

Probing QCD with Energy Flow Observables

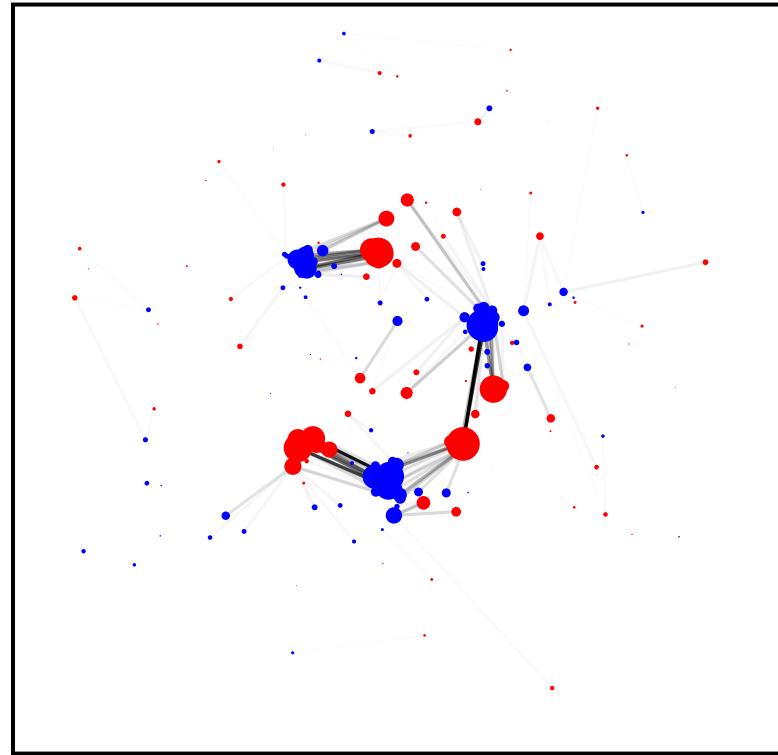
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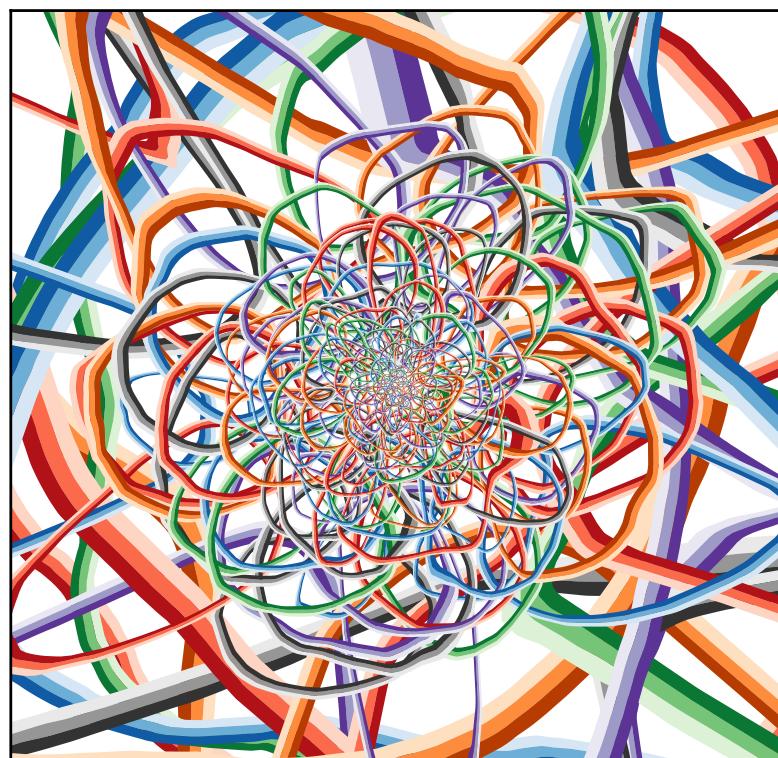
[1712.07124](#) [1810.05165](#) [1911.04491](#)

CEPC Workshop

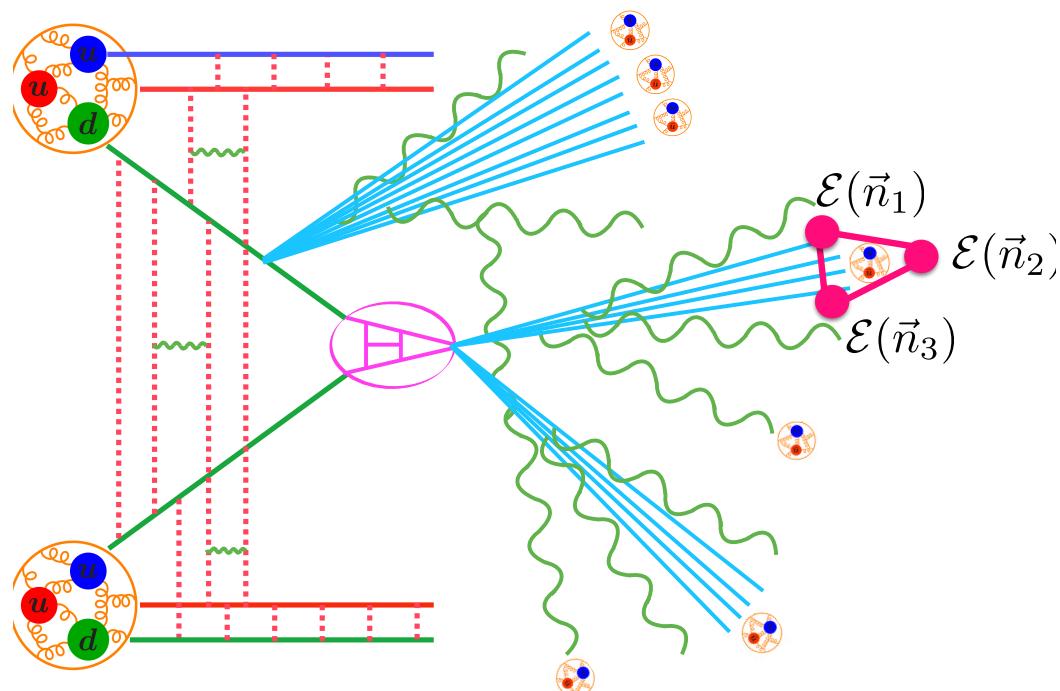
October 27, 2020



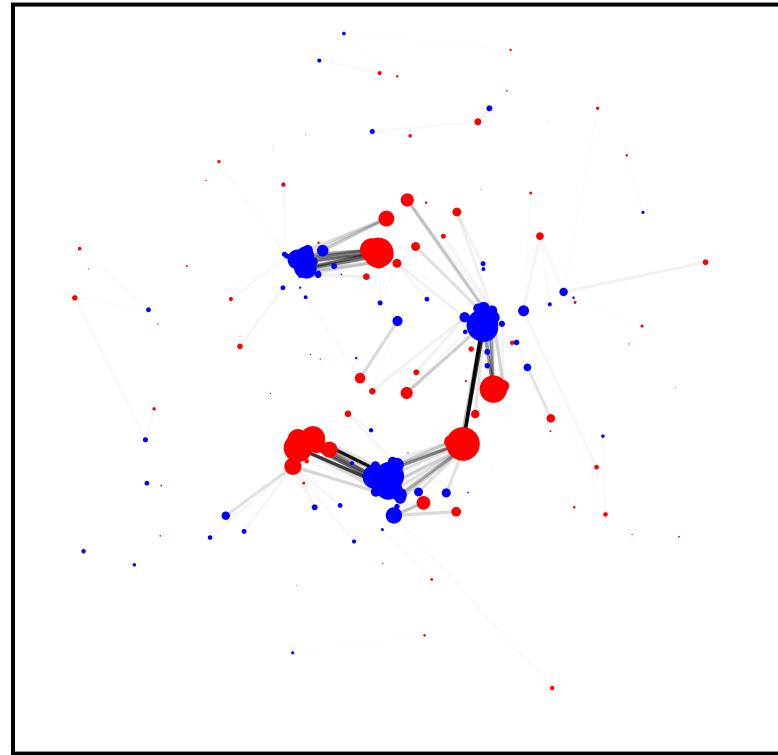
Collider Event Fundamentals



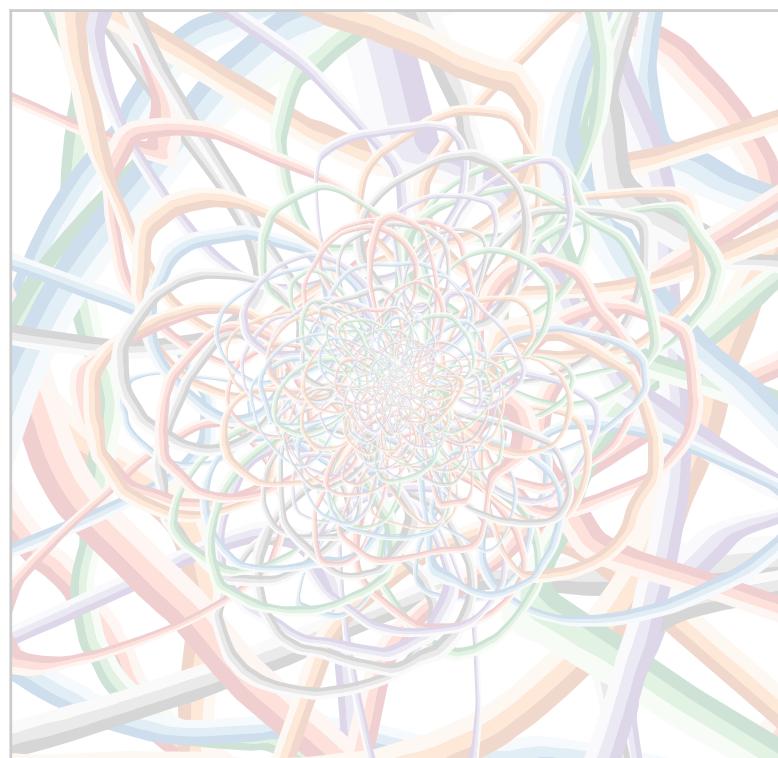
Energy Flow Observables



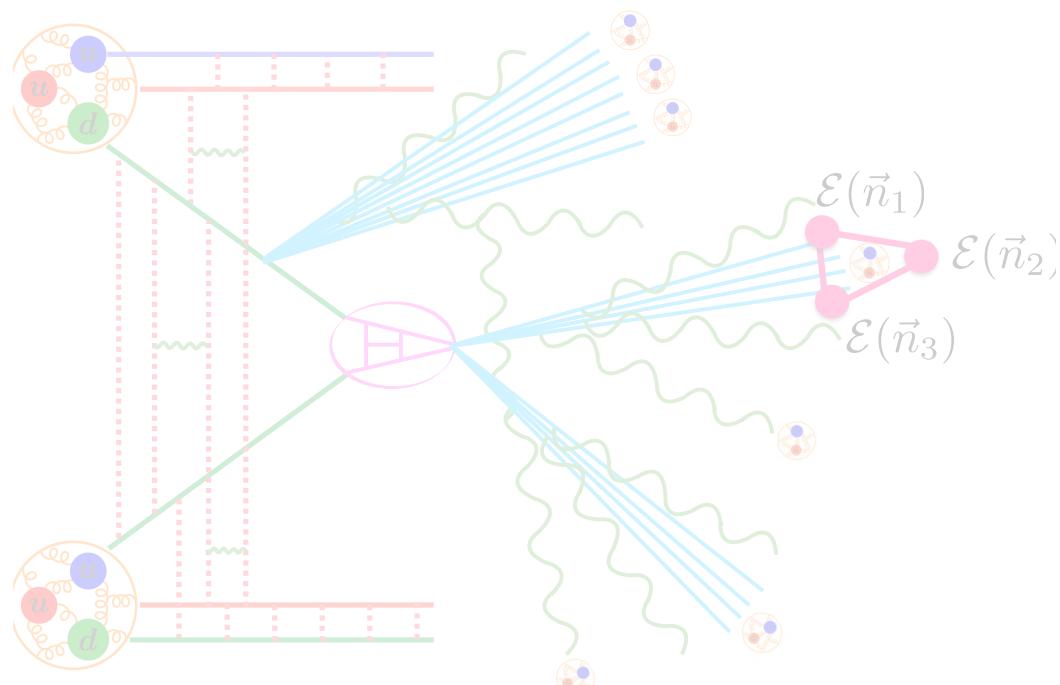
Energy-Energy Correlators



Collider Event Fundamentals



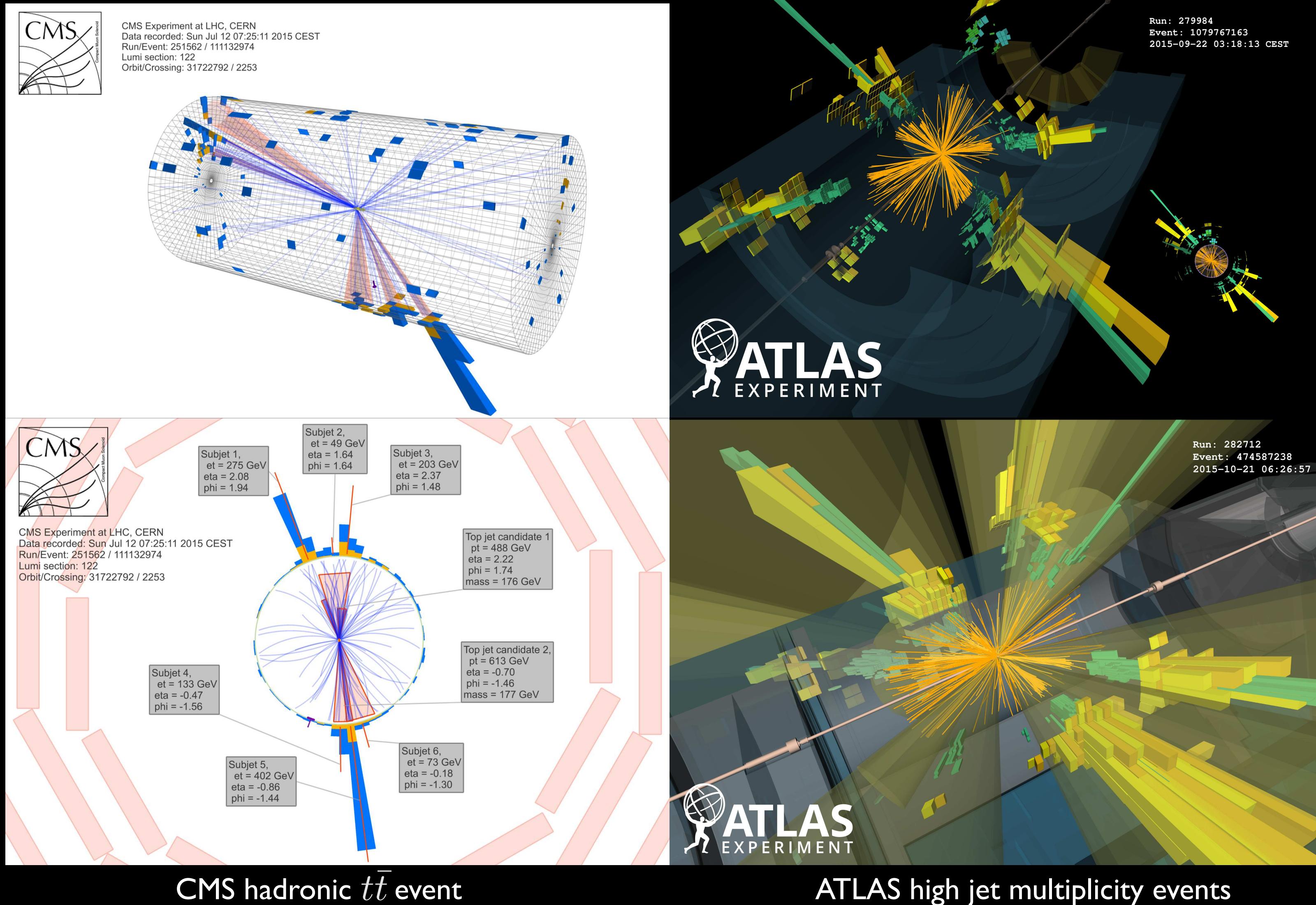
Energy Flow Observables



Energy-Energy Correlators

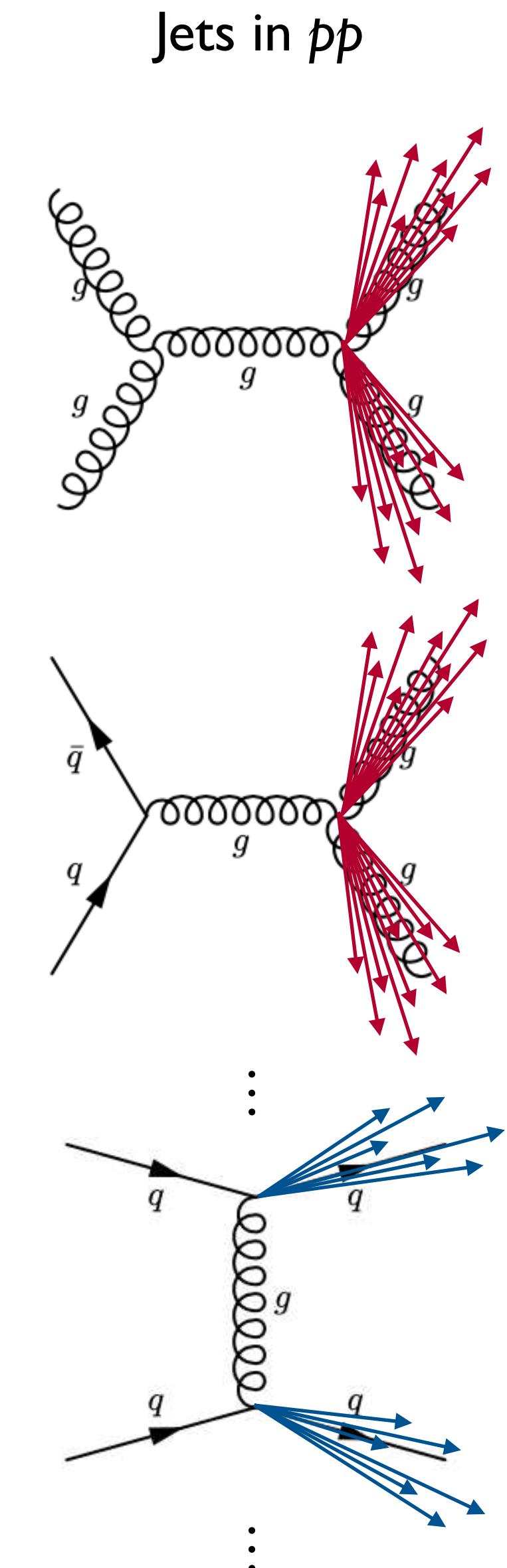
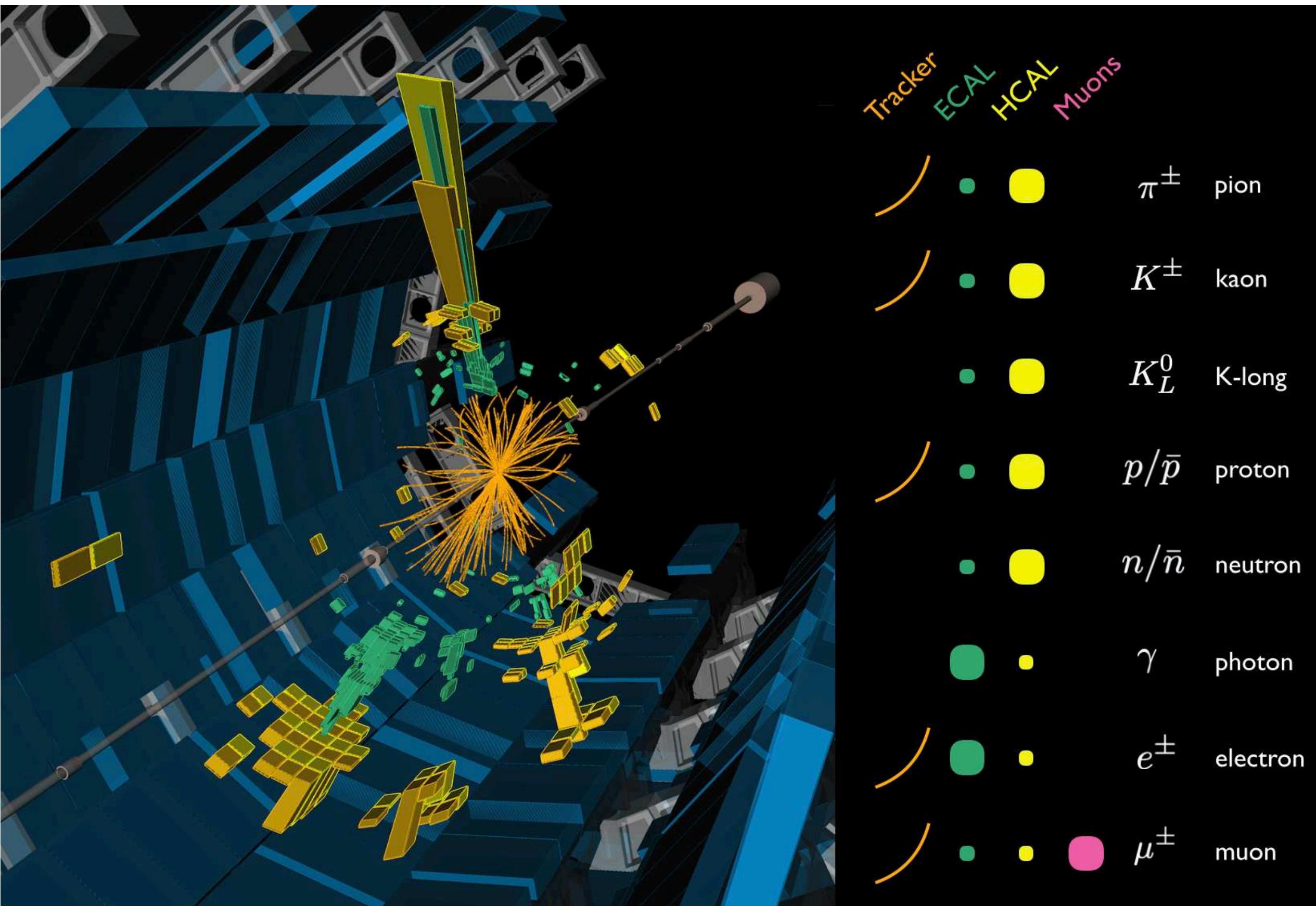
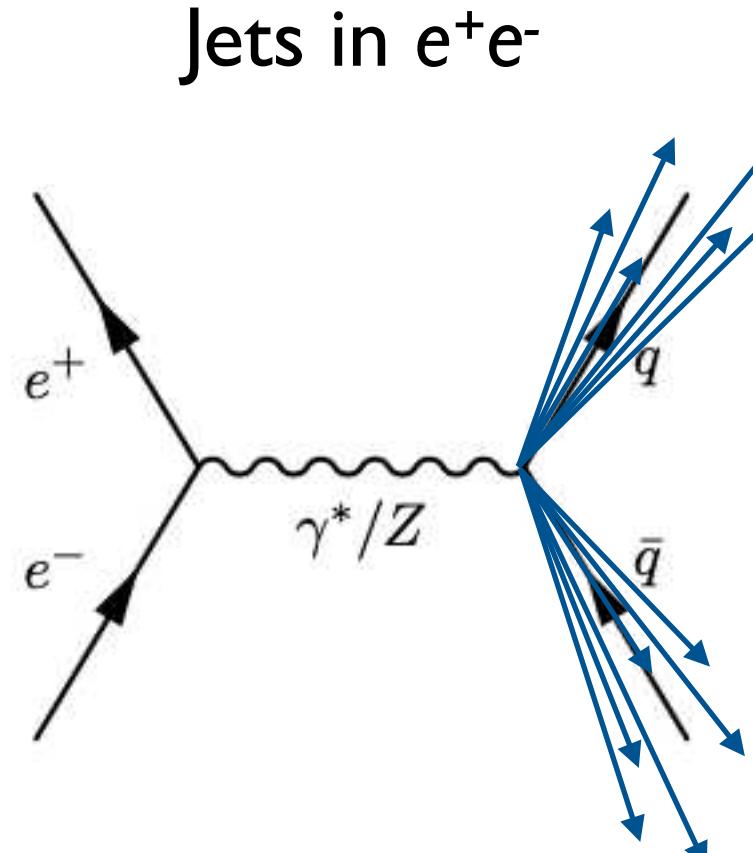
Hadronic Events at the Large Hadron Collider

Jets (*collimated sprays of color-neutral particles*) are ubiquitous at high-energy colliders



Hadronic Events as a Particle Point Cloud

High-energy collisions produce final state particles with
energy, direction, charge, flavor, and other quantum numbers



Jet Formation in Theory

Hard collision

Good understanding via perturbation theory

Fragmentation

Semi-classical parton shower, effective field theory

Hadronization

Poorly understood (non-perturbative), modeled empirically

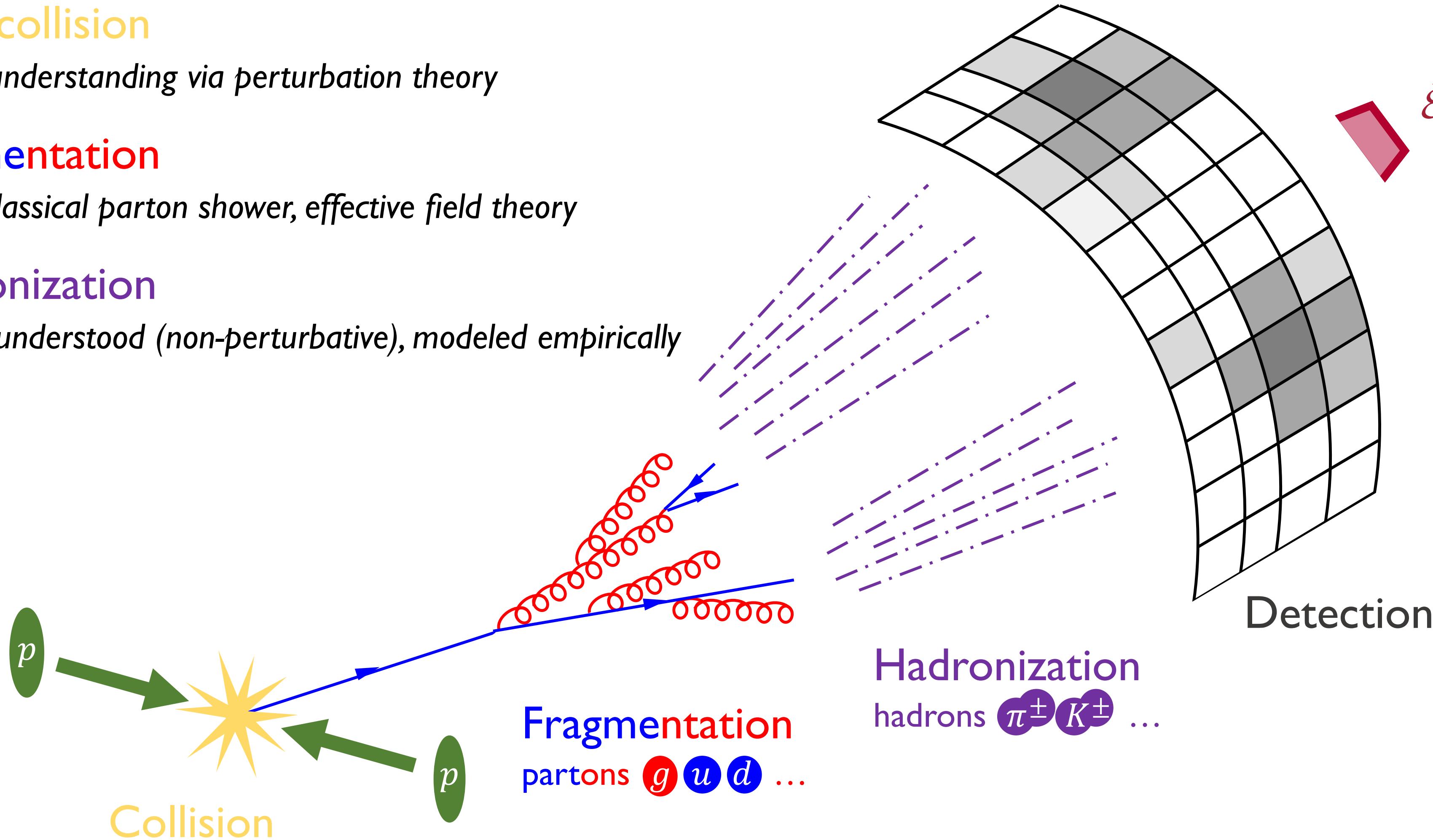


Diagram by Eric Metodiev

[Sveshnikov, Tkachov, [PLB 1996](#); Hofman, Maldacena, [JHEP 2008](#); Mateu, Stewart, Thaler, [PRD 2013](#); Belitsky, Hohenegger, Korchemsky, Sokatchev, Zhiboedov, [PRL 2014](#); Chen, Moult, Zhang, Zhu, [2004.11381](#); PTK, Moult, Thaler, Zhu, to appear soon]

$$\hat{\mathcal{E}} \simeq \lim_{t \rightarrow \infty} \hat{n}_i T^{0i}(t, vt\hat{n})$$

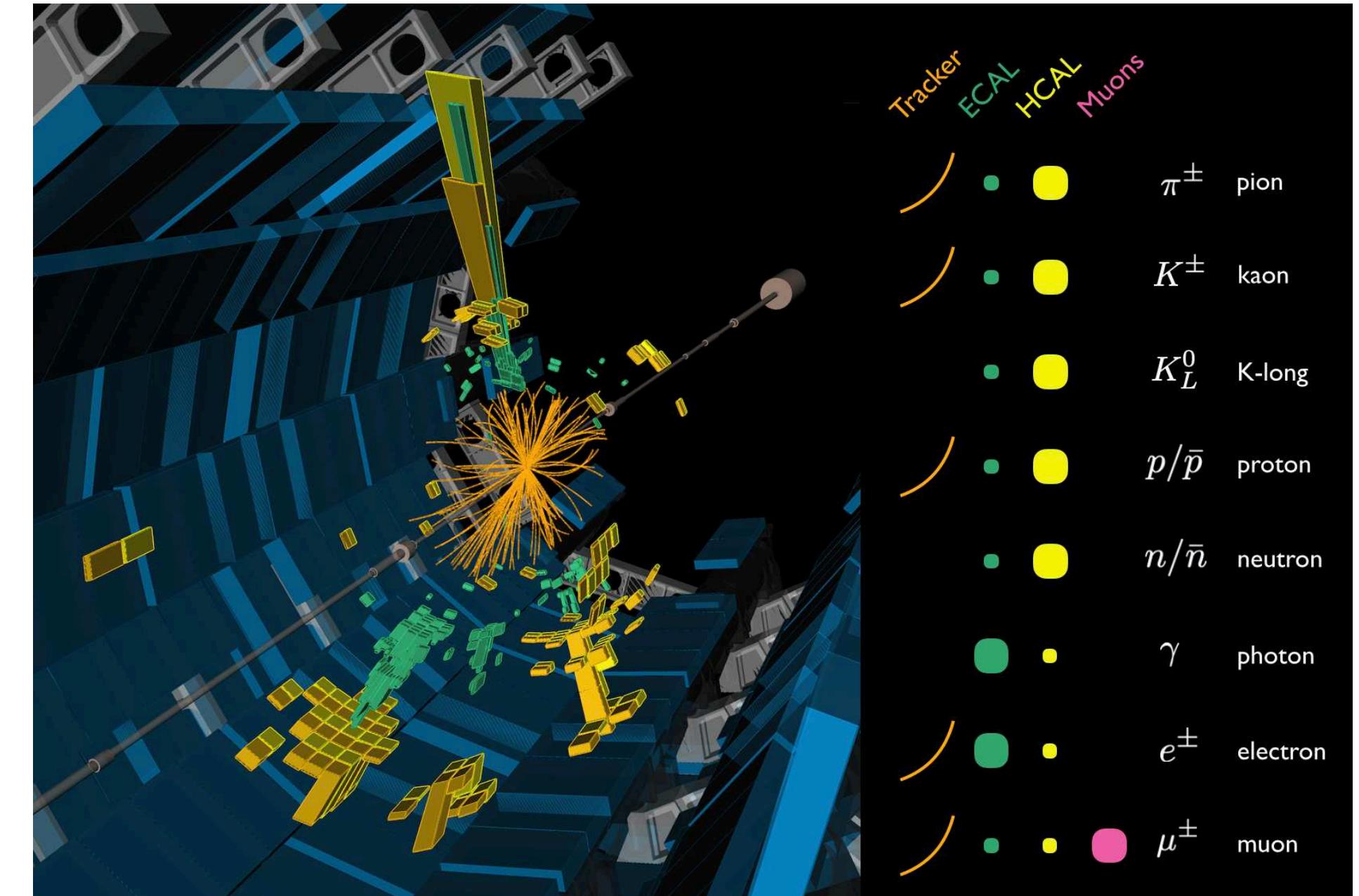
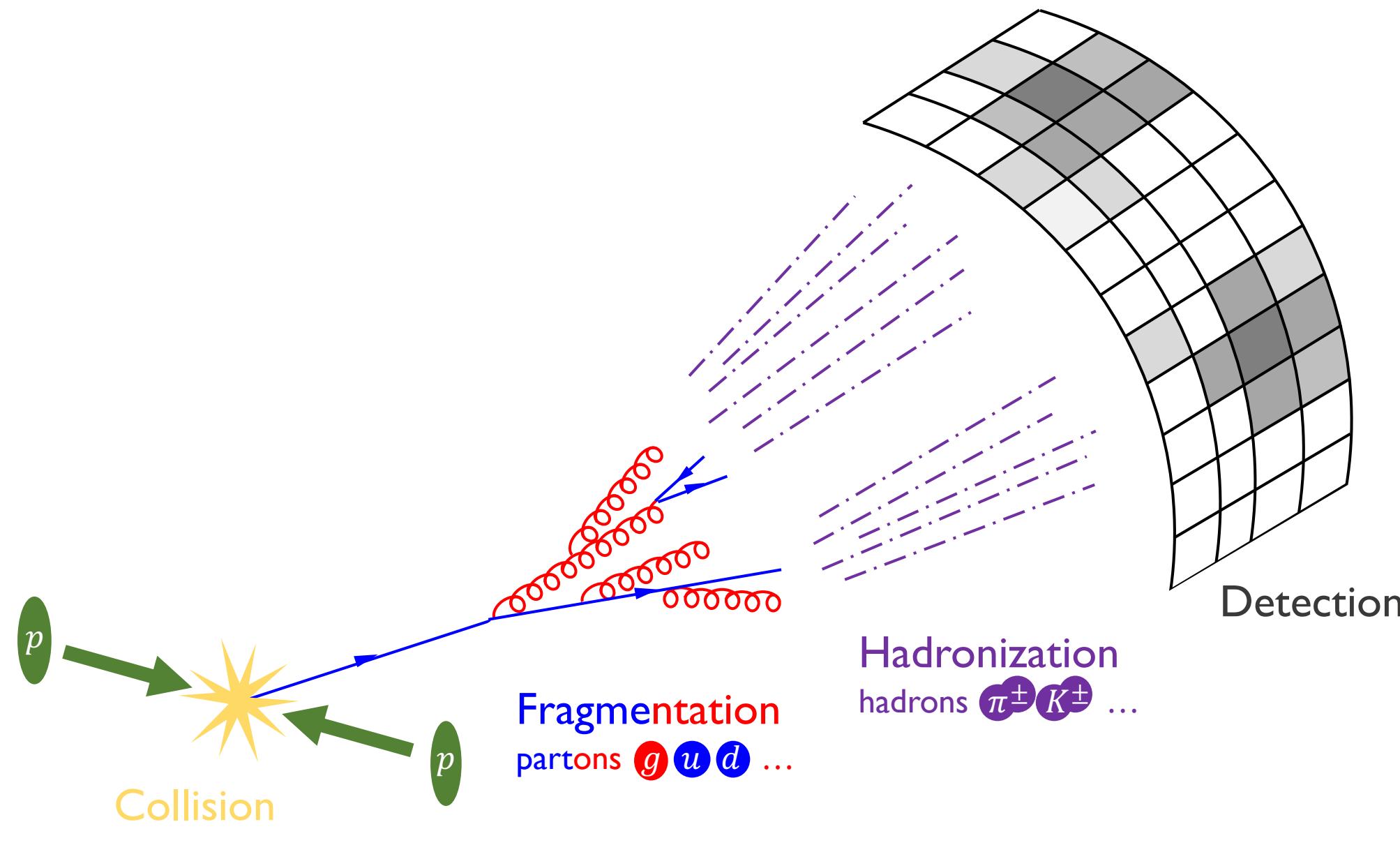
Stress-energy flow

Robust to non-perturbative and detector effects

Well-defined for massless gauge theories

Correlation functions calculated in $N=4$ SYM and QCD

Events in Theory vs. Experiment

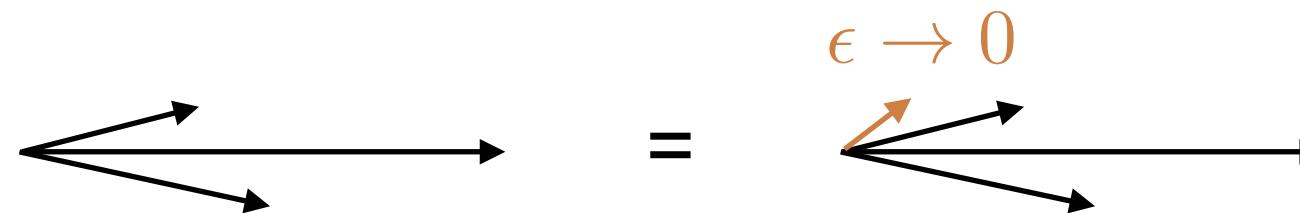


What information is both theoretically and experimentally robust?

