Bayesian Statistics Homework1

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1 Problem 1

$Y_1, Y_2, ..., Y_6$ are observations from a uniform distribution $(\theta - 1/2, \theta + 1/2)$ in order: 10.9, 11.0, 11.1, 11.4, 11.5, 11.7. The prior is uniform (10, 20).

$$P(\theta) \sim uniform(10, 20)$$

$$L(Y|\theta) = \prod_{i=1}^{6} \mathbb{1}(\theta - 1/2 < Y_i < \theta + 1/2)$$

$L(Y|\theta)$ would just be an intersection of the indicator functions. For example, $\mathbb{1}(\theta - 1/2 < 10.9 < \theta + 1/2) = \mathbb{1}(10.4 < \theta < 11.4)$.

The indicator functions give the following constraints:
$\mathbb{1}(10.4 < \theta < 11.4)$
$\mathbb{1}(10.5 < \theta < 11.5)$
$\mathbb{1}(10.6 < \theta < 11.6)$
$\mathbb{1}(10.9 < \theta < 11.9)$
$\mathbb{1}(11.0 < \theta < 12.0)$
which gives $\mathbb{1}(11.0 < \theta < 11.4)$

The posterior distribution is $P(\theta|Y) \propto L(\theta)P(\theta) = \mathbb{1}(11.0 < \theta < 11.4)\frac{1}{10}\mathbb{1}(10.0 < \theta < 20.0) = \mathbb{1}(11.0 < \theta < 11.4)\frac{1}{10}$

The posterior distribution $P(\theta|Y) \sim uniform(11.0, 11.4)$

2 Problem 2

2.1 i

$P(x|y,z) = \frac{P(y,z|x)P(x)}{P(y,z)} = \frac{P(x,y,z)}{P(y,z)}$
$= \frac{P(x,y,z)}{\int P(x,y,z)dx} = \frac{f(x,z)g(y,z)h(z)}{\int f(x,z)g(y,z)h(z)dx} = \frac{f(x,z)}{\int f(x,z)dx}$

2.2 ii

$P(y|x,z) = \frac{P(x,z|y)P(y)}{P(x,z)} = \frac{P(x,y,z)}{P(x,z)}$
$= \frac{P(x,y,z)}{\int P(x,y,z)dy} = \frac{f(x,z)g(y,z)h(z)}{\int f(x,z)g(y,z)h(z)dy} = \frac{g(y,z)}{\int g(y,z)dy}$

2.3 iii

We want to show, $P(x,y|z) = P(y|z)P(x|z)$

3 Problem 3

$$P(\theta) \sim gamma(x,k)$$

$$P(Y|\theta) \sim exp(\theta)$$

With $P(\theta) = constant * \theta^{k-1}e^{-\theta/x}$, with $E[\theta] = 0.2 = k\theta$ and $Var[\theta] = 1 = k\theta^2$, so it is gamma(5, 1/25) = gamma(x, k).

$L(\theta) = \prod_{i=1}^{N} P(Y_i|\theta) = \lambda^N e^{-\lambda \sum_{i=1}^{N} Y_i}$, and we know the average = 3.8, so $\sum i = 1^N Y_i = N * 3.8$.

The posterior distribution is $P(\theta|Y) \propto L(\theta)P(\theta) = L(\theta)gamma(x,k) = \theta^{k+N-1}e^{-\theta(\sum Y_i + 1/x)} * constant$.

The posterior distribution is $\sim gamma(\frac{1}{\sum Y_i + 1/x}, k+N)$. where k is known, x is known and the sum of Y_i are known.

4 Problem 4

The negative binomial distribution is defined for an unknown theta and r. k is defined as the number of successes from all samples.

Consider $Y_1, ..., Y_N$ observations that = 1 if successful and 0 otherwise, where $\sum Y_i = k$

$$P(\theta) \sim beta(a,b)$$

$$L(Y|\theta) = \prod_{i=1}^{N} \theta^{\sum Y_i}(1-\theta)^r * constant$$

The posterior distribution is $P(\theta|Y) \propto P(\theta)L(Y|\theta) = \theta^{a-1}(1-\theta)^{b-1}(1-\theta)^r\theta^{\sum Y_i} * constant = \theta^{a+\sum Y_i - 1}(1-\theta)^{b+r-1}$.

