$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$

bond price (B), ir (y)

$D_\$ = B\times D$ DV01 $= \frac{D_\$}{10\,000}$

$C_\$ = B\times C$ = D$ per basis pt

$D(0,3m) = (1+\Delta_{3m}L[3m,6m])D(3m,6m)$

if continuous $= e^{-\frac{3}{12}L(0,3m)}$; $\frac{1}{D(3m,6m)}$ if discrete compound

$D(3m,6m) = e^{-(\frac{6}{12}-\frac{3}{12})L(3m,6m)}$ if continuous compound

$L(3m,6m) = \frac{1}{\Delta_{(6-3)m}}\frac{D(0,3m)-D(0,6m)}{D(0,6m)}$

IR Parity discrete $FX_T = FX_0\left(\frac{1+r_d}{1+r_f}\right)^T$ FX: 1.59 usdsgd ∴ r_A is sg

IRS par swap $S(T_0,T_n) = \frac{D(0,T_0)-D(0,T_n)}{\sum_{i=1}^{n}\Delta_{i-1}D(T_0,T_i)}$ forward starting IRS

n payments; PV of basis pt (PVBP)

IRS payer pays fixed leg

Fixed leg PV $= k\sum_{i=1}^{n}D(0,T_i)\Delta_{i-1}$

Float leg PV $= 1 - D(0,T_n)$

interpolate discount rate $D(0,9m) = \frac{D(0,6m)+D(0,12m)}{2}$

continuous compound rates $D(0,T) = e^{-R(0,T)\cdot T}$ $R(0,T) = -\frac{\ln D(0,T)}{T}$

Q spot usdsgd 1.42, usd 6m libor 1.5%, 6m usdsgd 1.39, calc sgd 6m SOR

A $1+\Delta_{6m}L_{sgd}(0,6m) = \frac{usdsgd(6m)}{usdsgd(0)}[1+\Delta_{6m}L_{usd}(0,6m)]$

$L_{sgd}(0,6m) = -2.757\%$

$e^{F(0,1y)\cdot 1}\times e^{F(1y,2y)\cdot 1} = e^{R(0,2y)\cdot 2}$ continuous forward rate, continuous zero rate

$r_n = \frac{F(0,1)+F(1,2)+\ldots+F(n-1,n)}{n}$

IR Parity continuous $FX_T = FX_0\frac{D_f(0,T)}{D_d(0,T)}$

IRS Qn

continuous compounded zero rates

Maturity	Rate		Discount	
3m	1.1%	R(0,3m)	0.9973	D(0,3m)
6m	1.4%	R(0,6m)	0.993	D(0,6m)
12m	1.75%	R(0,12m)	0.9827	D(0,12m)
18m	1.9%	R(0,18m)	0.9719	D(0,18m)
24m	2%	R(0,24m)	0.9608	D(0,24m)

$e^{-1.4\%(\frac{6}{12})}$

IRS: 2y semi annual, fixed 1.75%, floating is 6m LIBOR rate semi annual, par swap rate = 2%

3 months later's continuous compounded zero rates

Maturity	Rate		DF	
3m	1.2%	R(0,3m)	0.997	D(0,3m)
6m	1.5%	R(0,6m)	0.99252	D(0,6m)
12m	1.85%	R(0,12m)	0.98167	D(0,12m)
18m	1.95%	R(0,18m)	0.97117	D(0,18m)
24m	2.05%	R(0,24m)	0.9598	D(0,24m)

calc value of receiver swap 3m later

Swap cashflows are 3m, 9m, 15m, 21m away

$R_{9m} = \frac{1.5+1.85}{2} = 1.675\%$

$R_{15m} = \frac{1.85+1.95}{2} = 1.9\%$

$R_{21m} = \frac{1.95+2.05}{2} = 2\%$ } new zero rates

Use zero rates (R) to calc forward LIBOR rates (L)

$L(0,6m) = R(0,6m) = 1.4\%$

$e^{1.2\%\times0.25}[1+0.5L(3,9)] = e^{1.675\%\times0.75}$ $L(3,9) = 1.92\%$

$e^{1.675\%\times0.75}[1+0.5L(9,15)] = e^{1.9\%\times1.25}$ $L(9,15) = 2.25\%$

$e^{1.9\%\times1.25}[1+0.5L(15,21)] = e^{2\%\times1.75}$ $L(15,21) = 2.265\%$

Float PV $= 0.5[D(0,3m)L(0,6m) + D(0,9m)L(3m,9m) + D(0,15m)L(9m,15m) + D(0,21m)L(15m,21m)]$ use old L(0,6m)

original

$D(0,9m) = \frac{D(0,6m)+D(0,12m)}{2} = \frac{0.993+0.9827}{2} = 0.98785$

$D(0,15m) = \frac{D(0,12m)+D(0,18m)}{2} = \frac{0.9827+0.9719}{2} = 0.9773$

$D(0,21m) = \frac{D(0,18m)+D(0,24m)}{2} = \frac{0.9719+0.9608}{2} = 0.96635$

Float PV $= 0.5[0.9973(1.4\%)+0.98785(1.92\%) + 0.9773(2.25\%)+0.96635(2.265\%)] = 0.035364$