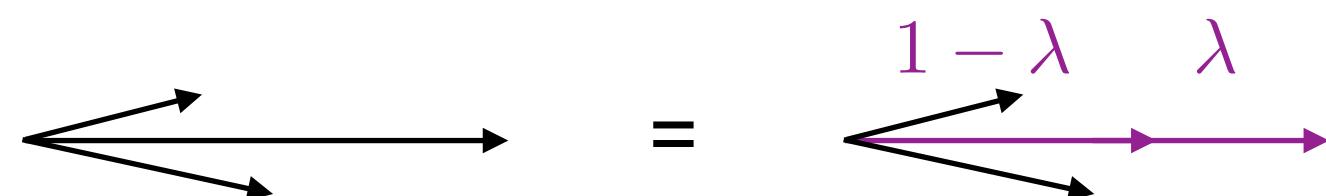
Theoretically and Experimentally Robust Information

Infrared and Collinear Safe Information

Infrared (IR) safety – observable is unchanged under addition of a soft particle



Collinear (C) safety – observable is unchanged under a collinear splitting of a particle



Theoretically

QCD has soft and collinear divergences associated with gluon radiation



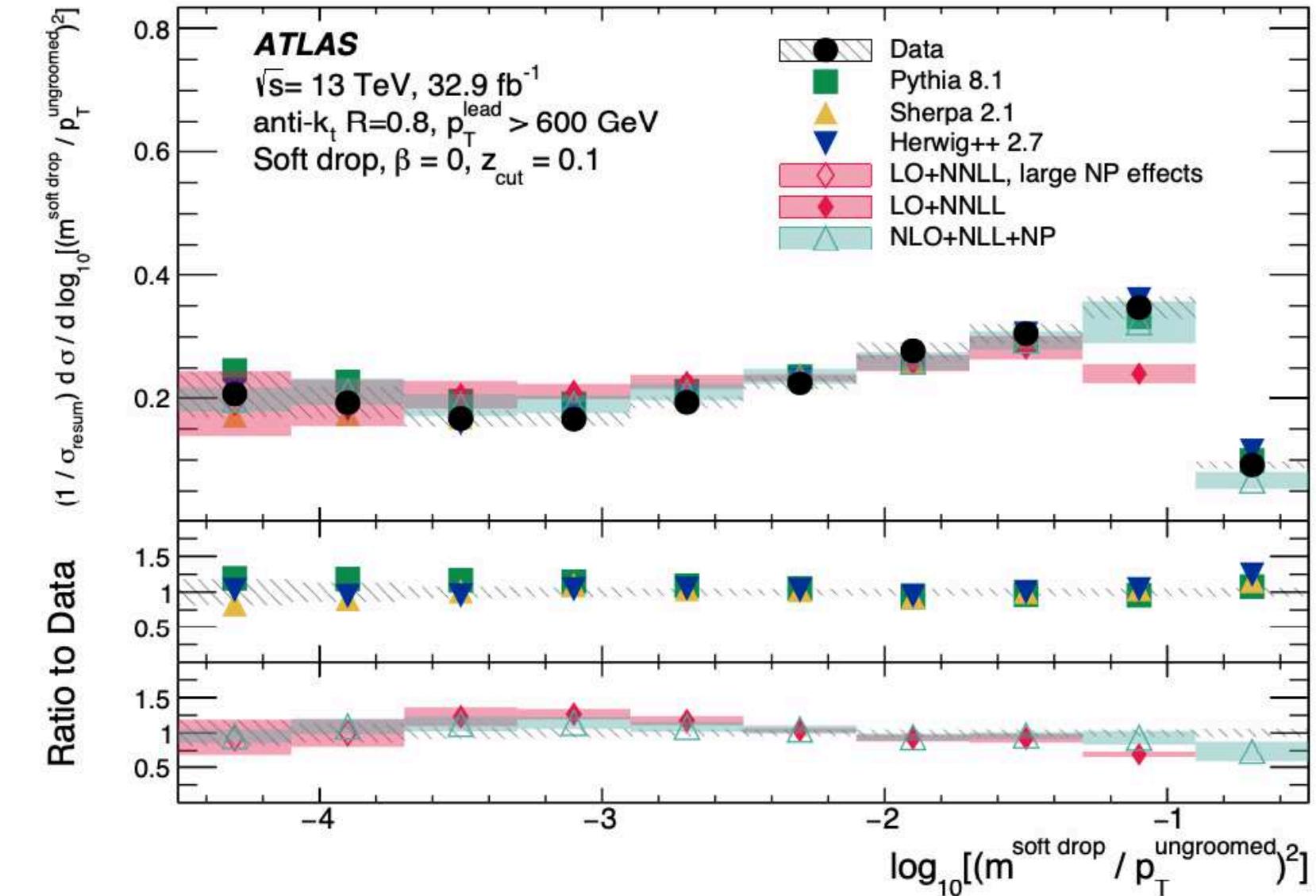
$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_a \frac{d\theta}{\theta} \frac{dz}{z}$$

$$\begin{aligned} C_q &= C_F = 4/3 \\ C_g &= C_A = 3 \end{aligned}$$

Experimentally

IRC safety is a statement of *linearity* in energy and *continuity* in geometry

e.g. groomed jet substructure at the LHC



[ATLAS, PRL 2018]

[See [backup](#) for more on **IRC** safety]

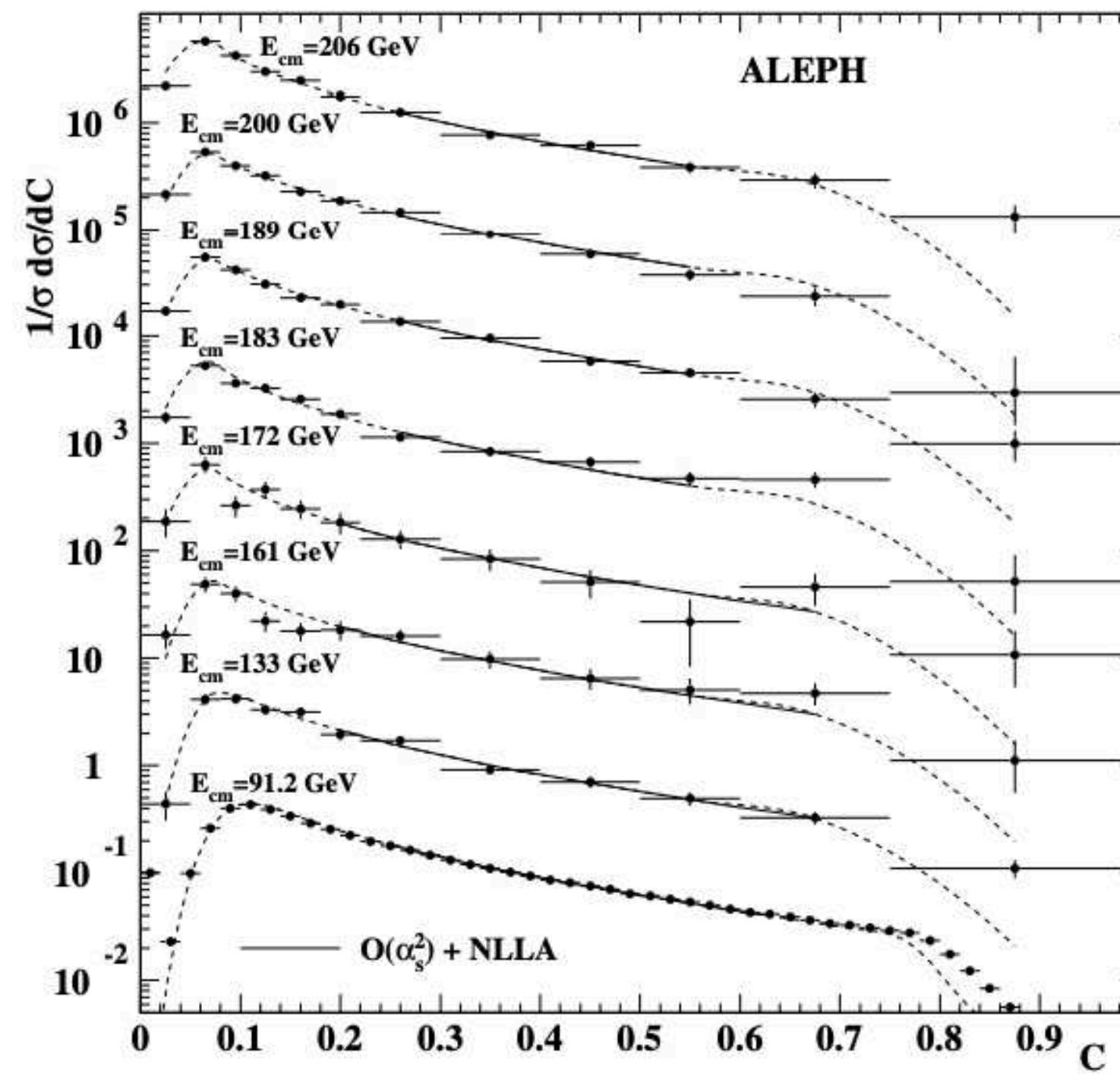
Event Shapes Past and Present

Ubiquitous collider observables that probe QCD radiation

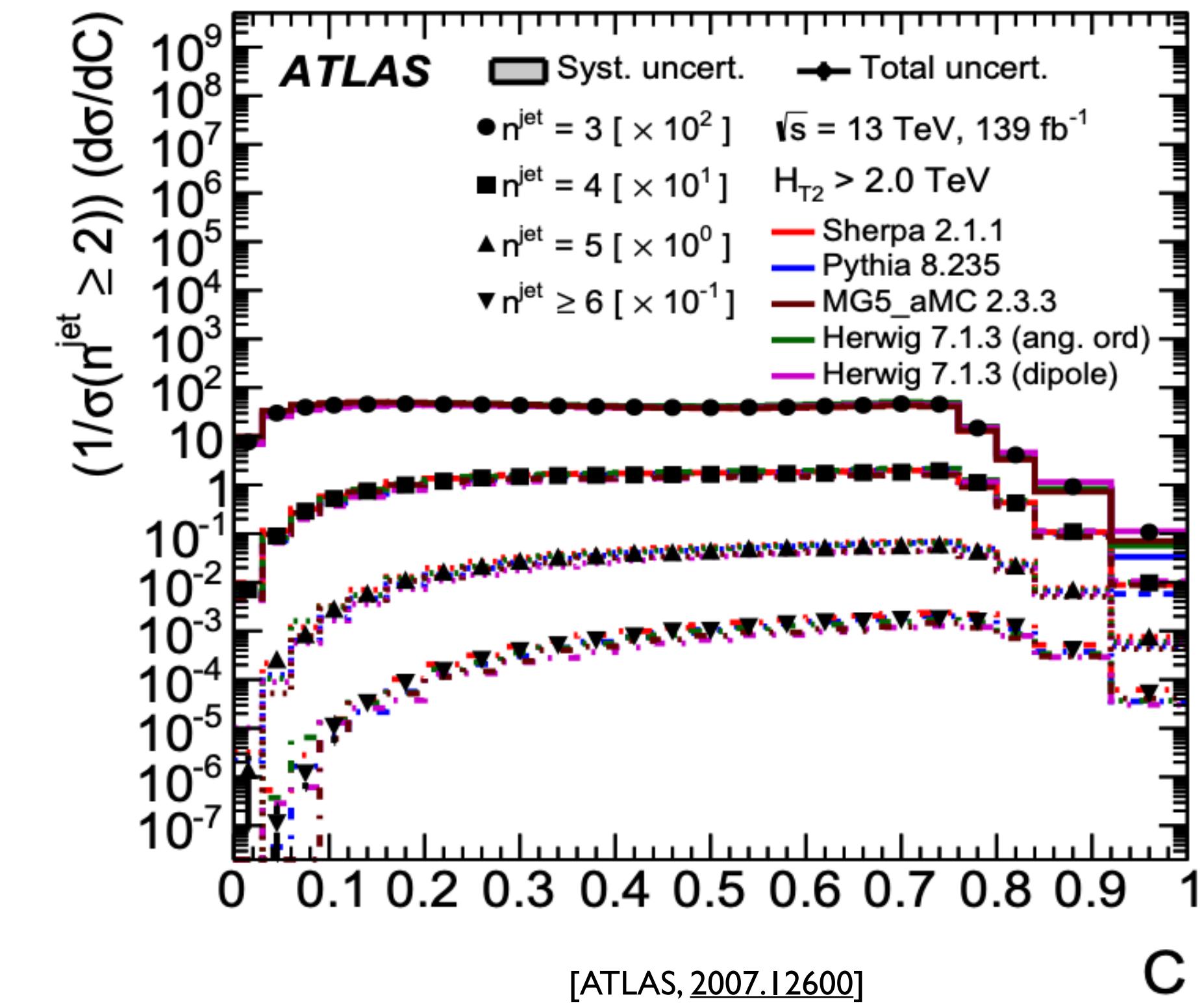
e.g. Sphericity tensor: $\Theta^{ij} = \sum_k z_k \hat{n}_k^i \hat{n}_k^j$, $z_k = \frac{|\vec{p}_k|}{\sum_i |\vec{p}_i|}$, $\hat{n}_k^i = \frac{(\vec{p}_k)^i}{|\vec{p}_k|}$

$$C = 3(\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3) \quad D = 27(\lambda_1 \lambda_2 \lambda_3)$$

Eigenvalues of Θ^{ij}



[ALEPH, EPJC 2004]



[ATLAS, 2007.I2600]

Have been used to extra α_s by comparing to precision QCD calculations

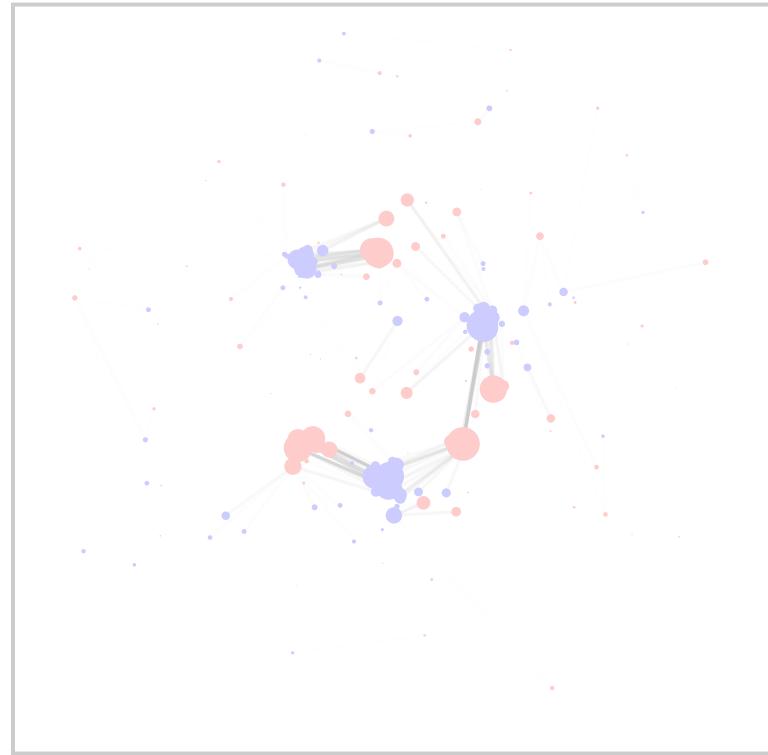
Master Formula for Collider Physics

$$\sigma_{\text{obs}} \sim \frac{1}{2E_{\text{CM}}^2} \sum_{n=2}^{\infty} \int d\Phi_n |\mathcal{M}_{AB \rightarrow 12\dots n}|^2 f_{\text{obs}}(\Phi_n)$$

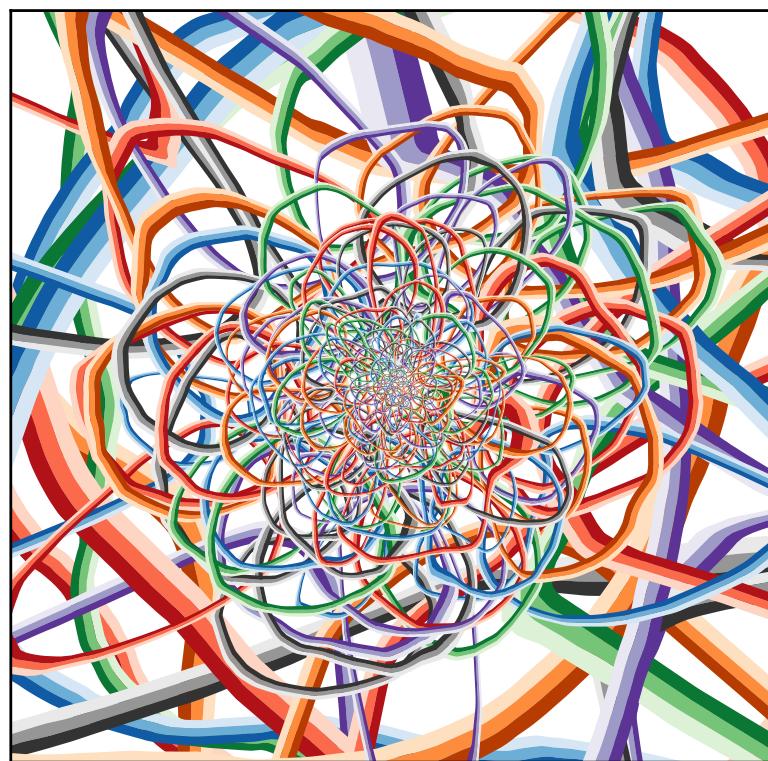
↑
Experiment phase space amplitude observable
 |
 Fixed-order calculations
 |
 Resummed calculations

Guiding question for rest of talk:

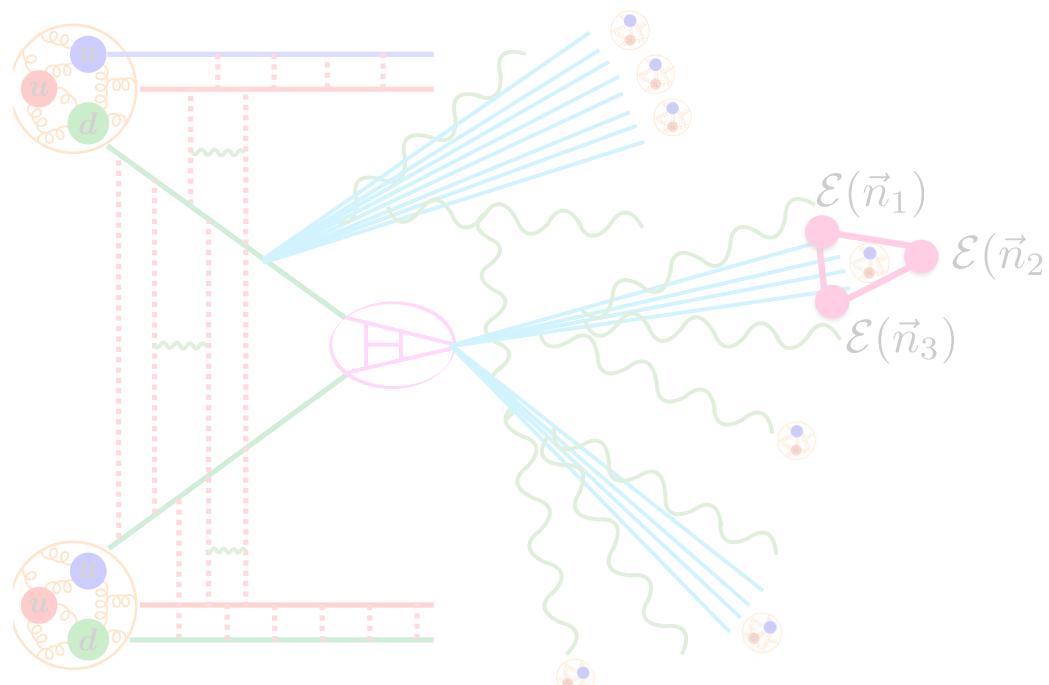
Can we develop a systematic understanding of observables to inform selection of quantities to measure/calculate?



Collider Event Fundamentals



Energy Flow Observables



Energy-Energy Correlators

Multiparticle Correlators

Expanded upon during the jet substructure revolution at the LHC

Definition of **energy factor** and pairwise angular

$$pp : \ z_i = \frac{p_{Ti}}{\sum_j p_{Tj}} \quad \theta_{ij}^2 = 2n_i^\mu n_{j\mu} = 2\frac{p_i^\mu}{p_{Ti}} \frac{p_{j\mu}}{p_{Tj}} \simeq (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$$e^+e^- : \ z_i = \frac{E_i}{\sum_j E_j} \quad \theta_{ij}^2 = 2n_i^\mu n_{j\mu} = 2\frac{p_i^\mu}{E_i} \frac{p_{j\mu}}{E_j}$$

Multiplicity

$$\sum_{i=1}^M 1$$

- Track multiplicity
- Subjet multiplicity
- Calo multiplicity

Mass

$$\frac{1}{2} \sum_{i=1}^M \sum_{j=1}^M z_i z_j \theta_{ij}^2$$

- Track mass
- Soft Drop mass
- Trimmed mass

$$\frac{1}{2}$$

Graphs represent correlators

vertex \leftrightarrow energy factor

$$j \longleftrightarrow \sum_{i,j=1}^M z_{ij}$$

edge \leftrightarrow pairwise angle

$$k \text{---} l \longleftrightarrow \theta_{i_k i_l}$$

Energy Correlation Functions (ECFs)

$$\sum_{i_1=1}^M \cdots \sum_{i_N=1}^M z_{i_1} \cdots z_{i_N} \prod_{j < k} \theta_{i_j i_k}^\beta$$

[Larkoski, Salam, Thaler, [JHEP 1305.0007](#)]
 [Larkoski, Moult, Neill, [JHEP 1409.6298](#)]

- Ratios typical, D_2 , C_2 , C_3 , etc.
- Used for multi-prong tagging
- Generalized ECFs also useful (angular part not monomial)

$$N = 1:$$

$$N = 2:$$

$$N = 3:$$

$$N = 4:$$

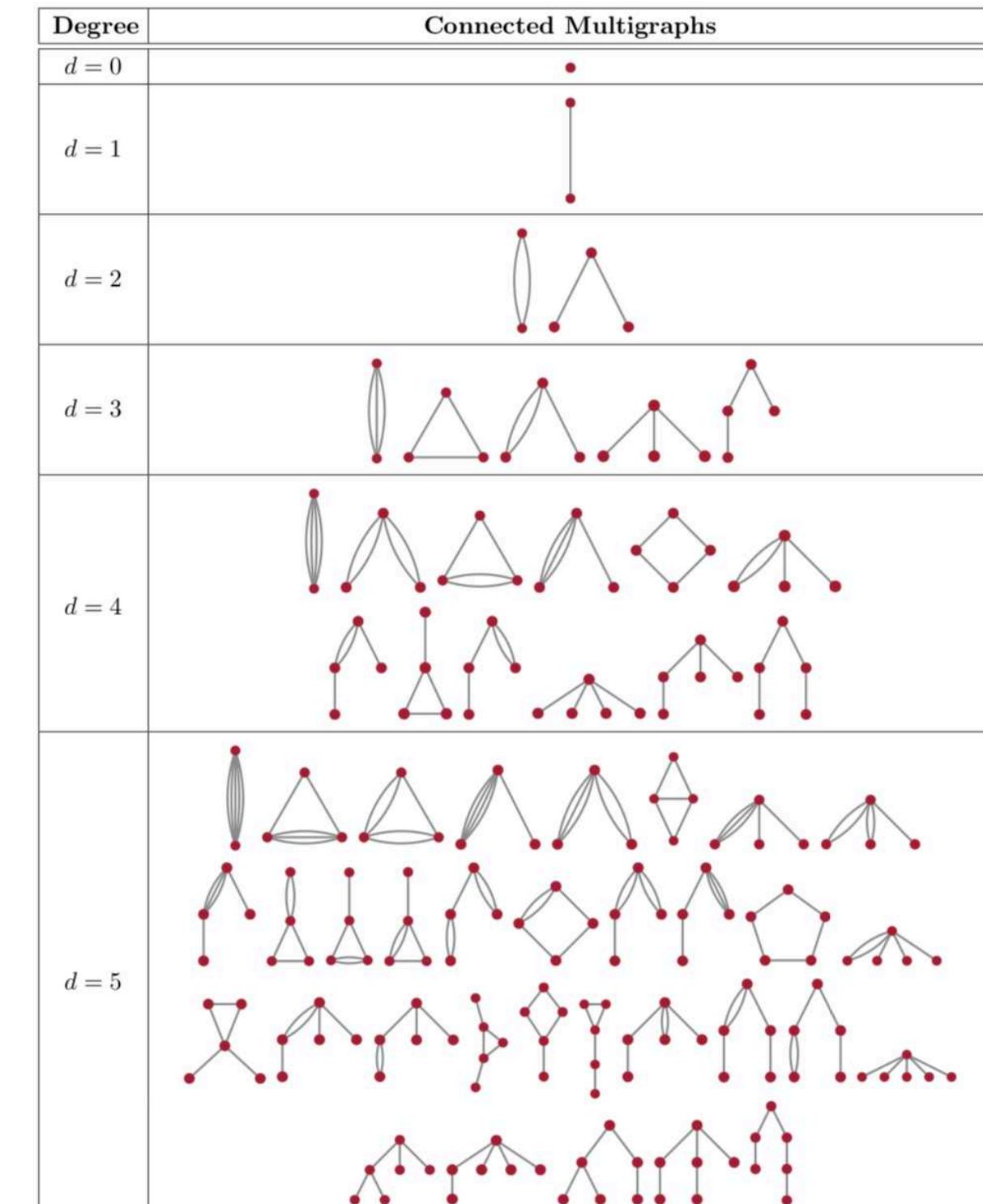
$$C_2 : \frac{(\bullet - \bullet)^2}{(\bullet - \bullet)^2}$$

$$D_2 : \frac{(\bullet - \bullet)^3}{(\bullet - \bullet)^3}$$

Energy Flow Polynomials (EFPs)

$$\sum_{i_1=1}^M \cdots \sum_{i_N=1}^M z_{i_1} \cdots z_{i_N} \prod_{(j,k) \in G} \theta_{i_j i_k}^\beta$$

[PTK, Metodiev, Thaler, [JHEP 2018](#)]



Energy Flow Polynomials (EFPs)

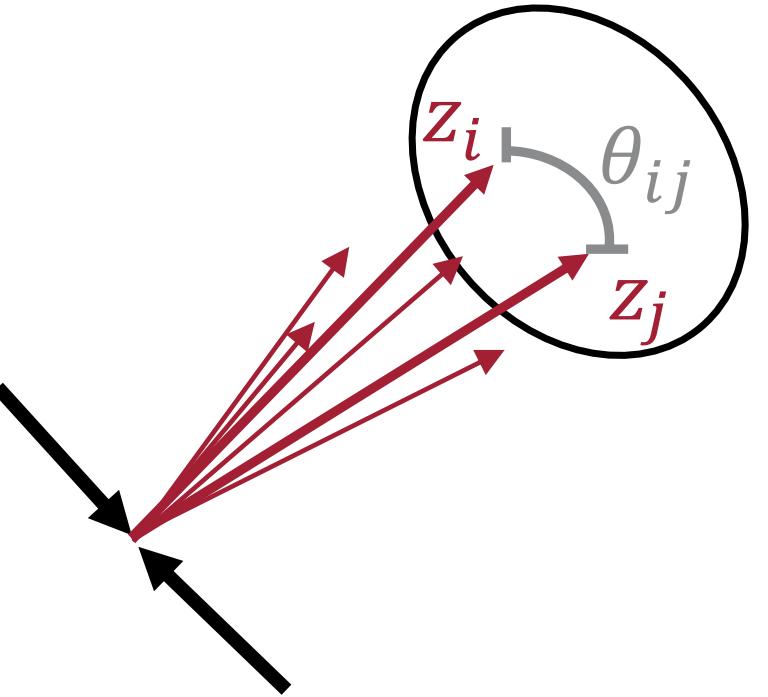
Obtained via systematically expanding in energies and angles

[PTK, Metodiev, Thaler, JHEP 2018]

Energy Flow Polynomials (EFPs)

$$\sum_{i_1=1}^M \cdots \sum_{i_N=1}^M z_{i_1} \cdots z_{i_N} \prod_{(j,k) \in G} \theta_{i_j i_k}^\beta$$

[PTK, Metodiev, Thaler, 1712.07124]



e.g.

$$= \sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M \sum_{i_4=1}^M \sum_{i_5=1}^M z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} \theta_{i_1 i_2} \theta_{i_2 i_3} \theta_{i_1 i_3} \theta_{i_1 i_4} \theta_{i_1 i_5} \theta_{i_4 i_5}^2$$

Graphs represent correlators

vertex \leftrightarrow energy factor edge \leftrightarrow pairwise angle

$$j \longleftrightarrow \sum_{i_j=1}^M z_{i_j}$$

$$k \text{---} l \longleftrightarrow \theta_{i_k i_l}$$

Definition of **energy factor** and pairwise angular

$$pp : z_i = \frac{p_{Ti}}{\sum_j p_{Tj}} \quad \theta_{ij}^2 = 2n_i^\mu n_{j\mu} = 2 \frac{p_i^\mu}{p_{Ti}} \frac{p_{j\mu}}{p_{Tj}} \simeq (y_i - y_j)^2 + (\phi_i - \phi_j)^2$$

$$e^+ e^- : z_i = \frac{E_i}{\sum_j E_j} \quad \theta_{ij}^2 = 2n_i^\mu n_{j\mu} = 2 \frac{p_i^\mu}{E_i} \frac{p_{j\mu}}{E_j}$$

Organized by number of edges d

Degree	Connected Multigraphs
$d = 0$	
$d = 1$	
$d = 2$	
$d = 3$	
$d = 4$	
$d = 5$	

Familiar Observables as EFPs

Energy correlation functions are complete graphs

Energy Correlation Functions:

$$e_N^{(\beta)} = \sum_{i_1}^M \cdots \sum_{i_N=1}^M z_{i_1} \cdots z_{i_N} \prod_{k < \ell \in \{1, \dots, N\}} \theta_{i_k i_\ell}^\beta$$

$e_2^{(\beta)} =$, $e_3^{(\beta)} =$, $e_4^{(\beta)} =$

with measure choice β

A.J. Larkoski, G.P. Salam, J. Thaler, [1305.0007](#)

Observables commonly used at LHC

$D_2 = \frac{\text{triangle}}{(\text{horizontal line})^3}$

[Larkoski, Moult, Neill, 2014]

$C_2 = \frac{\text{triangle}}{(\text{horizontal line})^2}$

[Larkoski, Salam, Thaler, 2013]

$m_j^2 =$

Even angularities are exact linear combinations of EFPs

Angularities:

$$\lambda^{(\alpha)} = \sum_{i=1}^M z_i \theta_i^\alpha \quad \text{using } p_T\text{-centroid axis}$$

C.F. Berger, T. Kucs, and G. Sterman, [hep-ph/0303051](#)
S.D. Ellis, et al., [1001.0014](#)
A.J. Larkoski, J. Thaler, and W. Waalewijn, [1408.3122](#)

$\lambda^{(4)} =$ $- \frac{3}{4} \times$
 $\lambda^{(6)} =$ $- \frac{3}{2} \times$ $+ \frac{5}{8} \times$

Even classic event shapes!

with $\beta = 2$

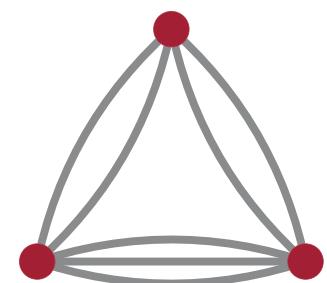
$C = -\frac{3}{8} \times$ $+ \frac{3}{2} \times$

$D = -\frac{9}{8} \times$ $+ \frac{27}{4} \times$ $- \frac{27}{8} \times$

EFPs as a Linear Basis of **IRC**-Safe Observables

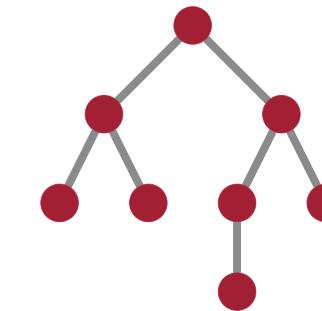
Any **IRC**-safe observable is a linear combination of EFPs (via the Stone-Weierstrass approximation theorem)!

[See [backup](#) for derivation]

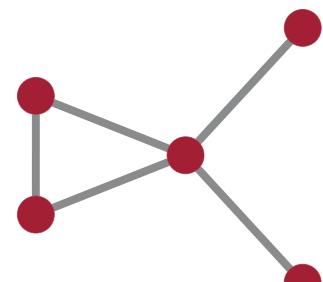


$$\mathcal{S} \simeq \sum_{G \in \mathcal{G}} s_G \text{EFP}_G, \quad \mathcal{G} \text{ a set of multigraphs}$$

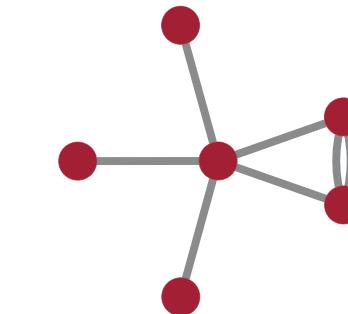
IRC-Safe Observable



Multivariate combinations of EFPs only require linear methods to achieve full generality

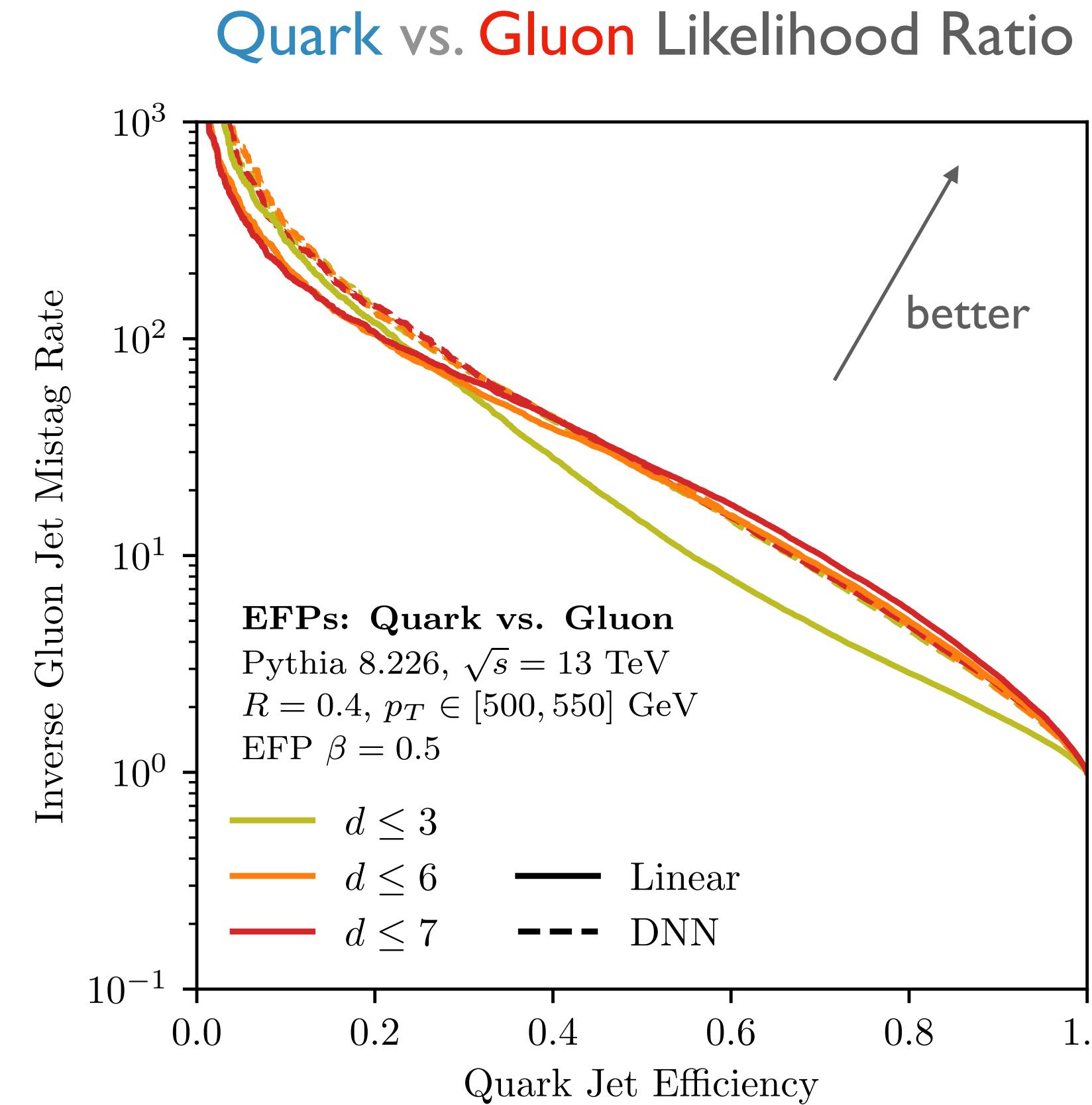


Strategy: Learn coefficients s_G via linear regression or classification



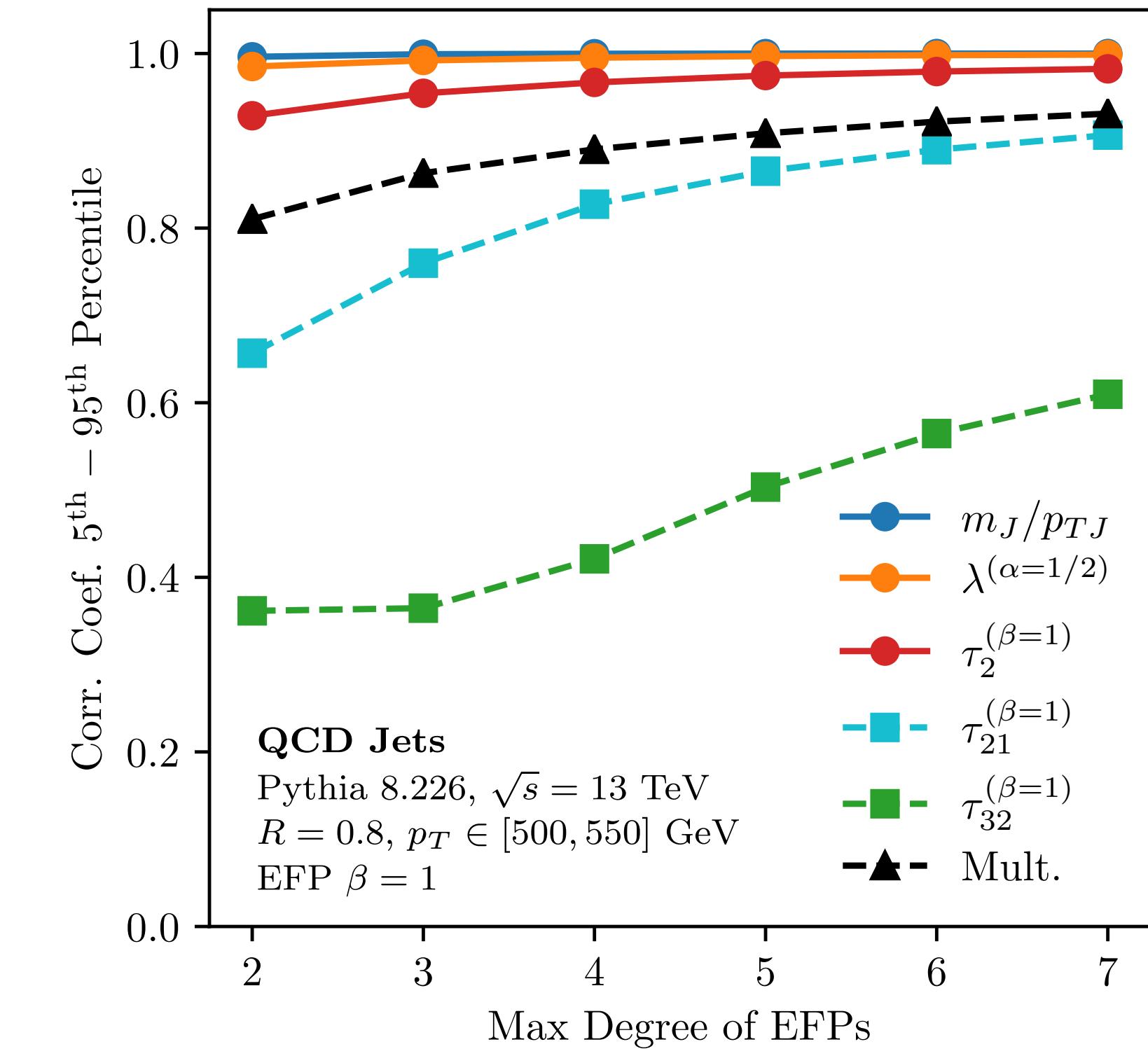
Testing EFPs on QCD Jets

[PTK, Metodiev, Thaler, JHEP 2018]



Convergence observed with more EFPs

DNN gets there faster but linear suffices



EFPs regress onto IRC-safe observables best

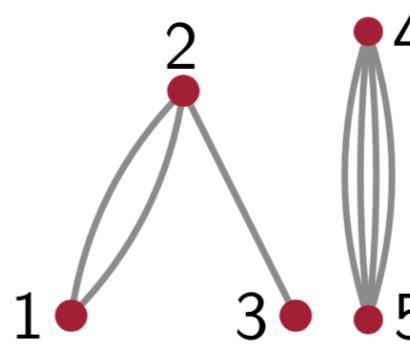
Computational Complexity

Naive computation complexity of an energy correlator is $\mathcal{O}(M^N)$

EnergyCorrelator fjcontrib solution:

```
// if N > 5, then throw error
if (_N > 5) {
    throw Error("EnergyCorrelator is only hard coded for N = 0,1,2,3,4,5");
}
```

Variable elimination (VE) algorithm: $\mathcal{O}(M^\chi)$, $\chi \lesssim N$



Disconnected is product of connected

$$= \left(\sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_2 i_3} \right) \left(\sum_{i_4=1}^M \sum_{i_5=1}^M z_{i_4} z_{i_5} \theta_{i_4 i_5}^4 \right)$$

