

Stiff Equations and Absolute Stability

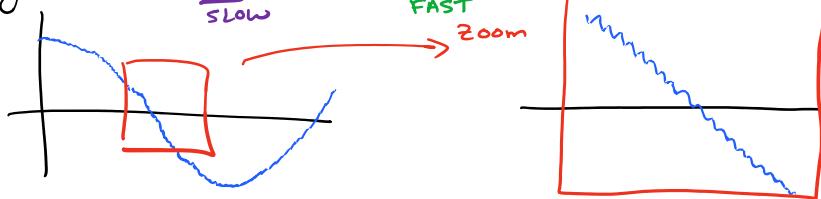
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Ch. 5.11 is "Stiff Differential Eq'n" but it's really also about **stability** of a new kind

Stiff Equations

There's not a precise definition of a **stiff** equation, and it's really a **scale**, just like how ill-conditioning is a scale. The basic phenomenon is when our solution $y(t)$ (to the ODE) has very fast and very slow timescales.

Ex. $y(t) = \cos(\underline{t}) - \cos(\underline{10^4 t})$



① Slow timescales

These are not problematic by themselves, but the issue is that it means we usually want a large T

② Fast timescales

Ex. $\cos(10^4 t)$ or $e^{-100 t}$

$$\begin{array}{|c|} \hline \text{IVP: } & y' = f(t, y) \\ & y(0) = y_0 \\ & 0 \leq t \leq T \\ \hline \end{array}$$

If we took a large stepsize h , we'd jump over these features and not resolve them at all!

The issue: large T , but due to **fast behaviour**, need small h

$$\Rightarrow n = \frac{T}{h} + 1 \quad (\# \text{ steps in our method}) \text{ is going to be huge!}$$

So very slow to solve
(Also, small h not good for roundoff)

Ex Simulating Earth's climate.

One reason it's hard are the different timescales

- Temperature might change 3°C in an hour, but we need to simulate for 100,000 years!

The name **stiff** comes from mass-spring systems

Ex Linear ODEs: $y'' + 150y' + 5000y = \cos(t)$

represents a driven mass-spring ($m = 1 \text{ kg}$)

$b = 150$ is damping

$k = 5000$ is spring constant

1) homogeneous ODE $y'' + 150y' + 5000y = 0$

has solutions of the form $e^{\lambda t}$. Plug this in to get

$$\lambda^2 + 150\lambda + 5000 = 0$$

$$(\lambda + 50)(\lambda + 100) = 0 \Rightarrow \lambda \in \{-50, -100\}$$

$$y_{\text{homogeneous}}(t) = c_1 e^{-50t} + c_2 e^{-100t}$$

"transient"
"fast" timescale solution

2) particular solution

$$y_p = c_3 \cos(t) + c_4 \sin(t)$$

"steady state" solution
"slow" timescale

$$3) y = \underbrace{c_1 e^{-50t} + c_2 e^{-100t}}_{\text{fast}} + \underbrace{c_3 \cos(t) + c_4 \sin(t)}_{\text{slow}}$$

What to do about stiffness?

Avoid making h extremely small (and give up on capturing all the fast dynamics) if we have to

But some methods can't handle a large value of h ...

Absolute Stability

Previously, "stability" meant the root condition, equivalent (under consistency) to " O -stability"

i.e., $0 \leq t \leq T$, take $h \rightarrow 0$ --- hence the name "O-stability"

Now, focus on absolute stability

Fix $h > 0$ (do not take $h \rightarrow 0$), solve on interval $0 \leq t \leq T$

take $T \rightarrow \infty$.

To define this notion, we need to introduce:

Def (Dahlquist's) Test Equation

$$y' = \lambda y, \quad y(0) = 1$$

our book allows $y(0) = \alpha$ but you can take $\alpha = 1$ wlog

Our book assumes $\lambda < 0$ (not all books do... it doesn't affect the math, but the interpretation makes the most sense if $\lambda < 0$. But $\lambda \in \mathbb{C}$ can make sense too)

True solution is $y(t) = e^{\lambda t}$.

Why? Think of a 1st order linear ODE $\vec{y}' = A \cdot \vec{y}$ for a matrix A

Then if A is diagonalizable, after a change of basis, each component of \vec{y} , y_i , will satisfy $y_i' = \lambda_i y_i$, where λ_i is an eigenvalue of A .

If A has eigenvalues of very disparate magnitudes, it's a stiff problem.