This is $\sim beta(a + \sum Y_i, b+r)$, which shows that the beta distribution is a conjugate family for the negative binomial distribution.

5 Problem 5

5.1 i

This is just the joint distribution of many Bernoulli's, which is the Binomial distribution.

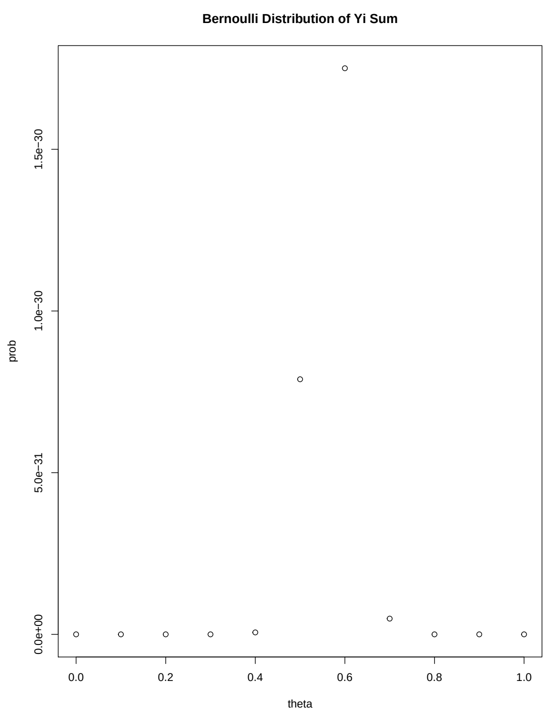
$P(Y_1 = y_1, ..., Y_{100} = y_{100}|\theta = \binom{100}{k}\theta^k(1-\theta)^{100-k}$

The probability of the sum given theta, though does not look at the order of Bernoulli trials, but just at the sum. It is therefore

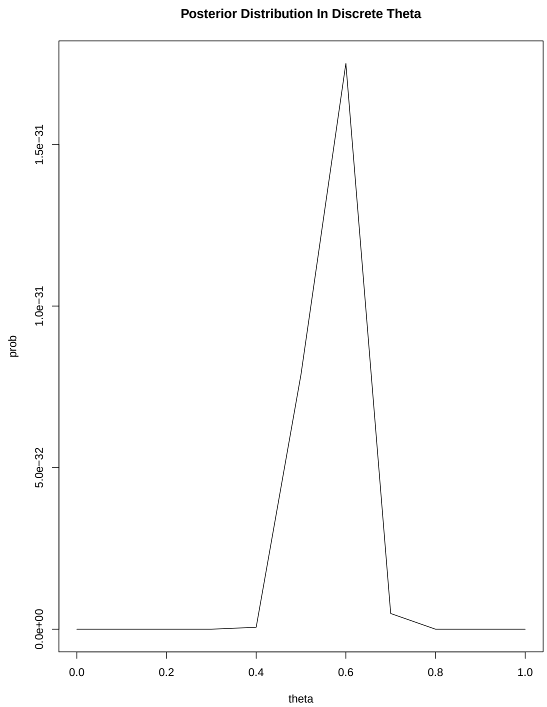
$P(\sum Y_i = y|\theta = \theta^k(1-\theta)^{100-k}$

5.2 ii

See R Code.