VE find clever parentheses placement to minimize computation

$$= \underbrace{\sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M \sum_{i_4=1}^M \sum_{i_5=1}^M \sum_{i_6=1}^M \sum_{i_7=1}^M \sum_{i_8=1}^M z_{i_1} z_{i_2} z_{i_3} z_{i_4} z_{i_5} z_{i_6} z_{i_7} z_{i_8}}_{\mathcal{O}(M^8)} \prod_{j=2}^7 \theta_{i_1 i_j}$$

$$= \underbrace{\sum_{i_1=1}^M z_{i_1} \left(\sum_{i_2=1}^M z_{i_2} \theta_{i_1 i_2} \right)}_{\mathcal{O}(M^2)}$$

All tree graphs become $\mathcal{O}(M^2)$

$\chi = N$ iff G is complete graph, ECFs still slow

Energy Flow Moments (EFMs)

[PTK, Metodiev, Thaler, PRD 2020]

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}} \quad \beta = 2 \text{ removes square root}$$

Factors of n_i^μ can be organized in optimal way

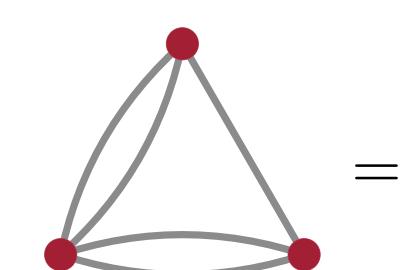
EFM_v is a little group tensor with v indices

$$\mathcal{T}^{\mu_1 \dots \mu_v} = \sum_{i=1}^M \cancel{z_i} n_i^{\mu_1} \dots n_i^{\mu_v}$$

v	0	1	2	3	4	5	6
$n_{\text{components}}^{(d=4)}$	1	4	10	20	35	56	84

$$\mathcal{T}^{j_1 j_2 \dots j_v} = 2^{v/2} \Theta^{j_1 j_2 \dots j_v}$$

spatial e^+e^- EFMs



$$= \sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_2 i_3}^2 \theta_{i_1 i_3}^2$$

linearized sphericity tensors

[Donoghue, Low, Pi, PRD 1979]

$$= 2^5 \underbrace{\left(\sum_{i_1=1}^M \cancel{z_{i_1}} n_{i_1}^\alpha n_{i_1}^\beta n_{i_1}^\gamma n_{i_1}^\delta \right)}_{\mathcal{I}^{\alpha\beta\gamma\delta}} \underbrace{\left(\sum_{i_2=1}^M \cancel{z_{i_2}} n_{i_2\alpha} n_{i_2\beta} n_{i_2}^\epsilon \right)}_{\mathcal{I}_{\alpha\beta}^\epsilon} \underbrace{\left(\sum_{i_3=1}^M \cancel{z_{i_3}} n_{i_3\gamma} n_{i_3\delta} n_{i_3\epsilon} \right)}_{\mathcal{I}_{\gamma\delta\epsilon}}$$

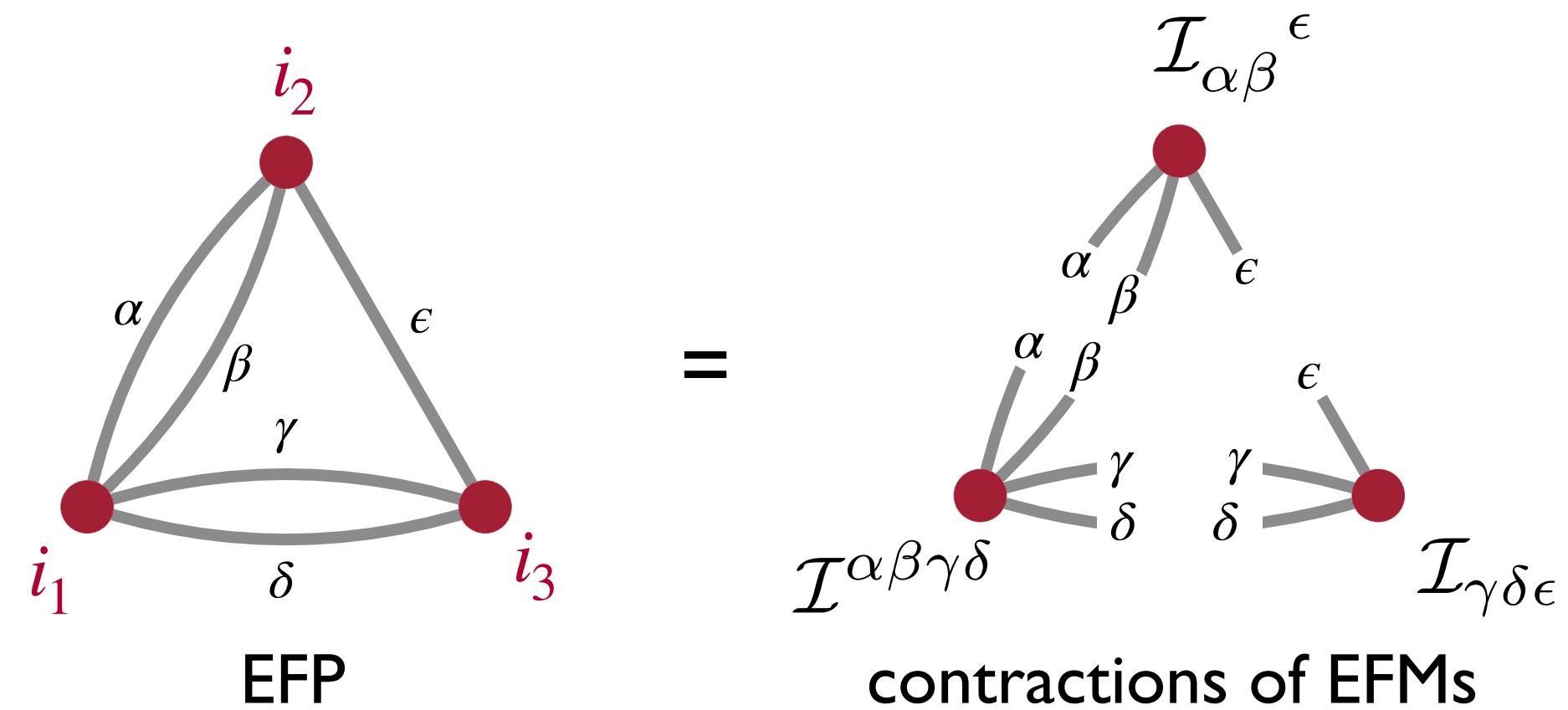
Naively $\mathcal{O}(M^3)$ EFP shown to be $\mathcal{O}(M)$

Computational Complexity

All $\beta = 2$ EFPs are $\mathcal{O}(M)$

- $\text{ECF}_N^{(\beta=2)}$ are all $\mathcal{O}(M)$
- $D_2^{(\beta=2)}, C_2^{(\beta=2)}$ are $\mathcal{O}(M)$

EFMs result from cutting edges of EFP graph



Energy Flow Moments (EFMs)

[PTK, Metodiev, Thaler, PRD 2020]

$$\theta_{ij} = \sqrt{2n_i^\mu n_{j\mu}} \quad \beta = 2 \text{ removes square root}$$

Factors of n_i^μ can be organized in optimal way

EFM_v is a little group tensor with v indices

$$\mathcal{I}^{\mu_1 \dots \mu_v} = \sum_{i=1}^M z_i n_i^{\mu_1} \dots n_i^{\mu_v}$$

v	0	1	2	3	4	5	6
$n_{\text{components}}^{(d=4)}$	1	4	10	20	35	56	84

$$\mathcal{I}^{j_1 j_2 \dots j_v} = 2^{v/2} \Theta^{j_1 j_2 \dots j_v}$$

spatial e^+e^- EFMs

linearized sphericity tensors
[Donoghue, Low, Pi, PRD 1979]

$$\begin{aligned}
 &= \sum_{i_1=1}^M \sum_{i_2=1}^M \sum_{i_3=1}^M z_{i_1} z_{i_2} z_{i_3} \theta_{i_1 i_2}^2 \theta_{i_2 i_3}^2 \theta_{i_1 i_3} \\
 &= 2^5 \underbrace{\left(\sum_{i_1=1}^M z_{i_1} n_{i_1}^\alpha n_{i_1}^\beta n_{i_1}^\gamma n_{i_1}^\delta \right)}_{\mathcal{I}^{\alpha\beta\gamma\delta}} \underbrace{\left(\sum_{i_2=1}^M z_{i_2} n_{i_2\alpha} n_{i_2\beta} n_{i_2}^\epsilon \right)}_{\mathcal{I}_{\alpha\beta}^\epsilon} \underbrace{\left(\sum_{i_3=1}^M z_{i_3} n_{i_3\gamma} n_{i_3\delta} n_{i_3}^\epsilon \right)}_{\mathcal{I}_{\gamma\delta\epsilon}}
 \end{aligned}$$

Naively $\mathcal{O}(M^3)$ EFP shown to be $\mathcal{O}(M)$

Understanding Linear Redundancies via EFM

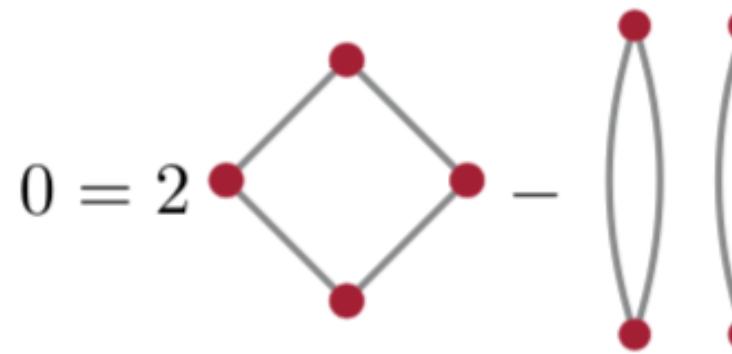
[PTK, Metodiev, Thaler, PRD 2020]

Linear redundancies among EFPs are troublesome

Studying coefficients of linear fit difficult $\mathcal{O} = \sum_G s_G \text{EFP}_G$

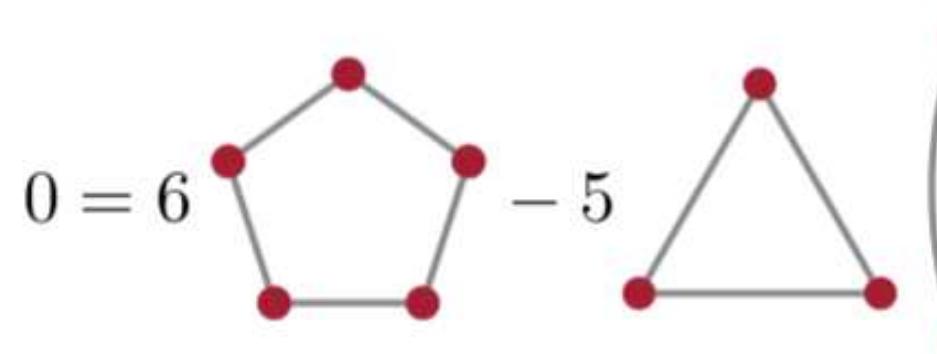
Examples of redundancies

in 3 or fewer spacetime dimensions

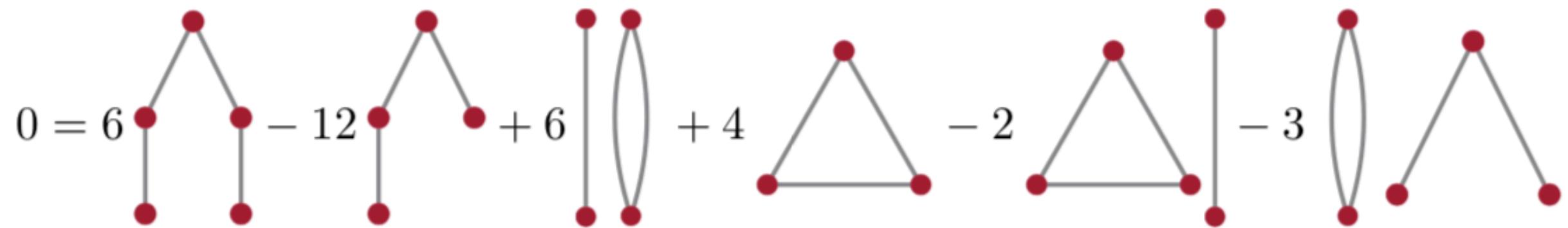


$$0 = \mathcal{I}_{[\alpha}^{\beta} \mathcal{I}_{\beta}^{\gamma} \mathcal{I}_{\gamma}^{\delta} \mathcal{I}_{\delta]}^{\alpha}$$

in 4 or fewer spacetime dimensions



$$0 = \mathcal{I}_{[\alpha}^{\beta} \mathcal{I}_{\beta}^{\gamma} \mathcal{I}_{\gamma}^{\delta} \mathcal{I}_{\delta}^{\epsilon} \mathcal{I}_{\epsilon]}^{\alpha}$$



How to obtain a tensor identity

Consider tensor over n dimensional vector space

Antisymmetrize $m > n$ indices

Result is zero because any assignment of n possible values to m slots has a repetition

$$T_{b_1 \dots b_\ell [c_1 \dots c_m]}^{a_1 \dots a_k} = 0$$

Bonus: all tensor identities up to ones governed by existing symmetries take above form

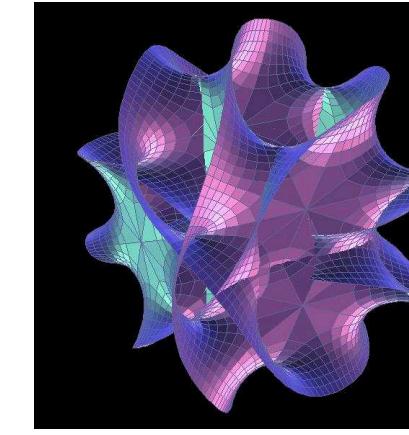
[Sneddon, Journal of Mathematical Physics]

In e^+e^- there are additional relations due to

$$n_i^\mu = (1, \hat{n})^\mu \implies \mathcal{I}^{0\mu_1 \dots \mu_v} = \sqrt{2} \mathcal{I}^{\mu_1 \dots \mu_v}$$

See backup for more on these “Euclidean” relations

Counting Superstring Amplitudes



New OEIS Entries!
[A307317](#), [A307316](#)

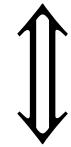
Constructing a basis of amplitudes – how large is it?

[Boels, [I304.7918](#); OEIS [A226919](#)]

non-isomorphic multigraph



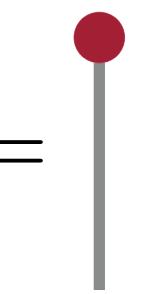
Q: What is the number of symmetric polynomials of degree d in kinematic variables up to momentum conservation?



$\theta_{ij}^2 = 2n_i \cdot n_j$
pairwise angular distance



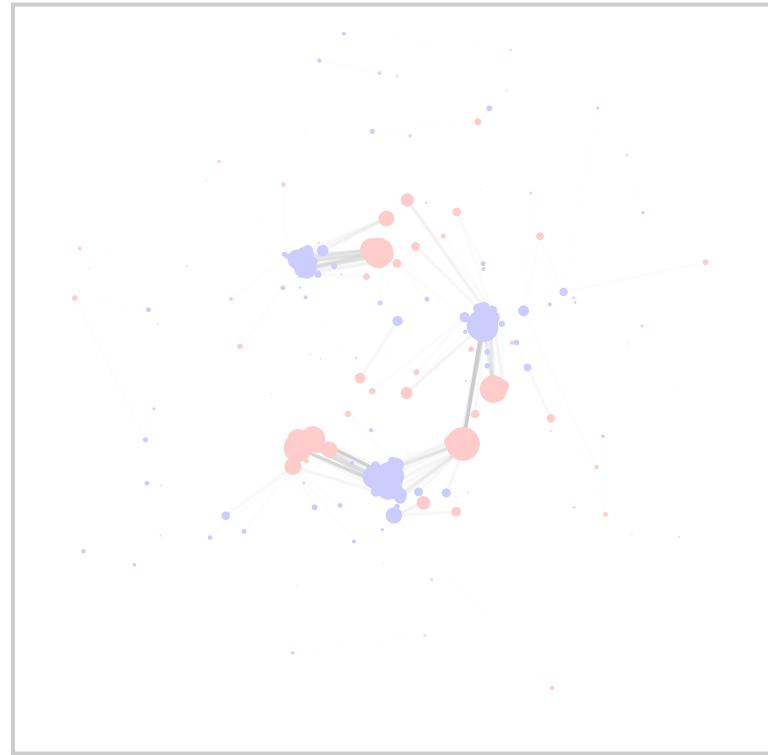
$$0 = \sum_{i=1}^M p_i^\mu = \mathcal{I}^\mu =$$



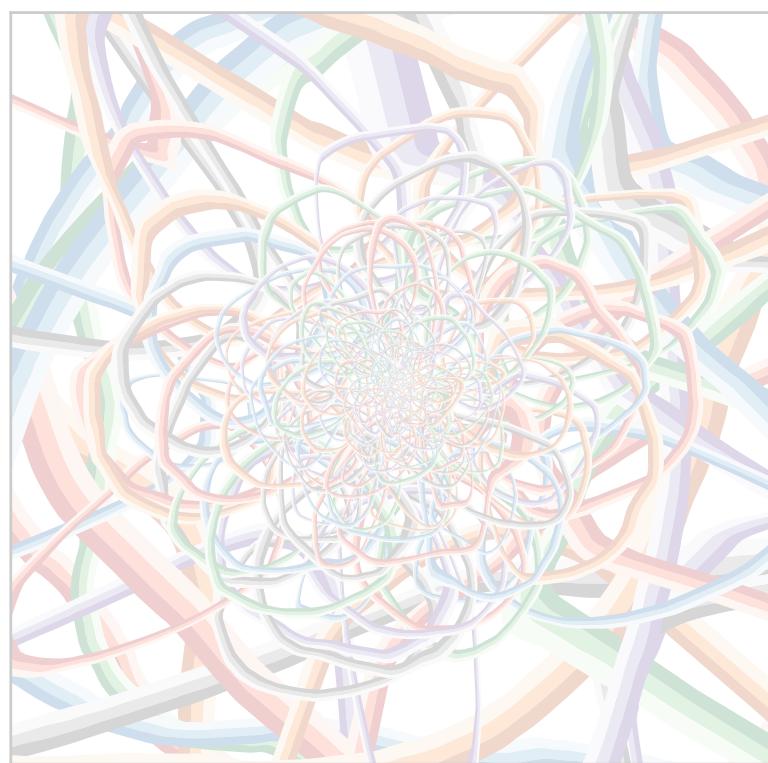
A: Same as the number of non-isomorphic multigraphs with no leaves (vertices of valency one)

Edges d	Leafless Multigraphs	
	Connected	All
Edges d	A307317	A307316
1	0	0
2	1	1
3	2	2
4	4	5
5	9	11
6	26	34
7	68	87
8	217	279
9	718	897
10	2 553	3 129
11	9 574	11 458
12	38 005	44 576
13	157 306	181 071
14	679 682	770 237
15	3 047 699	3 407 332
16	14 150 278	15 641 159

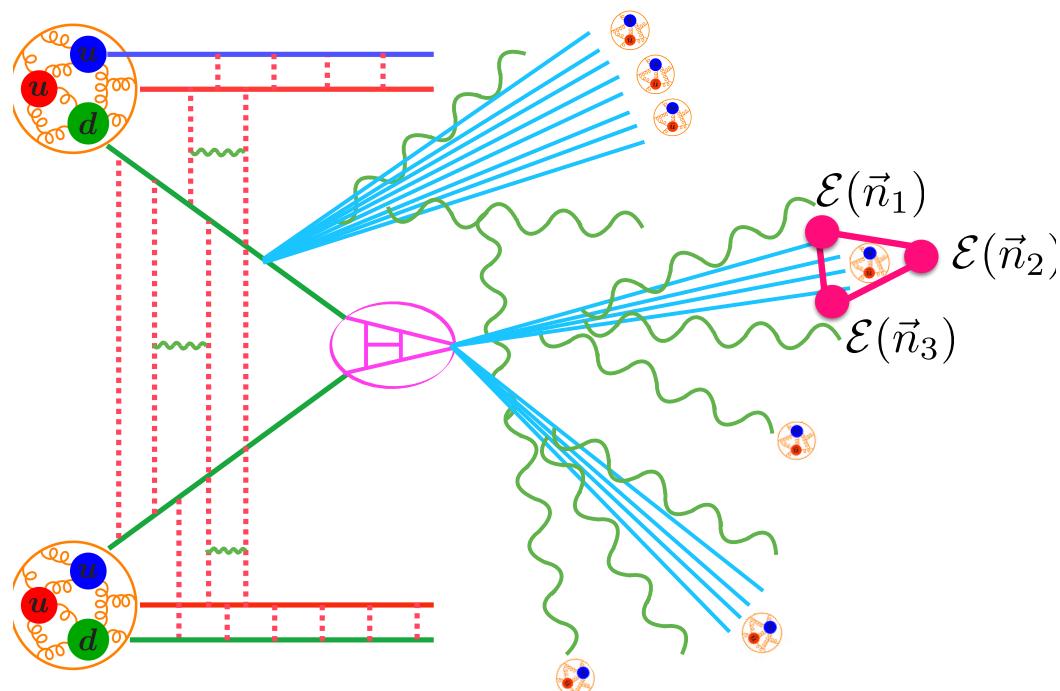
Bolded values previously unknown



Collider Event Fundamentals



Energy Flow Observables



Energy-Energy Correlators

Beyond Observables via Weighted Cross Sections

Standard observable (e.g. EFPs)

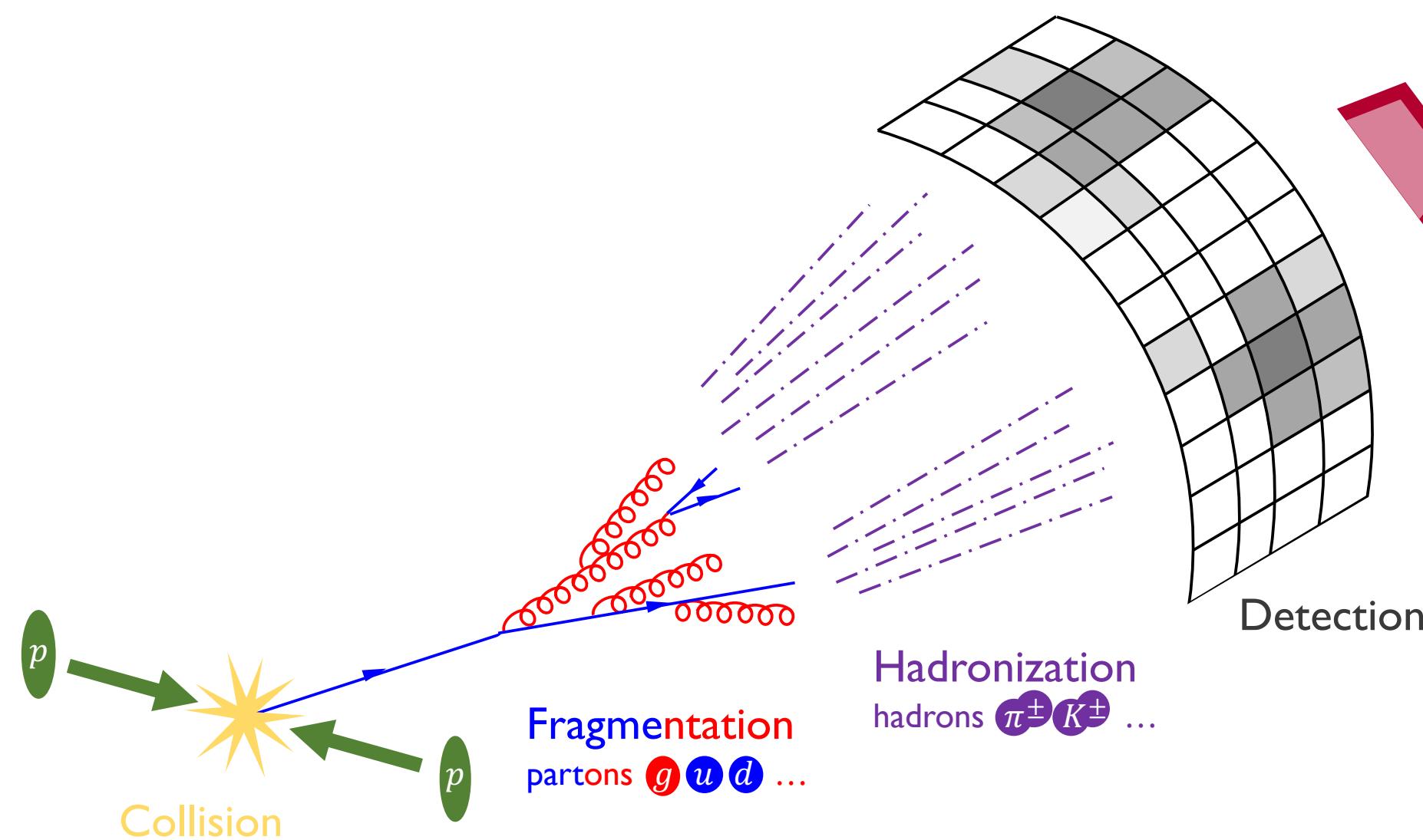
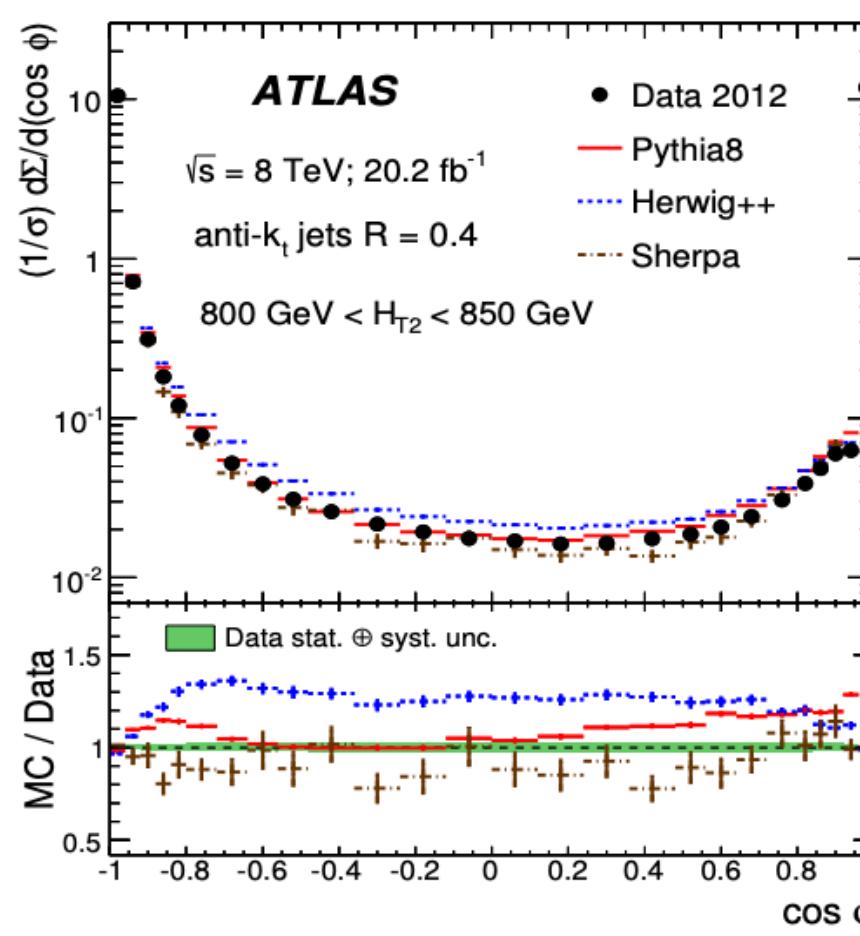
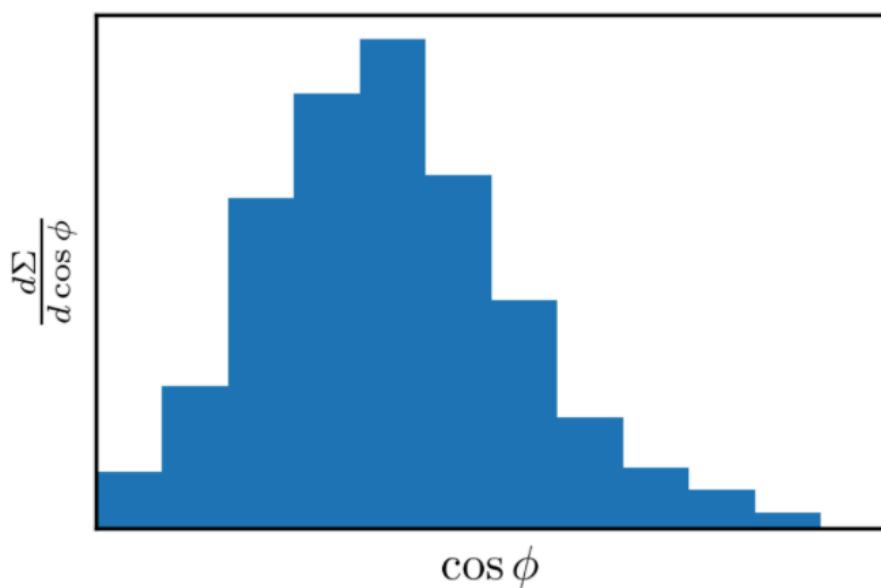
*Calculate a single number for each jet/event
and study distribution of values*

Weighted cross section

*Calculate a distributional quantity per event
and study the mean distribution*

e.g. energy-energy correlator (EEC)

$$\frac{d\Sigma}{d\cos\phi} = \sum_{i,j} \int d\sigma \frac{E_i E_j}{Q^2} \delta(\cos\theta_{ij} - \cos\phi)$$

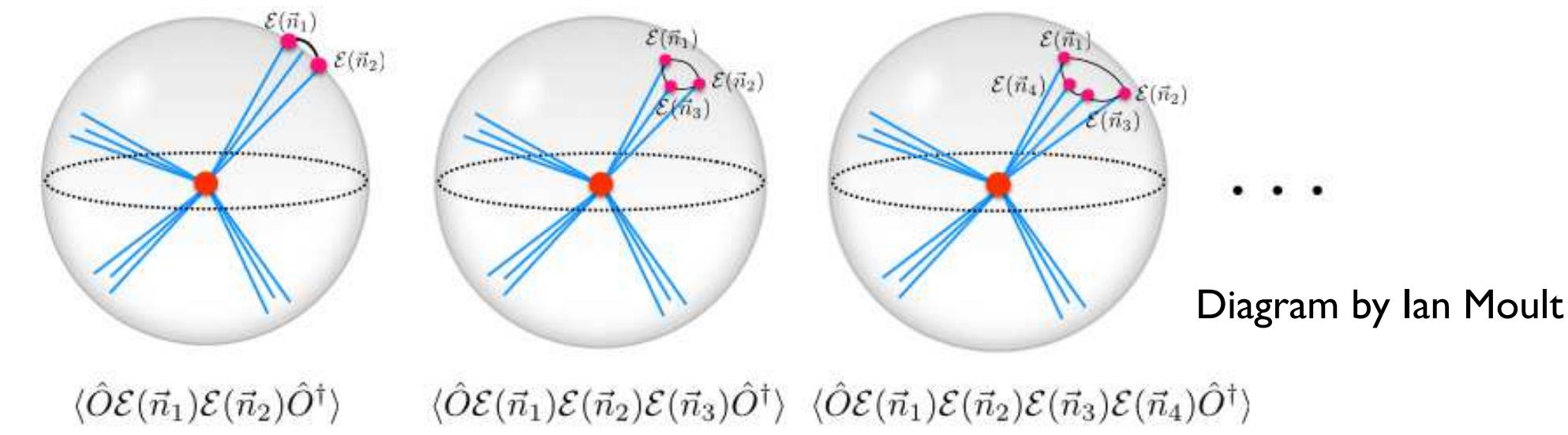


$$\mathcal{E}(\hat{n}) = \int_0^\infty dt \lim_{r \rightarrow \infty} r^2 n^i T_{0i}(t, r\hat{n})$$

Stress-energy flow

$$\frac{1}{\sigma_{\text{tot}}} \frac{d\sigma}{d\hat{n}_1 \cdots d\hat{n}_N} = \frac{\langle \mathcal{O}\mathcal{E}(\hat{n}_1) \cdots \mathcal{E}(\hat{n}_N)\mathcal{O}^\dagger \rangle}{\langle \mathcal{O}\mathcal{O}^\dagger \rangle}$$

Correlations of energy flow operators can be directly studied!



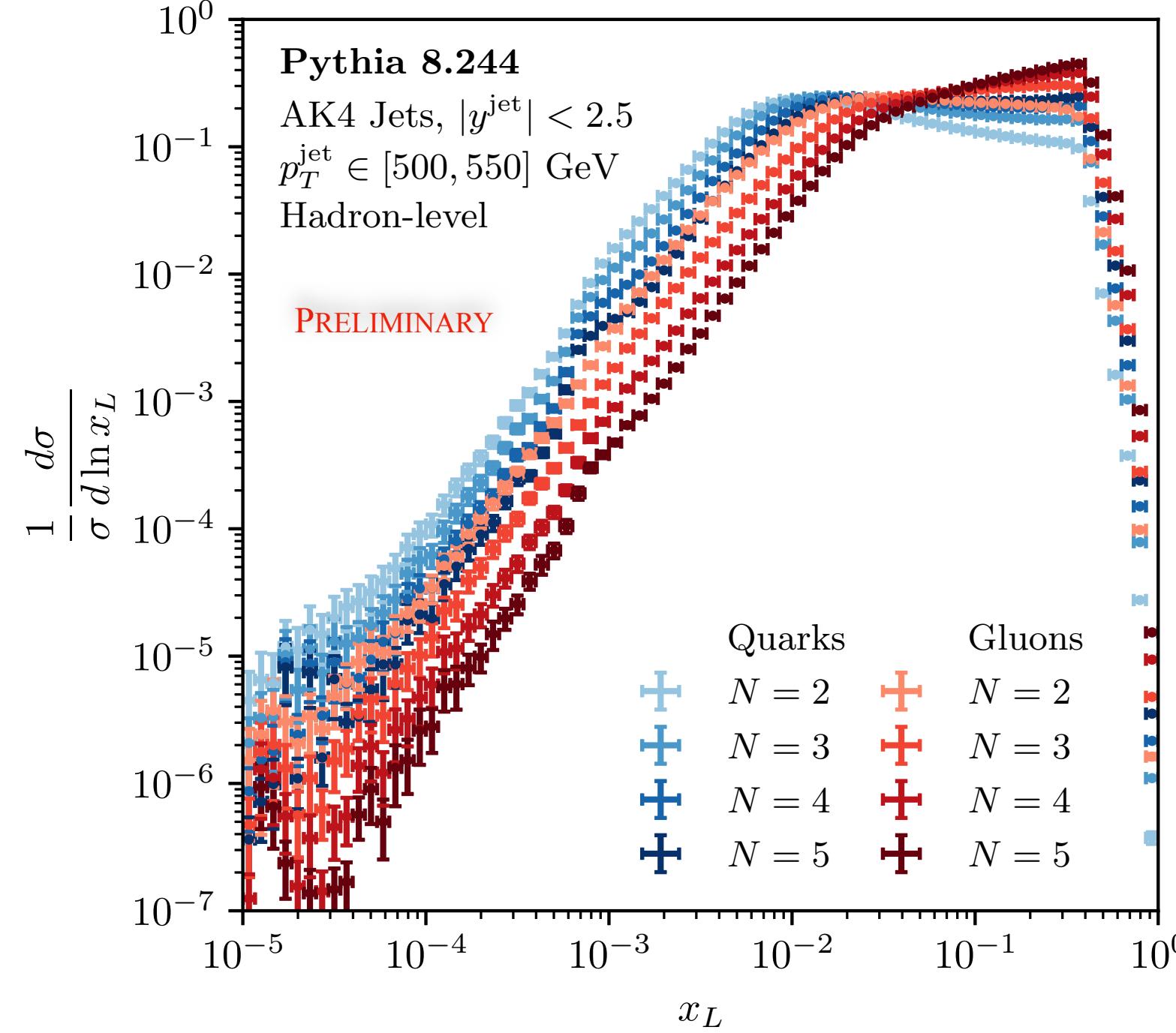
Energy-Energy Correlators – Projection to Longest Side

[PTK, Moult, Thaler, Zhu, to appear soon]

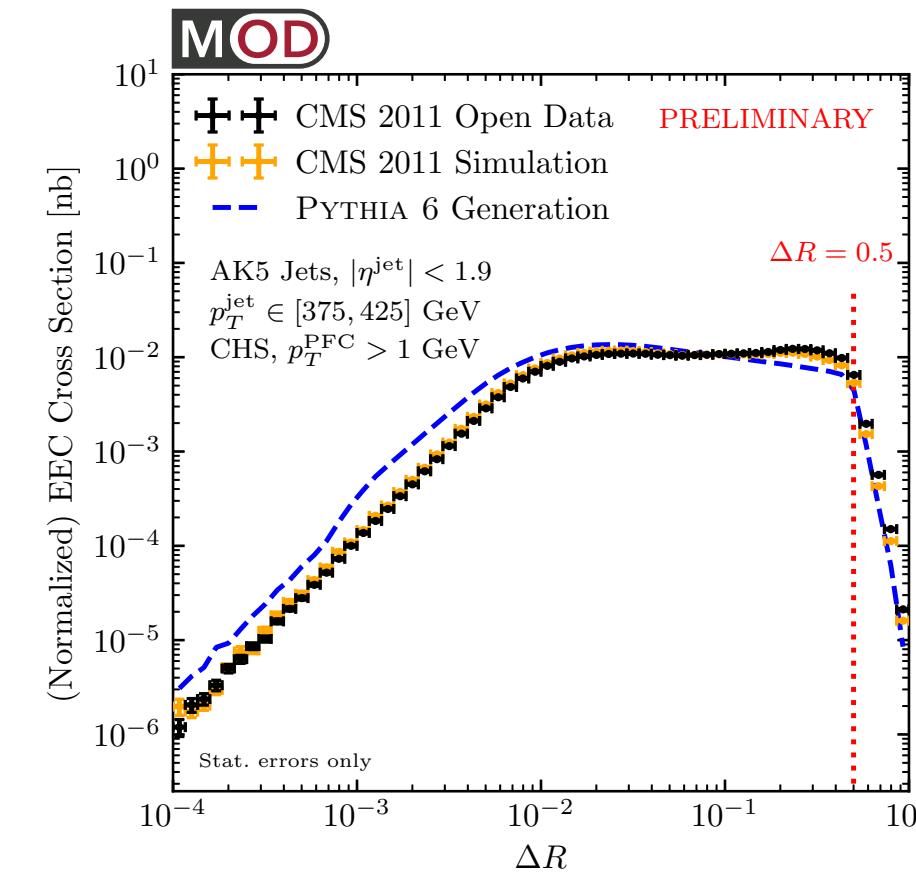
Integrate out shape dependence but keep overall size dependence

$$\frac{d\Sigma[N]}{dx_L} = \sum_n \sum_{1 \leq i_1 \leq \dots \leq i_N \leq n} \int d\sigma_n \frac{E_{i_1} \cdots E_{i_N}}{Q^N} \delta(x_L - \max_{1 \leq j < k \leq N} \{\theta_{i_j i_k}\})$$