Not all stiff ODEs are linear, but often linear approximations give us good rules-of-thumb

Let's solve the test equation using Euler's method w/ stepsize h

$$\begin{aligned} w_0 &= 1 \\ w_1 &= w_0 + h f(t_0, w_0) \\ &= w_0 + h \lambda w_0 \\ &= (1 + h \lambda) w_0 \end{aligned}$$

Similarly,

$$w_2 = (1 + h \lambda) w_1 = (1 + h \lambda)^2 w_0$$

so generally

$$w_n = (1 + h \lambda)^n w_0, \text{ which approximates } y_n = y(t_n) = e^{n h \lambda}$$

does this $\rightarrow 0$ as $n \rightarrow \infty$?

$$\begin{aligned} y(t) &= e^{\lambda t} \\ t_n &= n \cdot h \\ \rightarrow 0 &\text{ if } \lambda < 0 \\ \text{as } n \rightarrow \infty & \end{aligned}$$

$$\lim_{n \rightarrow \infty} a^n = \begin{cases} 1 & \text{if } a = 1 \\ \text{DNE} & \text{if } |a| \geq 1 \text{ and } a \neq 1 \\ 0 & \text{if } |a| < 1 \end{cases}$$

So want $|1 + h \lambda| < 1$, i.e., $h \lambda$ is in a ball centered at -1 , of radius 1

Repeat using backward (implicit) Euler

$$w_0 = 1 \quad f(t, y) = \lambda y \text{ as before}$$

$$\begin{aligned} w_1 &= w_0 + h f(t_1, w_1) \\ &= w_0 + h \lambda w_1 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{Solve implicit eq'n,} \quad (1 - h \lambda) w_1 = w_0$$

$$w_1 = \frac{1}{1 - h \lambda} w_0$$

$$\cdots w_n = \frac{1}{(1 - h \lambda)^n} w_0. \quad \text{Does } w_n \rightarrow 0 \text{ as } n \rightarrow \infty ?$$

i.e., is $\left| \frac{1}{1 - h \lambda} \right| < 1$? If $h \lambda$ is negative
(makes sense... $\lambda < 0, h > 0$)

$$\begin{aligned} \text{then } 1 - h \lambda &> 1 \\ \Rightarrow \frac{1}{1 - h \lambda} &< 1 \quad \checkmark \end{aligned}$$

What we've just done is find a
region of stability: values of $h\lambda$ for which $\lim_{n \rightarrow \infty} w_n = 0$
 $(\text{not } h \rightarrow 0)$

Def (part 1) For a one-step method $w_{i+1} = w_i + h\phi(t_i, w_i, h)$,
apply it to $f(t, y) = \lambda y$ and rewrite as $w_{i+1} = Q^{1\text{-step}}(h\lambda)w_i$.

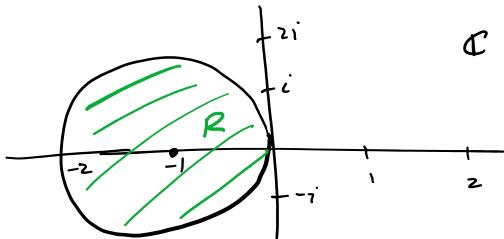
Then the region R of absolute stability is

$$R = \left\{ h\lambda \in \mathbb{C} : |Q^{1\text{-step}}(h\lambda)| < 1 \right\}.$$

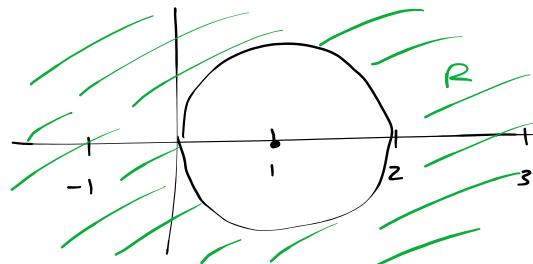
(why allow \mathbb{C} ? $y(t) = e^{\lambda t}$
if λ is imaginary, this is a sin or cos
Euler's id: $e^{ix} = \cos(x) + i \cdot \sin(x)$)

So we've derived

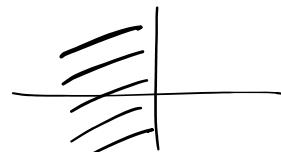
$$\text{Forward Euler: } Q^{1\text{-step}}(h\lambda) = 1 + h\lambda$$



$$\text{Backward Euler: } Q^{1\text{-step}}(h\lambda) = (1 - h\lambda)^{-1}$$



We call backward Euler
A-Stable
because R contains the
entire left-half plane



Multistep methods

Same principle but shortcut to find region R of abs. stability.

