$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$

bond price (B), ir (y)

$D_\$ = B \times D$ $\quad DV01 = \frac{D_\$}{10\,000}$

$C_\$ = B \times C$ $\quad = D_\$$ per basis pt

$D(0,3m) = (1+\Delta_{3m} L(3m,6m)) D(0,6m)$

if discrete compound

if continuous compound

IR parity

$FX_T = FX_0 \left(\frac{1+r_d}{1+r_f}\right)$

IRS

$S(T_0,T_n) = \frac{D(0,T_0) - D(0,T_n)}{\sum_{i=1}^{n} \Delta_{i-1} D(T_0,T_i)}$

n payments

PV of basis pt (PVBP)

forward starting IRS

IRS payer pays fixed leg

Fixed leg PV $= k \sum_{i=1}^{n} D(0,T_i)\Delta_{i-1}$

Float leg PV $= 1 - D(0,T_n)$

interpolate discount rate

$D(0,9m) = \frac{D(0,6m) + D(0,12m)}{2}$

continuous compound rates

$D(0,T) = e^{-R(0,T)\cdot T}$ $\quad R(0,T) = -\frac{\ln D(0,T)}{T}$

Q spot usd/sgd 1.42, usd 6m libor 1.5%, 6m usd/sgd 1.39, calc sgd 6m SOR

A $1+\Delta_{6m} L_{sgd}(0,6m) = \frac{usdsgd(6m)}{usdsgd(0)}\left[1+\Delta_{6m} L_{usd}(0,6m)\right]$

$L_{sgd}(0,6m) = -2.757\%$

$e^{f(0y,1y)\cdot 1} \times e^{f(1y,2y)\cdot 1} = e^{R(0,2y)\cdot 2}$

continuous forward rate $\quad$ continuous zero rate

$r_n = \frac{F(0,1) + F(1,2) + \dots + F(n-1,n)}{n}$

IR parity continuous

$FX_T = FX_0 \frac{D_f(0,T)}{D_d(0,T)}$

IRS Qn

$e^{-1.4\%\left(\frac{6}{12}\right)}$

continuous compounded zero rates

Maturity	Rate		Discount	
3m	1.1%	R(0,3m)	0.9973	D(0,3m)
6m	1.4%	R(0,6m)	0.993	D(0,6m)
12m	1.75%	R(0,12m)	0.9827	D(0,12m)
18m	1.9%	R(0,18m)	0.9719	D(0,18m)
24m	2%	R(0,24m)	0.9608	D(0,24m)

Q Futures Prices

Fed funds futures quote 98.1, implied probability of rate cut to 1.75% on 10 Sept when FOMC is held.

A $1.9 = \frac{10}{30}\times 2 + \frac{20}{30}\left[1.75P + 2(1-P)\right]$

Matching collateral & payment currency

$D(t,t+\delta) = \frac{1}{1+\delta f(t,t+\delta)}$

$f(\cdot)$ forward rate

Q 6m OIS 1.05%, 6m LIBOR 2.1%, you receive \$250k in 6m's time

A if collateralised PV of cashflow $= \frac{1}{1+\frac{6}{12}\times 1.05\%}$

if uncollateralised PV of cashflow $= \frac{1}{1+\frac{6}{12}\times 2.1\%}$

Portfolio discount rate

$D_{x,y}(t,T) = D_y(t,T)\frac{FX_{x,y}(t,T)}{FX_{x,y}(t,t)}$

$FX_{x,y}$ exchange rate

1 unit of x = n unit of y

if portfolio pay in x currency and collateral paid in y currency and portfolio can replicate by fixed payment

Q For collateralised interest rate swaps (annual payment)

1y IRS 2%

2y IRS 2.5%

Assume collateral posted in same currency as swap and daily OIS $f(t_{i-1},t_i) = 0.5\%$

A LIBOR 1y rate $= \frac{1}{\left(1+\frac{5\%}{360}\right)^{360}} = 2\%$

2y LIBOR rate $= \frac{1}{\left(1+\frac{5\%}{360}\right)^{720}} = 0.9048\%$

$D(0,1y)(2\%) + D(0,2y)L(1y,2y) = D(0,1y)(2.5\%) + D(0,2y)(2.5\%)$

$L(0,1y) = \frac{1-D(0,1y)}{D(0,1y)}$

$L(1y,2y) = \frac{D(0,1)-D(0,2y)}{D(0,2y)}$

Q uncollateralised interest rates and swap (annual payment)

Q collateralised IRS (annual payment) based on 1y LIBOR fixing

1y spot LIBOR 1.5%

2y swap 2.25%

3y swap 2.55%

A collateralised forward pays forward LIBOR $L(2y,3y)$ at end of 3y for a notional of \$1m. Assume collateral and IRS same currency. Overnight interest rate 0.25%