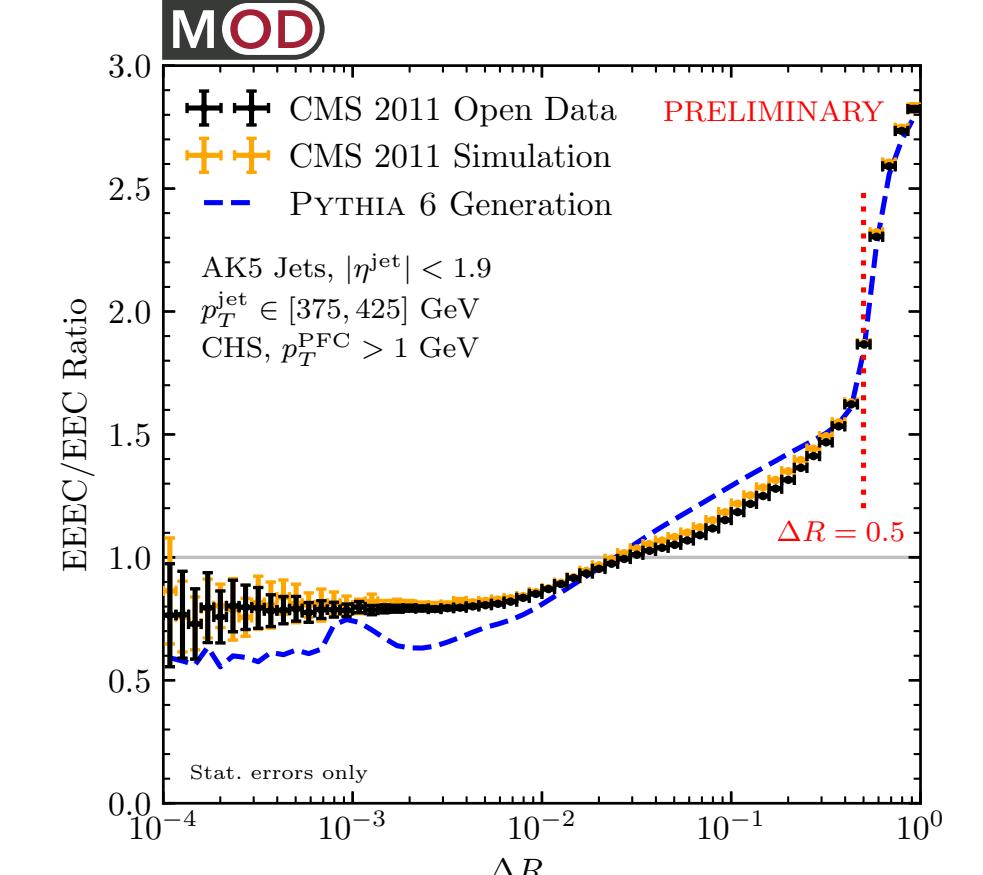
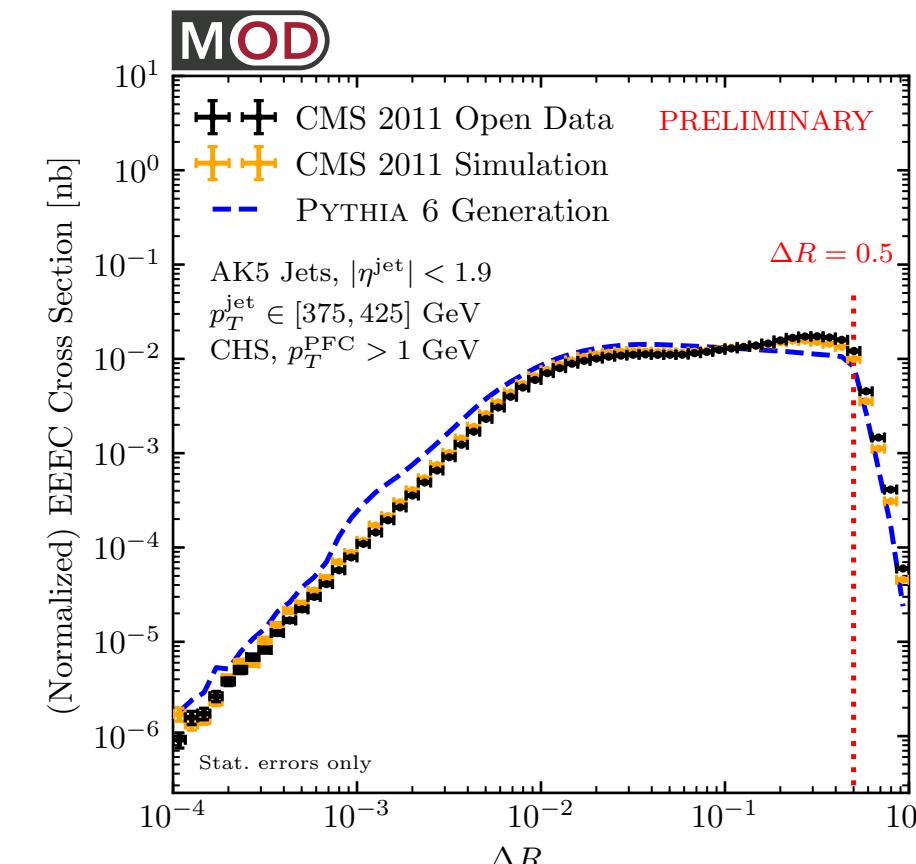
EEEC/EEC Ratio



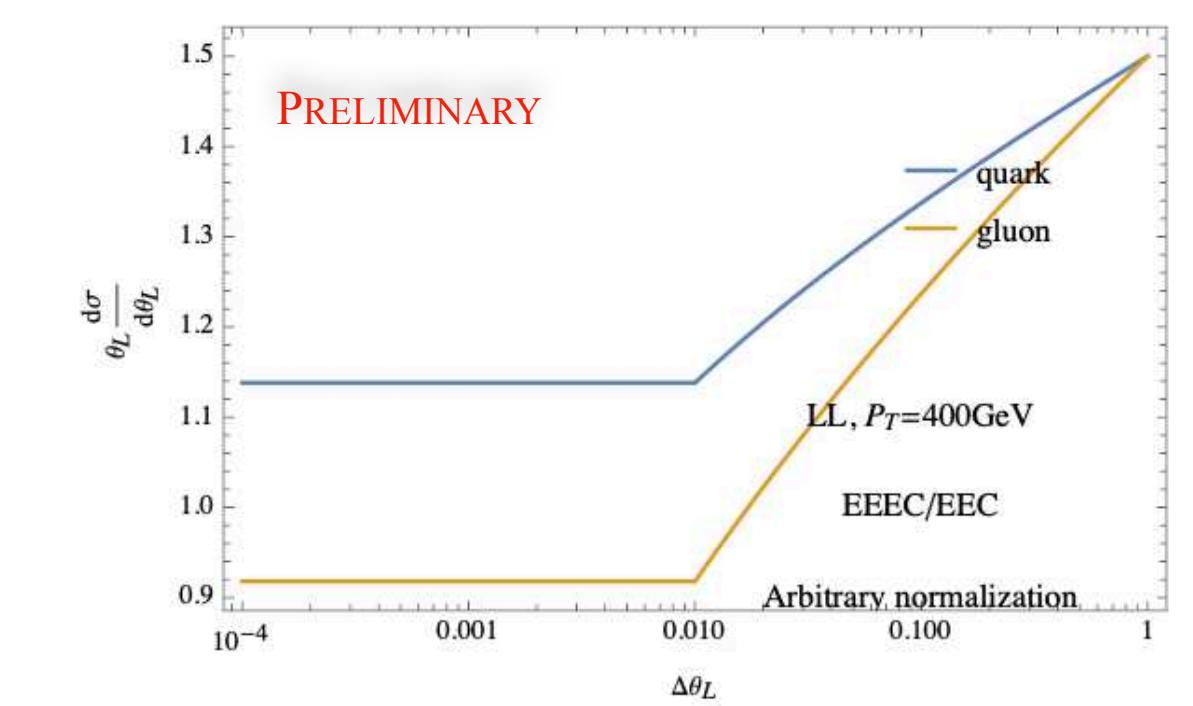
$N = 2$



$N = 3$

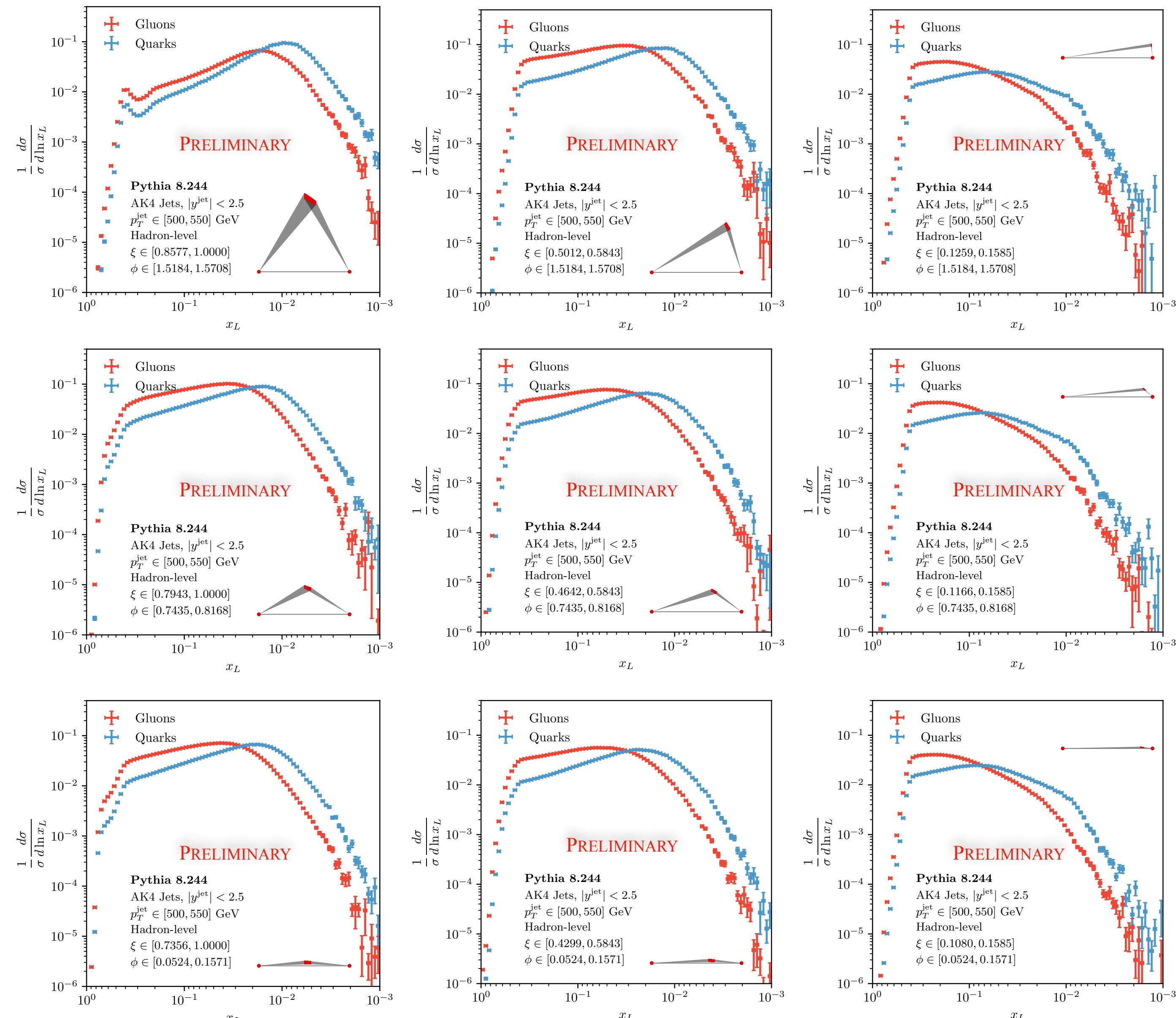
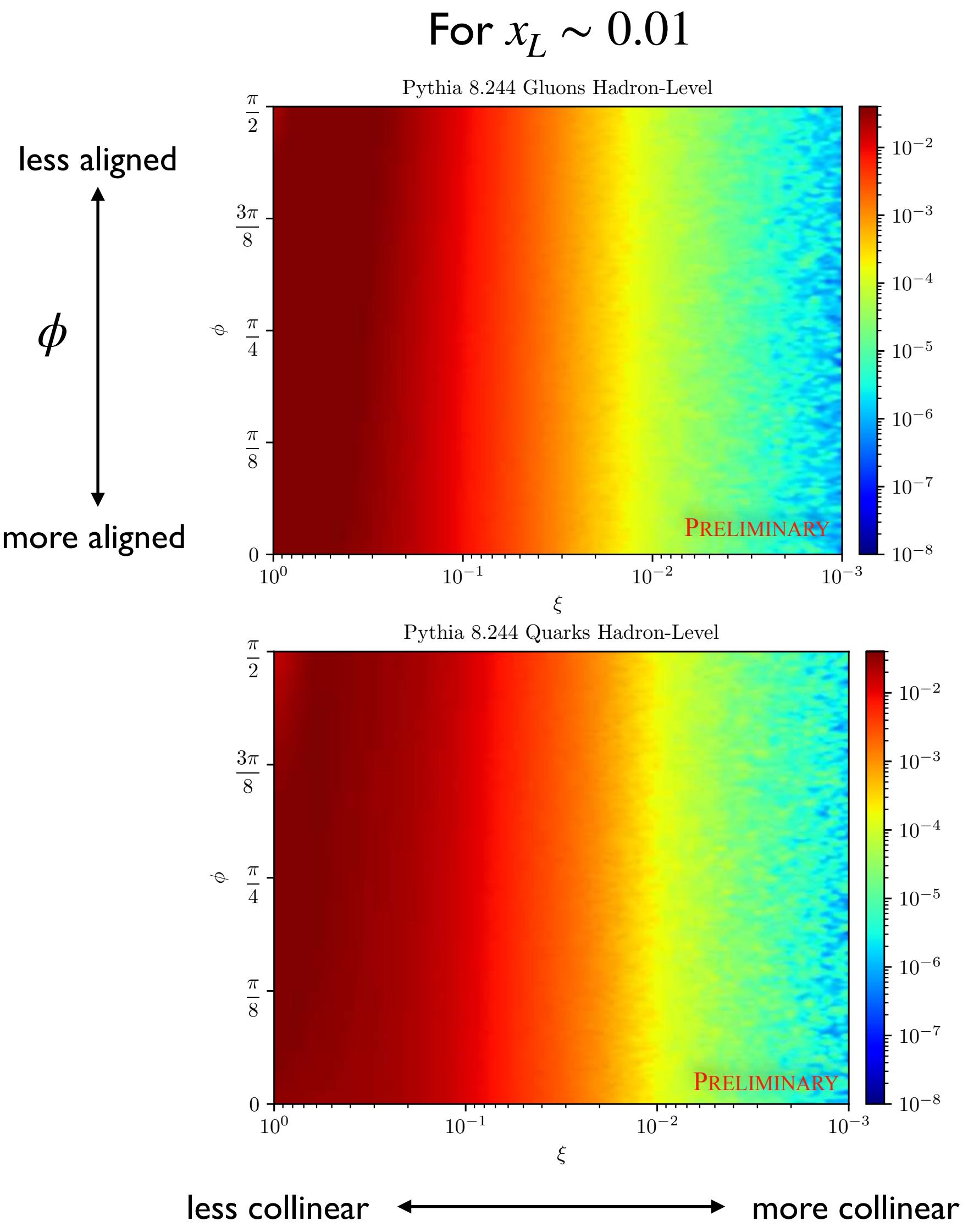


LL prediction of ratio



EEEC – Full Shape Dependence

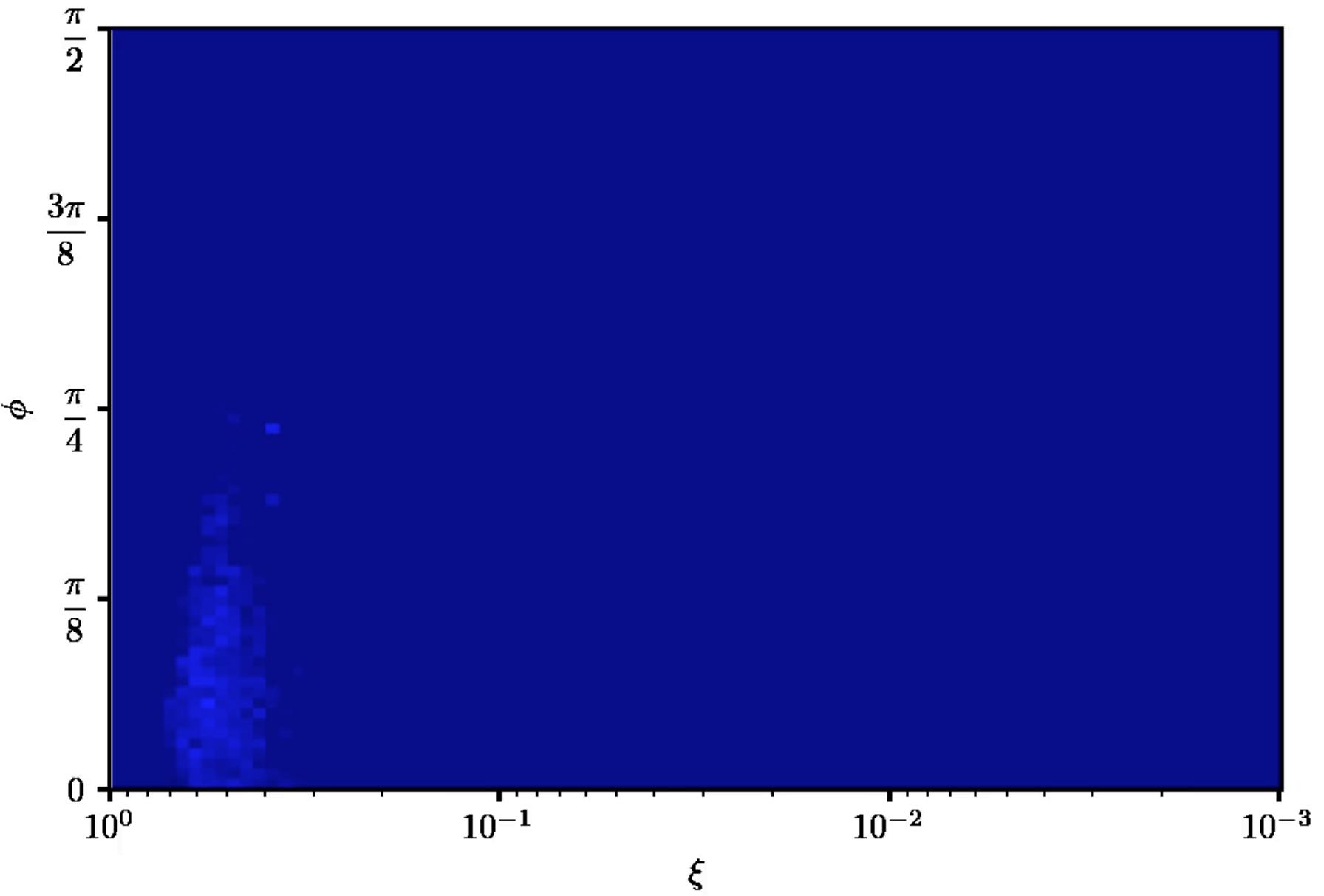
[PTK, Moult, Thaler, Zhu, to appear soon]



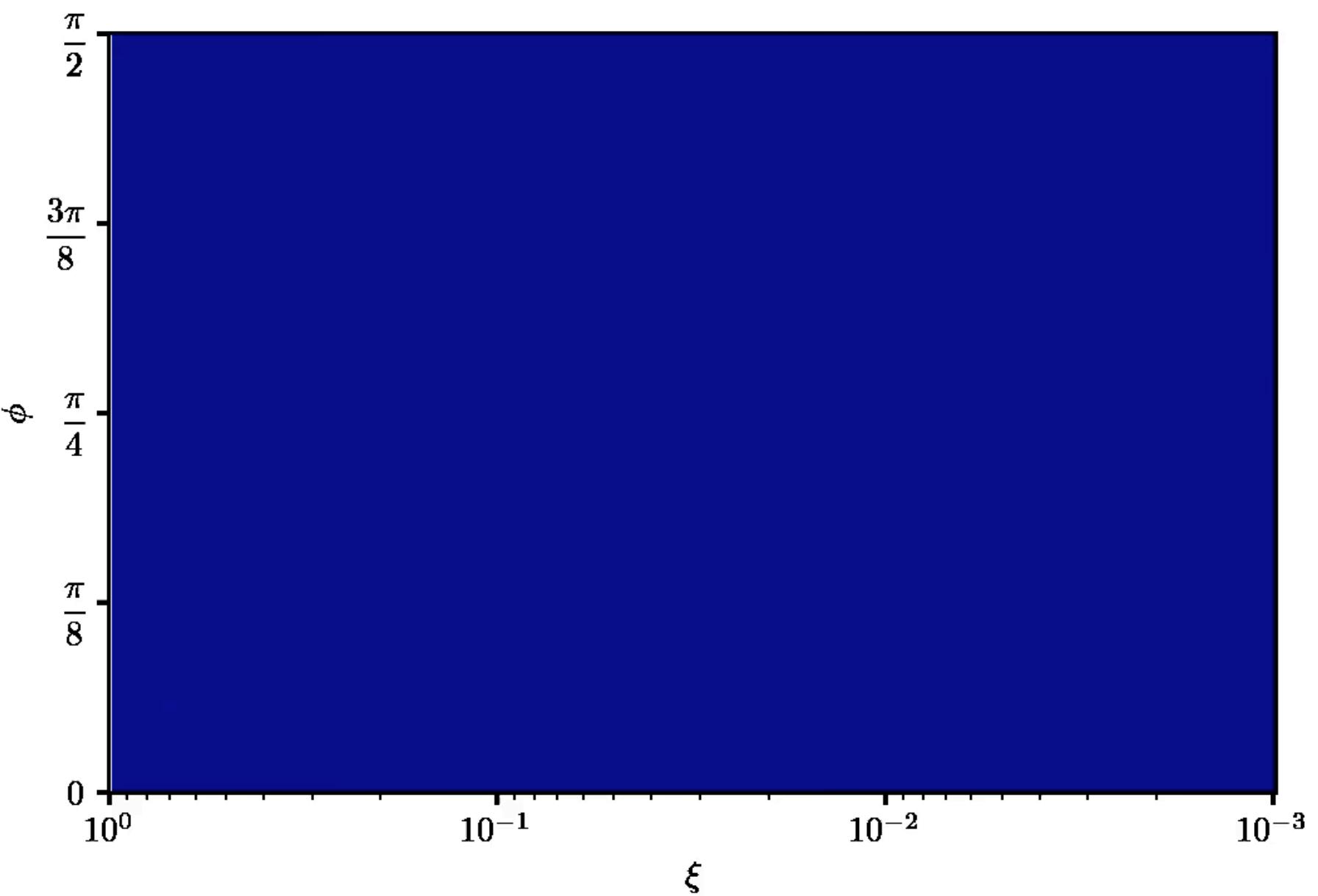
Visualizing the 3D EEEC

[PTK, Moult, Thaler, Zhu, to appear soon]

Pythia **Gluon** Jets
 $p_T \in [500,550]$ GeV



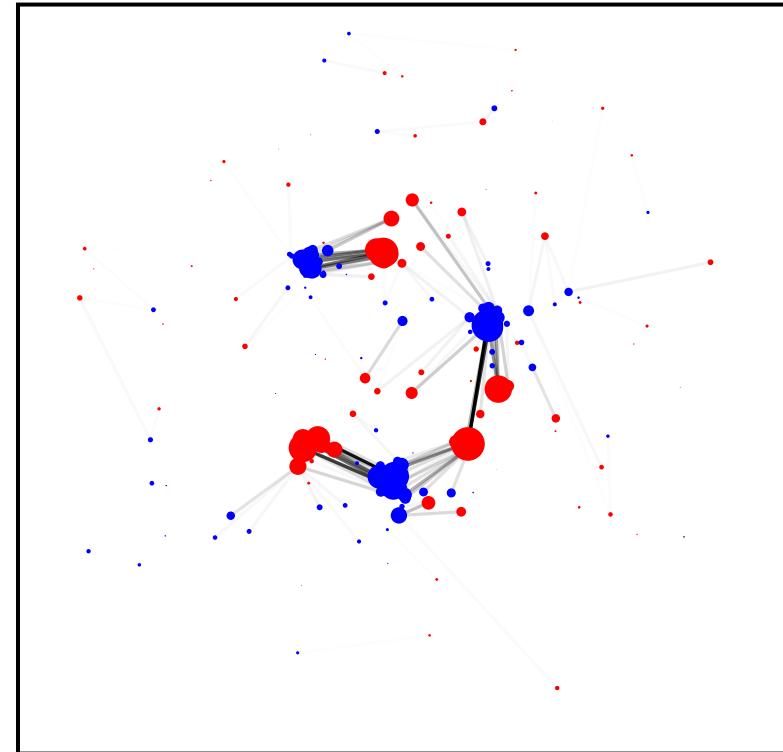
Pythia **Quark** Jets
 $p_T \in [500,550]$ GeV



Time in the videos corresponds to
 $\ln x_L$ going from 0 to $-\infty$

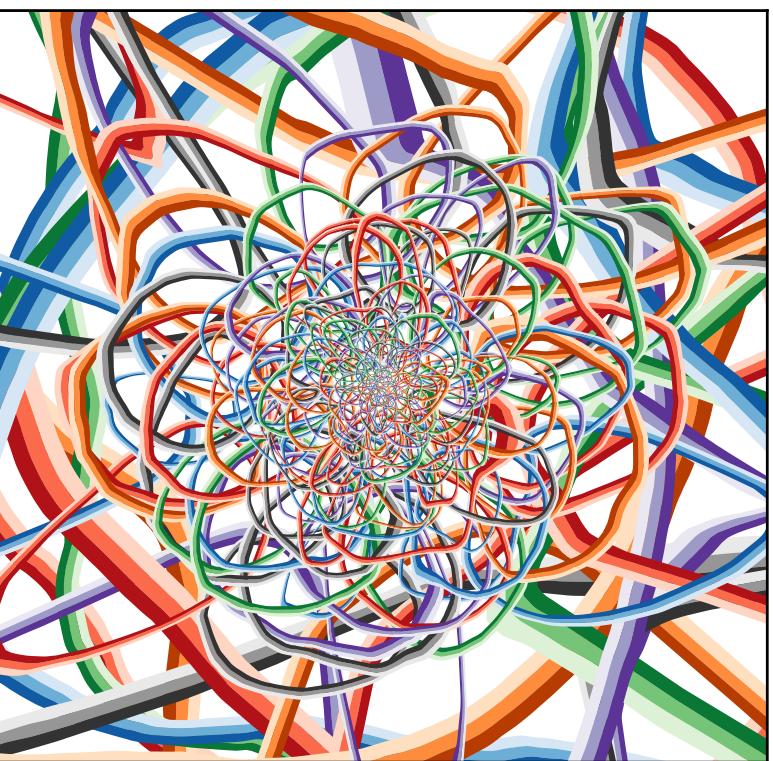
Color corresponds to log of EEC
(red is large, blue is small)

Uniformly persistent **red** is roughly the
perturbatively accessible region



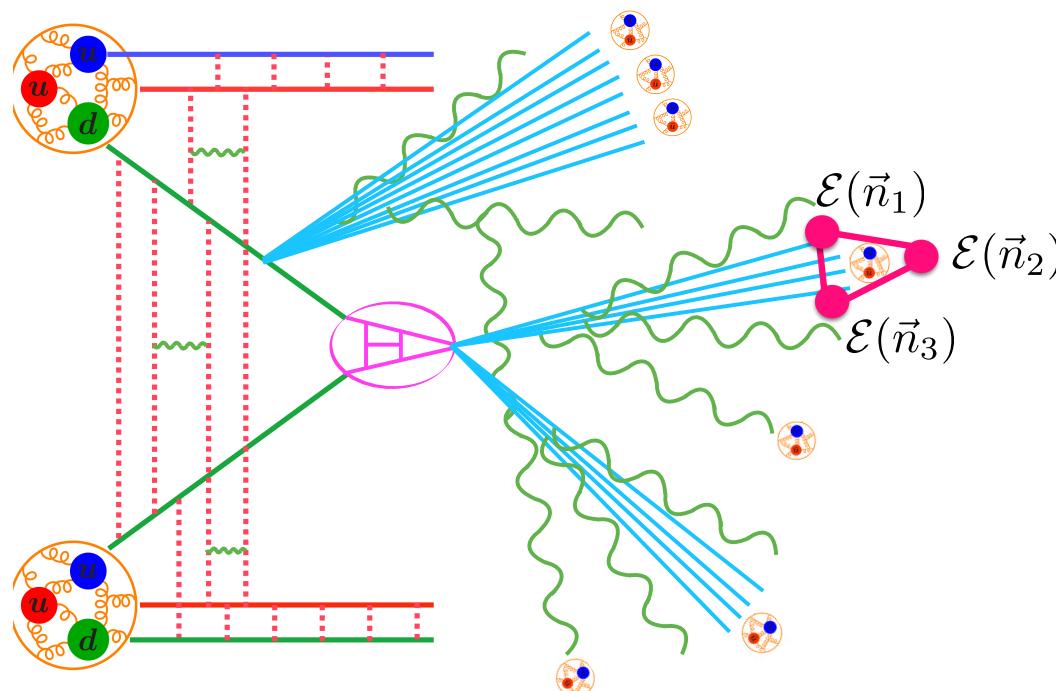
Collider Event Fundamentals

- Energy flow is theoretically and experimentally robust
- See [2004.04159](#) or [this talk](#) for a new geometric phrasing of many collider techniques



Energy Flow Observables

- EFPs provide a systematic basis of **IRC** observables and encompass
- EFMs allow for understanding EFP linear relations and fast computation
- EFNs ([1810.05165](#)) use ML to learn optimal **IRC**-safe observables automatically



Energy-Energy Correlators

Stay tuned!

- Single-log observables (no grooming!) probe detailed properties of **QCD**
- Benefit from jet substructure and conformal field theory insights
- Relatively simple to interface with track functions to compute on charged particles

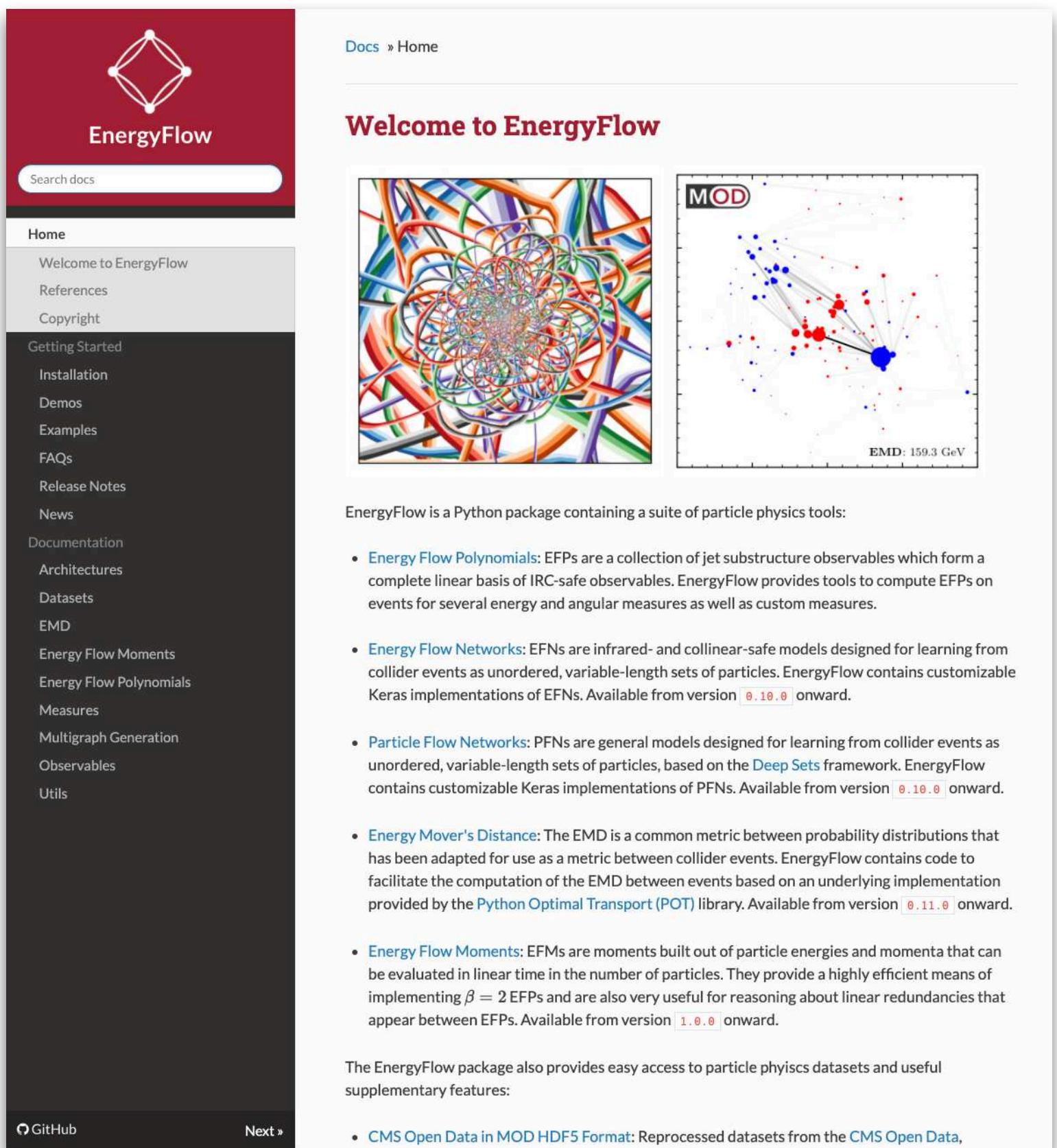
EnergyFlow Python Package

pip3 install energyflow

Computes EFPs efficiently via variable elimination and EFM s (for $\beta = 2$)

Detailed [examples](#), [demos](#), and [documentation](#)

Also computes EMD, constructs EFNs/PFNs, and loads datasets



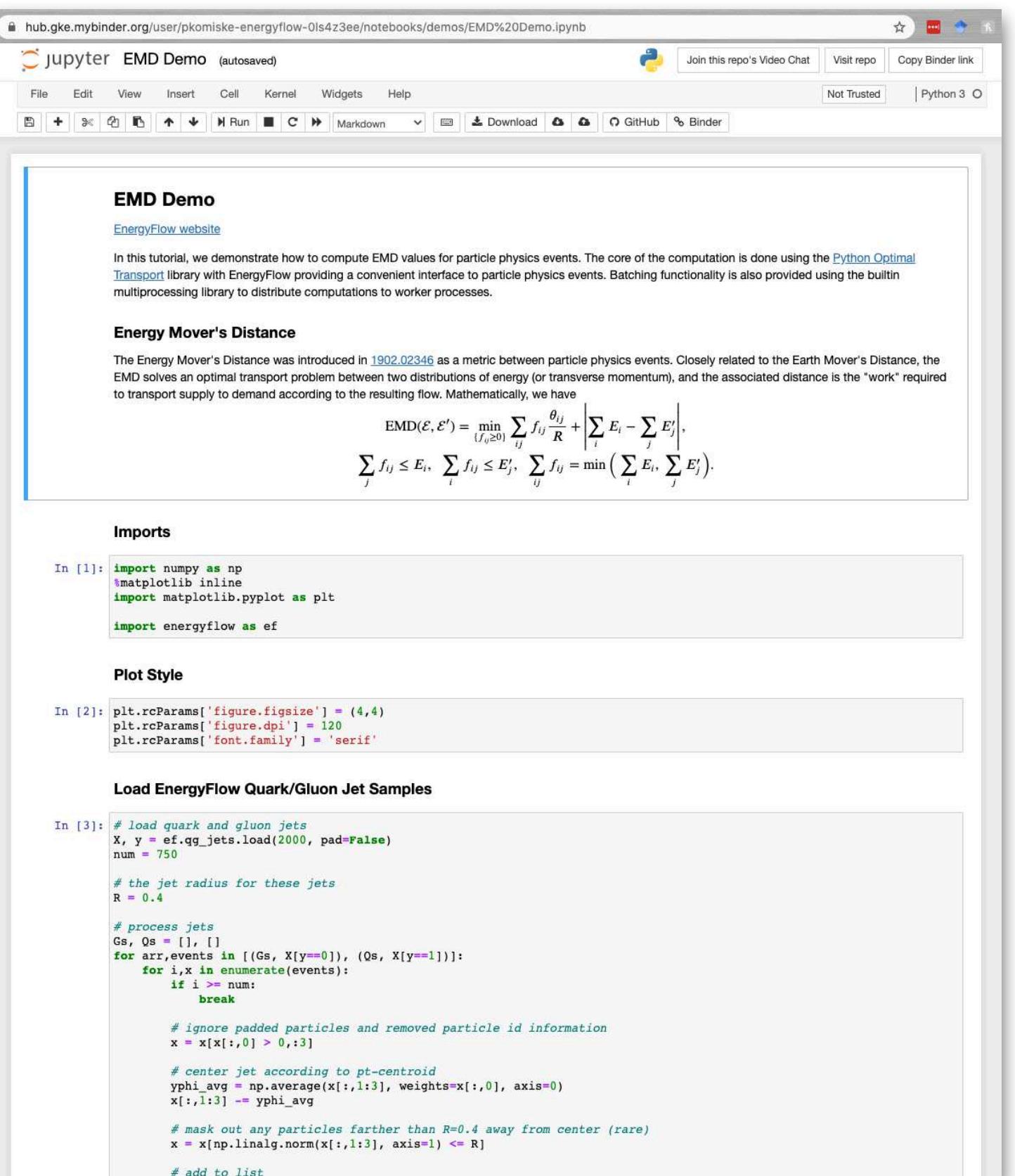
Welcome to EnergyFlow

EnergyFlow is a Python package containing a suite of particle physics tools:

- **Energy Flow Polynomials:** EFPs are a collection of jet substructure observables which form a complete linear basis of IRC-safe observables. EnergyFlow provides tools to compute EFPs on events for several energy and angular measures as well as custom measures.
- **Energy Flow Networks:** EFNs are infrared- and collinear-safe models designed for learning from collider events as unordered, variable-length sets of particles. EnergyFlow contains customizable Keras implementations of EFNs. Available from version 0.10.0 onward.
- **Particle Flow Networks:** PFNs are general models designed for learning from collider events as unordered, variable-length sets of particles, based on the Deep Sets framework. EnergyFlow contains customizable Keras implementations of PFNs. Available from version 0.10.0 onward.
- **Energy Mover's Distance:** The EMD is a common metric between probability distributions that has been adapted for use as a metric between collider events. EnergyFlow contains code to facilitate the computation of the EMD between events based on an underlying implementation provided by the Python Optimal Transport (POT) library. Available from version 0.11.0 onward.
- **Energy Flow Moments:** EFMs are moments built out of particle energies and momenta that can be evaluated in linear time in the number of particles. They provide a highly efficient means of implementing $\beta = 2$ EFPs and are also very useful for reasoning about linear redundancies that appear between EFPs. Available from version 1.0.0 onward.

The EnergyFlow package also provides easy access to particle physics datasets and useful supplementary features:

- CMS Open Data in MOD HDF5 Format: Reprocessed datasets from the CMS Open Data,



EMD Demo

In this tutorial, we demonstrate how to compute EMD values for particle physics events. The core of the computation is done using the [Python Optimal Transport](#) library with EnergyFlow providing a convenient interface to particle physics events. Batching functionality is also provided using the builtin multiprocessing library to distribute computations to worker processes.