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m} \quad (\text{generic multi-step method})$$

$$+ h \cdot (b_m f_{i+1} + b_{m-1} f_i + \dots + b_0 f_{i+1-m})$$

$$f_i = f(t_i, w_i). \quad \text{Apply to our test eqn. } y' = \lambda y$$

$$\begin{aligned} \text{so } f_i &= f(t_i, w_i) \\ &= \lambda w_i \end{aligned}$$

to get

$$w_{i+1} = a_{m-1} w_i + a_{m-2} w_{i-1} + \dots + a_0 w_{i+1-m}$$

$$+ h (b_m \lambda w_{i+1} + b_{m-1} \lambda w_i + \dots + b_0 \lambda w_{i+1-m})$$

so rewrite as

$$(1 - b_m h \lambda) w_{i+1} - (a_{m-1} + h \lambda b_{m-1}) w_i - \dots - (a_0 + h \lambda b_0) w_{i+1-m} = 0$$

This is a linear, homogeneous difference equation like we saw in ch 5.10

To solve it, guess $w_n = z^n$ for some $z \in \mathbb{C}$

$\Rightarrow z$ solves the polynomial equation

$$(1 - b_m h \lambda) z^m - (a_{m-1} + h \lambda b_{m-1}) z^{m-1} - \dots - (a_0 + h \lambda b_0) \cdot z^0 = 0$$

⚠ Confusingly, the book calls this $Q^{(h\lambda)}$ the "characteristic polynomial" $\rightarrow Q^{(h\lambda)}(z)$

but it's not the same "characteristic polynomial" as in ch 5.10
 and it turns out that if all roots β_j of $Q^{(h\lambda)}$ are simple,* then all sol'n to the difference equation can be written as $w_n = \sum_{k=1}^m c_k \beta_k^n$
 denotes that coefficients depend on $h\lambda$

So if we want to guarantee $\lim_{n \rightarrow \infty} w_n = 0$, we want all $|\beta_k| < 1$

* (if roots aren't simple it's more complicated, but end up with λ also)

Def (part 2) The region R of absolute stability

for a multi-step method is

$$R = \{ h \lambda \in \mathbb{C} : \text{all roots } \beta \text{ of } Q^{(h\lambda)} \text{ satisfy } |\beta| < 1 \}$$

Punchline

Stiff eqn often have a component like $e^{\lambda t}$ for $\lambda < 0$

The effect of this part in w_n should decay, so we want to pick h such that $h\lambda \in R$

In general, we prefer methods with a large R

-- especially A-stable methods
these are "unconditionally" stable: any $h>0$ is stable

Fact: implicit methods typically have larger regions of stability than comparable explicit methods

-- so for stiff problems, we usually prefer implicit methods despite their extra hassle.

Δ Finding roots of $Q^{(h\lambda)}$: λ is a parameter now, not a variable
(the variable is z)

Examples

① Implicit Trapezoidal ($= AM2$)

$$w_{i+1} = w_i + \frac{h}{2} (f_{i+1} + f_i)$$

$$\text{So } Q^{(h\lambda)} = (1 - \frac{1}{2} h\lambda) \cdot z - (1 + \frac{1}{2} h\lambda)$$

$$\Rightarrow \text{only one root, } \beta = \frac{1 + \frac{1}{2} h\lambda}{1 - \frac{1}{2} h\lambda} = \frac{2 + h\lambda}{2 - h\lambda}$$

$$\text{So } h\lambda \in R \text{ when } \left| \frac{2 + h\lambda}{2 - h\lambda} \right| < 1$$

$$\text{i.e., } |z + h\lambda| < |z - h\lambda| \quad \text{write as } a + i.b \in \mathbb{C}$$

$$\text{i.e., } \underbrace{|(z+a) + ib|^2}_{(z+a)^2 + b^2} < \underbrace{|(z-a) - ib|^2}_{(z-a)^2 + b^2}$$

$$(z+a)^2 + b^2 < (z-a)^2 + b^2 \quad b \text{ plays no role}$$

$$z^2 + 4a + a^2 < z^2 - 4a + a^2$$

$$\begin{aligned} a < -a \\ 2a < 0 \end{aligned} \quad \Rightarrow \quad a < 0$$

$$\text{So } R = \{ h\lambda \in \mathbb{C} : \operatorname{Re}(h\lambda) < 0 \} = \text{left-half plane}$$

\Rightarrow AM2/trapezoidal is A-stable

So, Backward Euler and AM2/trapezoidal are A-stable

... it turns out there are no other A-stable multistep methods

("2nd Dahlquist Barrier": All A-stable linear multistep methods must be implicit and be $O(h^2)$ or less)

More examples at the .ipynb demos
since involve a lot of plotting

Chapter Summary

- We always want consistent methods

- We like high-order local truncation error
- Explicit methods are simpler to work with, faster per iteration
- Want O-stability / root condition
- Want large region of absolute stability especially for **stiff** problems

These are often at odds with each other:

Higher order usually means smaller region of absolute stability
Explicit usually means smaller region of absolute stability

- Multistage (RK) have good O-stability, and easy to work w/
adaptive stepsizes \Rightarrow ode45 in Matlab is often your 1st choice

- Implicit multistep methods have good stability
(and predictor-corrector helps w/ implicitness)
 \Rightarrow after your 1st choice for stiff eq'n
ode113, ode15s, ode23s