A $D_{0,1} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{360} = 0.9975$

$D_{0,2} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{2\times 360} = 0.995$

$D_{0,3} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{3\times 360} = 0.9925$

$[D_0(0,1y) + D_0(0,2)] \times 2.25\% = D_0(0,1)\times 1.5\% + D_0(0,2)\times L(1,2)$

$\therefore L(1,2) = 0.03$

$[D_0(0,1) + D_0(0,2) + D_0(0,3)] \times 0.0255 = D_0(0,1)\times 0.015 + D_0(0,2)\times L(1,2) + D_0(0,3)\times L(2,3)$

$\Rightarrow L(2,3) = 0.0315$

$PV = D_0(0,3) \times L(2y \times 3y) \times 1\,000\,000 = 31\,263.75$

PV of payoff

A If uncollateralised

$D(0,1) = \frac{1}{1+0.05} = 0.985$

LIBOR

Q If real world measure we denote W_t Brownian motion. A stock prices follow SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$, risk free bond whose price is $dB_t = rB_t dt$. Use risk neutral measure with risk free bond as numeraire

A calc $E^*[S_T]$

$f(s) = \ln(x)$

A calc $E^*[S_T]$

Swap-settled swaptions

IRR-settled swaptions

payer swaption $= \left\{IRR(S_{n,N}(T))\,(S_{n,N}(T) - K)\right\}^+$

receiver swaption $= \left\{IRR(S_{n,N}(T))(K - S_{n,N}(T))\right\}^+$

where $IRR(S) = \sum_{i=1}^{(T_N - T_n)m} \frac{\frac{1}{m}}{\left(1+\frac{S}{m}\right)^i}$

$\frac{1}{m} = \Delta$ is day count fraction where m is payment frequency of swap

$V^{Pay}_{IRR}(0) = D(0,T)\,IRR(S_{n,N}(0)) \cdot \mathbb{E}^{n+1,N}\left[(S_{n,N}(T) - K)^+\right]$

market model to value IRR-settled swaption

$V_{n,N}(0) = D(0,T)\,IRR(S_{n,N}(0)) \cdot \text{Black}(S_{n,N}(0), K, \sigma_{n,N}, T)$

Static replication to value payoff of constant maturity swap

Q value contract paying $g(S_{n,N}(T))$ at time T

A $h(K) = \frac{g(K)}{IRR(K)}$ — payoff of swaption for strike K — internal rate of return for strike K

swap rate

let $F = S_{n,N}(0)$ be forward swap rate

$V_0 = D(0,T)\,\mathbb{E}^T[g(S)]$

PV of swaption at time

$= D(0,T)\int_0^\infty g(K) f(K)\, dK$ $\quad F =$ forward swap rate

CMS caplet

CMS floorlet

Asset-or-nothing digital

use stock as numeraire and price option under risk-neutral measure associated to the stock Q^S

$V_{\text{asset digital}}(0) = e^{-rT}\mathbb{E}^*\left[S_T \mathbb{1}_{S_T > K}\right]$

$= S_0 \mathbb{E}^*\left[\mathbb{1}_{S_T>K} \cdot \frac{S_T/S_0}{e^{rT}}\right]$

Martingale, $\mathbb{E}^{N}[G_T] = \mathbb{E}^{M}\left[G_T \frac{N_T/N_0}{M_T/M_0}\right]$

LIBOR-in-arrears (LIA)

Interest payment at time T_i is based on $L_i(T_i)$ forward LIBOR rate

$L_i(T_i) = \frac{D_i(t) - D_{i+1}(t)}{\Delta_i D_{i+1}(t)}$

day count — discount factor maturing at time T_i

$V^{LIA}(0) = D_i(0)\,\mathbb{E}^i[L_i(T_i)]$

L_i is not martingale under Q^i measure

$\mathbb{E}^i[L_i(T_i)] = \mathbb{E}^{i+1}\left[L_i(T)\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}\right]$

$= L_i(0)\cdot\frac{1+\Delta_i L_i(0)e^{\sigma_i^2 T_i}}{1+\Delta_i L_i(0)}$

> 1

spot exchange rate

domestic perspective $\quad dX_t = (r^D - r^F)X_t dt + \sigma_x X_t dW_t^D$

foreign perspective $\quad d\left(\frac{1}{X_t}\right) = (r^F - r^D)\left(\frac{1}{X_t}\right)dt + \sigma_x\left(\frac{1}{X_t}\right)dW_t^F$