Energy Mover's Distance

The Energy Mover's Distance was introduced in [1902.02346](#) as a metric between particle physics events. Closely related to the Earth Mover's Distance, the EMD solves an optimal transport problem between two distributions of energy (or transverse momentum), and the associated distance is the "work" required to transport supply to demand according to the resulting flow. Mathematically, we have

$$\text{EMD}(\mathcal{E}, \mathcal{E}') = \min_{\{f_{ij} \geq 0\}} \sum_{ij} f_{ij} \frac{\theta_{ij}}{R} + \left| \sum_i E_i - \sum_j E'_j \right|,$$
$$\sum_j f_{ij} \leq E_i, \quad \sum_i f_{ij} \leq E'_j, \quad \sum_{ij} f_{ij} = \min \left(\sum_i E_i, \sum_j E'_j \right).$$

Imports

```
In [1]: import numpy as np
%matplotlib inline
import matplotlib.pyplot as plt
import energyflow as ef
```

Plot Style

```
In [2]: plt.rcParams['figure.figsize'] = (4,4)
plt.rcParams['figure.dpi'] = 120
plt.rcParams['font.family'] = 'serif'
```

Load EnergyFlow Quark/Gluon Jet Samples

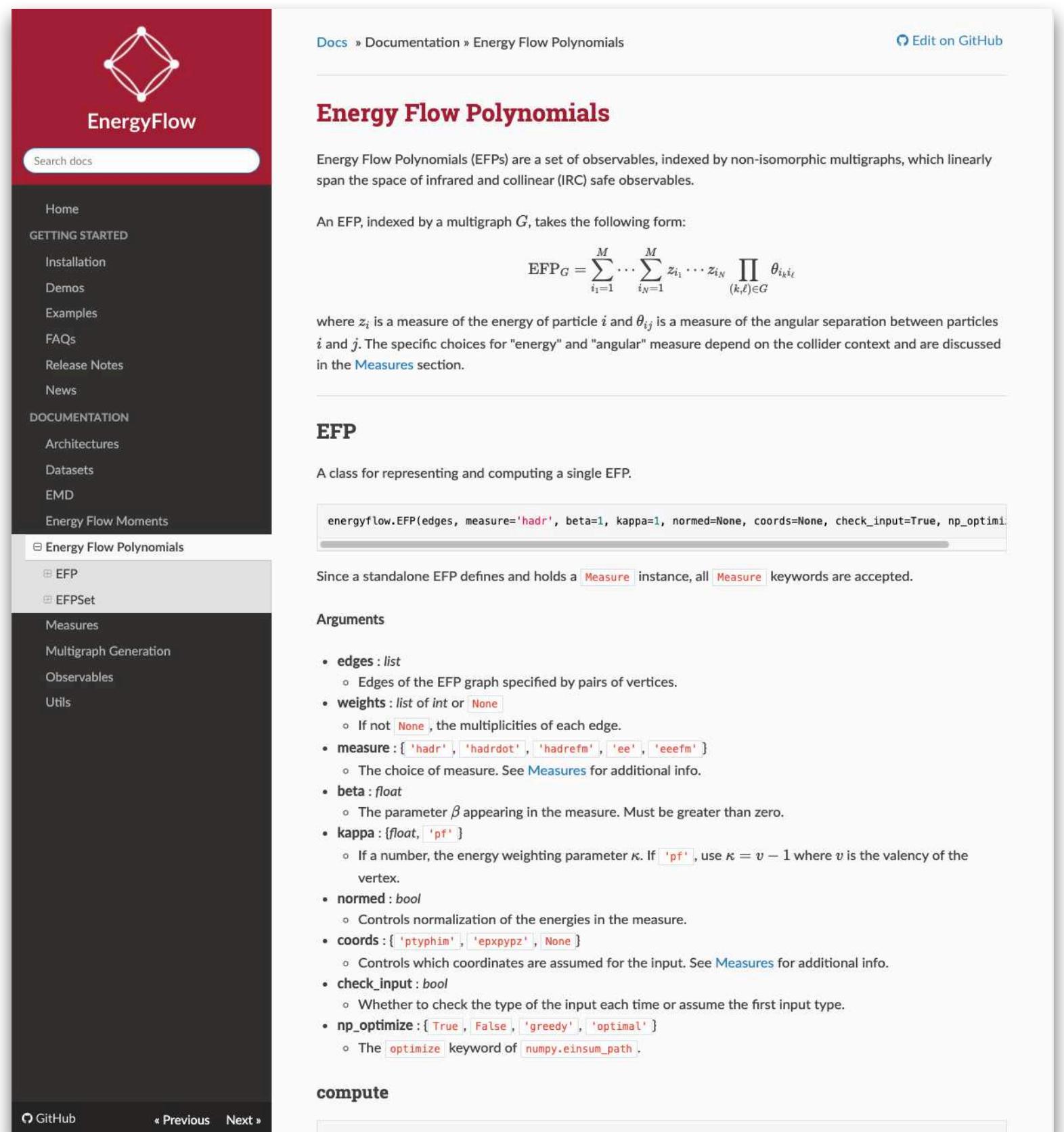
```
In [3]: # load quark and gluon jets
X, y = ef.qg_jets.load(2000, pad=False)
num = 750

# the jet radius for these jets
R = 0.4

# process jets
Gs, Qs = [], []
for arr,events in [(Gs, X[y==0]), (Qs, X[y==1])]:
    for i,x in enumerate(events):
        if i >= num:
            break
        # ignore padded particles and removed particle id information
        x = x[x[:,0] > 0,:3]

        # center jet according to pt-centroid
        yphi_avg = np.average(x[:,1:3], weights=x[:,0], axis=0)
        x[:,1:3] -= yphi_avg

        # mask out any particles farther than R=0.4 away from center (rare)
        x = x[np.linalg.norm(x[:,1:3], axis=1) <= R]
    # add to list
    Gs.append(Gs)
    Qs.append(Qs)
```



Energy Flow Polynomials

An EFP, indexed by a multigraph G , takes the following form:

$$\text{EFP}_G = \sum_{i_1=1}^M \cdots \sum_{i_N=1}^M z_{i_1} \cdots z_{i_N} \prod_{(k,l) \in G} \theta_{i_k i_l}$$

where z_i is a measure of the energy of particle i and θ_{ij} is a measure of the angular separation between particles i and j . The specific choices for "energy" and "angular" measure depend on the collider context and are discussed in the [Measures](#) section.

EFP

A class for representing and computing a single EFP.

```
energyflow.EFP(edges, measure='hadr', beta=1, kappa=1, normed=None, coords=None, check_input=True, np_optimi
```

Since a standalone EFP defines and holds a `Measure` instance, all `Measure` keywords are accepted.

Arguments

- **edges : list**
 - Edges of the EFP graph specified by pairs of vertices.
- **weights : list of int or None**
 - If not `None`, the multiplicities of each edge.
- **measure : { 'hadr', 'hadrdot', 'hadrefm', 'ee', 'eefm' }**
 - The choice of measure. See [Measures](#) for additional info.
- **beta : float**
 - The parameter β appearing in the measure. Must be greater than zero.
- **kappa : float**
 - If a number, the energy weighting parameter κ . If `'pf'`, use $\kappa = v - 1$ where v is the valency of the vertex.
- **normed : bool**
 - Controls normalization of the energies in the measure.
- **coords : { 'pythia', 'exppyz', 'None' }**
 - Controls which coordinates are assumed for the input. See [Measures](#) for additional info.
- **check_input : bool**
 - Whether to check the type of the input each time or assume the first input type.
- **np_optimize : { True, False, 'greedy', 'optimal' }**
 - The `optimize` keyword of `numpy.einsum_path`.

compute

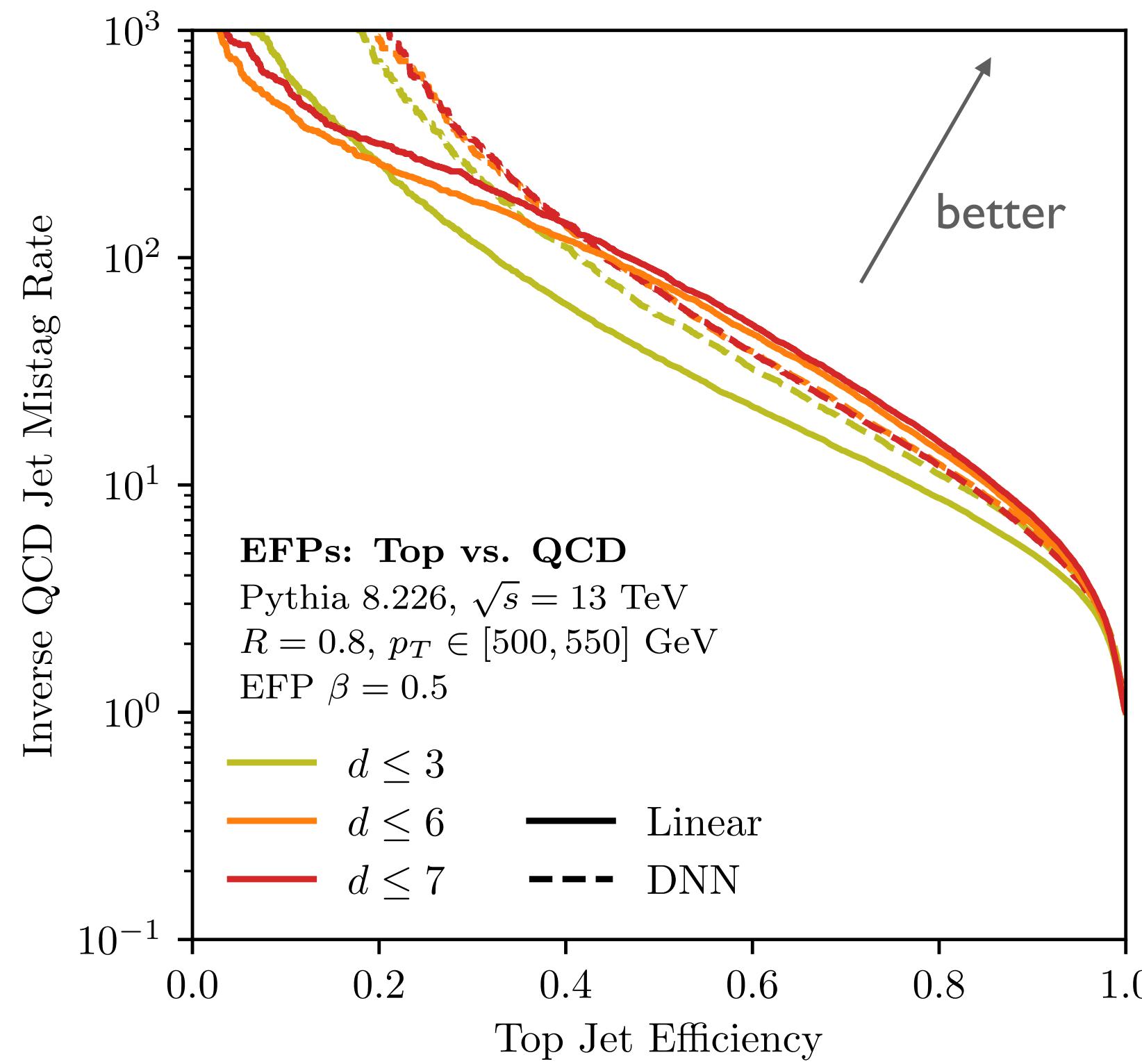
```
compute(event=None, zc=None, theta=None, obeta=None)
```

Additional Slides

Rewriting General EFP as Contraction of EFMs

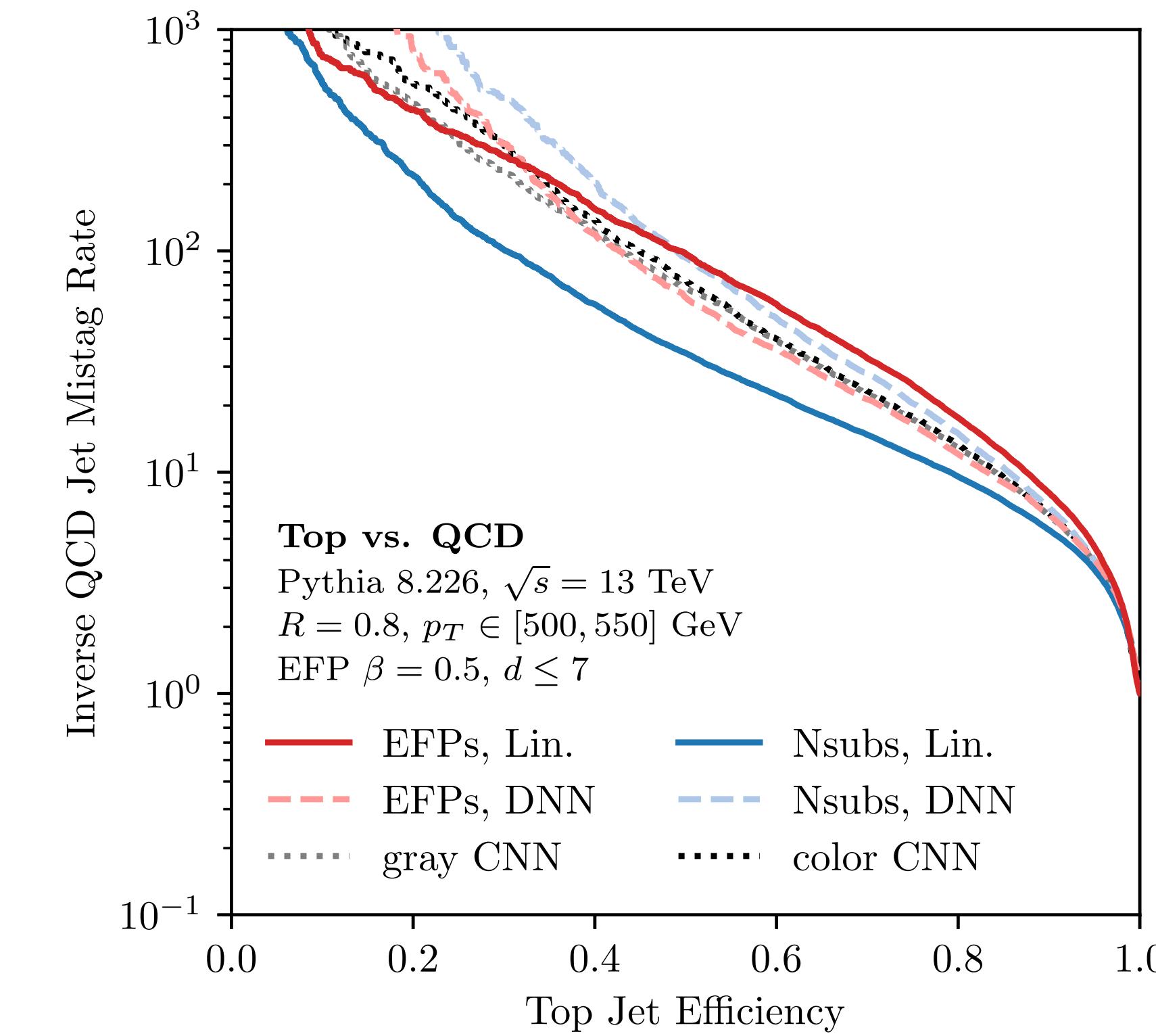
$$\begin{aligned}\text{EFP}_G &= \sum_{i_1=1}^M \cdots \sum_{i_N=1}^M E_{i_1} \cdots E_{i_N} \prod_{(k,\ell) \in G} 2\eta_{\mu\nu} n_{i_k}^\mu n_{i_\ell}^\nu \\ &= \left(\prod_{j=1}^N \sum_{i_j=1}^M z_{i_j} n_{i_j}^{\mu_1^j} n_{i_j}^{\mu_2^j} \cdots n_{i_j}^{\mu_{v_j}^j} \right) \prod_{(k,\ell) \in G} 2\eta_{\mu_{A_{k\ell}}^k \mu_{A_{\ell k}}^\ell} \\ &= \left(\prod_{j=1}^N \mathcal{I}^{\mu_1^j \mu_2^j \cdots \mu_{v_j}^j} \right) \prod_{(k,\ell) \in G} 2\eta_{\mu_{A_{k\ell}}^k \mu_{A_{\ell k}}^\ell},\end{aligned}$$

Boosted Top: EFP Classification Performance Comparison



Saturation observed with more EFPs

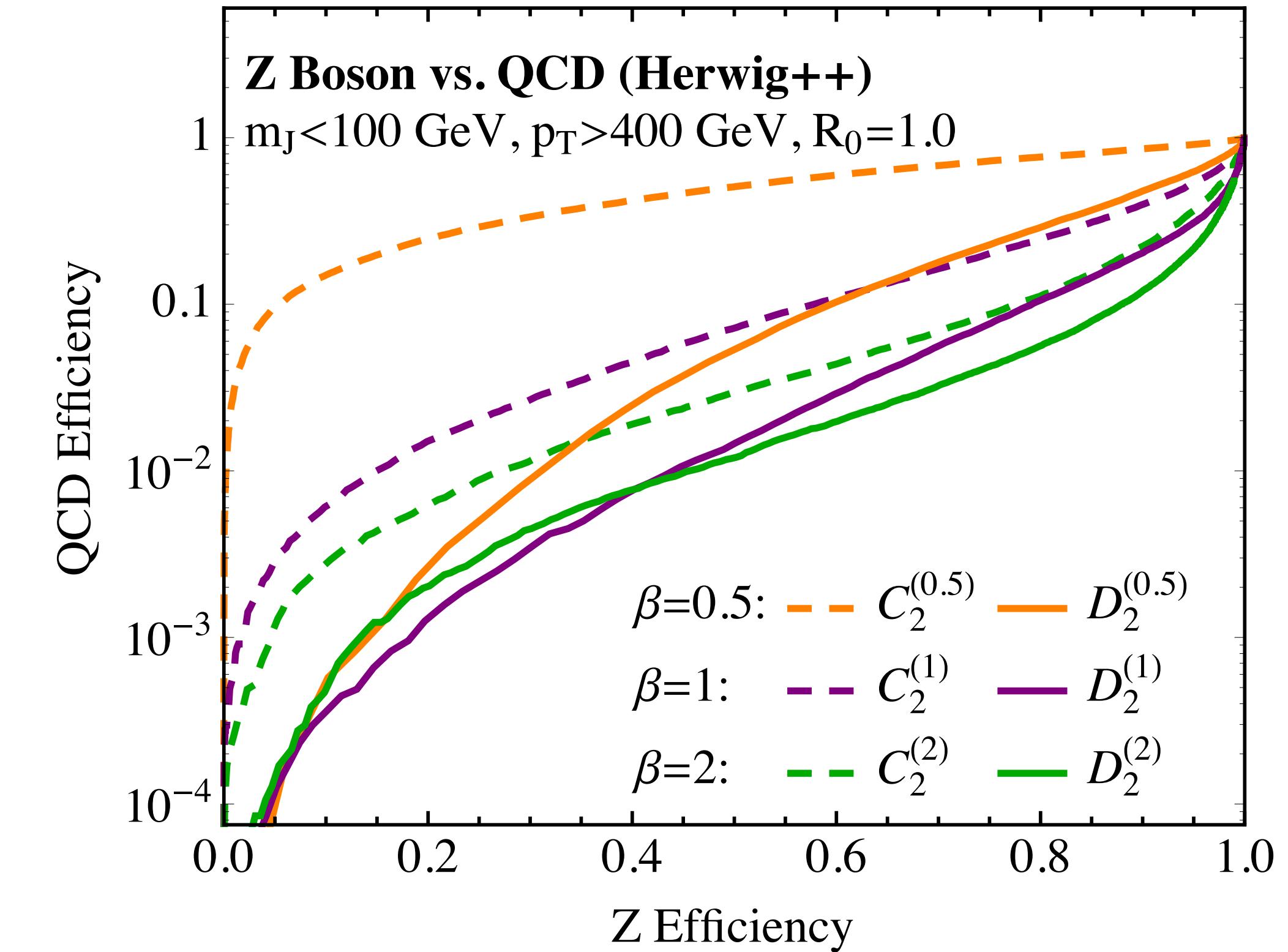
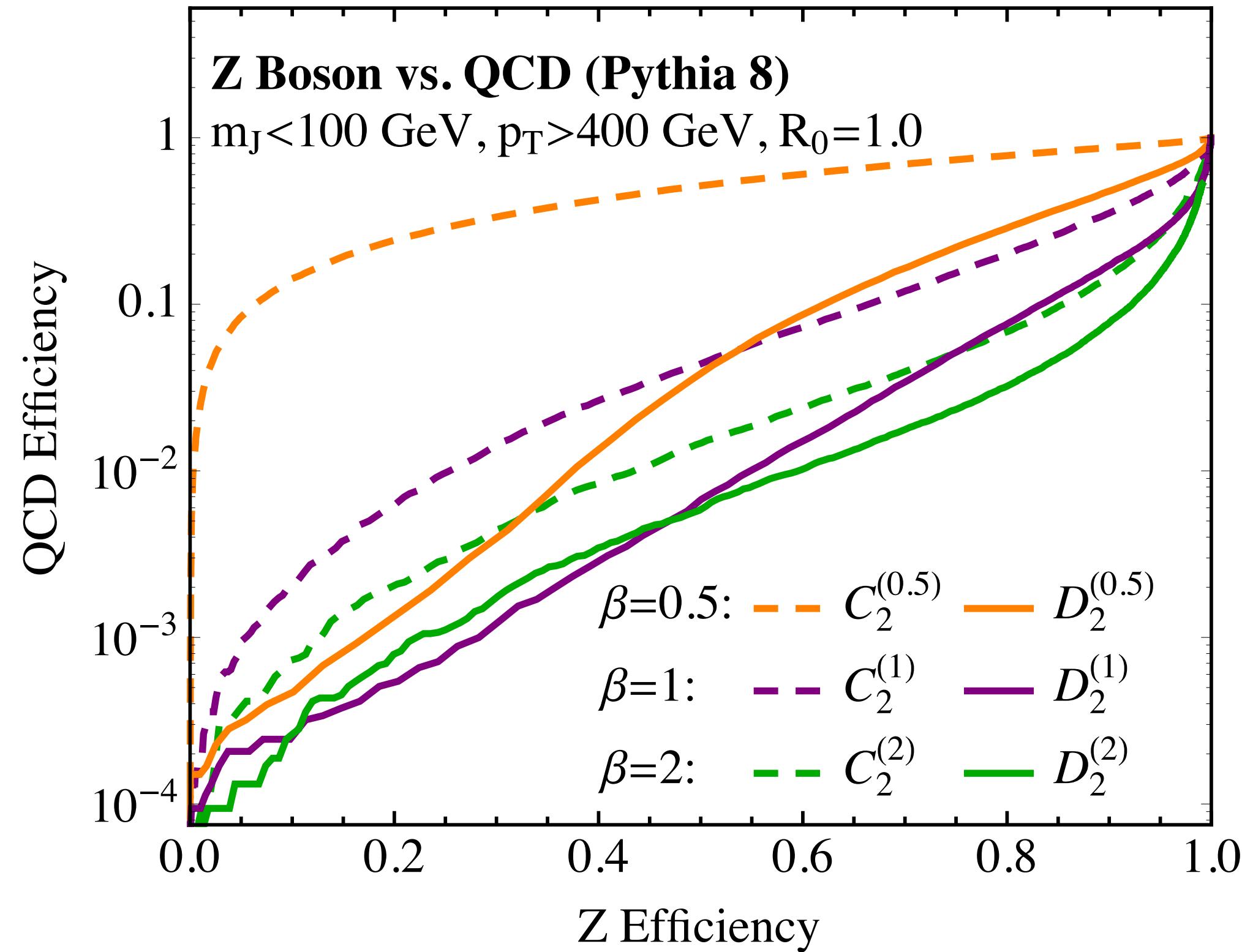
DNN gets there faster but linear suffices



Linear EFPs excel at high efficiency

[de Oliveira, Kagan, Mackey, Nachman, Schwartzman, 2015; PTK, Metodiev, Schwartz, 2016; Datta, Larkoski, 2017]

Two-Prong Classification with Varying β



$\beta = 2$ for both D_2 and C_2 for both Pythia 8 and Herwig++ works better than $\beta = 1$ for Z vs. QCD

[Larkoski, Moult, Neill, [JHEP09\(2014\)029](#)]

Table of Euclidean-Lorentz Identities

$$0 = 6 \begin{array}{c} \text{diamond} \\ \text{yellow} \end{array} - 16 \begin{array}{c} \text{triangle} \\ \text{yellow} \end{array} - 3 \begin{array}{c} \text{double loop} \\ \text{yellow} \end{array} + 24 \begin{array}{c} \text{double loop} \\ \text{yellow} \end{array} - 16,$$

$$0 = 6 \begin{array}{c} \text{diamond} \\ \text{yellow} \end{array} - 12 \begin{array}{c} \text{triangle} \\ \text{yellow} \end{array} - 3 \begin{array}{c} \text{double loop} \\ \text{yellow} \end{array} - 2 \begin{array}{c} \text{triangle} \\ \text{black} \end{array} + 12 \begin{array}{c} \text{triangle} \\ \text{black} \end{array} + 6 \begin{array}{c} \text{double loop} \\ \text{black} \end{array} - 8 \begin{array}{c} \text{double loop} \\ \text{black} \end{array},$$

$$0 = 6 \begin{array}{c} \text{diamond} \\ \text{red} \end{array} + 16 \begin{array}{c} \text{triangle} \\ \text{red} \end{array} - 3 \begin{array}{c} \text{double loop} \\ \text{red} \end{array} - 48 \begin{array}{c} \text{triangle} \\ \text{red} \end{array} + 24 \begin{array}{c} \text{double loop} \\ \text{red} \end{array},$$

$$0 = 6 \begin{array}{c} \text{diamond} \\ \text{red} \end{array} - 12 \begin{array}{c} \text{triangle} \\ \text{red} \end{array} - 3 \begin{array}{c} \text{double loop} \\ \text{red} \end{array} - 2 \begin{array}{c} \text{triangle} \\ \text{black} \end{array} + 4 \begin{array}{c} \text{triangle} \\ \text{black} \end{array} + 6 \begin{array}{c} \text{double loop} \\ \text{black} \end{array}.$$

	$d=0$	$d=1$	$d=2$	$d=3$	\dots
	1				
	2	-1			
	4	-4	1		
	4	-4	1		
	4	-4	1		
	8	-12	6	-1	
	8	-12	6		
	8	-12	2 4	-1	
	8	-12	6		
	8	-12	4 2		
	8	-12	2 4		
	8	-12	2 4	-1	
	8	-12	6		
	16	-32	24	-8	1
	16	-32	8 16		
	16	-32	4 20	-4 -4	
	16	-32	12 12	-2 -6	
	16	-32	16 8		
	16	-32	20 4	-2 -2 -4	
	16	-32	4 16 4	-4 -4 -4	
	16	-32	4 12 8	-2 -4 -4	
	16	-32	4 20	-4 -4 -8	
	16	-32	24		
	16	-32	16 8	-2 -4 -2	
	16	-32	12 12	-4 -4 -4	
	16	-32			
	16	-32			

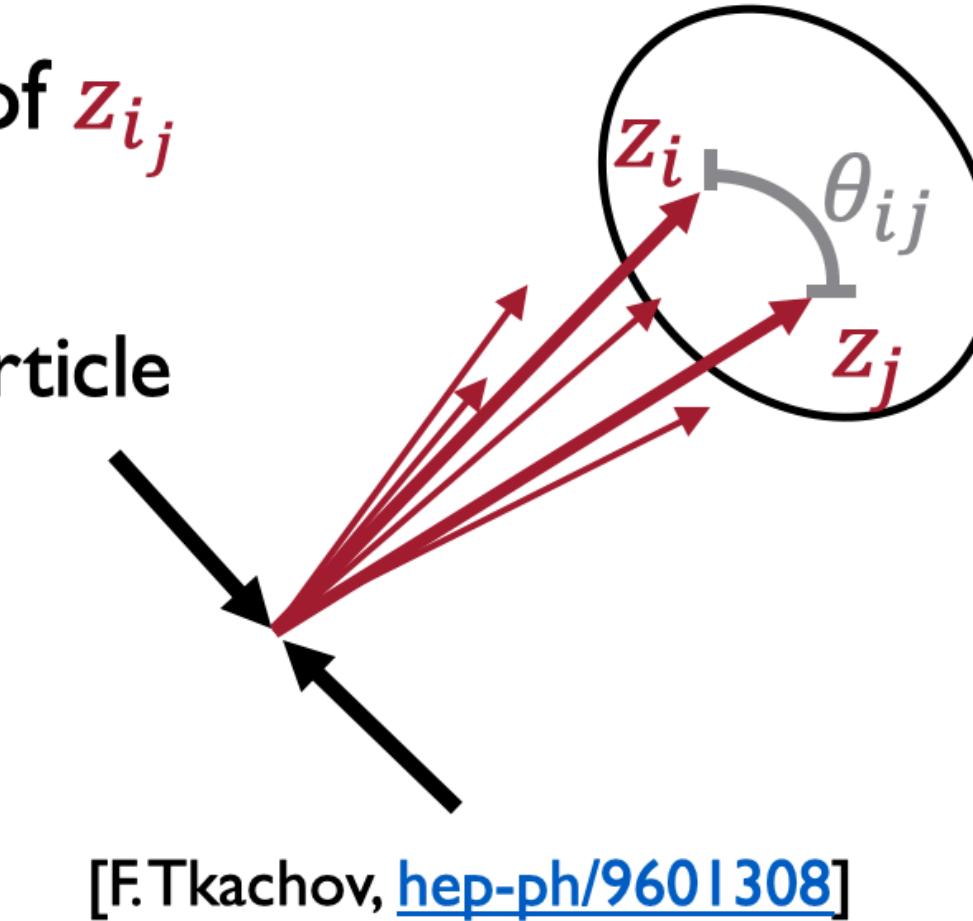
Derivation of EFP Linear Spanning Basis

Arbitrary **IRC**-safe observable: $S(p_1^\mu, \dots, p_M^\mu)$

- **Energy expansion***: Approximate S with polynomials of z_{ij}
 - **IR safety**: S is unchanged under addition of soft particle
 - **C safety**: S is unchanged under collinear splitting of a particle
 - **Relabeling symmetry**: Particle index is arbitrary

Energy correlator parametrized
by angular function f

$$\sum_{i_1=1}^M \dots \sum_{i_N=1}^M z_{i_1} \dots z_{i_N} f(\hat{p}_{i_1}, \dots, \hat{p}_{i_N})$$



→ Energy correlators linearly span **IRC**-safe observables

- **Angular expansion***: Approximate f with polynomials in θ_{ij}
- **Simplify**: Identify unique analytic structure that emerge

→ Linear spanning basis in terms of “EFPs” has been found!

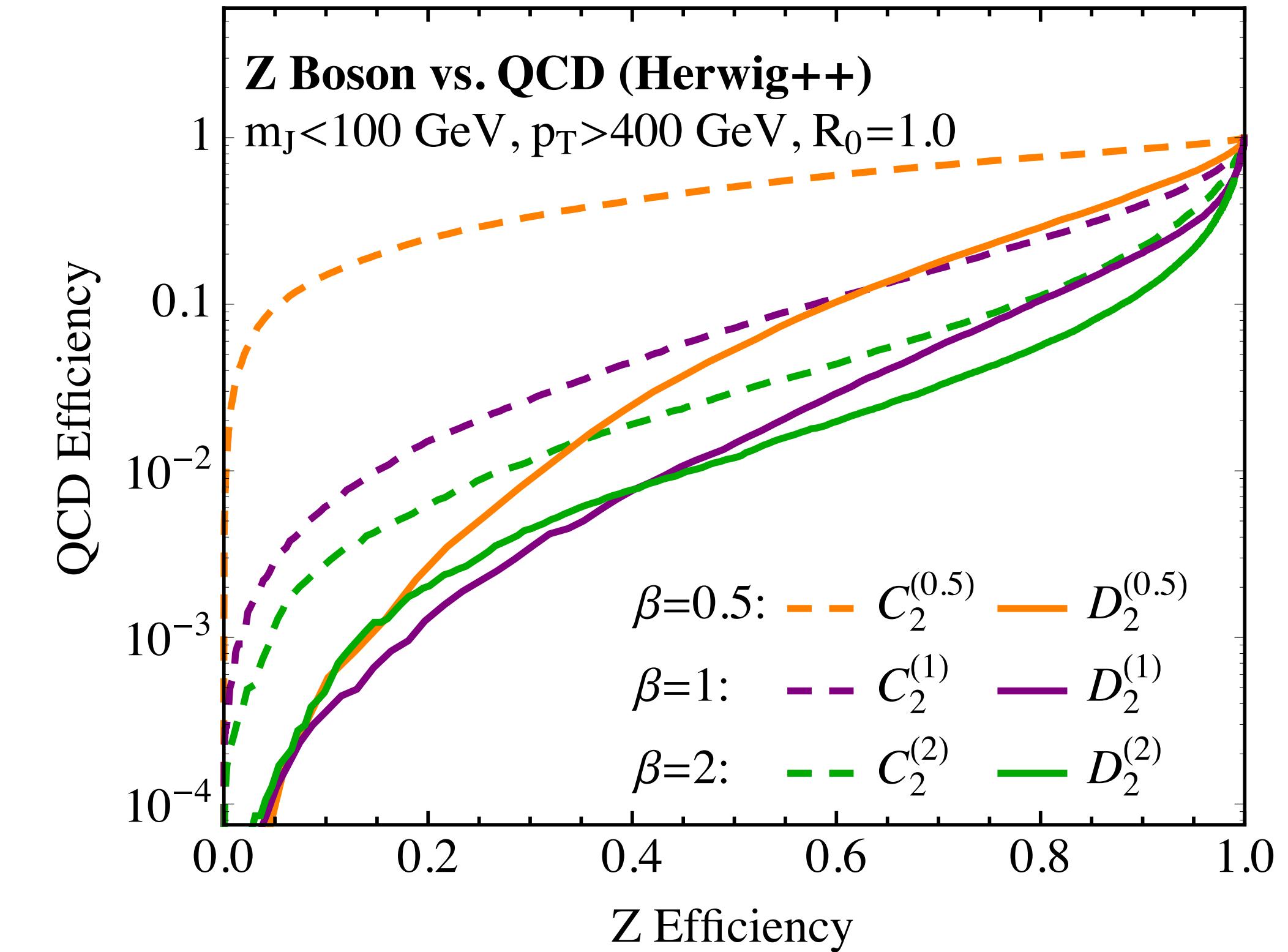
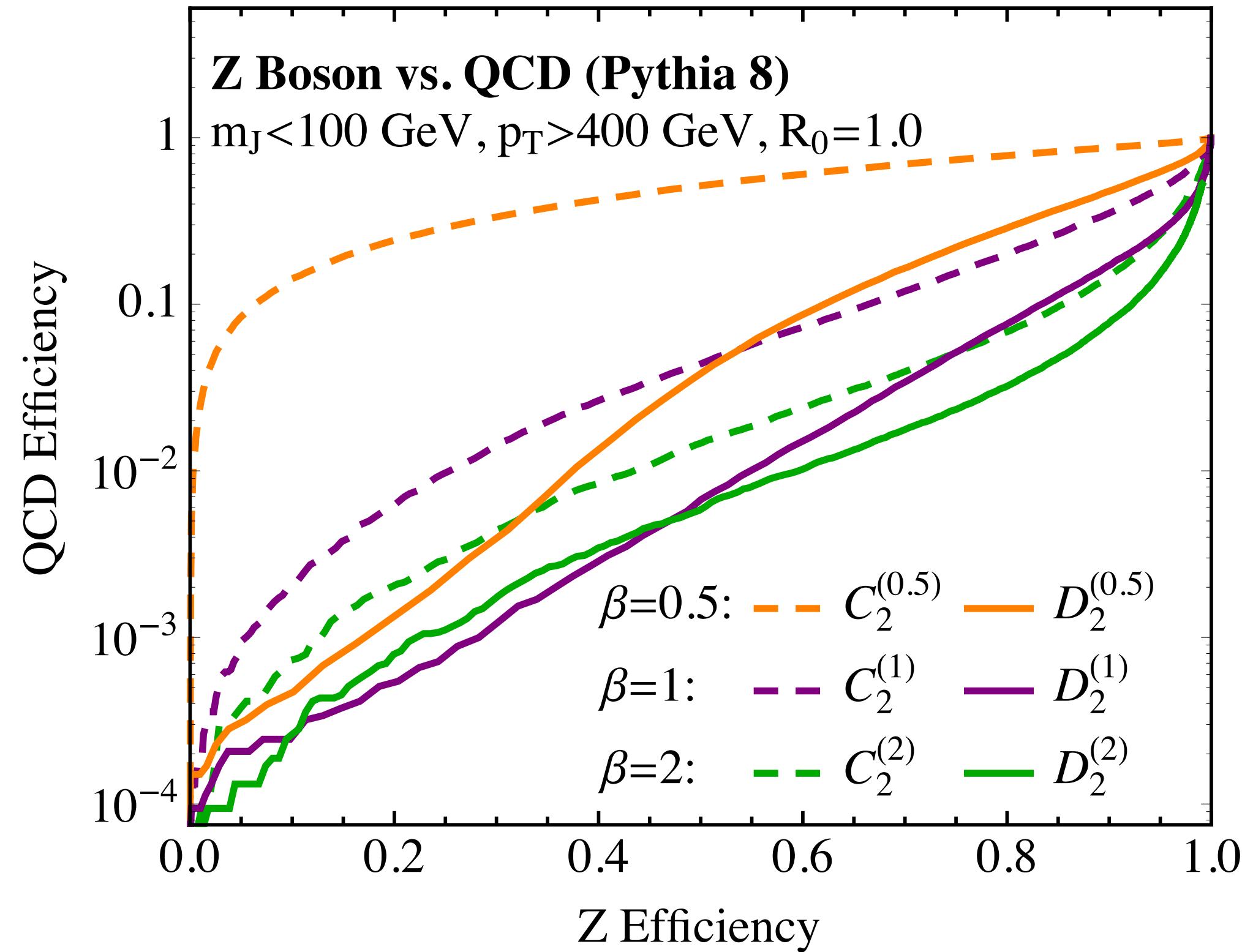
$$S \simeq \sum_{G \in G} s_G \text{EFP}_G, \quad \text{EFP}_G \equiv \sum_{i_1=1}^M \dots \sum_{i_N=1}^M z_{i_1} \dots z_{i_N} \prod_{(k,\ell) \in G} \theta_{i_k i_\ell}$$

*Generically, approximations exist by the Stone-Weierstrass theorem

Rewriting General EFP as Contraction of EFMs

$$\begin{aligned}
 \text{EFP}_G &= \sum_{i_1=1}^M \cdots \sum_{i_N=1}^M E_{i_1} \cdots E_{i_N} \prod_{(k,\ell) \in G} 2\eta_{\mu\nu} n_{i_k}^\mu n_{i_\ell}^\nu \\
 &= \left(\prod_{j=1}^N \sum_{i_j=1}^M z_{i_j} n_{i_j}^{\mu_1^j} n_{i_j}^{\mu_2^j} \cdots n_{i_j}^{\mu_{v_j}^j} \right) \prod_{(k,\ell) \in G} 2\eta_{\mu_{A_{k\ell}}^k \mu_{A_{\ell k}}^\ell} \\
 &= \left(\prod_{j=1}^N \mathcal{I}^{\mu_1^j \mu_2^j \cdots \mu_{v_j}^j} \right) \prod_{(k,\ell) \in G} 2\eta_{\mu_{A_{k\ell}}^k \mu_{A_{\ell k}}^\ell},
 \end{aligned}$$

Two-Prong Classification with Varying β



$\beta = 2$ for both D_2 and C_2 for both Pythia 8 and Herwig++ works better than $\beta = 1$ for Z vs. QCD

[Larkoski, Moult, Neill, [JHEP09\(2014\)098](#)]

Table of Euclidean-Lorentz Identities

Valid only in e^+e^- collisions

$$0 = 6 \text{ (diagram)} - 16 \text{ (diagram)} - 3 \text{ (diagram)} + 24 \text{ (diagram)} - 16,$$

$$0 = 6 \text{ (diagram)} - 12 \text{ (diagram)} - 3 \text{ (diagram)} - 2 \text{ (diagram)} + 12 \text{ (diagram)} + 6 \text{ (diagram)} - 8 \text{ (diagram)},$$

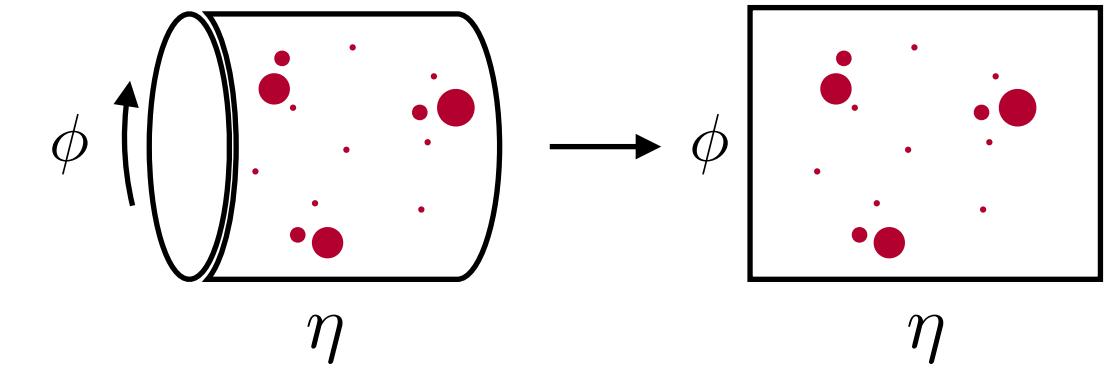
$$0 = 6 \text{ (diagram)} + 16 \text{ (diagram)} - 3 \text{ (diagram)} - 48 \text{ (diagram)} + 24 \text{ (diagram)},$$

$$0 = 6 \text{ (diagram)} - 12 \text{ (diagram)} - 3 \text{ (diagram)} - 2 \text{ (diagram)} + 4 \text{ (diagram)} + 6 \text{ (diagram)}.$$