Fix PV $= 0.5[D(0,3m)+D(0,9m)+D(0,15m) + D(0,21m)]\times2\% = 0.0392878$

Value of receiver swap $V_{rec} = PV_{fix} - PV_{float} = 0.001$ per $1 notional

---

Q **Futures Prices** today 16 aug, futures quote 98.1 (=1.9%), Fed rate 2%, calc implied probability of rate cut to 1.75% on 10 sept when a FOMC is held.

A $1.9 = \frac{10}{30}\times2 + \frac{20}{30}[1.75P + 2(1-P)]$

Matching collateral & payment currency

$D(t,t+\delta) = \frac{1}{1+\delta f(t,t+\delta)}$ f(.) forward rate

Q 6m OIS 1.05%, 6m LIBOR 2.1%. You receive $250k in 6m's time

A If collateralised PV of cashflow $= \frac{1}{1+\frac{6}{12}\times1.05\%}$

if uncollateralised PV of cashflow $= \frac{1}{1+\frac{6}{12}\times2.1\%}$

Portfolio discount rate $D_{x,y}(t,T) = D_y(t,T)\frac{FX_{x,y}(t,T)}{FX_{x,y}(t,t)}$ $FX_{x,y}$ exchange rate

Q 1 unit of x = n unit of y

if portfolio pay in x currency and collateral paid in y currency and portfolio can replicate by fixed payment

Q For collateralised interest rate swaps (annual payment)

1y IRS 2%
2y IRS 2.5%

Assume collateral posted in same currency as swap and daily OIS $f(t_{i-1},t_i) = 0.5\%$

A 1y LIBOR rate $= \frac{1}{(1+\frac{5\%}{360})^{360}} = 2\%$

2y LIBOR rate $= \frac{1}{(1+\frac{5\%}{360})^{720}} = 0.9048\%$

$D(0,1y)(2\%) + D(0,2y)L(1y,2y) = D(0,1y)(2.5\%) + D(0,2y)(2.5\%)$

$L(0,1y) = \frac{1-D(0,1y)}{D(0,1y)}$

$L(1y,2y) = \frac{D(0,1)-D(0,2y)}{D(0,2y)}$

Q uncollateralised interest rates and swaps (annual payment)

6m CD 1.2%
1y IRS 2%
2y IRS 2.5%

A $L(6m,1y) = \frac{1}{\Delta_{6m}}\frac{D_{0,6m}-D_{0,1y}}{D_{0,1y}}$

undiscount discount rate use LIBOR

$D(0,6m) = \frac{1}{1+\frac{6}{12}\times1.2\%} = 0.994$

$0.02 = \frac{D_{0,0}-D_{0,1}}{D_{0,1}} \to D_{0,1y} = 0.984$

$L(6m,1y) = 2.47\%$

B $0.025 = \frac{D_{0,0}-D_{0,2}}{D_{0,1}+D_{0,2}}$

$D_{0,2y} = 0.9517$

$D_{0,1.5y} = \frac{D_{0,1}+D_{0,2}}{2} = 0.9661$

$L(1y,1.5y) = \frac{1}{\frac{6}{12}}\frac{D(0,1)-D_{0,1.5}}{D_{0,1.5}} = 2.96\%$

Q spot USD LIBOR

1m 1.15% 6m 1.4%
2m 1.2% 9m 1.45%
3m 1.25% 12m 1.75%

overnight interest rate (Fed fund) 0.7%
Index basis swap spread 0%

A LIBOR discount factor

$D(0,3m) = \frac{1}{1+0.25\times0.0125} = 0.99688$

$D_{OIS}(0,3m) = \prod_{i=1}^{90}\frac{1}{1+\frac{1}{360}\cdot0.007} = 0.99825$

Q collateralised IRS (annual payment) based on 1y LIBOR fixing

1y spot LIBOR 1.5%
2y Swap 2.25%
3y Swap 2.55%

A collateralised forward pays forward LIBOR L(2y,3y) at end of 3y for a notional of $1m. Assume collateral and IRS same currency. Overnight interest rate 0.25%

A $D_{0,1} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{360} = 0.9975$

$D_{0,2} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{2\times360} = 0.995$