Foreign exchange rate $= E^D[X_T] = X_t e^{(r^D - r^F)(T-t)}$

FX rate process $\quad F_t = X_t e^{(r^D-r^F)(T-t)}$

Ito's formula shows that $dF_t = \sigma_x F_t dW_t^D$

forward rate mature at T, decided at t

spot FX rate at t

$\frac{1}{F_t} = \frac{1}{X_t}\frac{D^D(t,T)}{D^F(t,T)}$

spot FX at time 0

$F_T = F_0 \frac{D^F(0,T)}{D^D(0,T)}$

foreign discount bond maturing time T

FX forward

domestic investor FX process $\quad dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D$

(Q) calc price of FX forward for domestic investor

(A) let $y_t = \ln(X_t)$

univariate Ito's lemma to solve SDE

$dX_t = \mu_t dt + \sigma_t dW_t \quad y_t = f(X_t)$

$dy_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$

multivariate

$y_t = g(t, X_t)$

$dy_t = g_t(t,X_t)dt + g_x(t,X_t)dX_t + 0.5\,g_{xx}(t,X_t)(dX_t^2)$

$f(x) = \ln(X_t),\ f'(x) = \frac{1}{X_t},\ f''(x) = -\frac{1}{X_t^2}$

$dy_t = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2$

$dy_t = (r^D - r^F)dt + \sigma dW_t^D - \frac{1}{2X_t^2}(\sigma^2 X_t^2)dt$

$= (r^D - r^F)dt + \sigma dW_t^D - \frac{1}{2}\sigma^2 dt$

$= (r^D - r^F - \frac{1}{2}\sigma^2)dt + \sigma dW_t^D$

$\ln\left(\frac{X_T}{X_0}\right) = (r^D - r^F - \frac{1}{2}\sigma^2)T + \sigma dW_T^D$

$X_T = X_0 \exp\left[(r^D - r^F - \frac{1}{2}\sigma^2)T + \sigma dW_T^D\right]$

Take expectation under risk-neutral measure associated to domestic risk free bond

$\mathbb{E}^D[X_T] = X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T}\,\mathbb{E}^D\left[e^{\sigma W_T^D}\right]$

MGF: $E[e^{\theta x}] = e^{\mu\theta + 0.5\sigma^2\theta^2}$

$= X_0 e^{(r^D - r^F - \frac{\sigma^2}{2})T} e^{0.5\sigma^2 T}$ $\quad W_T \sim N(0,T)$

$= X_0 e^{(r^D - r^F)T}$

(Q) show that foreign investor see SDE

$d\frac{1}{X_t} = (r^F - r^D + \sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D$

$d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F$

(A) $dX_t = (r^D - r^F)X_t dt + \sigma X_t dW_t^D$

let $Y_t = f(X_t) = \frac{1}{X_t}$

$d\left(\frac{1}{X_t}\right) = (r^F - r^D + \sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D$

Next, start with B_t^D

$dX_t = \mu X_t dt + \sigma X_t dW_t$

let $y_t = \frac{B_t^D}{X_t} = f(B_t^D, X_t)$

$\therefore dy_t = (r^D + \sigma^2 - \mu)X_t dt - \sigma y_t dW_t$

$Z_t = \frac{y_t}{B_t^F} = g(y_t, B_t^F)$

$dZ_t = (r^D - r^F + \sigma^2 - \mu)Z_t dt - \sigma Z_t dW_t$

$= -\sigma Z_t\left[dW_t - \frac{r^D - r^F + \sigma^2 - \mu}{\sigma}(dt)\right]$ —(1)

$= -\sigma Z_t\, dW_t^F$

(Q) L_i^F foreign LIBOR observed at T_i and paid T_{i+1} with volatility σ_i. Foreign FX process $\frac{dF_t}{F_t} = \sigma_x F_t dW_t^D$ from foreign perspective, and LIBOR market model is lognormal. Show that domestic perspective

(A)

convexity-adjusted price of quanto LIBOR

convexity correction to account for how FX affects asset price

Term structure models

drift in short rate models

LIBOR in arrears

(Q) LIBOR rate L_i observed at time T_i and paid at T_i. Use LIBOR market model, evaluate $\mathbb{E}^i[L_i(T_i)]$ by performing single convexity correction

(A) LIBOR market model, forward LIBOR $L_i(t)$ is martingale under risk-neutral measure with numeraire $D_{i+1}(t)$

$L_i(T) = L_i(0)\exp\left(-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}\right)$

Radon-Nikodym to change the measure

$= L_i(0)\left[\frac{1+\Delta_i L_i(0)\exp(\sigma_i^2 T)}{1+\Delta_i L_i(0)}\right]$