	$d=0$	$d=1$	$d=2$	$d=3$	\dots
•	1				
	2	-1			
○	4	-4	1		
△	4	-4	1		
	4	-4	1		
○○	8	-12	6	-1	
△○	8	-12	6		
○△	8	-12	2	-1	
○○○	8	-12	6		
○○△	8	-12	4	-1	
○△○	8	-12	2	-1	
○○○○	8	-12	4		
○○○△	8	-12	2	-1	
○○△○	8	-12	4		
○△○○	8	-12	2	-1	
○○○○○	8	-12	6		
○○○○△	16	-32	24	-8	1
○○○△○	16	-32	8	16	
○○△○○	16	-32	4	20	
○○○○○○	16	-32	12	12	
○○○○○△	16	-32	16	8	
○○○○△○	16	-32	20	4	
○○○△○○	16	-32	4	16	
○○○○○○○	16	-32	4	12	
○○○○○○△	16	-32	4	20	
○○○○○△○	16	-32	24		
○○○○△○○	16	-32	16	8	
○○○○○○○○	16	-32	12	12	
⋮	⋮	⋮	⋮	⋮	⋮

Explicit Geometry – Events as Distributions of Energy

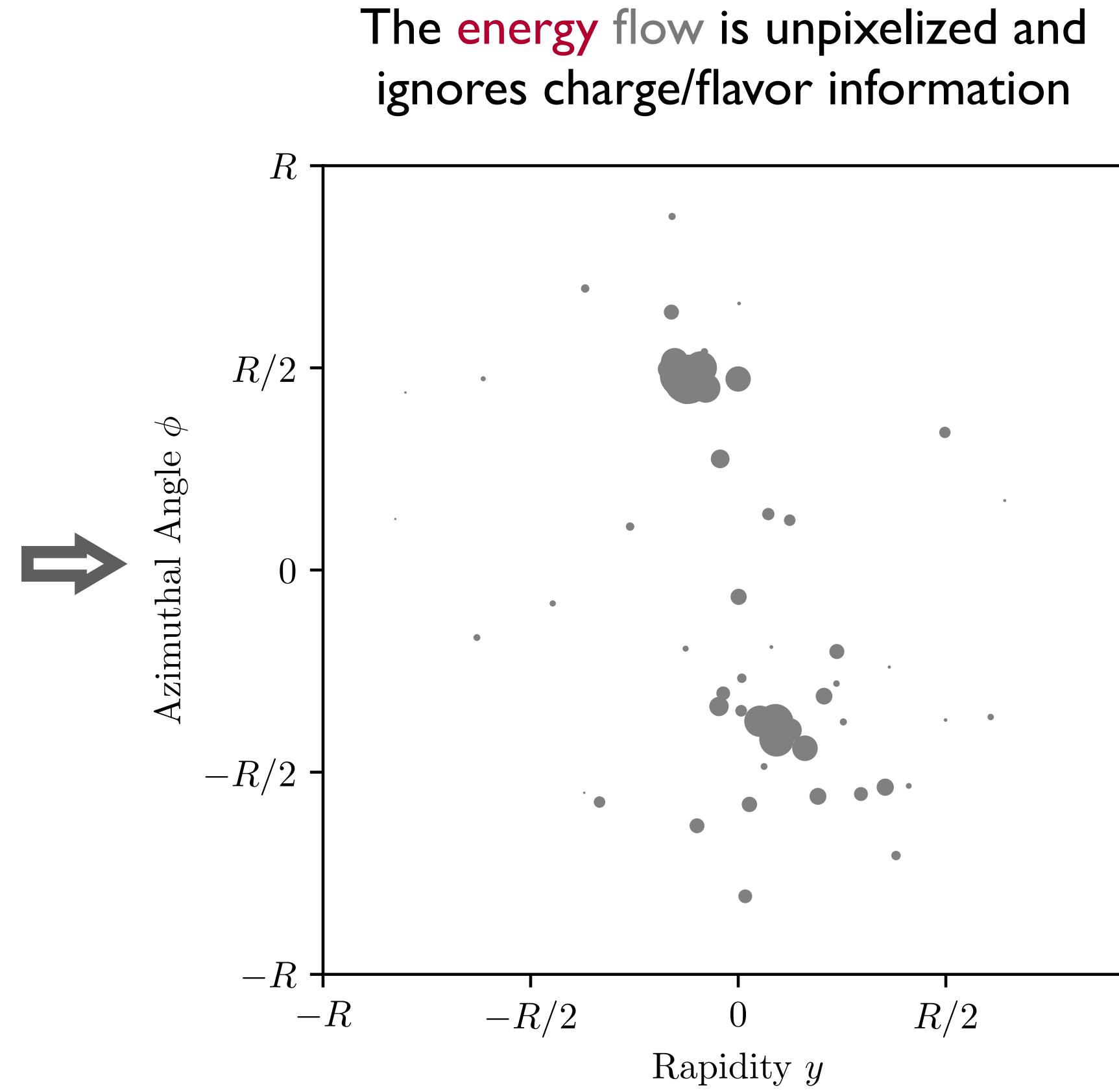
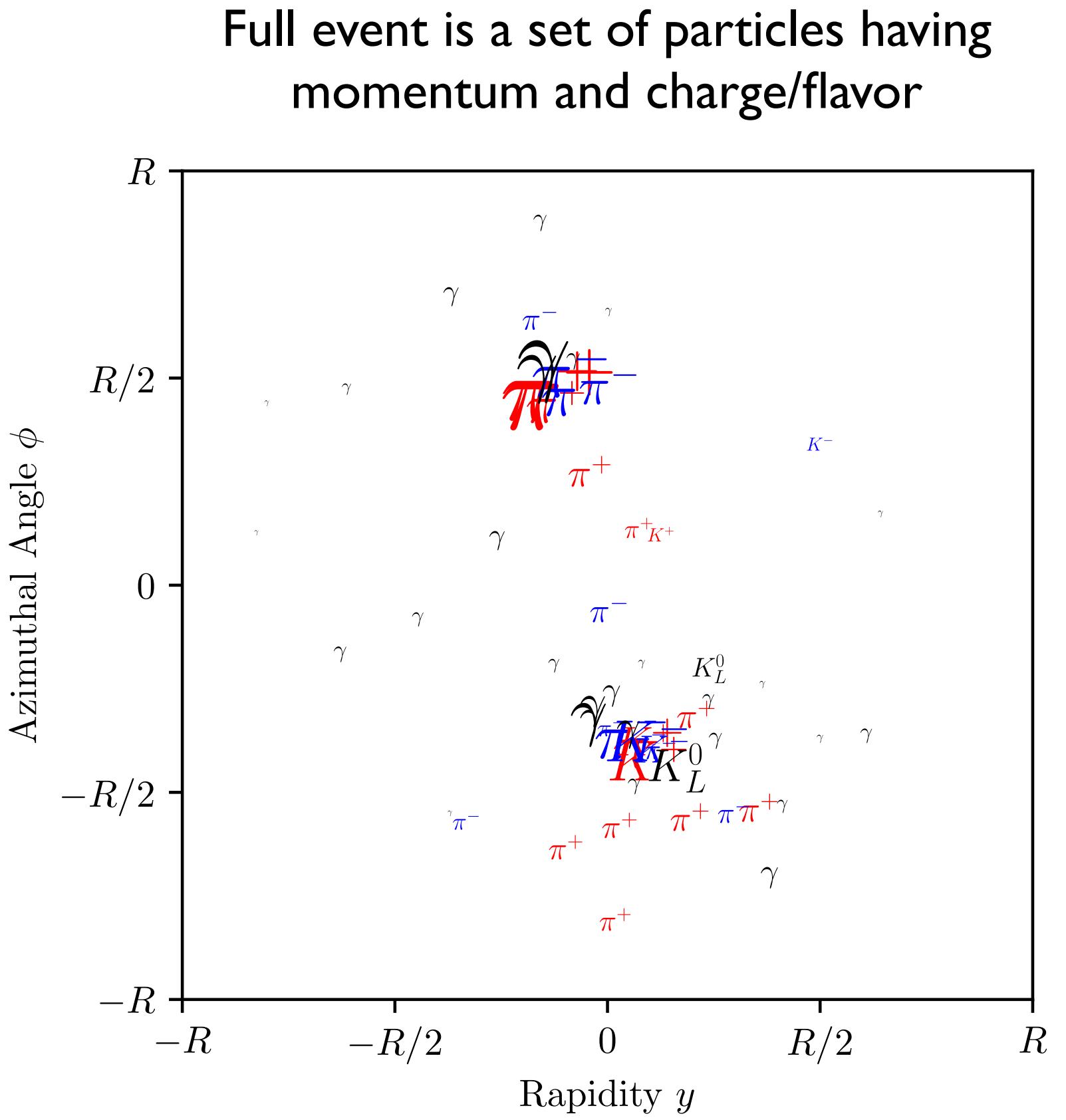
[PTK, Metodiev, Thaler, JHEP 2019; PTK, Metodiev, Thaler, JHEP 2020]



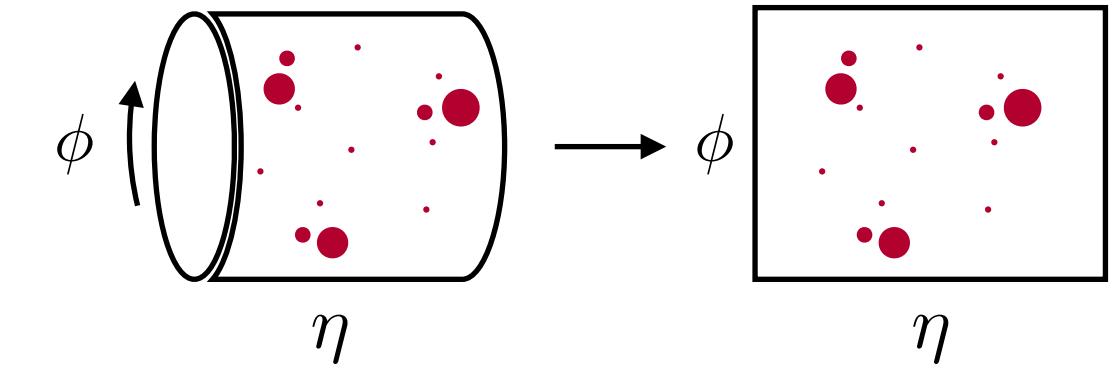
Energy flow distribution fully captures IRC-safe information

$$\mathcal{E}(\hat{n}) = \sum_{i=1}^M E_i \delta(\hat{n} - \hat{n}_i)$$

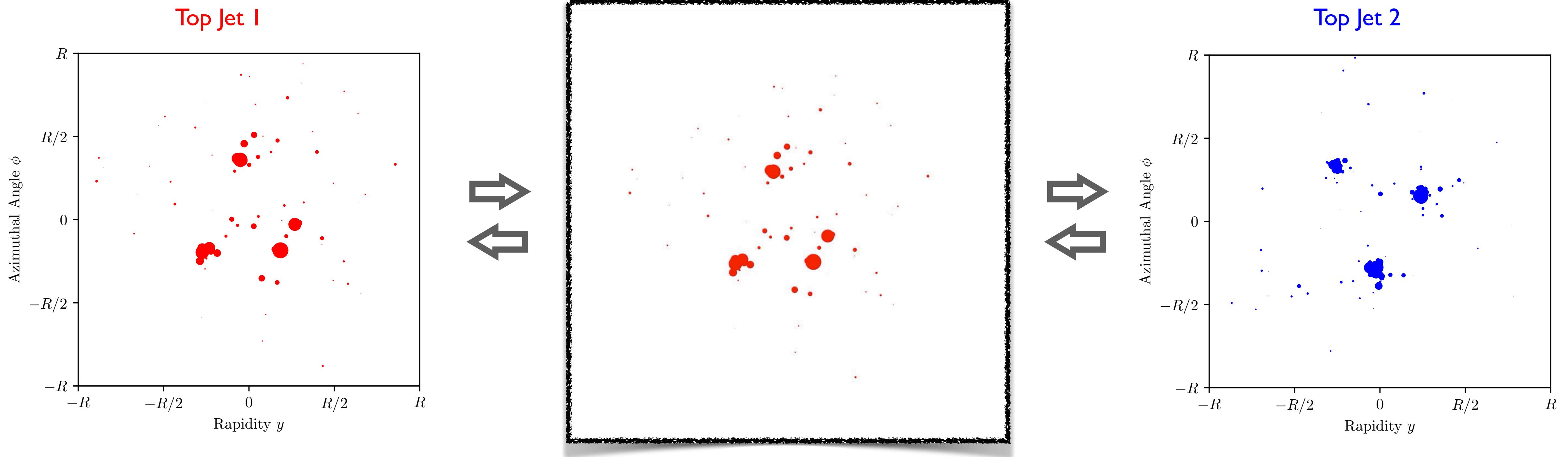
↑
 Energy Flow
 Distribution ↑
 Energy (p_T) Direction
 (y, φ)



Towards a Hidden Geometry – When are two events similar?



Optimal transport minimizes the “**work**” (**stuff** x distance) required to transport **supply** to **demand**



$$\mathcal{E}(\hat{n}) = \sum_{i=1}^M E_i \delta(\hat{n} - \hat{n}_i)$$

Provides a **metric** on **normalized distributions** in a space with a ground distance measure

↳ symmetric, non-negative, triangle inequality, zero iff identical

$$\theta_{ij} = \sqrt{\Delta y_{ij}^2 + \Delta \phi_{ij}^2}$$

[Peleg, Werman, Rom, [IEEE 1989](#); Rubner, Tomasi, Guibas, [ICCV 1998](#), [ICCV 2000](#); Pele, Werman, [ECCV 2008](#); Pele, Taskar, [GSI 2013](#)]

The Energy Mover's Distance (EMD)

[PTK, Metodiev, Thaler, PRL 2019]

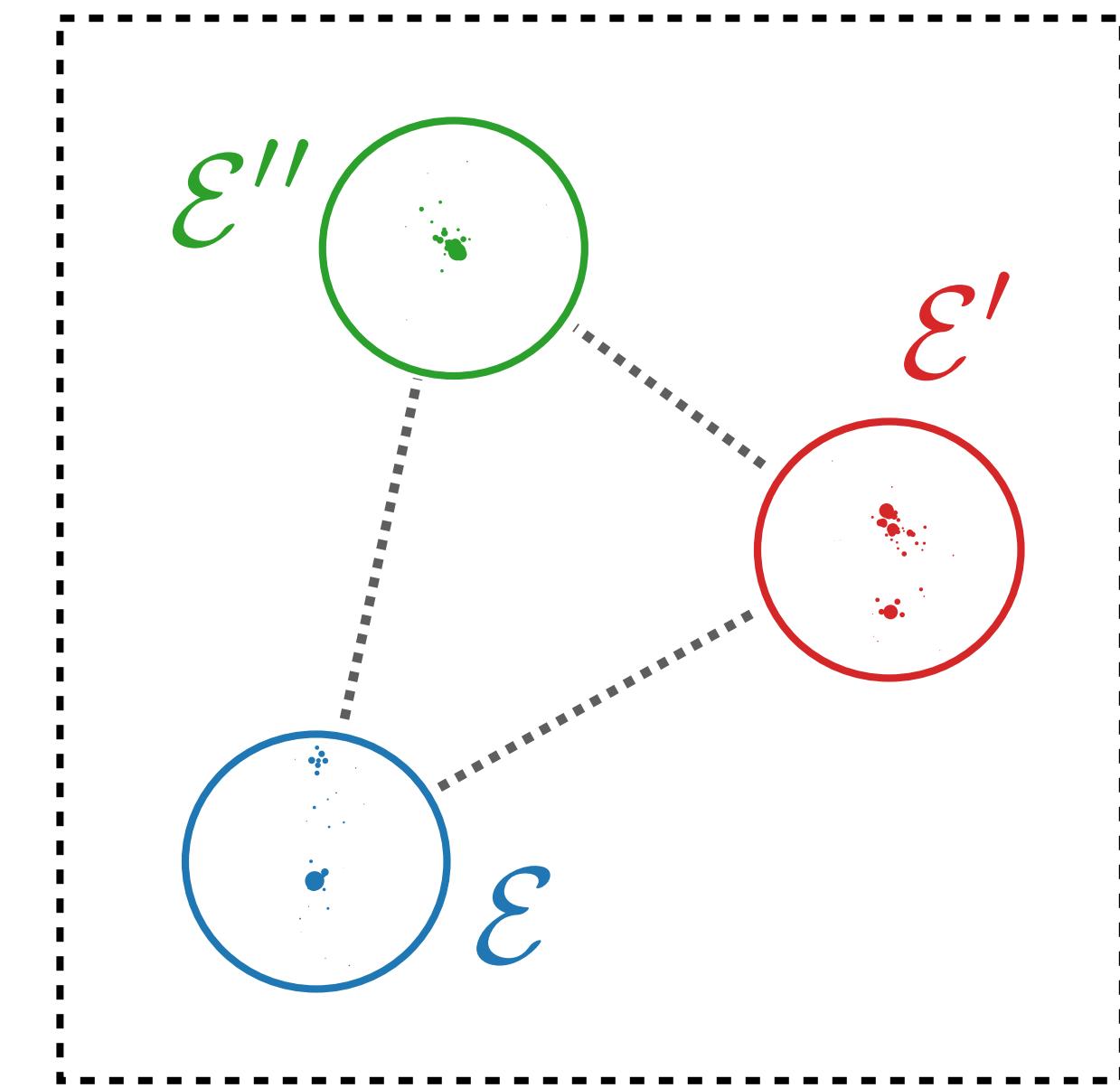
EMD between energy flows defines a metric on the space of events

$$\text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}') = \min_{\{f_{ij} \geq 0\}} \sum_i \sum_j f_{ij} \left(\frac{\theta_{ij}}{R} \right)^\beta + \left| \sum_i E_i - \sum_j E'_j \right|$$

Cost of optimal transport Cost of energy creation

$$\sum_j f_{ij} \leq E_i, \quad \sum_i f_{ij} \leq E'_j, \quad \sum_{ij} f_{ij} = \min \left(\sum_i E_i, \sum_j E'_j \right)$$

Capacity constraints to ensure proper transport



R : controls cost of transporting energy vs. destroying/creating it

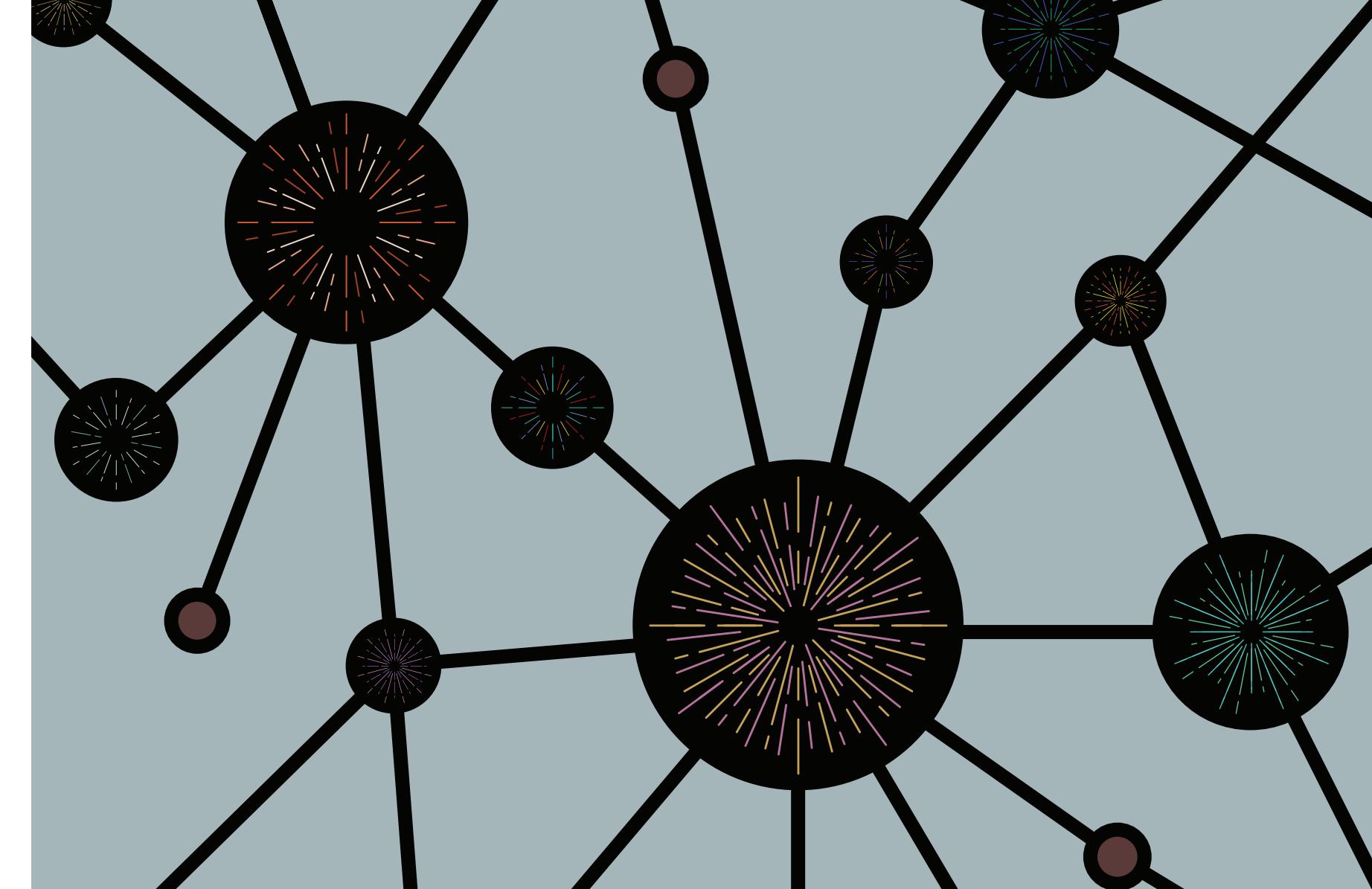
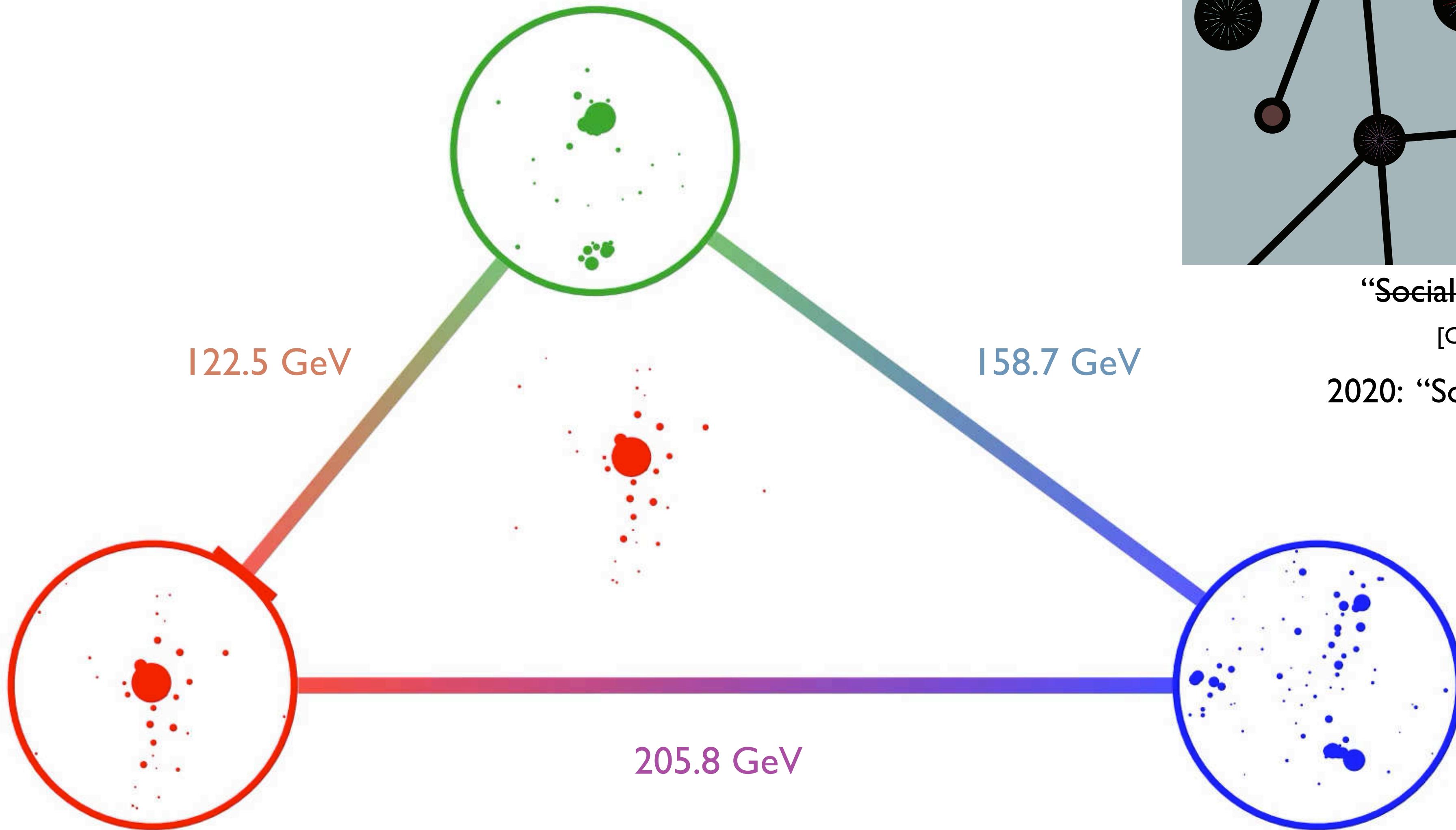
β : angular weighting exponent

Triangle inequality satisfied for $R \geq d_{\max}/2$

$$0 \leq \text{EMD}(\mathcal{E}, \mathcal{E}') \leq \text{EMD}(\mathcal{E}, \mathcal{E}'') + \text{EMD}(\mathcal{E}'', \mathcal{E}')$$

i.e. $R \geq$ jet radius for conical jets

Geodesics in the Space of Events



“Social networking of jets”

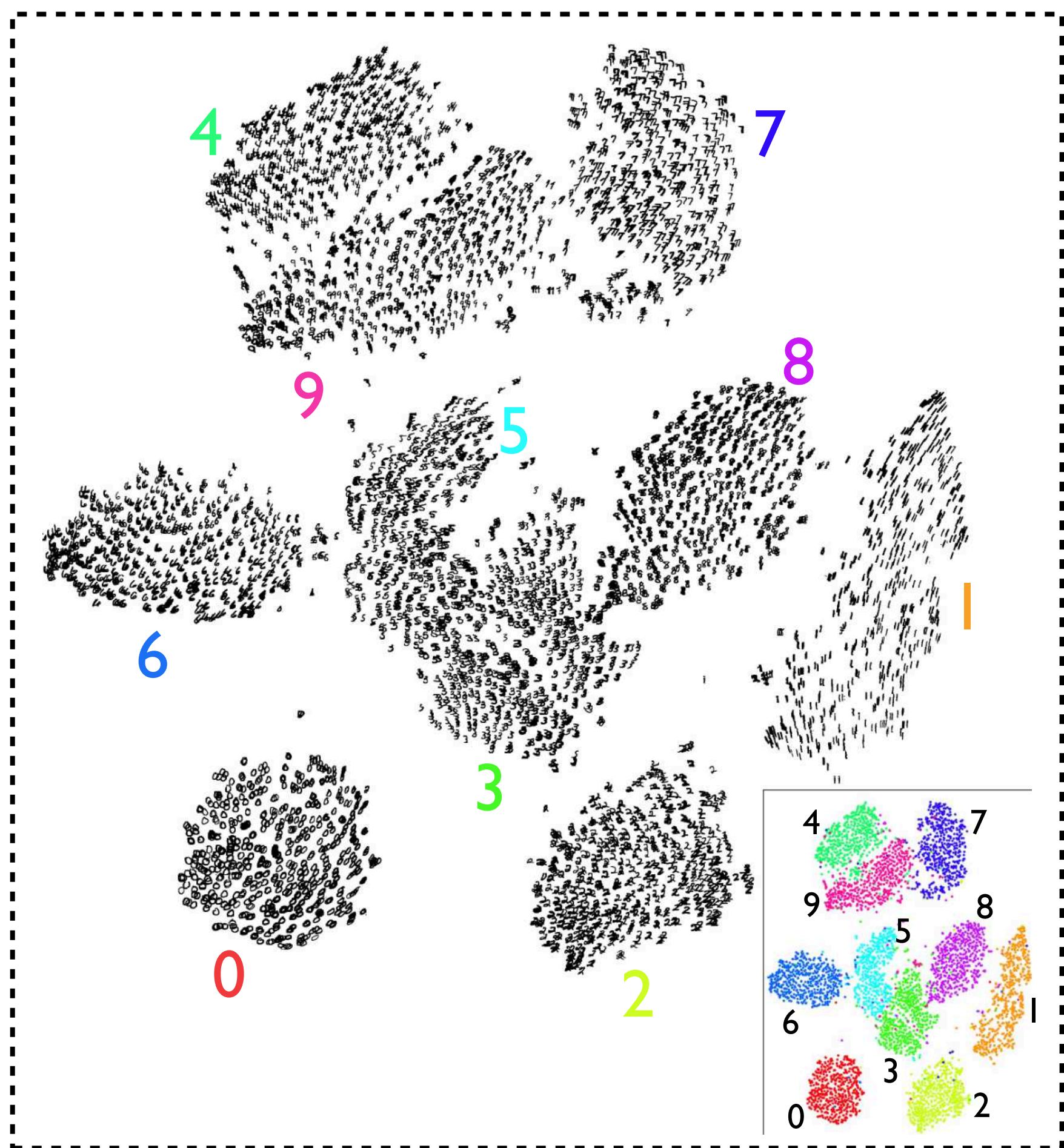
[Chu, MIT News 2019]

2020: “Social distancing of jets”

Visualizing Geometry in the Space of Events

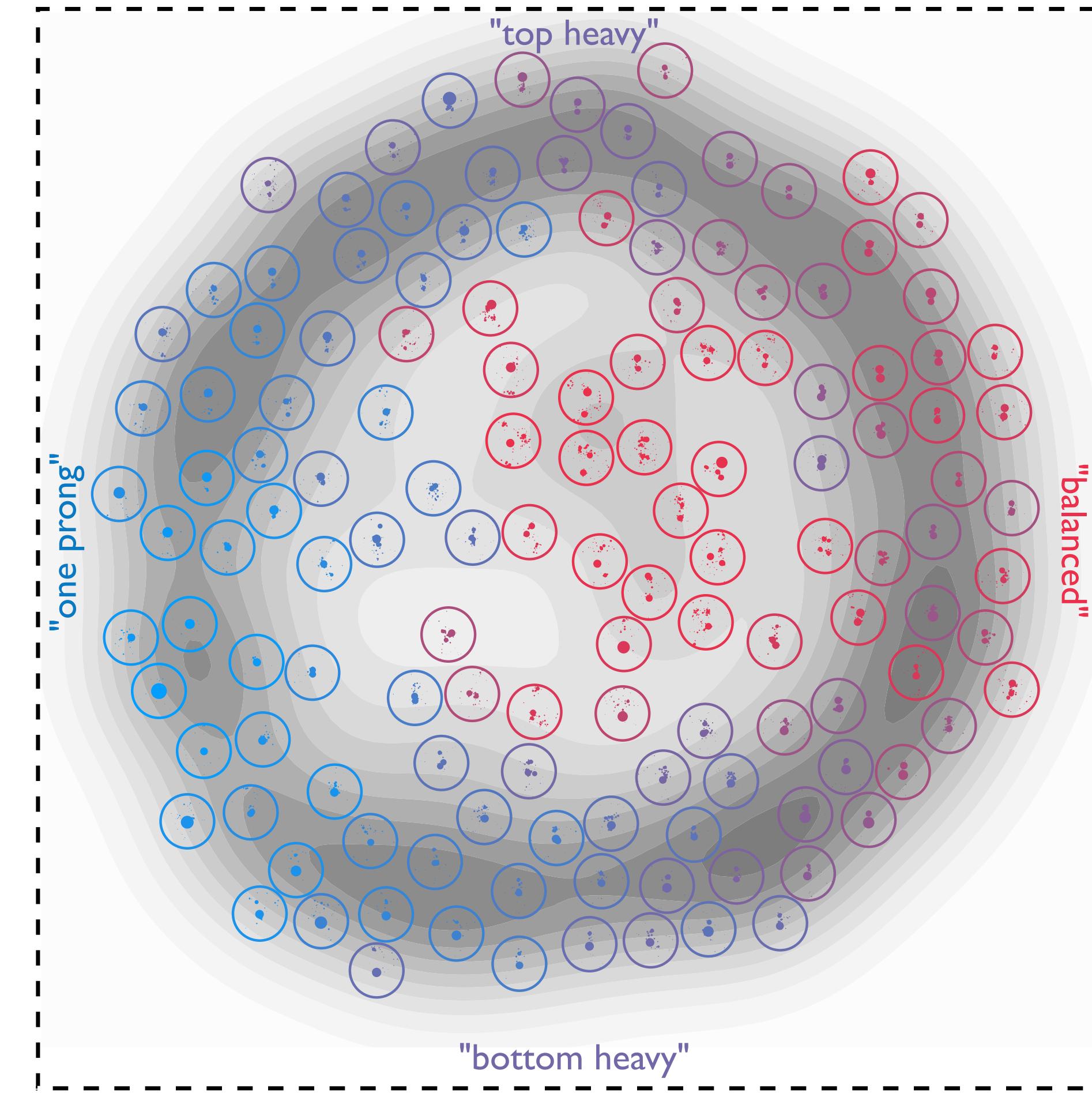
[PTK, Metodiev, Thaler, PRL 2019]

t-Distributed Stochastic Neighbor Embedding (t-SNE)
MNIST handwritten digits

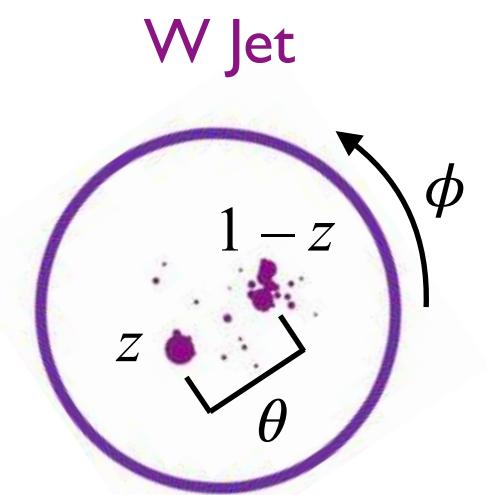


[L. van der Maaten, G. Hinton, JMLR 2008]

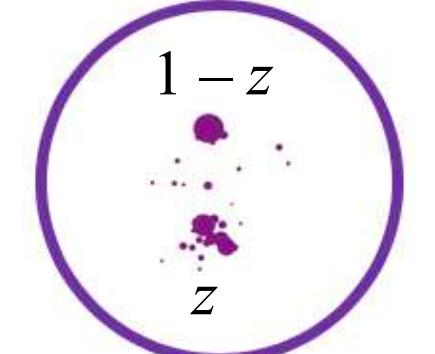
Geometric space of W jets



Gray contours represent the density of jets
Each circle is a particular W jet

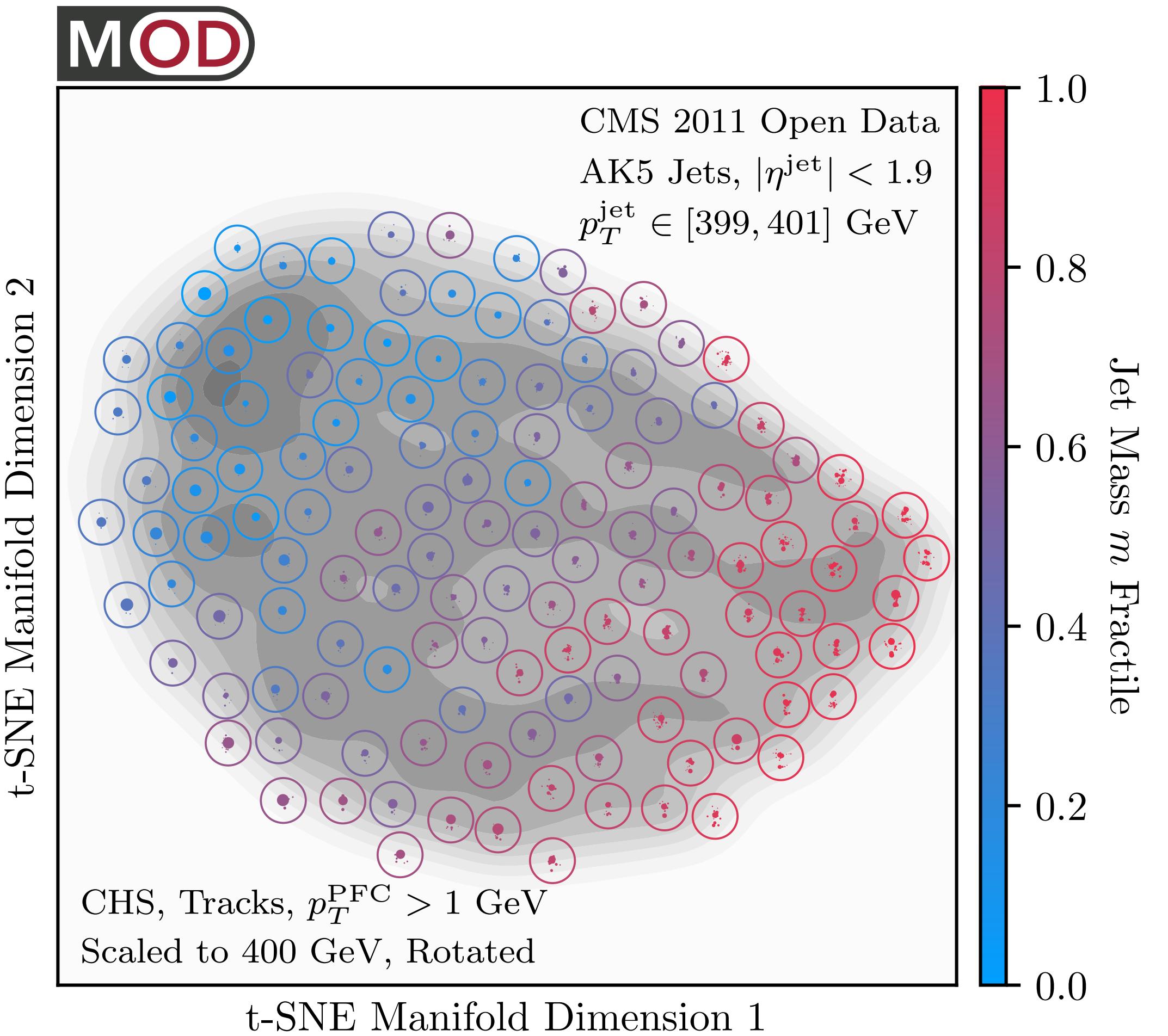


Constraints: W Mass and
 $\phi = 0$ preprocessing

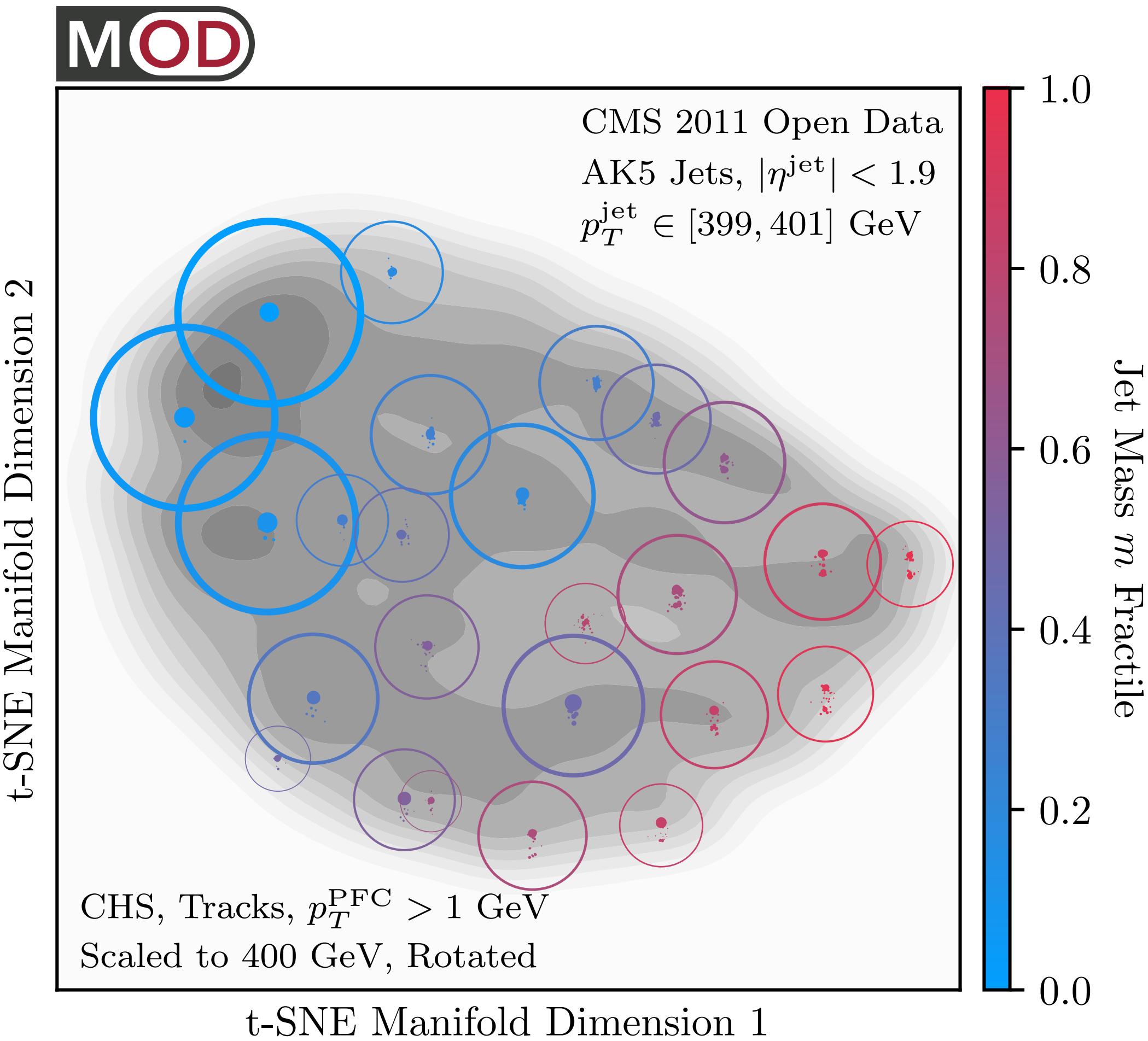


Visualizing Geometry in CMS Open Data

[PTK, Mastandrea, Metodiev, Naik, Thaler, PRD 2019; code and datasets at energyflow.network]