$D_{0,3} = \left(\frac{1}{1+\frac{0.25\%}{360}}\right)^{3\times360} = 0.9925$

$[D_0(0,1y)+D_0(0,2)]\times2.25\% = D_0(0,1)\times1.5\% + D_0(0,2)\times L(1,2)$

$\therefore L(1,2) = 0.03$

$[D_0(0,1)+D_0(0,2)+D_0(0,3)]\times0.0255 = D_0(0,1)\times0.015 + D_0(0,2)\times L(1,2) + D_0(0,3)\times L(2,3)$

$\Rightarrow L(2,3) = 0.0315$

PV of payoff

$PV = D_0(0,3)\times L(2y\times3y)\times1\,000\,000 = 31\,263.75$

A If uncollateralised

PV of swaption at time

$D(0,1) = \frac{1}{1+0.015} = 0.985$

Swap$(0,2y) = 0.0225 = \frac{1-D_{0,2}}{D_{0,1}+D_{0,2}}$

$D_{0,2} = 0.9563$

Swap$(0,3) = 0.0255 = \frac{1-D_{0,3}}{D_{0,1}+D_{0,2}+D_{0,3}}$

$D_{0,3} = 0.927$

discount factor from 0 to T

LIBOR

$L_{0,1} = 0.015$

$L_{1,2} = \frac{D_{0,1}-D_{0,2}}{D_{0,2}} = 0.03$

$L_{2,3} = \frac{D_{0,2}-D_{0,3}}{D_{0,3}} = 0.0316$

PV of uncollateralised forward $= D_{0,3}\times L_{2,3}\times1\,000\,000 = 29293.2$

Q P real world measure, W_t denote P-brownian motion. A stock prices follow SDE $dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

C

risk free bond whose price is $dB_t = rB_t\,dt$

Q be risk neutral measure with risk free bond as numeraire

A calc $E^P[S_T]$

$f(S)=\ln(S)$ $x=\ln(S)$ $f'(S) = \frac{1}{S}$ $f''(x) = -\frac{1}{S^2}$

Ito's $f(S) = f'(S)dS + \frac{1}{2}f''(S)(dS)^2$

$= \frac{1}{S}dS - \frac{1}{2S^2}(dS)^2$

$= \frac{1}{S}\mu S_t\,dt - \frac{1}{S}\sigma S_t\,dW_t - \frac{1}{2S^2}\sigma^2S_t^2\,dt$

$= \mu dt - \frac{1}{2}\sigma^2dt - \sigma dW_t$

$f(S) = (\mu-\frac{1}{2}\sigma^2)dt - \sigma dW_t$

↓ integrate from 0 to T

$f(S_T) - f(S_0) = \int_0^T(\mu-\frac{1}{2}\sigma^2)dt + \int_0^T\sigma\,dW_t$

$\ln(S_T) = \ln(S_0) + (\mu-\frac{1}{2}\sigma^2)T + \sigma W_T$

↓ exponential both sides

$S_T = S_0e^{(\mu-\frac{1}{2}\sigma^2)T+\sigma W_T}$

$E^P[S_T] = S_0e^{\mu T}$

A calc $E^Q[S_T]$

Q measure: $\mu S_t\,dt + \sigma S_t\,dW_t$

change numeraire to money mkt instrument

$rS_t\,dt - rS_t\,dt + \sigma S_t\,dW_t + \mu S_t\,dt$

$= rS_t\,dt + \sigma S_t(dW_t + \frac{\mu-r}{\sigma}dt)$

$= rS_t\,dt + \sigma S_t\,dW_t^*$

$\therefore dW_t^* = dW_t + \boxed{\frac{\mu-r}{\sigma}}dt$ k

$dS_t = \mu S_t\,dt + \sigma S_t\,dW_t$

sub in → $dS_t = \mu S_t\,dt + \sigma S_t(dW_t^* - \frac{\mu-r}{\sigma}dt)$

$= rS_t\,dt + \sigma S_t\,dW_t^*$

$S_T = S_0\exp[(r-\frac{\sigma^2}{2})T + \sigma W_T^Q]$

$E^Q[S_T] = S_0e^{rT}$

Swap-settled swaptions

Payer payoff

$= \{P_{n+1,N}(T)[S_{n,N}(T)-K]\}^+$

receiver leg $= \{P_{n+1,N}(T)[K-S_{n,N}(T)]\}^+$

where $P_{n+1,N}(T) = \sum_{i=n+1}^{N}\Delta_{i-1}D_i(T)$

$V^{Pay}(0) = P_{n+1,N}(0)\mathbb{E}^{n+1,N}[(S_{n,N}(T)-K)^+]$

IRR-settled swaptions

payer swaption $= \{IRR(S_{n,N}(T))(S_{n,N}(T)-K)\}^+$

receiver swaption $= \{IRR(S_{n,N}(T))(K-S_{n,N}(T))\}^+$

where $IRR(S) = \sum_{i=1}^{(T_N-T_n)m}\frac{\frac{1}{m}}{(1+\frac{S}{m})^i}$ $\frac{1}{m}=\Delta$ is day count fraction where m is payment frequency of swap