(Q) contract pay $\Delta_i L_i(T)$ at T_{i+1}, use LIBOR market model to value

(A) LIBOR market model $dL_i(t) = \sigma_i L_i(t)\,dW_t^{i+1}$

(Q) Contract pays $\Delta_i L_i(T)$ at T_i. Derive valuation formula using LIBOR mkt model

(Q) L_i^D be forward domestic LIBOR at T_i and paid at T_{i+1}, with volatility σ_i. Forward exchange process is $dF_t = \sigma_x F_t dW_t^D$ for domestic investor. Suppose correlation of W_t^D and LIBOR mkt model W_t^{i+1} is ρ. Evaluate (from foreign investor perspective) $\mathbb{E}^{i+1,F}[L_i^D(T)]$

(A) $dF_t = \sigma_x F_t dW_t^D \Rightarrow F_T = F_0 e^{-\frac{\sigma_x^2 T}{2} + \sigma_x W_T^D}$

$L_i^D(T) = L_i^D(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}}$

multi-currency change of numeraire theorem

$\mathbb{E}^{i+1,F}[L_i^D(T)] = \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{dQ^{i+1,F}}{dQ^{i+1,D}}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_T D^F_{i+1}(T)/X_0 D^F_{i+1}(0)}{D^D_{i+1}(T)/D^D_{i+1}(0)}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_T D^F_{i+1}(T)/D^D_{i+1}(T)}{X_0 D^F_{i+1}(0)/D^D_{i+1}(0)}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{F_T}{F_0}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(0)e^{-\frac{\sigma_i^2 T}{2} + \sigma_i W_T^{i+1}} e^{-\frac{\sigma_x^2 T}{2} + \sigma_x W_T^D}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2 T}{2}}e^{-\frac{\sigma_x^2 T}{2}}\mathbb{E}^{i+1,D}\left[e^{\sigma_i W_T^{i+1} + \sigma_x W_T^D}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2 T}{2} - \frac{\sigma_x^2 T}{2}}\mathbb{E}^{i+1,D}\left[e^{\sigma_i Z_T^1 + \sigma_x(\rho Z_T^1 + \sqrt{1-\rho^2}Z_T^2)}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2 T}{2} - \frac{\sigma_x^2 T}{2}}\mathbb{E}^{i+1,D}\left[e^{(\sigma_i + \rho\sigma_x)Z_T^1 + \sigma_x\sqrt{1-\rho^2}Z_T^2}\right]$

$= L_i^D(0)\exp\left[T\left\{-\frac{\sigma_i^2}{2} - \frac{\sigma_x^2}{2} + \frac{1}{2}\sigma_i^2 + \sigma_i\rho\sigma_x + \frac{1}{2}\rho^2\sigma_x^2 + \frac{1}{2}\sigma_x^2 - \frac{1}{2}\sigma_x^2\rho^2\right\}\right]$

$= L_i^D(0)\exp(\sigma_i\sigma_x\rho T)$

(Q) 3 correlated Brownian motion W_t^f, W_t^g, W_t^h where $dW_t^f dW_t^g = \rho_{fg}dt$, $dW_t^g dW_t^h = \rho_{gh}dt$, $dW_t^f dW_t^h = \rho_{fh}dt$. Calculate Cholesky decomposition

(A) $dW_t^f = dZ_t^1$, $dW_t^g = \rho_{fg}dZ_t^1 + \rho_{fh}dZ_t^2$

$dW_t^h = \rho_{fh}dZ_t^1 + \frac{\rho_{gh} - \rho_{fg}\rho_{fh}}{\sqrt{1-\rho_{fh}^2}}dZ_t^2 + \sqrt{1 - (\rho_{fh})^2 - \left[\frac{\rho_{gh} - \rho_{fg}\rho_{fh}}{\sqrt{1-\rho_{fh}^2}}\right]^2}\,dZ_t^3$

(Q) Contract pays $\Delta_i\sqrt{L_i(T)}$ at time T_{i+1}, derive valuation formula using LIBOR market model

(A) $dL_i(t) = \sigma_i L_i(t)\,dW_t^{i+1}$

where $L_i(t) = L_i(t; T_i, T_{i+1})$ and W_t^{i+1} is Brownian motion under risk neutral measure

Affine short rate model

(Q) Swap mkt model

A floating-leg-or-nothing digital option

(Q) contract pays $S_{n,N}(T)$ on maturity T, explain why we cannot value this contract directly using swap market model without applying convexity correction

