$

$$l_0 = \frac{(x-2h)}{(0-2h)} = \frac{x-2h}{-2h} = 1 - \frac{x}{2h}$$

$$l_1 = \frac{(x-0)}{(2h-0)} = \frac{x}{2h}$$

Now: $$\int_0^{2h} l_0(x)\,dx = \int_0^{2h} 1 - \frac{1}{2h}x\,dx = \frac{1}{2h}\left[x - \frac{x^2}{2}\right]_0^{2h} = \frac{1}{2h}\left[2h - 2h^2\right]$$
$$= \frac{1}{h}\left[h - h^2\right]$$
$$= 1 - h$$

$$\int_0^{2h} l_1(x)\,dx = \frac{1}{2h}\int_0^{2h} x\,dx = \frac{1}{2h}\left[\frac{x^2}{2}\right]_0^{2h} = \frac{1}{2h}\left[2h^2\right] = h$$

finaly: $$\int_0^{2h} p(x)\,dx = f(0)\int_0^{2h} l_0(x) + f(2h)\int_0^{2h} l_1(x)$$

$$\boxed{= f(0)(1-h) + f(2h)\,h}$$

Problem 3

I think I would use a RK-4 process.

Firstly I would choose a 4^{th} order RK over other higher order methods because they require more smoothness in the solution, Y, and they are much more complicated. RK-4 processs also have the advantage of having local truncation error of $O(h^4)$ and because its a one-step problem I can work w/ adapative stepsizes and they are usually stable (under mild conditions).

Problem 4

a) From the problem we can see that we have a system of equations equivalent to:

(1) $1x_1 = 1$

(2) $\frac{1}{2}x_1 + 1x_2 = 1$

(3) $\frac{1}{4}x_1 + \frac{1}{2}x_2 + 1x_3 = 1$

(4) $\frac{1}{8}x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 + 1x_4 = 1$

$\vdots$

from this we can see a pattern:

$x_1 = 1$ (from (1))

$x_2 = \frac{1}{2}$ (from (2))

$x_3 = \frac{1}{2}$ (from (3))

$x_4 = \frac{1}{2}$ (from (4))

So we can say that $x_n = \frac{1}{2}$ is the n^{th} component of $\vec{x}$ that satisfies $A\vec{x} = \vec{b}$

4b) Recal that swapping rows (pivoting) means permuting the rows. So we would take a permutation matrix, P to get the following: $PA = LU$

So when solving a several $A\vec{x} = \vec{b}$ we get:

$$PA\vec{x} = P\vec{b} := \tilde{b}$$

$$\Rightarrow LU\vec{x} = \tilde{b}$$

Then we solve like a normal LU factorization.

i.e solve $L\vec{y} = \tilde{b}$ via forward substitution

then solve $U\vec{x} = \vec{y}$ via backward substitution.

this is much more efficient than using $\vec{x} = A^{-1}\vec{b}$.

Further pivoted LU factorization is more accurate than this as well.