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Midterm 1
Math 4650

1) $X_{n+1} = r X_n (1 - X_n)$

$g(x) = r x (1-x)$

$g'(x) = r - 2rx$

Because $0 \leq r < 1$ the max value of $|g'(x)|$ on $x \in [0,1]$ is less than 1 because r never actually takes the value of 1. Thus $|g'(x)| < 1$, so $g(x)$ is a contraction on $x \in [0,1]$.

We also know $\forall x \in [0,1]$, $g(x) \in [0,1]$.

So by the contraction mapping theorem there is a unique fixed point of $g(x)$ in $[0,1]$; call it p.

Further we know $|g'(p)| < 1$ by the same logic of the contraction, so we know the iteration will converge.

2) $\cos(x)-1 = f(x)$

a) By Taylor Remainder theorm;

$$\cos(x)-1 = ((\cos(0)-1) - x\sin(0) - \frac{x^2\cos(0)}{2} + \dots + \frac{\cos^{(n+1)}(\xi)x^{n+1}}{(n+1)!}$$

for some ξ.

So our error is $\left|\frac{x^{n+1}}{(n+1)!}\right| \leq \frac{1}{(n+1)!}$ for $x \in [-1, 1]$

if we want to find n such that our error is less than $2e^{-16}$

we have $2e^{-16} > \frac{1}{(n+1)!} \geq \frac{x^{n+1}}{(n+1)!}$

if $n=10$ this holds, so we know a taylor polynomial of degree 10 of $f(x)$ will be accurate to at least $2e^{-16}$.

$$T_{10}(x) = \frac{-x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$$

$$T_{10}(2e^{-8}) = -2.000\,e^{-16}$$

Thus the answer provided by the computer is incorrect.

2) b) $K_f^{rel} = \left| \frac{x}{f(x)} f'(x) \right|$

$$\lim_{x \to 0} K_f^{rel} = \lim_{x \to 0} \left| \frac{-x \sin(x)}{\cos(x)-1} \right| = \lim_{x \to 0} \left| \frac{x \sin(x)}{\cos(x)-1} \right|$$

$$\overset{\text{l'Hopital}}{=} \lim_{x \to 0} \left| \frac{-x\cos(x) - \sin(x)}{\sin(x)} \right|$$

$$= \lim_{x \to 0} \left| \frac{-x\cos(x)}{\sin(x)} - 1 \right|$$

$$= \left| \lim_{x \to 0} (-1) - \lim_{x \to 0} (\cos(x)) \lim_{x \to 0} \frac{x}{\sin x} \right|$$

$$= \left| -1 - 1 \cdot \lim_{x \to 0} \frac{x}{\sin x} \right|$$

$$\overset{\text{l'Hopital}}{=} \left| -1 - 1 \cdot \lim_{x \to 0} \frac{1}{\cos(x)} \right|$$

$$= |-1-1(1)|$$

$$= |-2|$$

$$= 2$$

2 c) for $x \approx 0$ we could say that $f(x)$ is modertly ill conditioned, so small changes in the input correspond to modertly large changes in the output. The only way to compute the answer accurately in floating point would be to approximate $f(x)$.

d) I think a better way to calculate $f(x)$ would be to use $\cos(x)$'s taylor approximation to callulate $f(x)$ when $x \approx 0$.

3) a) Newtons Method: $P_n = P_{n-1} - \dfrac{f(P_{n-1})}{f'(P_{n-1})}$

or

$$P_{n+1} = P_n - \frac{f(P_n)}{f'(P_n)}$$

$f'(x) = 2x$, let $P_1 = 1$

so $P_2 = 1 - \dfrac{(1)^2 - 2}{2(1)}$

$= 1 + \frac{1}{2}$

$= 1.5$

3 b) let $a_1 = 1$, $b_1 = 2$

1^{st}: $P_1 = \frac{1+2}{2} = 3/2$

$f(P_1) = 1/4$; $f(P_1) \cdot f(a) < 0$ so:

$a_2 = a_1 = 1 \qquad b_2 = P_1 = 3/2$

2nd: $P_2 = \frac{1 + 3/2}{2} = 5/4$

$f(P_2) = -7/16$; $f(P_2) \cdot f(b) < 0$ so:

$a_3 = P_2 = 5/4 \qquad b_3 = b_2 = 3/2$

3^{rd}: $P_3 = \frac{5/4 + 3/2}{2} = 11/8$

$f(P_3) = \frac{-7}{64}$

3 c) error $= \frac{a+b}{2^n}$

$$2^n = \frac{a+b}{\text{error}}$$

$$n = \frac{\log\left(\frac{a+b}{\text{error}}\right)}{\log(2)} = \frac{\log(a+b) - \log(\text{error})}{\log(2)}$$

So if we have error $= 1^{-6}$, $a = 1$ $b = 2$

we want

$$n \geq \frac{\log(3) - \log(2^{-20})}{\log(2)}$$

$n \geq 21.5849$

So we would need at least 22 iterations.