Vasicek model for short rate

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

mean reversion coefficient $\quad$ long run mean

see pg2 for full steps

Ito's $f(r_t,t) = r_t e^{kt}$

$r_t = r_0 e^{-kt} + \theta(1-e^{-kt}) + \sigma\int_0^t e^{k(u-t)}dW_u^*$

$E[r_t]$ $\quad V[r_t] = \frac{\sigma^2}{2k}(1 - e^{-2kt})$

reconstruct discount factor $D(t,T) = E\left[e^{-\int_t^T r_u du}\right]$

$D(t,T) = e^{-R(t,T)(T-t)} = e^{A(t,T) - B(t,T)r_t}$

$R(t,T) = \frac{1}{T-t}\left[-A(t,T) + B(t,T)r_t\right]$

where $B(t,T) = \frac{1}{k}\left[1 - e^{-k(T-t)}\right]$

$A(t,T) = \frac{[B(t,T) - (T-t)](k^2\theta - \frac{1}{2}\sigma^2)}{k^2} - \frac{\sigma^2 B(t,T)^2}{4k}$

Cox-Ingersoll-Ross model

r_t is chi-square distributed

$dr_t = k(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t^*$

No-arbitrage short rate affine model

Ho-Lee: $dr_t = \theta(t)dt + \sigma dW_t^*$

choose θ to match the observed spot curve

integrate both sides

$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$

integrate again

$\int_0^T r_u du = \int_0^T r_0 du + \int_0^T\int_0^u \theta(s)ds\,du + \int_0^T\int_0^u \sigma dW_s^* du$

$= r_0 T + \int_0^T \theta(s)[T-s]ds + \int_0^T \sigma[T-s]dW_s^*$

$= E\left[\int_0^T r_u du\right]$ $\quad V\left[\int_0^T r_u du\right]$

$= \int_0^T \sigma^2(T-s)^2 ds = \frac{1}{3}\sigma^2 T^3$

Reconstruct zero-coupon discount factor

$D(0,T) = E\left[e^{-\int_0^T r_u du}\right]$

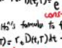
$= \exp\left[-r_0 T - \int_0^T \theta(s)(T-s)ds + \frac{1}{6}\sigma^2 T^3\right]$

This proves that Ho-Lee is an affine model i.e. $D(t,T) = e^{A(t,T) - r_t B(t,T)}$

constant $\quad$ linear function

apply Ito's formula to find SDE

$dD(t,T) = r_t D(t,T)dt - (T-t)\sigma D(t,T)dW_t^*$



(Q) Ho-Lee model, 2 period binomial tree, every step, rate can move up or down by 0.5%, risk-neutral probabilities of up or down move are both 0.5.

Hull White trinomial model

Vasicek short rate model

Question

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

Apply Itô's formula to $f(r,t) = e^{kt} r_t$

Solution: $r_t = r_0 e^{-kt} + \theta(1-e^{-kt}) + \sigma\int_0^t e^{k(u-t)} dW_u^* \, dt$

$D(t,T) = \exp\begin{bmatrix} A(t,T) \\ -r_t B(t,T) \end{bmatrix}$

Integrate both sides from 0 to T:

$\int_0^T r_t dt = \int_0^T r_0 e^{-kt} dt + \int_0^T \theta(1-e^{-kt}) dt + \int_0^T\int_0^t \sigma e^{k(u-t)} dW_u^* \, dt$

$= \int_0^T\int_u^T \sigma e^{k(u-t)} dt \, dW_u^*$

$= \int_0^T \left[-\frac{\sigma}{k} e^{k(u-t)}\right]_u^T dW_u^*$

$= \frac{\sigma}{k}\int_0^T (1-e^{k(u-T)}) dW_u^*$

$\therefore \int_0^T r_t dt = \int_0^T r_0 e^{-kt} + \int_0^T \theta(1-e^{-kt}) dt + \frac{\sigma}{k}\int_0^T (1-e^{k(u-T)}) dW_u^*$

$E^*[\int_0^T r_t dt] = \int_0^T r_0 e^{-kt} dt + \int_0^T \theta(1-e^{-kt}) dt$

$= \frac{r_0}{k}(1-e^{-kT}) + \theta T - \frac{\theta}{k}(1-e^{-kT})$

$V[\int_0^T r_t dt] = V[\frac{\sigma}{k}\int_0^T (1-e^{k(u-T)}) dW_u^*]$

$= \frac{\sigma^2}{k^2}\int_0^T (1-e^{k(u-T)})^2 du$

$= \frac{\sigma^2}{k^2}\int_0^T [1-2e^{k(u-T)} + e^{2k(u-T)}] du$

$= \frac{\sigma^2}{k^2}[T - \frac{2}{k}(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})]$