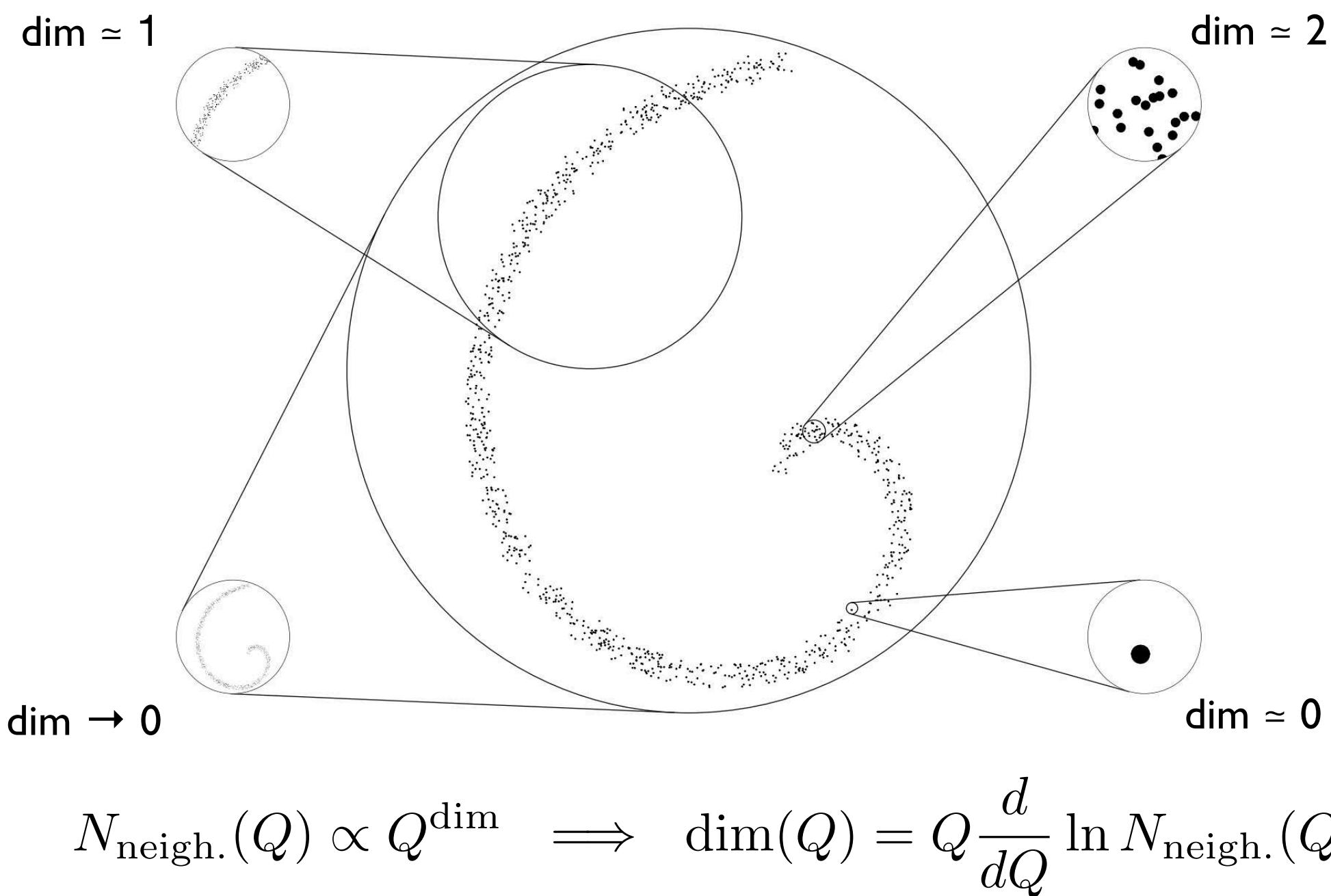
Example jets sprinkled throughout



25 most representative jets ("medoids")
Size is proportional to number of jets associated to that medoid

Quantifying Event-Space Manifolds

Correlation dimension: how does the # of elements within a ball of size Q change?



Correlation dimension lessons:

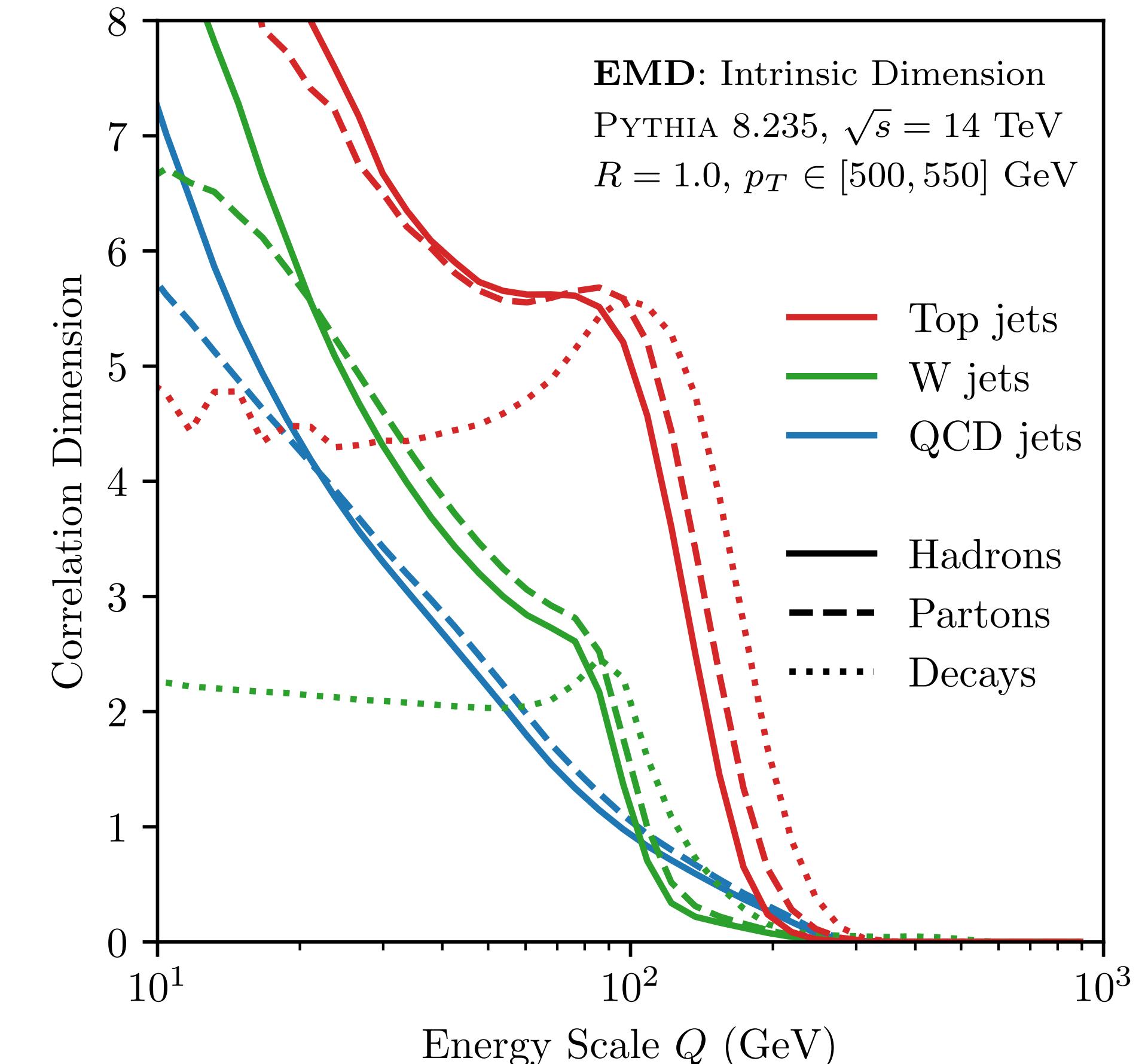
Decays are "constant" dim. at low Q

Complexity hierarchy: QCD < W < Top

Fragmentation increases dim. at smaller scales

Hadronization important around 20-30 GeV

$$\text{dim}(Q) = Q \frac{\partial}{\partial Q} \ln \sum_i \sum_j \Theta(\text{EMD}(\mathcal{E}_i, \mathcal{E}'_j) < Q)$$

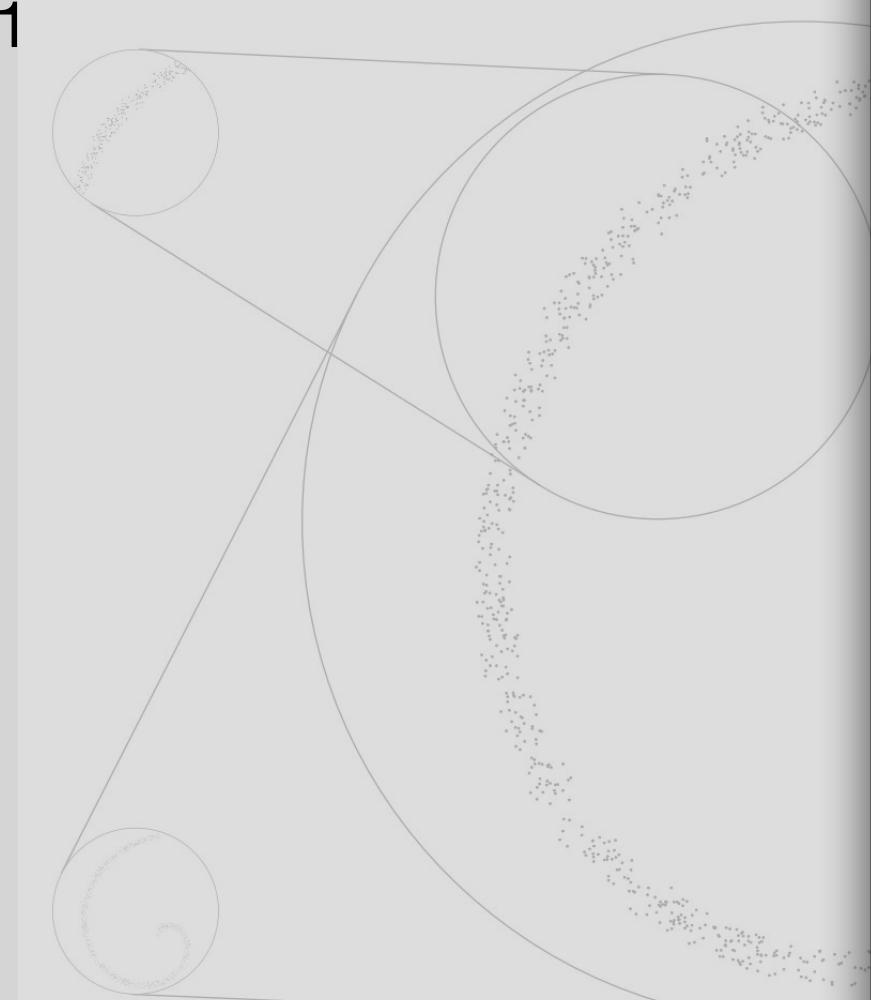


[Grassberger, Procaccia, [PRL 1983](#); PTK, Metodiev, Thaler, [PRL 2019](#)]

Quantifying Event-Space Manifolds

Correlation dimension
elements within a ball

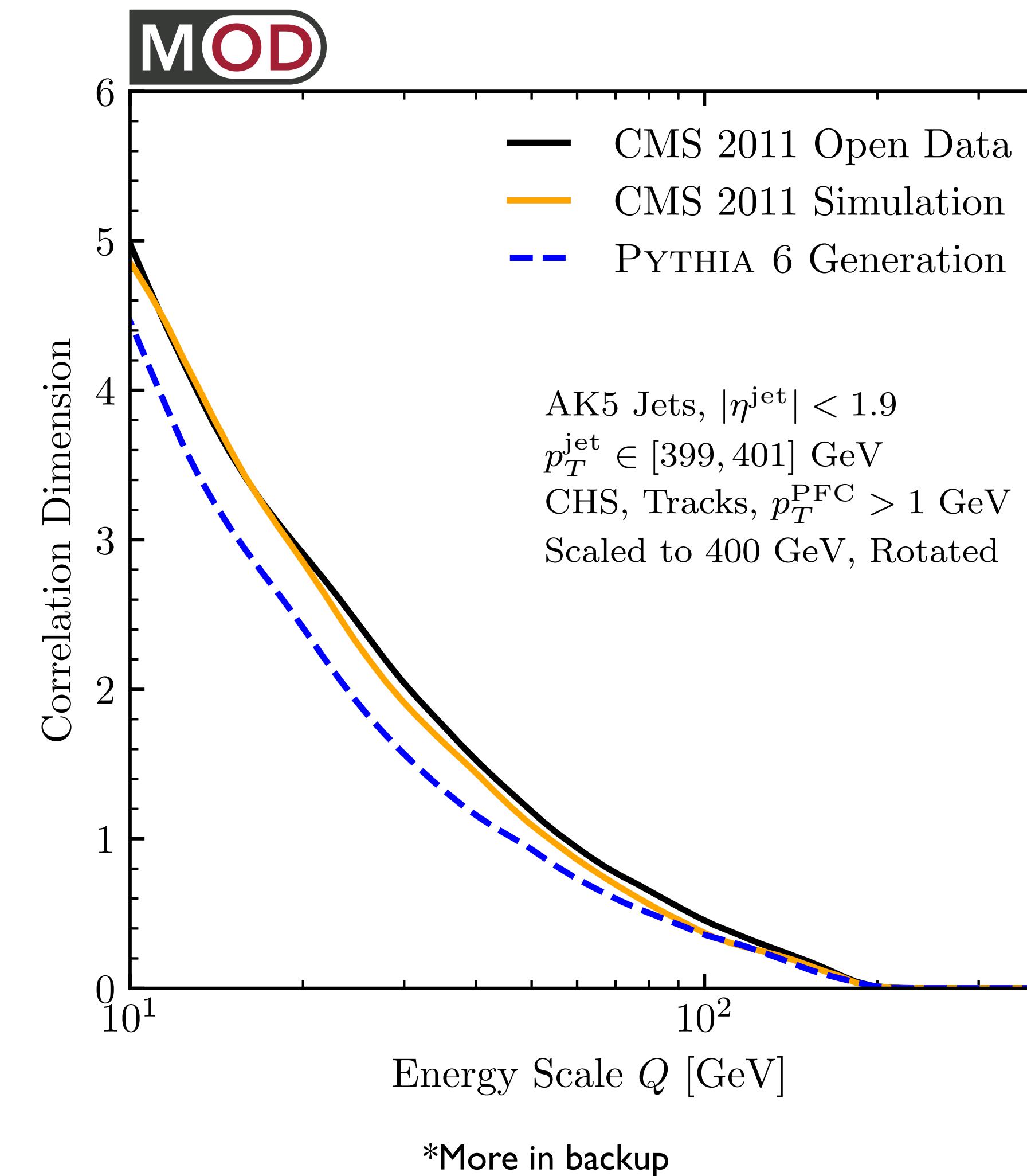
$\text{dim} \approx 1$



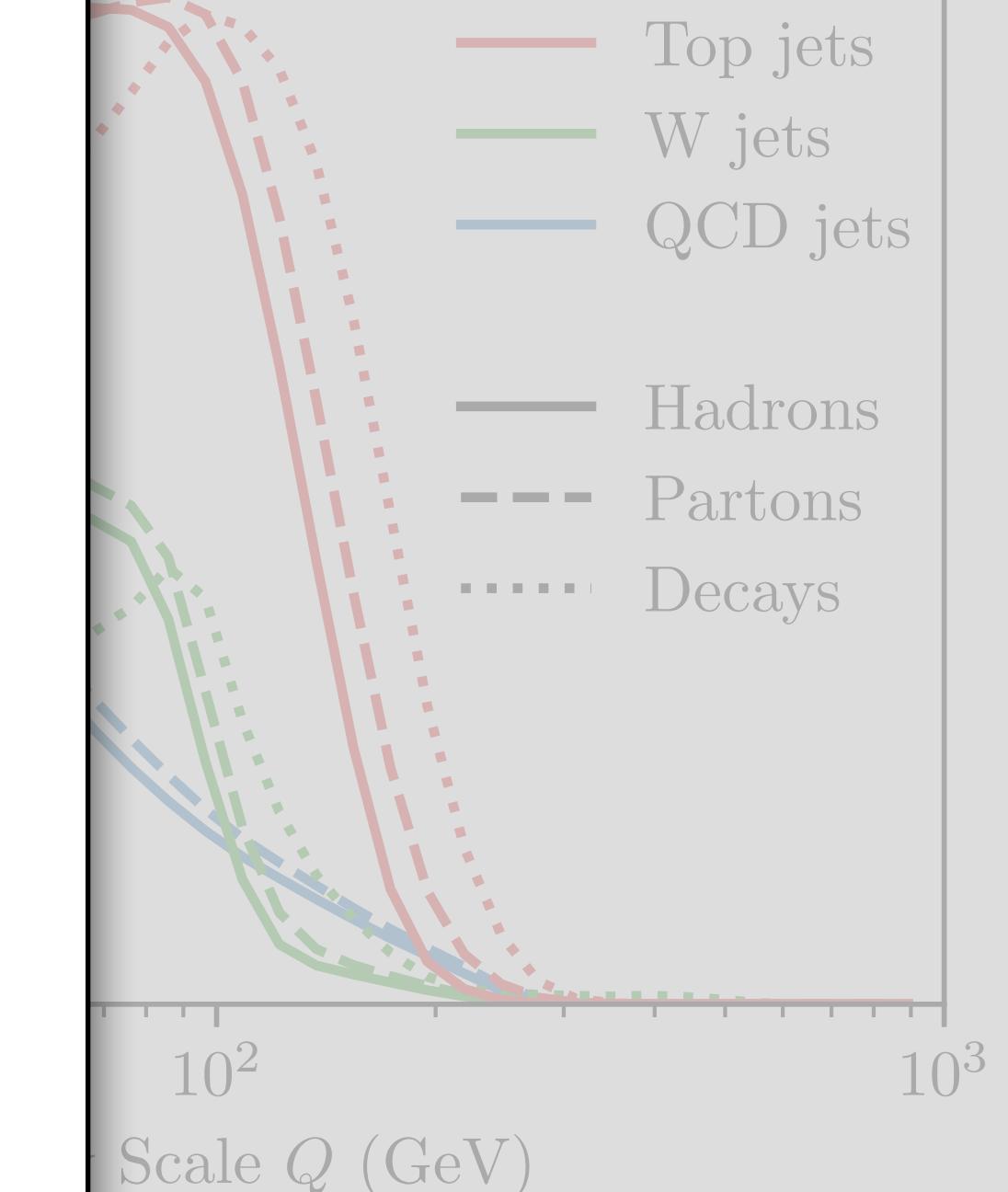
$$N_{\text{neigh.}}(Q) \propto Q^{\text{dim}} \implies \text{dim} \approx 1$$

Correlation dimension
Decays are "constant"
Complexity hierarchy
Fragmentation increases
Hadronization important

... in CMS Open Data



EMD: Intrinsic Dimension
PYTHIA 8.235, $\sqrt{s} = 14$ TeV
 $R = 1.0$, $p_T \in [500, 550]$ GeV

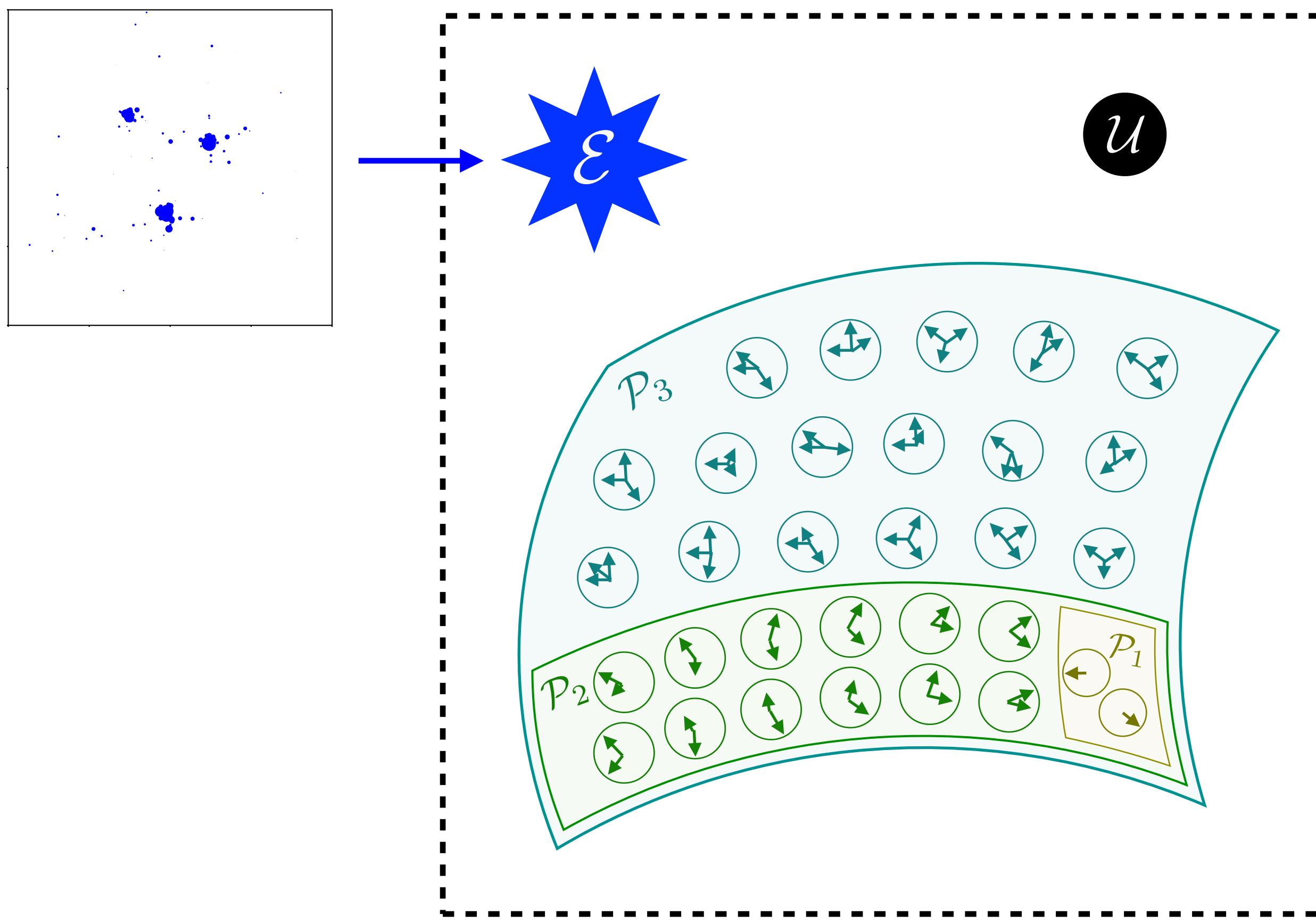


[Bocca, PRL 1983; PTK, Metodiev, Thaler, PRL 2019]

N-particle Manifolds in the Space of Events

[PTK, Metodiev, Thaler, 2004.04.159]

$$\mathcal{P}_N = \text{set of all N-particle configurations} = \left\{ \sum_{i=1}^N E_i \delta(\hat{n} - \hat{n}_i) \mid E_i \geq 0 \right\}$$



\mathcal{P}_1 : manifold of events with one particle

\mathcal{P}_2 : manifold of events with two particles

\mathcal{P}_3 : manifold of events with three particles

⋮

$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \dots \supset \mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1$

by soft and collinear limits

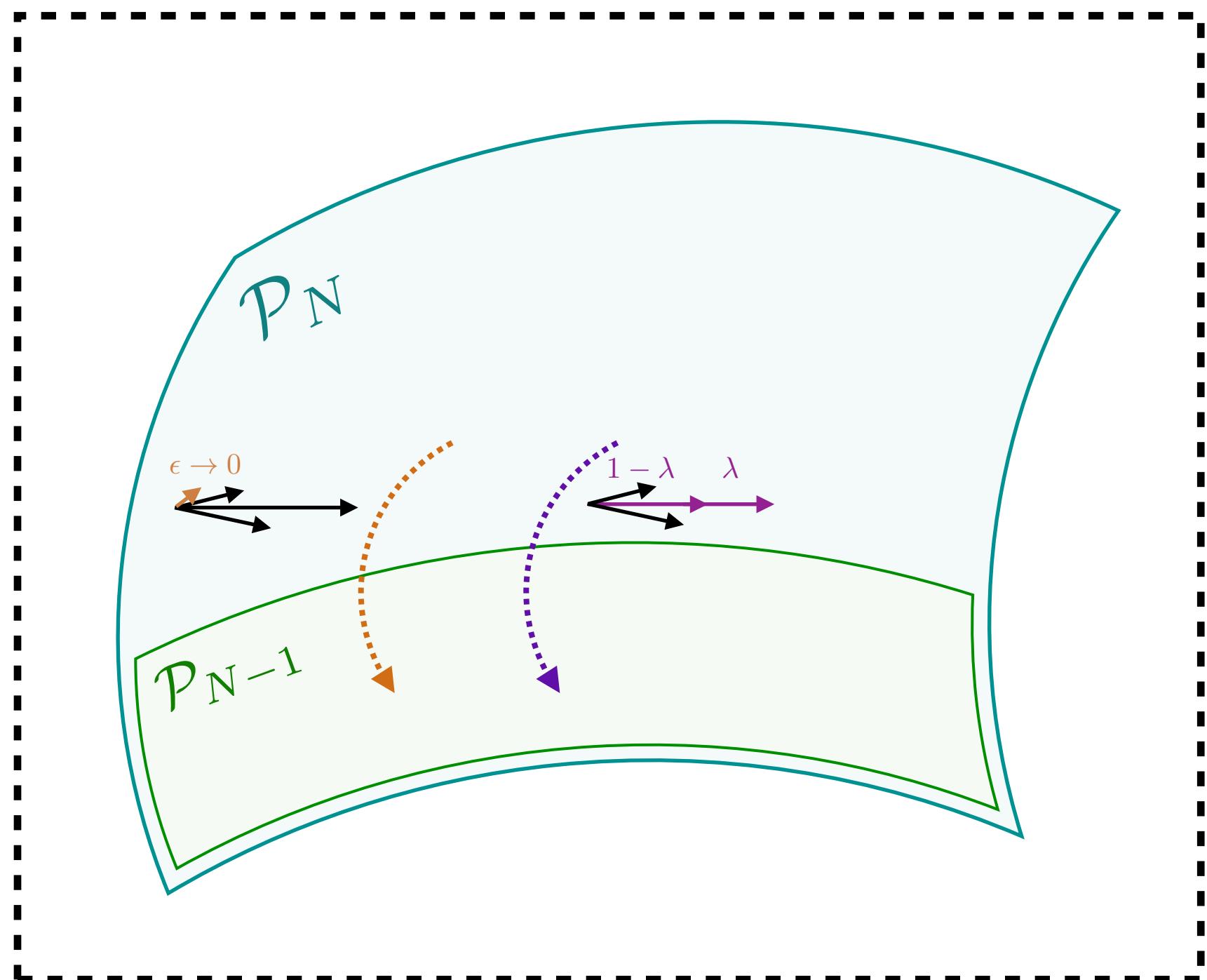


Uniform event, not contained in any \mathcal{P}_N

N-particle Manifolds in the Space of Events – Infrared Divergences

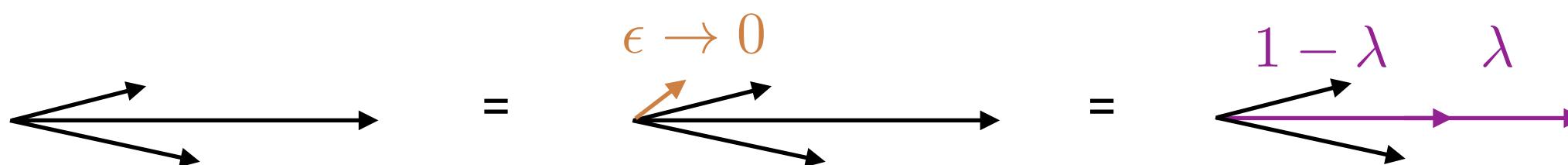
[PTK, Metodiev, Thaler, 2004.04.159]

$$\mathcal{P}_N = \text{set of all } N\text{-particle configurations} = \left\{ \sum_{i=1}^N E_i \delta(\hat{n} - \hat{n}_i) \mid E_i \geq 0 \right\}$$



Energy flow is unchanged by exact soft/collinear emissions

$$dP_{i \rightarrow ig} \simeq \frac{2\alpha_s}{\pi} C_a \frac{d\theta}{\theta} \frac{dz}{z}$$



Functions of energy flow automatically satisfy exact IRC invariance!

Real and virtual divergences appear naturally together

$$\mathcal{P}_N \supset \mathcal{P}_{N-1} \supset \cdots \supset \mathcal{P}_3 \supset \mathcal{P}_2 \supset \mathcal{P}_1$$

by soft and collinear limits

Defining IRC Safety Precisely

[Sterman, Weinberg, [PRL 1997](#); Sterman, [PRD 1978](#); Banfi, Salam, Zanderighi, [JHEP 2005](#)]

Infrared and collinear safety is a proxy for perturbative calculability of an observable

Exact **IRC** invariance

$$\mathcal{O}(p_1^\mu, \dots, p_M^\mu) = \mathcal{O}(0p_0^\mu, p_1^\mu, \dots, p_M^\mu)$$

$$\mathcal{O}(p_1^\mu, \dots, p_M^\mu) = \mathcal{O}(\lambda p_1^\mu, (1 - \lambda)p_1^\mu, \dots, p_M^\mu)$$

Guarantees observable is well-defined on **energy** flows

Allows for pathological observables, e.g. pseudo-multiplicity

Smooth **IRC** invariance

$$\mathcal{O}(p_1^\mu, \dots, p_M^\mu) = \lim_{\epsilon \rightarrow 0} \mathcal{O}(\epsilon p_0^\mu, p_1^\mu, \dots, p_M^\mu)$$

$$\mathcal{O}(p_1^\mu, \dots, p_M^\mu) = \lim_{p_0^\mu \rightarrow p_1^\mu} \mathcal{O}(\lambda p_0^\mu, (1 - \lambda)p_1^\mu, \dots, p_M^\mu)$$

Eliminates common observables with hard boundaries

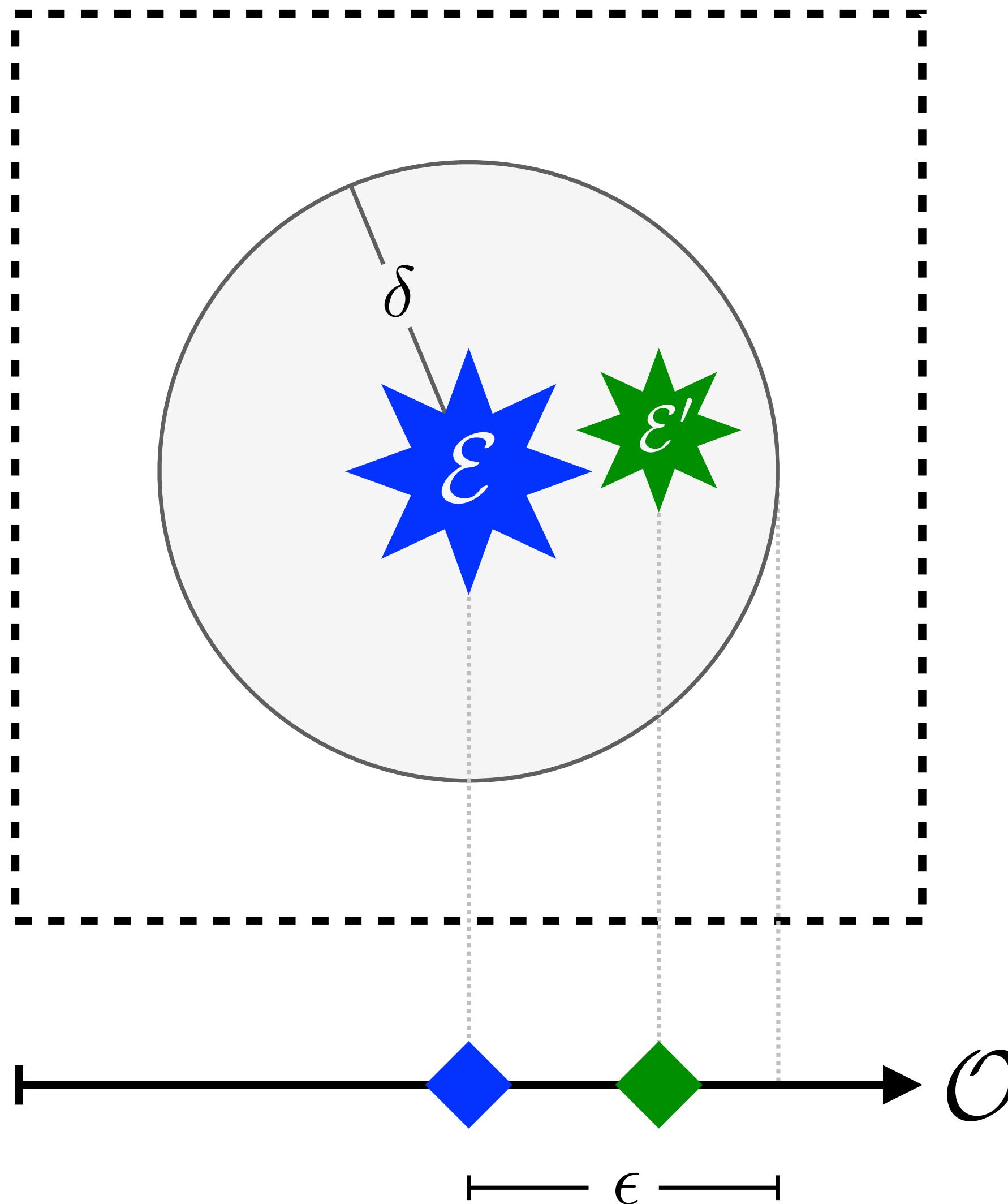
All Observables	Comments
Multiplicity ($\sum_i 1$)	IR unsafe and C unsafe
Momentum Dispersion [65] ($\sum_i E_i^2$)	IR safe but C unsafe
Sphericity Tensor [66] ($\sum_i p_i^\mu p_i^\nu$)	IR safe but C unsafe
Number of Non-Zero Calorimeter Deposits	C safe but IR unsafe

Defined on Energy Flows		
Pseudo-Multiplicity ($\min\{N \mid \mathcal{T}_N = 0\}$)		Robust to exact IR or C emissions

Infrared & Collinear Safe		
Jet Energy ($\sum_i E_i$)		Disc. at jet boundary
Heavy Jet Mass [67]		Disc. at hemisphere boundary
Soft-Dropped Jet Mass [38, 68]		Disc. at grooming threshold
Calorimeter Activity [69] (N_{95})		Disc. at cell boundary

More EMD Geometry – Continuity in the Space of Events

[PTK, Metodiev, Thaler, 2004.04.159]



Classic $\epsilon - \delta$ definition of continuity in a metric space

An observable \mathcal{O} is **EMD continuous** at an event \mathcal{E} if, for any $\epsilon > 0$, there exists a $\delta > 0$ such that for all events \mathcal{E}' :

$$\text{EMD}(\mathcal{E}, \mathcal{E}') < \delta \implies |\mathcal{O}(\mathcal{E}) - \mathcal{O}(\mathcal{E}')| < \epsilon.$$

Towards a geometric definition of **IRC Safety**

IRC Safety = EMD Continuity*

*on all but a negligible set[‡] of events

[‡]a negligible set is one that contains no positive-radius EMD-ball

⋮

Perturbation Theory in the Space of Events

[PTK, Metodiev, Thaler, 2004.04.159]

Sudakov safety

[Larkoski, Thaler, JHEP 2014; Larkoski, Marzani, Thaler, PRD 2015]

Some observables have discontinuities on P_N for some N

A resummed **IRC-safe companion** can mitigate the divergences

$$p(\mathcal{O}_{\text{Sudakov}}) = \int d\mathcal{O}_{\text{Comp.}} p(\mathcal{O}_{\text{Sudakov}} | \mathcal{O}_{\text{Comp.}}) p(\mathcal{O}_{\text{Comp.}})$$

Event geometry suggests N -(sub)jettiness as universal companion

Fixed-order calculability

[Sterman, PRD 1979; Banfi, Salam, Zanderighi, JHEP 2005]