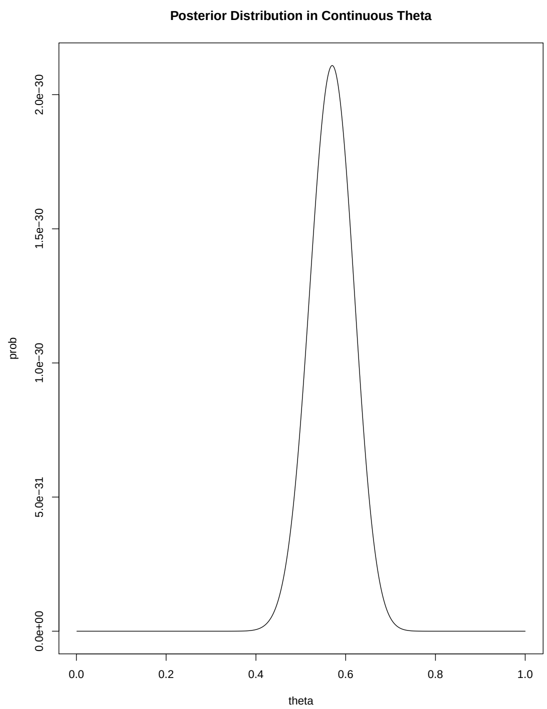


5.3 iii

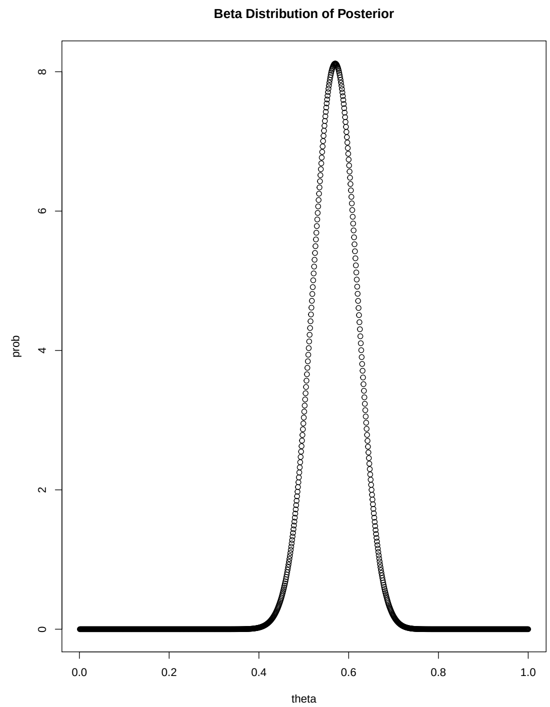


Posterior Distribution In Discrete Theta
prob
1.5e−31
1.0e−31
5.0e−32
0.0e+00
0.0
0.2
0.4
0.6
0.8
1.0
theta

5.4 iv


Posterior Distribution in Continuous Theta
prob
0.0e+00
5.0e-31
1.0e-30
1.5e-30
2.0e-30
0.0
0.2
0.4
0.6
0.8
1.0
theta

5.5 v

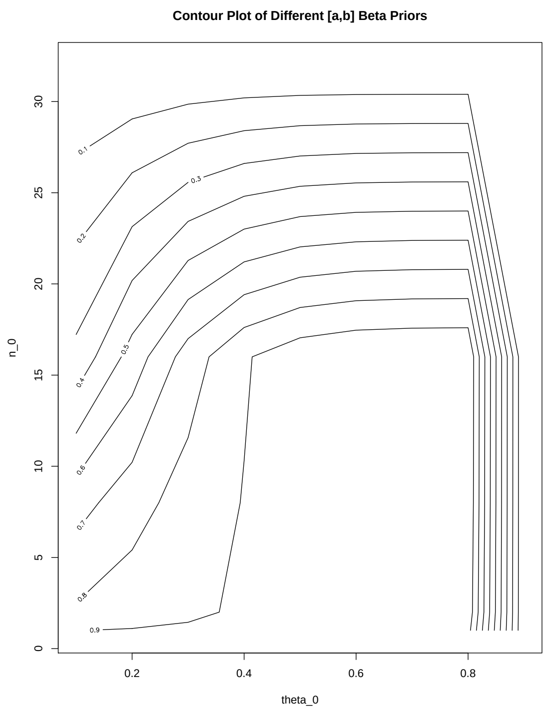


The different plots show different variations of a "posterior distribution". In part ii), this shows a discretized plot of the likelihood function, whereas in part iii), there is now the introduction of a prior. IN part iv), this shows a more continuous versino of the posterior distribution. And then in v), this is the actual posterior distribution based on conjugate prior analysis.

6 Problem 6

We know that the beta priors for a binomial/bernoulli distribution is a conjugate prior. So the posterior distribution is of the form beta(A, B), where in this case, we solve and obtain:
$A = a + 57$
$B = b + 100 - 57$

The contour plots are shown here:


Contour Plot of Different [a,b] Beta Priors
n_0
theta_0
0
5
10
15
20
25
30
0.2
0.4
0.6
0.8
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9