$\therefore D(0,T) = E^*[\exp(-\int_0^T r_t dt)]$

$= \exp\{-E^*[\int_0^T r_t dt] + \frac{1}{2}V[\int_0^T r_t dt]\}$

(Q) Ho-Lee interest rate model

$dr_t = \theta(t)dt + \sigma dW_t^*$

show that $\theta(t) = -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2 t$

(A) integrate both sides

$r_t = r_0 + \int_0^t \theta(s)ds + \int_0^t \sigma dW_s^*$

$\int_0^t r_u du = \int_0^t r_0 du + \int_0^t\int_0^u \theta(s) ds \, du + \int_0^t\int_0^u \sigma dW_s^* du$

$= r_0 t + \int_0^t \theta(s)(t-s) ds + \int_0^t \sigma(t-s) dW_s^*$

$E[\int_0^t r_u du] = r_0(t-0) + \int_0^t \theta(s)(t-s) ds$

$V[\int_0^t r_u du] = \int_0^t \sigma^2(t-s)^2 dt$

$= \frac{1}{3}\sigma^2(t-0)^3$

$D(0,t) = E[\exp\{-\int_0^t r_u du\}]$

$= \exp[\underbrace{-r_0 t - \int_0^t \theta(s)(t-s)ds}_{-E} + \underbrace{\frac{1}{6}\sigma^2 t^3}_{\frac{1}{2}V}]$

$\log D(0,t) = -r_0 t - \int_0^t \theta(s)(t-s) ds + \frac{1}{6}\sigma^2 t^3$

Ho-Lee model: zero coupon bond option price under Q^*

$dD(t,T) = r_t D(t,T)dt - (T-t)\sigma D(t,T) dW_t^*$

solution

$D(t,T) = D(0,T)\exp\begin{bmatrix} \int_0^t r_u du \\ -\frac{\sigma^2}{2}\int_0^t (T-u)^2 du \\ -\sigma\int_0^t (T-u) dW_u^* \end{bmatrix}$

$\int_0^t r_u du = -\log D(0,t) + \frac{1}{6}\sigma^2 t^3 + \sigma\int_0^t (t-u) dW_u^*$

$\int_0^t r_u du = -\log D(0,t) + \frac{\sigma^2}{2}\int_0^t (t-u)^2 du + \sigma\int_0^t (t-u) dW_u^*$

examples $dS_t = rS_t dt + \sigma S_t dW_t^*$

$S_T = S_0 \exp[(r - \frac{\sigma^2}{2})T + \sigma W_T^*]$

$dS_t = r_t S_t dt + \sigma_t S_t dW_t^*$

$S_T = S_0 \exp[\int_0^T (r_u - \frac{\sigma_u^2}{2}) du + \int_0^T \sigma_u dW_u^*]$

$D(t,T) = \frac{D(0,T)}{D(0,t)}\exp\begin{bmatrix} -\frac{\sigma^2}{2}\int_0^t [(T-u)^2 - (t-u)^2] du \\ -\sigma\int_0^t [(T-u)-(t-u)] dW_u^* \end{bmatrix}$

$= dW_u^* - \sigma(t-u) du$

$= \frac{D(0,T)}{D(0,t)}\exp\begin{bmatrix} -\frac{\sigma^2}{2}\int_0^t [(T-u)-(t-u)]^2 du \\ -\sigma\int_0^t [(T-u)-(t-u)] dW_u^* \end{bmatrix}$

$= \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2}(T-t)^2 t - \sigma(T-t)W_t^*]$

(Q) zero coupon bond option with option expiry t, and maturity T is given by $[D(t,T)-k]^+$. Value it

(A) $V_0 = E[\frac{B_0(D[t,T]-k)^+}{B_t}]$

$= E^*[(D(t,T)-k)^+ \cdot \frac{D(t,t)/D(0,t)}{B_t/B_0}] \cdot D(0,t)$

$= D(0,t)E^T[(D(t,T)-k)^+]$

$= D(0,T)\Phi\left[\frac{\log\frac{D(0,T)}{D(0,t)k} + \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}\right]$

$- D(0,t)k\Phi\left[\frac{\log\frac{D(0,T)}{D(0,t)k} - \frac{1}{2}\sigma^2(T-t)^2 t}{\sigma(T-t)\sqrt{t}}\right]$

and let $\sigma_{HL} = \sigma(T-t)\sqrt{t}$ which is the Ho-Lee integrated volatility