$V^{Pay}_{IRR}(0) = D(0,T)IRR(S_{n,N}(0))\cdot\mathbb{E}^{n+1,N}[(S_{n,N}(T)-K)^+]$

market model to value IRR-settled swaption

$V_{n,N}(0) = D(0,T)IRR(S_{n,N}(0))\cdot Black(S_{n,N}(0),K,\sigma_{n,N},T)$

Static replication to value payoff of Constant maturity swap

Q value contract paying $g(S_{n,N}(T))$ at time T

A $h(K) = \frac{g(K)}{IRR(K)}$ payoff of swaption for strike K; swap rate; internal rate of return for strike K

let $F = S_{n,N}(0)$ be forward swap rate

$V_0 = D(0,T)\mathbb{E}^T[g(S)]$

$= D(0,T)\int_0^\infty g(K)f(K)\,dK$ F = forward swap rate

$= D(0,T)g(F) + h'(F)[V^{Pay}(F) - V^{rec}(F)]$ = 0 at time zero

$+\int_0^F h''(K)V^{rec}(K)\,dK + \int_F^\infty h''(K)V^{Pay}(K)\,dK$

where $h(K) = \frac{g(K)}{IRR(K)}$

$h'(K) = \frac{IRR(K)g'(K) - g(K)IRR'(K)}{IRR(K)^2}$

$h''(K) = \frac{IRR(K)g''(K) - IRR''(K)g(K) - 2IRR'(K)g'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2g(K)}{IRR(K)^3}$

CMS caplet static replication

$V_0 = V^{Pay}(L)h'(L) + \int_L^\infty h''(K)V^{Pay}(K)\,dK$

payoff $= (S-L)^+$ strike price of caplet for ATM caplet $L = F = S_{n,N}(0)$ swap rate at maturity Time T

payer holder profit if CMS rate > Caplet strike. Is call option on yield.

PV of Payer swaption to enter into a IRS to pay fixed(L) and receive floating

CMS caplet = many payer swaption with increasing strike which profit if floating > fixed leg

CMS floorlet

$V_0 = V^{rec}(L)h'(L) + \int_0^L h''(K)V^{rec}(K)\,dK$

---

asset-or-nothing digital

use stock as numeraire and price option under risk-neutral measure associated to the stock Q^S

$V_{asset\,digital}(0) = e^{-rT}\mathbb{E}^*[S_T\mathbb{1}_{S_T>K}]$

$= S_0\mathbb{E}^*[\mathbb{1}_{S_T>K}\cdot\frac{S_T/S_0}{e^{rT}}]$

Martingale, $\mathbb{E}^N[G_T] = \mathbb{E}^M[G_T\frac{N_T/N_0}{M_T/M_0}]$

LIBOR-in-arrear (LIA)

Interest payment at time T_i is based on $L_i(T_i)$ forward LIBOR rate

$L_i(T_i) = \frac{D_i(t)-D_{i+1}(t)}{\Delta_iD_{i+1}(t)}$ day count; discount factor maturing at time T_i

$V^{LIA}(0) = D_i(0)\mathbb{E}^i[L_i(T_i)]$

L_i is not martingale under Q^i measure

$\mathbb{E}^i[L_i(T_i)] = \mathbb{E}^{i+1}\left[L_i(T)\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}\right]$

$= L_i(0)\cdot\frac{1+\Delta_iL_i(0)e^{\sigma_i^2T_i}}{1+\Delta_iL_i(0)}$ >1

spot exchange rate

domestic perspective $dX_t = (r^D-r^F)X_t\,dt + \sigma_XX_t\,dW_t^D$

foreign perspective $d(\frac{1}{X_t}) = (r^F-r^D)(\frac{1}{X_t})dt + \sigma_X(\frac{1}{X_t})dW_t^F$

Foreign exchange rate $= E^D[X_T] = X_te^{(r^D-r^F)(T-t)}$

FX rate process $= F_t = X_te^{(r^D-r^F)(T-t)}$

Ito's formula shows that $dF_t = \sigma_XF_t\,dW_t^D$

forward rate mature at T, decided at t; spot FX rate at t

$\frac{1}{F_t} = \frac{1}{X_t}\frac{D^D(t,T)}{D^F(t,T)}$

spot F_T at time 0 $F_T = F_0\frac{D^F(0,T)}{D^D(0,T)}$ foreign discount bond maturing time T