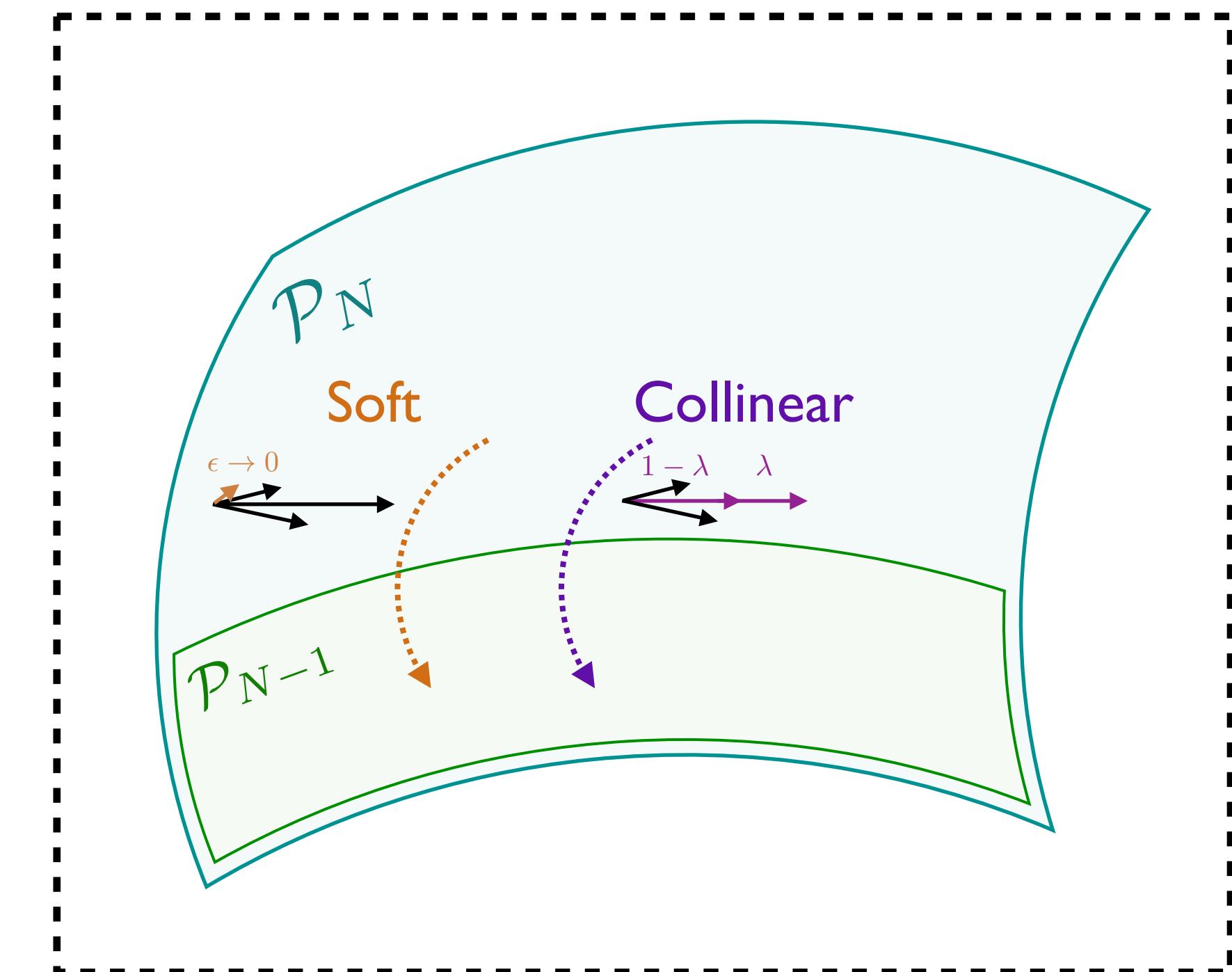
Is a statement of integrability on each P_N

EMD continuity must be upgraded to EMD-Hölder continuity on each P_N

$$\lim_{\mathcal{E} \rightarrow \mathcal{E}'} \frac{\mathcal{O}(\mathcal{E}) - \mathcal{O}(\mathcal{E}')}{\text{EMD}(\mathcal{E}, \mathcal{E}')^c} = 0, \quad c > 0$$

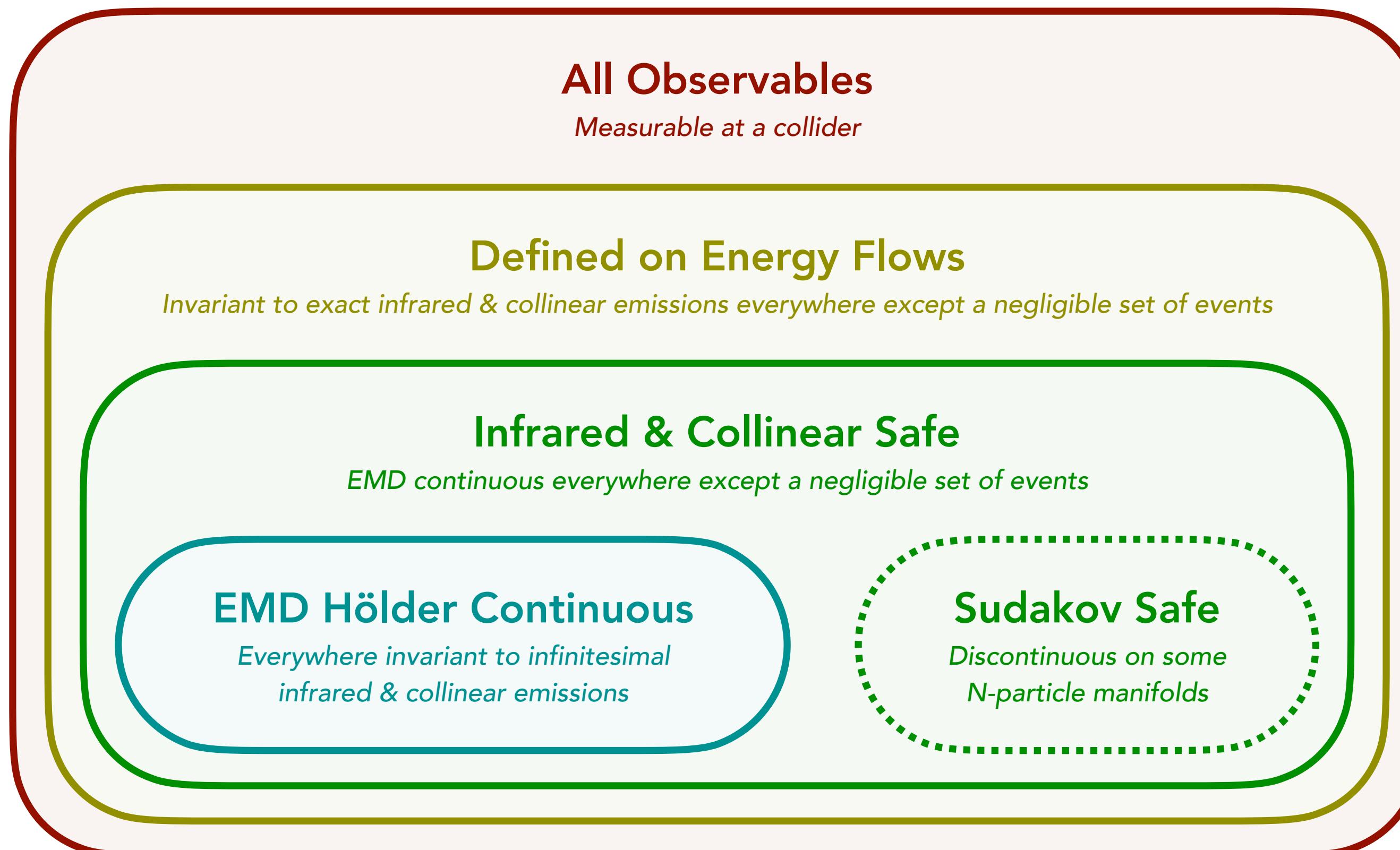
Example: $V(\mathcal{E}) = \mathcal{T}_2(\mathcal{E}) \left(1 + \frac{1}{\ln E(\mathcal{E})/\mathcal{T}_3(\mathcal{E})} \right)$ is EMD continuous but not EMD Hölder continuous (it is Sudakov safe)

Infrared singularities of massless gauge theories appear on each P_N



Hierarchy of IRC Safety Definitions

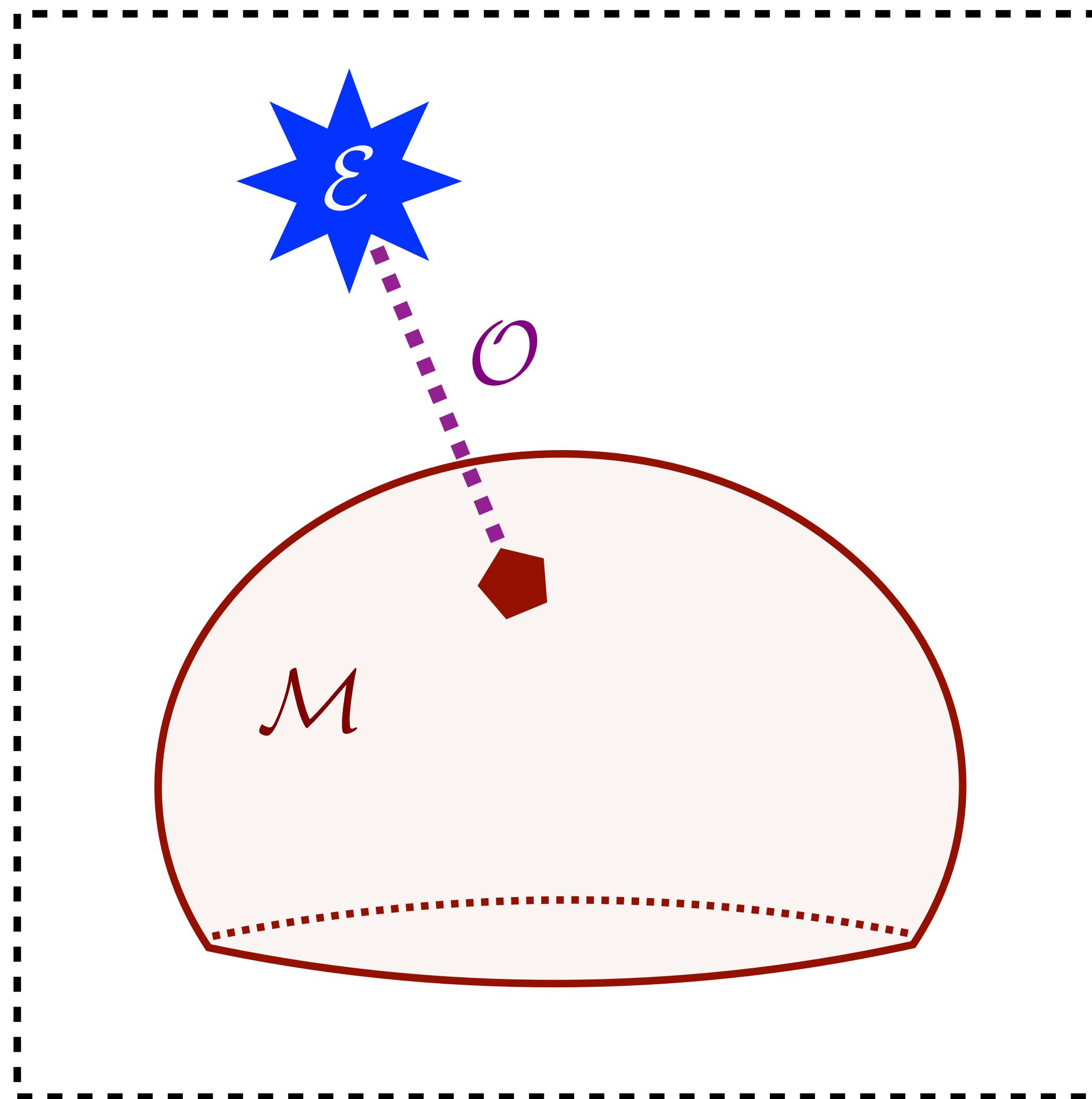
[PTK, Metodiev, Thaler, 2004.04.159]



All Observables	Comments
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Calorimeter Activity [69] (N_{95})	Disc. at cell boundary
Sudakov Safe	
Groomed Momentum Fraction [39] (z_g)	Disc. on 1-particle manifold
Jet Angularity Ratios [37]	Disc. on 1-particle manifold
N -subjettiness Ratios [47, 48] (τ_{N+1}/τ_N)	Disc. on N -particle manifold
V parameter [36] (Eq. (2.11))	Hölder disc. on 3-particle manifold
EMD Hölder Continuous Everywhere	
Thrust [40, 41]	
Sphericity [42]	
Angularities [70]	
N -jettiness [44] (\mathcal{T}_N)	
C parameter [71–74]	Resummation beneficial at $C = \frac{3}{4}$
Linear Sphericity [72] ($\sum_i E_i n_i^\mu n_i^\nu$)	
Energy Correlators [36, 75–77]	
Energy Flow Polynomials [15, 17]	

Defining Observables via Event Space Geometry

[PTK, Metodiev, Thaler, 2004.04.159]



Many common *observables* are distance of closest approach from event to a specific manifold

$$\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}')$$

EMD variant for equal-energy events

$$\text{EMD}_\beta(\mathcal{E}, \mathcal{E}') = \lim_{R \rightarrow \infty} R^\beta \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}') = \min_{\{f_{ij} \geq 0\}} \sum_{i=1}^M \sum_{j=1}^{M'} f_{ij} \theta_{ij}^\beta$$

Enforces equal energy (else infinity)

on equal-energy events

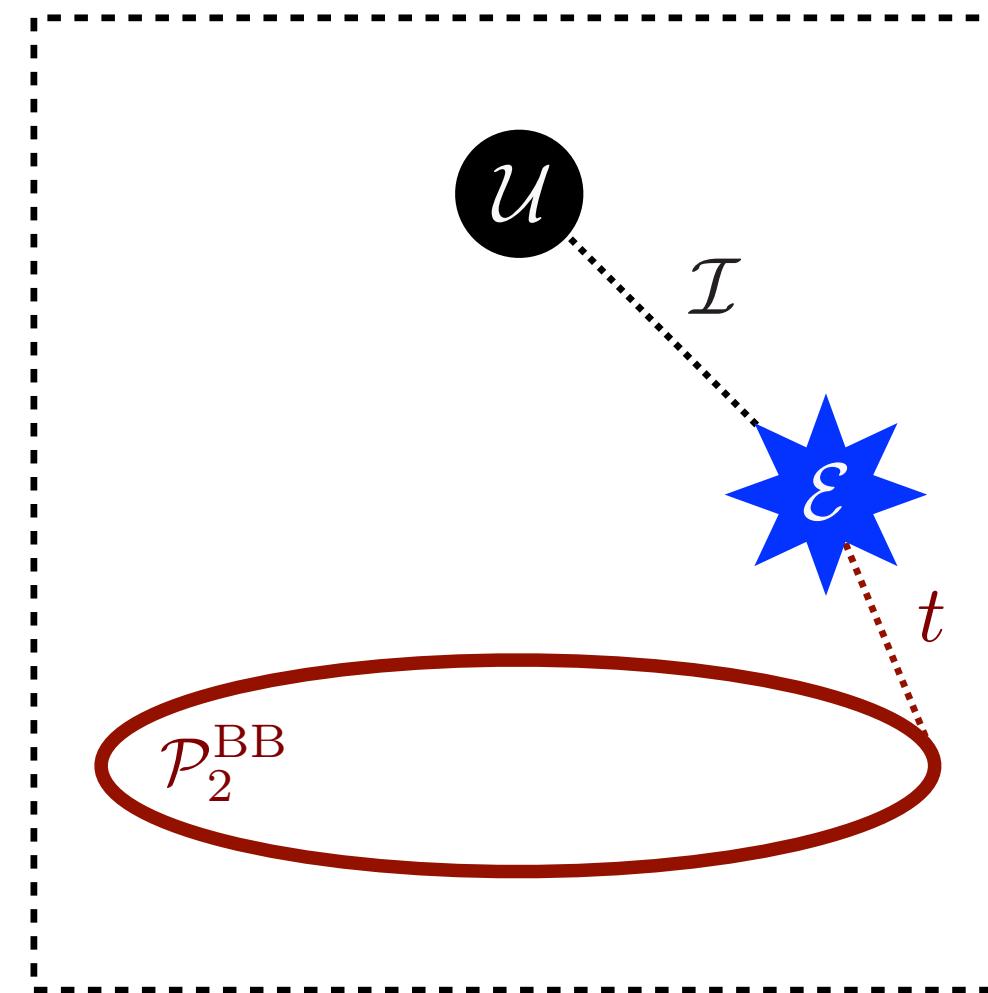
Defining Observables via Event Space Geometry

$$\mathcal{O}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{M}} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}')$$

[PTK, Metodiev, Thaler, 2004.04159]

Thrust, spherocity, isotropy*

*Distance of closest approach
to a specific manifold*



$$t(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_2(\mathcal{E}, \mathcal{E}')$$

$$\sqrt{s(\mathcal{E})} = \min_{\mathcal{E}' \in \mathcal{P}_2^{\text{BB}}} \text{EMD}_1(\mathcal{E}, \mathcal{E}')$$

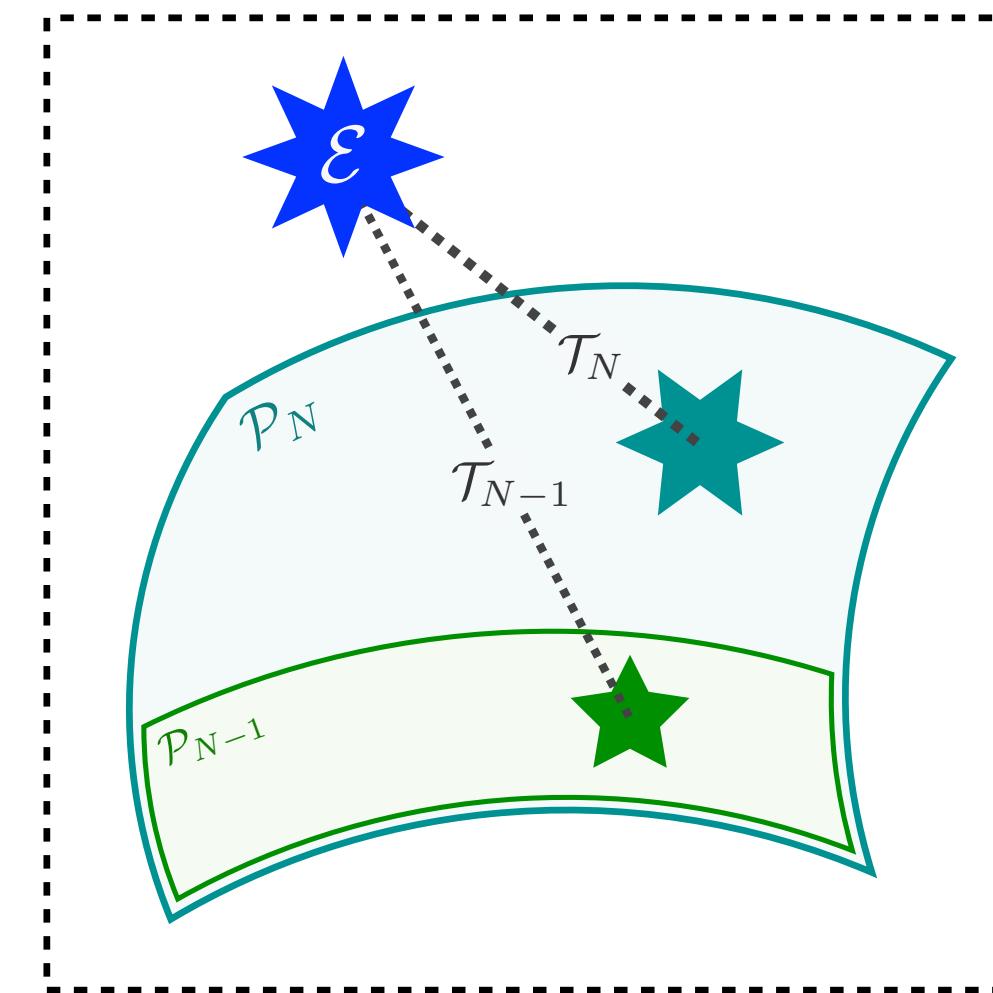
$$\mathcal{I}^{(\beta)}(\mathcal{E}) = \min_{\mathcal{E}' \in M_U} \text{EMD}_\beta(\mathcal{E}, \mathcal{E}')$$

[Farhi, [PRL 1977](#); Georgi, Machacek, [PRL 1977](#)]

*New! [Cesarotti, Thaler, [2004.06125](#)]

N-jettiness

*Minimum distance from event
to N-particle manifold*



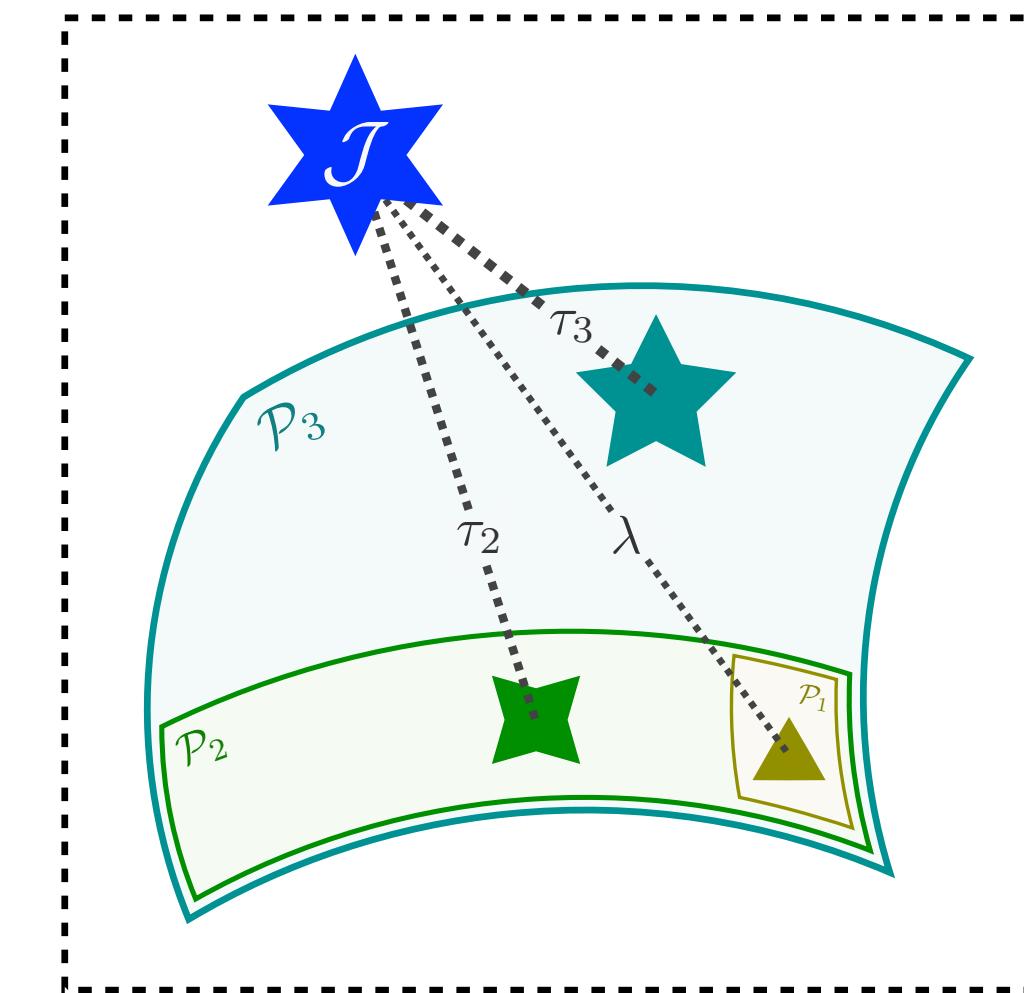
$$\mathcal{T}_N^{(\beta)}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_N} \text{EMD}_\beta(\mathcal{E}, \mathcal{E}')$$

$$\mathcal{T}_N^{(\beta,R)}(\mathcal{E}) = \min_{\mathcal{E}' \in \mathcal{P}_N} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{E}')$$

[Brandt, Dahmen, [Z. Phys 1979](#);
Stewart, Tackmann, Waalewijn, [PRL 2010](#)]

N-subjettiness, angularities

*Smallest distance from jet to
N-particle manifold*



for recoil-free angularity

$$\lambda_\beta(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_1} \text{EMD}_\beta(\mathcal{J}, \mathcal{J}')$$

$$\tau_N^{(\beta)}(\mathcal{J}) = \min_{\mathcal{J}' \in \mathcal{P}_N} \text{EMD}_\beta(\mathcal{J}, \mathcal{J}')$$

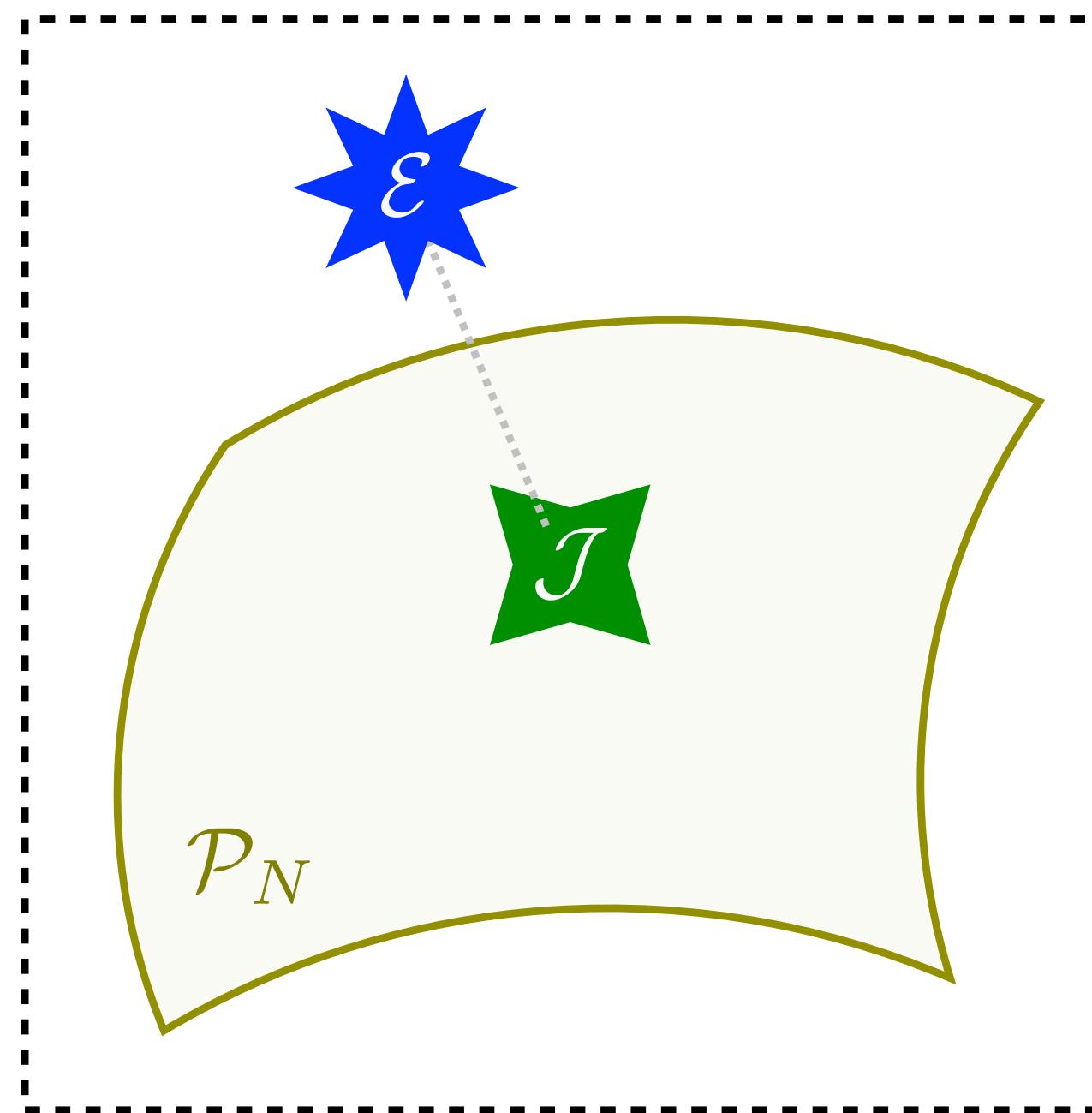
[Ellis, Vermilion, Walsh, Hornig, Lee, [JHEP 2010](#);
Thaler, Van Tilburg, [JHEP 2011](#), [JHEP 2012](#)]

Jets in the Space of Events – The Closest N -particle Description of an M -particle Event

[PTK, Metodiev, Thaler, 2004.04159]

Exclusive cone finding

XCone finds N jets by minimizing N -jettiness

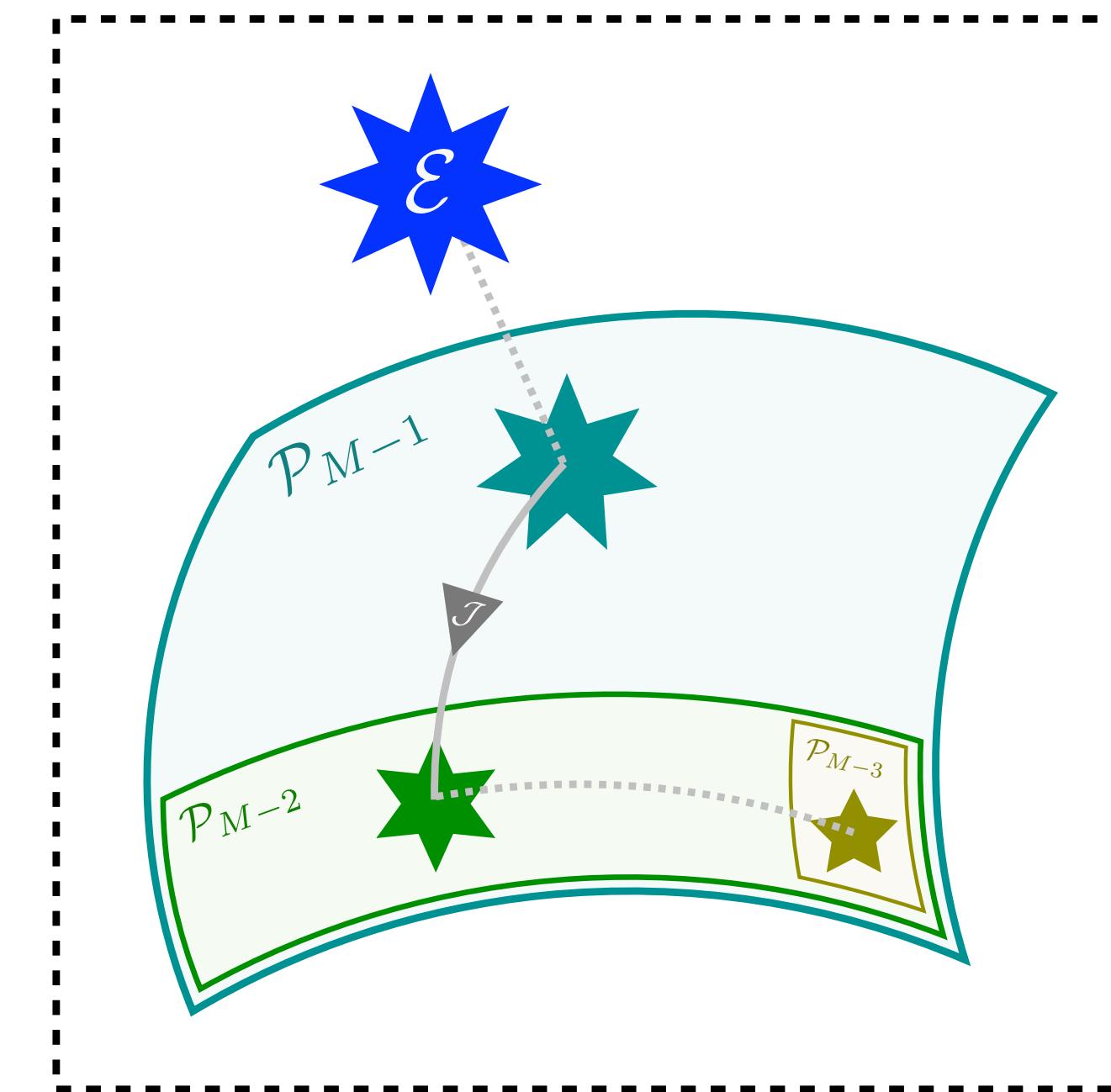


$$\mathcal{J}_{N,\beta,R}^{\text{XCone}}(\mathcal{E}) = \arg \min_{\mathcal{J} \in \mathcal{P}_N} \text{EMD}_{\beta,R}(\mathcal{E}, \mathcal{J})$$

[Stewart, Tackmann, Thaler, Vermilion, Wilkason, [JHEP 2015](#);
Thaler, Wilkason, [JHEP 2015](#)]

Sequential recombination

Iteratively merges particles or identifies a jet



event with one fewer particle after one step

$$\mathcal{E}_{M-1}^{(\beta,R)}(\mathcal{E}_M) = \arg \min_{\mathcal{E}'_{M-1} \in \mathcal{P}_{M-1}} \text{EMD}_{\beta,R}(\mathcal{E}_M, \mathcal{E}'_{M-1})$$

[Catani, Dokshitzer, Seymour, Webber, [Nucl. Phys. B 1993](#);
Ellis, Soper, [PRD 1993](#);
Dokshitzer, Leder, Moretti, Webber, [JHEP 1997](#);
Cacciari, Salam, Soyez, [JHEP 2008](#)]

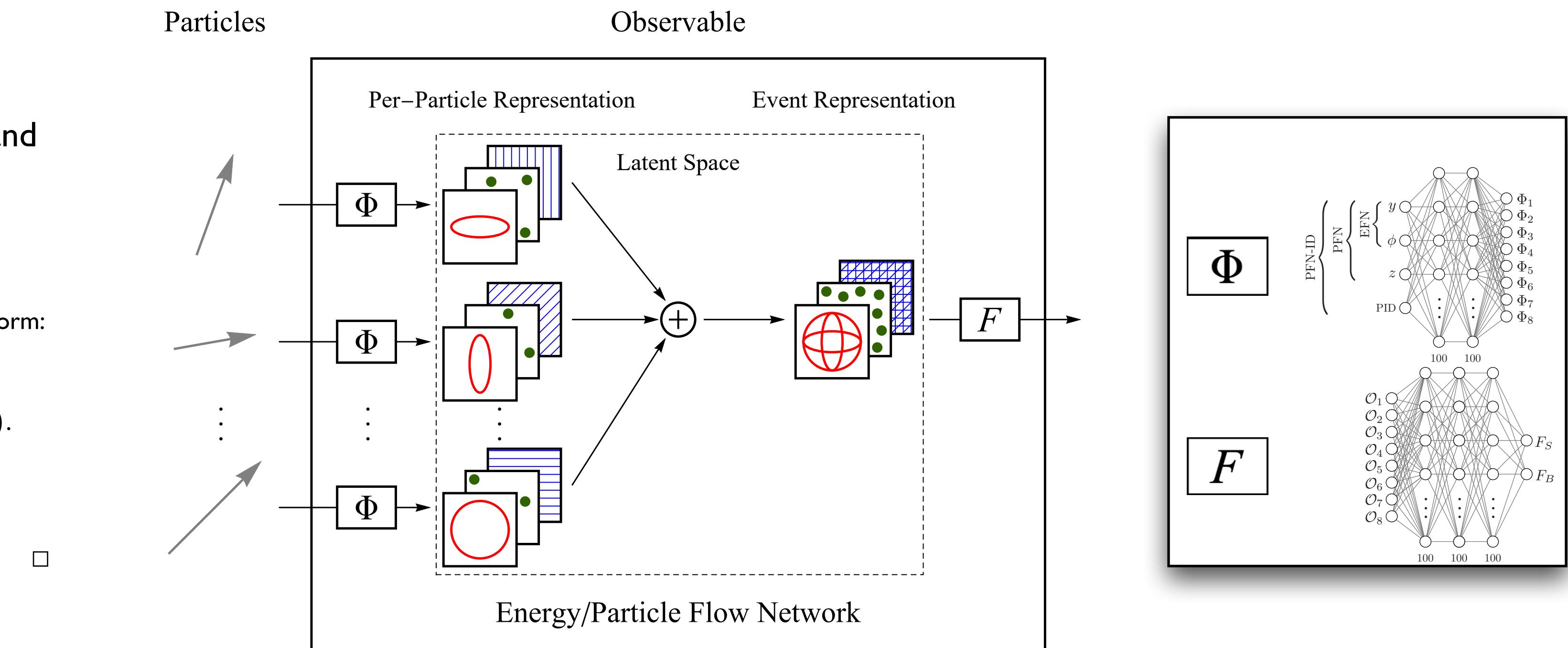
Energy Flow Networks (EFN)

IRC safety is a statement of *linearity* in energy and *continuity* in geometry

Theorem: Any IRC-safe observable can be written in the following form:

$$f(\{p_1^\mu, \dots, p_M^\mu\}) = F \left(\sum_{i=1}^M z_i \vec{\Phi}(\hat{p}_i) \right), \quad \hat{p}_i = (y_i, \phi_i).$$

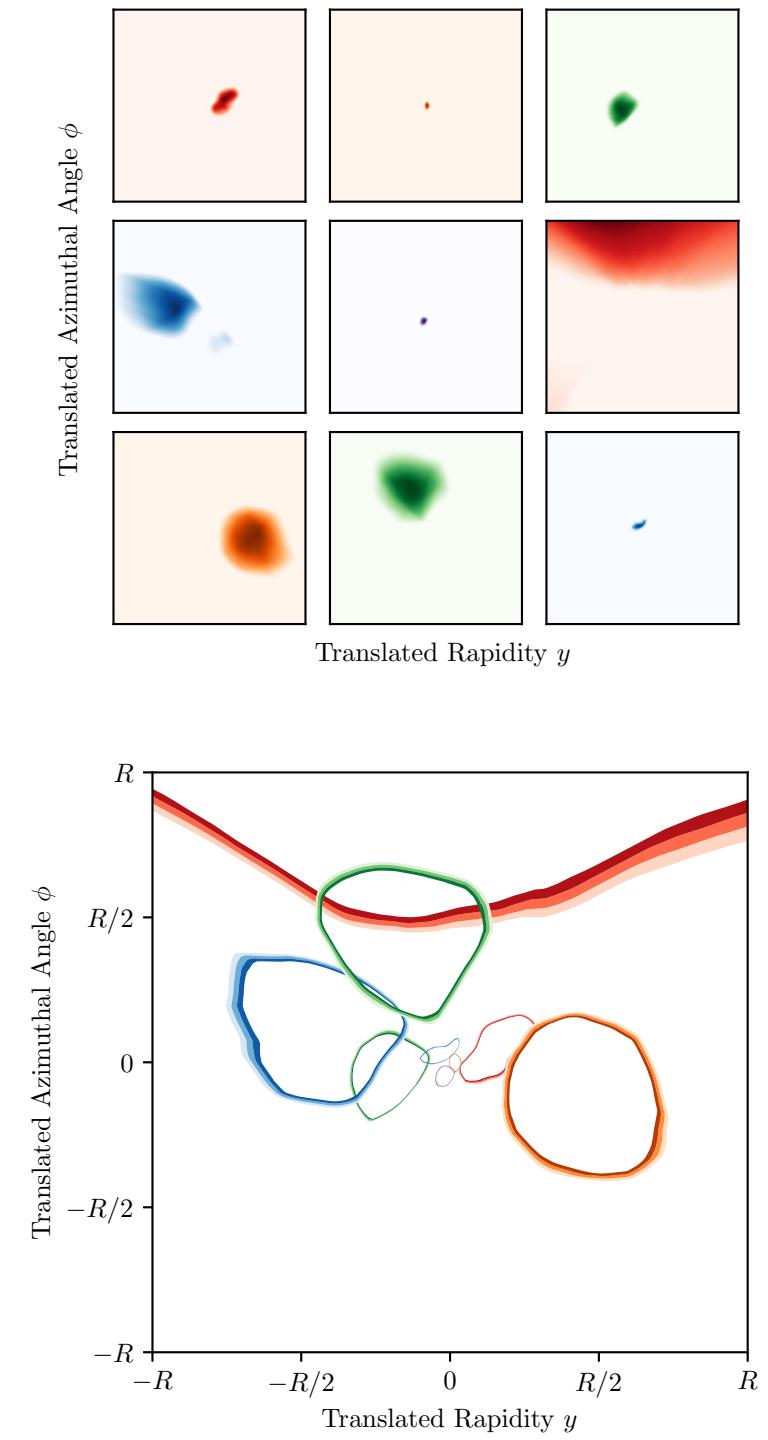
Proof: In [1810.05165](#).



$$\text{EFN : } \mathcal{O}_a = \sum_{i=1}^M z_i \Phi_a(y_i, \phi_i)$$

$$\text{PFN : } \mathcal{O}_a = \sum_{i=1}^M \Phi_a(z_i, y_i, \phi_i, [\text{PID}_i])$$

Visualizing EFNs



Simultaneous visualization strategy

Contour

Overlay