Vasicek model

integrated short rate

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

$\int_0^t r_u du = \int_0^t r_0 e^{-ku} du + \int_0^t \theta(1-e^{-ku}) du$

(Q) IRS mkt

6m LIBOR	2%	IRS pays semi-annual pat
1y IRS	2.2%	overnight interest rate
2y IRS	2.5%	is $f = 0.8\%$

calc value of forward LIBOR rate $L(1.5,2)$ if swap mkt is collateralised & uncollateralised

express forward LIBOR using interpolated discount factors

$\Delta(2.5\%)\begin{bmatrix} D_{OIS}(0,0.5) + D_{OIS}(0,1) + D_{OIS}(0,1.5) \\ + D_{OIS}(0,2) \end{bmatrix}$

day count fraction = 0.5

OIS discount factor

$= D_{OIS}(0,0.5)\Delta L(0,0.5) + D_{OIS}(0,1)\Delta L(0.5,1) + D_{OIS}(0,1.5)\Delta L(1,1.5) + D_{OIS}(0,2)\Delta L(1.5,2)$

use LIBOR discount factor to calc what 1.5y par swap rate

LIBOR discount factor

$IRS\ 1.5y = \frac{1-\tilde{D}(0,1.5)}{\Delta[\tilde{D}(0,0.5) + \tilde{D}(0,1) + \tilde{D}(0,1.5)]} = 2.3\%$

$2.39\%\,\Delta(D_0(0,0.5) + D_0(0,1) + D_0(0,1.5))$

$= D_0(0,0.5)\Delta L(0.5) + D_0(0,1)\Delta L(0.5,1)$

$+ D_0(0,1.5)\Delta L(1,1.5)$

(Q) IRS mkt

6m LIBOR	2%	IRS pays semi-annual pat
1y IRS	2.2%	overnight interest rate
2y IRS	2.5%	is $f = 0.8\%$

(A) calc value of forward LIBOR rate $L(1.5,2)$ if swap mkt is collateralised & uncollateralised

$D_{OIS}(0,0.5) = \frac{1}{(1+\frac{0.8\%}{360})^{180}} = \frac{1}{(1+\frac{0.8\%}{360})^{180}} = 0.9607937$

$D_{OIS}(0,1) = 0.992032 \qquad D_{OIS}(0,1.5) = 0.988042$

$D_{OIS}(0,2) = 0.984127$

$\Delta = 0.5 =$ day count fraction, $\tilde{D}$ (libor discount factor), D_{OIS} (OIS discount factor), L (forward libor rate)

uncollateralised

$\tilde{D}(0,0.5) = \frac{1}{1+\Delta L(0,0.5)} = \frac{1}{1+0.5(2\%)} = 0.990099$

6m LIBOR 2%

$1y\ IRS = 2.2\%\ (given) = \frac{1-\tilde{D}(0,1)}{\Delta[\tilde{D}(0,0.5) + \tilde{D}(0,1)]}$

$2.2\% = \frac{1-\tilde{D}(0,1)}{0.5[0.990099 + \tilde{D}(0,1)]}$

$0.01089109 + 0.011\tilde{D}(0,1) = 1 - \tilde{D}(0,1)$

$\tilde{D}(0,1) = 0.978347$

$2y\ IRS = 2.5\%\ (given) = \frac{1-\tilde{D}(0,2)}{\Delta\begin{bmatrix}\tilde{D}(0,0.5) + \tilde{D}(0,1) \\ + \tilde{D}(0,1.5) + \tilde{D}(0,2)\end{bmatrix}}$

$2.5\% = \frac{1-\tilde{D}(0,2)}{\Delta[\tilde{D}(0,0.5) + \frac{3}{2}\{\tilde{D}(0,1) + \tilde{D}(0,2)\}]}$

$2.5\% = \frac{1-\tilde{D}(0,2)}{0.5[0.990099 + \frac{3}{2}\{0.978347 + \tilde{D}(0,2)\}]}$

$0.0123762 + 0.018344 + 0.01875\tilde{D}(0,2) = 1 - \tilde{D}(0,2)$

$\tilde{D}(0,2) = 0.95144$

$\tilde{D}(0,1.5) = \frac{1}{2}[\tilde{D}(0,2) + \tilde{D}(0,1)] = \frac{1}{2}\begin{bmatrix}0.95144 + \\ 0.978347\end{bmatrix}$

$= 0.9648935$

$L(1.5,2) = \frac{1}{\Delta}\frac{D(0,1.5) - D(0,2)}{D(0,2)} = \frac{1}{0.5}\left[\frac{0.9648935 - 0.95144}{0.95144}\right]$

$= 0.02828$ (Ans)

collateralised

$2.2\%[D_{OIS}(0,0.5) + D_{OIS}(0,1)] = L(0,0.5)D_{OIS}(0,0.5) + L(0.5,1)D_{OIS}(0,1)$

1y IRS given

$2.2\%[0.9607937 + 0.992032] = 0.02(0.9607937) + L(0.5,1)(0.992032)$

$\therefore L(0.5,1) = 0.0239702$

$1.5y\ IRS = \frac{1-\tilde{D}(0,1.5)}{\Delta[\tilde{D}(0,0.5) + \tilde{D}(0,1) + \tilde{D}(0,1.5)]}$