(FX forward)

domestic investor FX process $= dX_t = (r^D-r^F)X_t\,dt + \sigma X_t\,dW_t^D$

(Q) calc Price of FX forward for domestic investor

(A) let $y_t = \ln(X_t)$

univariate itos lemma to solve SDE $dX_t = \mu_t\,dt + \sigma_t\,dW_t$ $y_t = f(X_t)$

$dy_t = f'(X_t)dX_t + \frac{1}{2}f''(X_t)(dX_t)^2$

multivariate $y_t = g(t,X_t)$

$dy_t = g_t(t,X_t)dt + g_x(t,X_t)dX_t + 0.5g_{xx}(t,X_t)(dX_t^2)$

$f(x) = \ln(X_t)$, $f'(x) = \frac{1}{X_t}$, $f''(x) = -\frac{1}{X_t^2}$

$dy_t = \frac{1}{X_t}dX_t - \frac{1}{2X_t^2}(dX_t)^2$

$dy_t = (r^D-r^F)dt + \sigma dW_t^D - \frac{1}{2X_t^2}(\sigma^2X_t^2)dt$

$= (r^D-r^F)dt + \sigma dW_t^D - \frac{1}{2}\sigma^2dt$

$= (r^D-r^F-\frac{1}{2}\sigma^2)dt + \sigma dW_t^D$

$\ln(\frac{X_T}{X_0}) = (r^D-r^F-\frac{1}{2}\sigma^2)T + \sigma dW_T^D$

$X_T = X_0\exp[(r^D-r^F-\frac{1}{2}\sigma^2)T + \sigma dW_T^D]$

Take expectation under risk-neutral measure associated to domestic risk free bond

$\mathbb{E}^D[X_T] = X_0e^{(r^D-r^F-\frac{\sigma^2}{2})T}\mathbb{E}[e^{\sigma W_T^D}]$

MGF: $E[e^{\theta x}] = e^{\mu\theta+0.5\sigma^2\theta^2}$

$= X_0e^{(r^D-r^F-\frac{\sigma^2}{2})T}e^{0.5\sigma^2T}$ $W_T\sim N(0,T)$

$= X_0e^{(r^D-r^F)T}$

(Q) show that foreign investor see SDE

$d\frac{1}{X_t} = (r^F-r^D+\sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D$

$d\frac{1}{X_t} = (r^F-r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F$

(A) $dX_t = (r^D-r^F)X_t\,dt + \sigma X_t\,dW_t^D$

let $Y_t = f(X_t) = \frac{1}{X_t}$

$d(\frac{1}{X_t}) = (r^F-r^D+\sigma^2)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^D$

Next, start with B_t^D

$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$

let $y_t = \frac{B_t^D}{X_t} = f(B_t^D,X_t)$

$\therefore dy_t = (r^D+\sigma^2-\mu)y_t\,dt - \sigma y_t\,dW_t$

$Z_t = \frac{y_t}{B_t^F} = g(y_t,B_t^F)$

$dZ_t = (r^D-r^F+\sigma^2-\mu)Z_t\,dt - \sigma Z_t\,dW_t$

$= -\sigma Z_t\left[dW_t - \frac{r^D-r^F+\sigma^2-\mu}{\sigma}(dt)\right]$ —(1)

$= -\sigma Z_t\,dW_t^F$

$dX_t = \mu X_t\,dt + \sigma X_t\,dW_t$

Sub (1) in:

$dX_t = (r^D-r^F+\sigma^2)X_t\,dt + \sigma X_t\,dW_t^F$

$Y_t = \frac{1}{X_t}$ ⇒ solve: $d(\frac{1}{X_t}) = (r^F-r^D)(\frac{1}{X_t})dt - \sigma(\frac{1}{X_t})dW_t^F$

(Q) calc FX forward price of foreign perspective and show its expectation ($\mathbb{E}^F[\frac{1}{X_T}]$) is consistent with forward price of domestic perspective

$dX_t = (r^D-r^F)X_t\,dt + \sigma X_t\,dW_t^D$

for foreign investor

$d(\frac{1}{X_t}) = (r^F-r^D)X_t\,dt + \sigma(\frac{1}{X_t})dW_t^F$

$d\log(\frac{1}{X_t}) = (r^F-r^D-\frac{\sigma^2}{2})dt + \sigma dW_t^F$

$\frac{1}{X_T} = \frac{1}{X_0}\exp[(r^F-r^D-\frac{\sigma^2}{2})T - \sigma W_T]$

$\mathbb{E}[\frac{1}{X_T}] = \frac{1}{X_0}\exp(r^F-r^D-\frac{\sigma^2}{2})T\,\mathbb{E}[e^{-\sigma dW_T^F}]$ MGF