3d) lets see if $\sqrt{2}$ is a simple root:

$$h'(x) = 4x^3 - 8x, \quad h'(\sqrt{2}) = 4(\sqrt{2})^3 - 8(\sqrt{2}) \neq 0$$

So $\sqrt{2}$ is a simple root of $h(x)$, so if Newtons method is initialized sufficently close to $\sqrt{2}$, then it will converge to it. Same as $f(x)$, at a quadric rate.

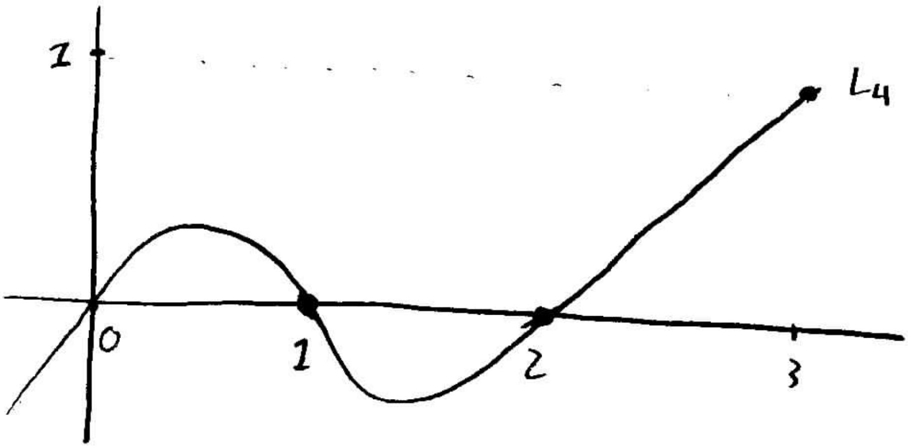
3 e) $K^{abs}(r) = |f'(r)|^{-1}$

$$K^{abs}(\sqrt{2}) = |2(\sqrt{2})|^{-1}$$

$$= \frac{1}{2\sqrt{2}} \approx 0.35355$$

4) $x_1 = 0,\ x_2 = 1,\ x_3 = 2,\ x_4 = 3$

a) $L_4(x) = \dfrac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$



4 b) We know P(x) will have a min degree of 3 because we have 4 nodes.

Lets use the Barycentric Interpolation method to find P(x):

$$L_1(x) = (x-1)(x-2)(x-3)\frac{1}{(0-1)(0-2)(0-3)} = (x-1)(x-2)(x-3)\left(-\frac{1}{6}\right)$$

$$L_2(x) = (x-0)(x-2)(x-3)\frac{1}{(1-0)(1-2)(1-3)} = (x-0)(x-2)(x-3)\left(\frac{1}{2}\right)$$

$$L_3(x) = (x-0)(x-1)(x-3)\frac{1}{(2-0)(2-1)(2-3)} = (x-0)(x-1)(x-3)\left(-\frac{1}{2}\right)$$

$$L_4(x) = (x-0)(x-1)(x-2)\frac{1}{(3-0)(3-1)(3-2)} = (x-0)(x-1)(x-2)\left(\frac{1}{6}\right)$$

$$p(x) = 1L_1 + 2L_2 + 3L_3 - 4L_4$$

$$= \left(-\frac{1}{6}\right)(x-1)(x-2)(x-3) + (x-0)(x-2)(x-3) - \left(\frac{3}{2}\right)(x-0)(x-1)(x-3) - \left(\frac{2}{3}\right)(x-0)(x-1)(x-2)$$

$p(1/2) = 1$, see code bellow.

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In [2]: p = lambda x: (-1/6)*(x-1)*(x-2)*(x-3) + (x-0)*(x-2)*(x-3) - (3/2)*(x-0)*(x-1)*(x-3) - (2/3)*(x-0)*(x-1)*(x-2)
        p( 1/2 )

Out[2]: 1.0
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4 c)

in order to solve this we need $\vec{c} = V^{-1}\vec{y}$

but generally we dont invert the Vandermonde matrix

because it is ill-conditioned.