$= \frac{1-0.9648935}{0.5[0.990099 + 0.978347 + 0.9648935]}$

$= 0.02392831$

$1.5y\ IRS[D_{OIS}(0,0.5) + D_{OIS}(0,1) + D_{OIS}(0,1.5)]$

$= L(0,0.5)D_{OIS}(0,0.5) + L(0.5,1)D_{OIS}(0,1)$

(Q) Spot LIBOR rates, uncollateralised

1m	1.25%
2m	1.5%
3m	1.65%
6m	1.95%
12m	1.80%

calc forward rate for 9x12 FRA

(A) $D(0,6m) = \frac{1}{1+\frac{6}{12}(1.95\%)} = 0.9903$

$D(0,12m) = \frac{1}{1+1.8\%} = 0.982318$

$\therefore L(9m,12m) = \frac{1}{\Delta_{3m}}\frac{D(0,9m) - D(0,12m)}{D(0,12m)} = 1.834\%$

(Q) calc 1y par swap rate for IRS with quarter payment

(A) $S(0,12m) = \frac{1-D(0,12m)}{\Delta_{3m}[D(0,3m) + D(0,6m) + \ldots + D(0,12m)]}$

$\frac{1}{1+\frac{3}{12}(1.65\%)} = 0.99589 \qquad = 1.7877\%$

(Q) calc continuous compounded zero rates $R(0,3m)$, $R(0,6m)$ and $R(0,12m)$

(A) $D(0,T) = e^{-R(0,T)\cdot T} \qquad R(0,T) = -\frac{\ln D(0,T)}{T}$

$R(0,3m) = -\frac{\ln(0.99589)}{3/12} = 1.6466\%$

$R(0,6m) = 1.7424\% \qquad R(0,9m) = 1.7687\%$

(Q) spot FX rate for USD/SGD is 1.42, and USD 6m LIBOR rate is 1.5%. If 6m forward FX rate for USD/SGD is 1.39. Calc implied 6m interest rate (SOR) in SGD

(A) $FX_T = FX_0\left(\frac{1+r_d}{1+r_f}\right)^T$

$1 + \Delta_{6m}L_{SGD}(0,6m) = \frac{usdsgd(6m)}{usdsgd(0)}[1 + \Delta_{6m}L_{USD}(0,6m)]$

$L_{SGD}(0,6m) = -2.757\%$

(Q) 6m LIBOR	1.5%	uncollateralised, IRS have semi-annual payment
1y IRS	1.8%	
2y IRS	2.0%	calc 1.5y IRS
3y IRS	2.05%	

(A) $L(0,6m) \Rightarrow D(0,0.5y) = \frac{1}{1+0.5(1.5\%)} = 0.99256$

$1.8\% = \frac{1-D(0,1y)}{0.5[D(0,0.5y) + D(0,1y)]} \Rightarrow D(0,1y) = 0.98223$

$2\% = \frac{1-D(0,2)}{0.5[D(0,0.5) + D(0,1.5) + D(0,2)]} \qquad = \frac{D_{01} + D_{02}}{2}$

$D(0,2) = 0.960917 \qquad D(0,1.5) = 0.97158$

$IRS(0,1.5y) = \frac{1-D(0,1.5y)}{0.5[D(0,0.5y) + D(0,1y) + D(0,1.5y)]}$

$= 1.92936\%$

(Q) A forward starting swap with 2y tenor start at t = 1y has these cashflows

Time (t)	Pay	Receive
1.5	Par swap rate	6m LIBOR
2.0		
2.5		
3.0		

what is the par swap rate for the forward starting swap?

(A) $IRS(1y,3y) = \frac{D(0,1y) - D(0,3y)}{0.5[D_{0,1.5} + D_{0,2} + D_{0,2.5} + D_{0,3}]}$

$IRS(0,3y) = \frac{1-D(0,3)}{0.5[D_{0,0.5} + D_{0,1} + D_{0,1.5} + 1.5D_{0,2} + 1.5D_{0,3}]} = 2.149\%$

$D_{0,3} = 0.940564$

$D_{0,2.5} = 0.950746$

the forward starting swap has a par swap rate of $S = \frac{D(0,1y) - D(0,3y)}{0.5[D_{0,1.5} + D_{0,2} + D_{0,2.5} + D_{0,3}]} = 2.18\%$

(Q)