$= \frac{1}{X_0}\exp[e(r^F-r^D)T]$

LIBOR in arrear

(Q) LIBOR rate L_i observed at time T_i and paid at T_i. use LIBOR market model, evaluate $\mathbb{E}^i[L_i(T_i)]$ by performing single convexity correction

adjust values for convexity

(A) LIBOR market model, forward LIBOR $L_i(t)$ is martingale under risk-neutral measure with numeraire $D_{i+1}(t)$

$L_i(T) = L_i(0)\exp(-\frac{\sigma_i^2T}{2} + \sigma_iW_T^{i+1})$

Radon-Nikodym to change the measure

$\mathbb{E}^i[L_i(T)] = \mathbb{E}^{i+1}[\frac{dQ^i}{dQ^{i+1}}L_i(T)]$

$= \mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}L_i(T)\right]$

$= \frac{1}{1+\Delta_iL_i(0)}\mathbb{E}^{i+1}[\{1+\Delta_iL_i(T)\}L_i(T)]$

$= \frac{1}{1+\Delta_iL_i(0)}[L_i(0) + \Delta_iL_i(0)^2\exp(\sigma_i^2T)]$

$= L_i(0)\left[\frac{1+\Delta_iL_i(0)\exp(\sigma_i^2T)}{1+\Delta_iL_i(0)}\right]$

(Q) L_i^F foreign LIBOR observed at T_i and paid T_{i+1} with volatility σ_i. Foreign FX process $d\frac{1}{F_t} = \sigma_X\frac{1}{F_t}dW_t^F$ from foreign perspective. dW_t^F and LIBOR market model is correlated with ρ. Show that domestic perspective see $\mathbb{E}^{i+1,D}[L_i^F(T)] = L_i^F(0)e^{\rho\sigma_X\sigma_iT}$

(A) W_t^F have correlation ρ.

$\mathbb{E}^{i+1,D}[L_i^F(T)] = \mathbb{E}^{i+1,F}[L_i^F\cdot\frac{dQ^{i+1,D}}{dQ^{i+1,F}}]$

convexity-adjusted price of quanto LIBOR

both are convexity correction to account for how ir affects asset price

$= \mathbb{E}^{i+1,F}\left[L_i^F(T)\frac{\frac{D^D_{i+1}(T_{i+1})}{D^D_{i+1}(0)}}{\frac{X_TD^F_{i+1}(T_i+1)}{X_0D^F_{i+1}(0)}}\right]$

$= \mathbb{E}^{i+1,F}\left[L_i^F(T)\cdot\frac{\frac{D^D_{i+1}(T_i+1)}{X_TD^F_{i+1}(T_i+1)}}{\frac{D^D_{i+1}(0)}{X_0D^F_{i+1}(0)}}\right]$

domestic investor see ρ, foreign investor see -ρ correlation

$= \mathbb{E}^{i+1,F}\left[L_i^F(T)\frac{\frac{1}{F_{T_{i+1}}}}{\frac{1}{F_0}}\right]$

Sub in $\frac{1}{F_T} = \frac{1}{F_0}\exp(\frac{\sigma_X^2T}{2} + \sigma_XW_T^F)$

$= L_i^F(0)\exp(-\frac{\sigma_i^2T}{2})\exp(-\frac{\sigma_X^2T}{2})$

$\mathbb{E}^{i+1,F}[e^{\sigma_iW_T^{i+1}}e^{\sigma_XW_T^F}]$

Sub in $W^{i+1} = Z_1$ and $W^F = \rho Z_1 + \sqrt{1-\rho^2}Z_2$ cholesky decomposition of brownian motion

$= L_i^F(0)\exp(-\frac{\sigma_i^2T}{2})\exp(-\frac{\sigma_X^2T}{2})$

$\mathbb{E}^{i+1,F}[\exp(\sigma_iZ_T^1 + \sigma_X\rho Z_T^1 + \sigma_X\sqrt{1-\rho^2}Z_T^2)]$

expectation under foreign risk-neutral measure at i+1

$= L_i^F(0)\exp(-\frac{\sigma_i^2T}{2} - \frac{\sigma_X^2T}{2})$

$\exp\left(\frac{[\sigma_i+\sigma_X\rho]^2T}{2} + \frac{\sigma_X^2(1-\rho^2)T}{2}\right)$

$= L_i^F(0)\exp(\sigma_i\sigma_X\rho T)$

convexity corrected foreign

$= L_i^F(T)$ = LIBOR rate paid in domestic denomination

Term structure models

Drift in short rate models

Assume $dr_t = \mu dt \to r_t = r_0 + \mu t$

$\int_t^Tr_u\,du = r_0\int_0^Tdt + \mu\int_0^Tt\,dt$

$= r_0(T-t) + \frac{1}{2}\mu(T^2-t^2)$

$= r_t\Delta T$

Initial 3m rates with continuous compounding is 5%. Next 3m rates = 5.1%, 5.2%, 5.3%

$D(0,3m) = \exp(-0.05\times0.25) \to R(0,3m) = 0.05$

$R(0,T) = -\frac{\ln D(0,T)}{T} \Rightarrow -\frac{\ln(\exp(-0.05\times0.25))}{\frac{3}{12}} = 0.05$

$D(0,6m) = \exp(-[0.05+0.051]\times0.25) \to R(0,6m) = 0.0505$

$D(0,9m) = \ldots \Rightarrow R(0,9m) = 0.051$

$D(0,12m) = \exp(-[5\%+5.1\%+5.2\%+5.3\%]\times0.25) \to R(0,12m) = 0.0515$

R is increasing, hence upward sloping term structure. Hence drift in short rate model (μ) is positive

(b) derive valuation formula for LIBOR in arrear caplet paying $(L_i(T_i)-K)^+$ observed and paid at T

(A) $V_0 = D_i(0)\mathbb{E}^i[(L_i(T)-K)^+]$

$= D_i(0)\mathbb{E}^{i+1}\left[\frac{dQ^i}{dQ^{i+1}}\{L_i(T)-K\}^+\right]$ Radon nikodym

$= D_i(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)/D_i(0)}{D_{i+1}(T)/D_{i+1}(0)}\{L_i(T)-K\}^+\right]$

change measure by changing numeraire from Q^{i+1} to Q^i

$= \frac{D_i(0)}{D_i(0)}D_{i+1}(0)\mathbb{E}^{i+1}\left[\frac{D_i(T)}{D_{i+1}(T)}\{L_i(T)-K\}^+\right]$

$1+\frac{D_i(T)-D_{i+1}(T)}{D_{i+1}(T)}$

$= D_{i+1}(0)\mathbb{E}^{i+1}[(1+\Delta_iL_i(T))\cdot(L_i(T)-K)^+]$

(Q) contract pay $\Delta_iL_i(T)$ at T_{i+1}, use LIBOR market model to value

(A) LIBOR market model $dL_i(t) = \sigma_iL_i(t)\,dW^{i+1}(t)$ where $W^{i+1}(t)$ is standard brownian motion under risk neutral measure Q^{i+1} associated to zero coupon discount bond $D_{i+1}(t) = D(t,T_{i+1})$

Solution of model is $L_i(T) = L_i(0)\exp(-\frac{\sigma_i^2T}{2} + \sigma_iW_T^{i+1})$

$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}[\frac{V_T}{D_{i+1}(T)}]$

$V_0 = D(0,T_{i+1})\mathbb{E}^{i+1}[\Delta_iL_i(T)]$

$= D(0,T_{i+1})\Delta_i\mathbb{E}^{i+1}[L_i(0)e^{-\frac{\sigma_i^2T}{2}+\sigma_iW_{T_i}^{i+1}}]$

$= D(0,T_{i+1})\Delta_iL_i(0)e^{-\frac{\sigma^2T}{2}+\frac{\sigma^2T}{2}}$

$= D(0,T_{i+1})\Delta_iL_i(0)$

(Q) Contract pays $\Delta_iL_i(T)$ at T_i. Derive valuation formula using LIBOR mkt model

(A) $V_0 = D(0,T_i)\mathbb{E}^i[\Delta_iL_i(T_i)]$

$= D(0,T_i)\mathbb{E}^{i+1}[\frac{dQ^i}{dQ^{i+1}}\Delta_iL_i(T)]$

$= D(0,T_i)\Delta_i\mathbb{E}^{i+1}\left[\frac{1+\Delta_iL_i(T_i)}{1+\Delta_iL_i(0)}L_i(T_i)\right]$

$= \frac{D(0,T_i)\Delta_i}{1+\Delta_iL_i(0)}\mathbb{E}^{i+1}\left[L_i(0)e^{-\frac{\sigma^2T}{2}+\sigma_iW_{T_i}^{i+1}} + \Delta_iL_i(0)^2e^{-\sigma^2T+2\sigma_iW_{T_i}^{i+1}}\right]$

$= \frac{D(0,T_i)\Delta_i}{1+\Delta_iL_i(0)}[L_i(0) + \Delta_iL_i(0)^2e^{\sigma^2T_i}]$

$= D_{i+1}(0)\{\mathbb{E}^{i+1}[(L_i(T)-K)^+] + \Delta_i\mathbb{E}^{i+1}[L_i(T)(L_i(T)-K)^+]\}$

$\int_{x^*}^\infty\left[L_i(0)e^{-\frac{1}{2}\sigma_i^2T+\sigma_i\sqrt{T}x} - K\right]\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

let $L_i(T_i) > K \Rightarrow L_i(0)e^{-\frac{1}{2}\sigma_i^2T+\sigma_i\sqrt{T}x} > K$

$\Rightarrow x^* = \frac{\ln[\frac{K}{L_i(0)}] + \frac{1}{2}\sigma_i^2T}{\sigma_i\sqrt{T}} < x$

$= L_i(0)e^{-\frac{1}{2}\sigma^2T}\int_{x^*}^\infty\frac{1}{\sqrt{2\pi}}e^{\sigma_i\sqrt{T}x-\frac{x^2}{2}}dx - K\Phi(-x^*)$

$e^{-\frac{1}{2}[(\sigma_i\sqrt{T}-x)^2-\sigma_i^2T]}$

$= L_i(0)\Phi(\sigma_i\sqrt{T}-x^*) - K\Phi(-x^*)$

$= L_i(0)\Phi\left(\frac{\sigma_i^2T+\ln(\frac{L_i(0)}{K})-\frac{1}{2}\sigma^2T}{\sigma_i\sqrt{T}}\right) - K\Phi(-x^*)$

$= L_i(0)\Phi\left(\frac{\ln\frac{L_i(0)}{K}+\frac{1}{2}\sigma^2T}{\sigma_i\sqrt{T}}\right) - K\Phi(-x^*)$

$= (1) - K\Phi\left(\frac{\ln(\frac{L_i(0)}{K})-\frac{1}{2}\sigma_i^2T}{\sigma_i\sqrt{T}}\right)$

$= L_i(0)\Phi(d_1) - K\Phi(d_2)$

where $d_1 = \frac{\ln(\frac{L_i(0)}{K})+\frac{1}{2}\sigma^2T}{\sigma_i\sqrt{T}}$ and $d_2 = d_1 - \sigma_i\sqrt{T}$

let $L_i(T) > K \to x^* = \frac{\ln(\frac{K}{L_i(0)})+\frac{1}{2}\sigma^2T}{\sigma_i\sqrt{T}} < x$

$= \int_{x^*}^\infty\left(L_i(0)^2e^{-\sigma_i^2T+2\sigma_i\sqrt{T}x}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$-K\int_{x^*}^\infty\left(L_i(0)e^{-\frac{1}{2}\sigma_i^2T+\sigma_i\sqrt{T}x}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$= L_i(0)^2e^{-\sigma_i^2T}\int_{x^*}^\infty e^{2\sigma_i\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$-KL_i(0)e^{-\frac{1}{2}\sigma_i^2T}\int_{x^*}^\infty e^{\sigma_i\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

$= -\frac{-4\sigma_i\sqrt{T}x+x^2}{2} = -\frac{1}{2}[(-2\sigma_i\sqrt{T}+x)^2 - 4\sigma_i^2T]$

$= -\frac{1}{2}(-2\sigma_i\sqrt{T}x+x^2) = -\frac{1}{2}[(\sigma_i\sqrt{T}-x)^2 - \sigma_i^2T]$

$= L_i(0)^2e^{-\sigma_i^2T+2\sigma^2T}\int_{x^*}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(-2\sigma_i\sqrt{T}+x)^2}dx$

$-KL_i(0)\int_{x^*}^\infty\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(\sigma_i\sqrt{T}+x)^2}dx$

$= L_i(0)^2e^{\sigma_i^2T}\Phi(-x^*+2\sigma_i\sqrt{T})$

$-KL_i(0)\Phi(-x^*+\sigma_i\sqrt{T})$

∴ Sol (a+b) ⇒

$V_0 = D_{i+1}(0)\{L_i(0)\Phi(d_1) - K\Phi(d_2)\}$

$+\Delta_iD_{i+1}(0)\{L_i(0)^2e^{\sigma_i^2T}\Phi(-x^*+2\sigma_i\sqrt{T}) - KL_i(0)\Phi(-x^*+\sigma_i\sqrt{T})\}$

Q) L_i^D be forward domestic LIBOR at T_i and paid at T_{i+1}, with volatility σ_i. Forward exchange process is $dF_t = \sigma_XF_t\,dW_t^D$ for domestic investor. Suppose correlation of W_t^D and LIBOR mkt model W_t^{i+1} is ρ. Evaluate (from foreign investor perspective) $\mathbb{E}^{i+1,F}[L_i^D(T)]$

(A) $dF_t = \sigma_XF_t\,dW_t^D \Rightarrow F_T = F_0e^{-\frac{\sigma_X^2T}{2}+\sigma_XW_T^D}$

$L_i^D(T) = L_i^D(0)e^{-\frac{\sigma_i^2T}{2}+\sigma_iW_T^{i+1}}$

multi-currency change of numeraire theorem

$\mathbb{E}^{i+1,F}[L_i^D(T)] = \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{dQ^{i+1,F}}{dQ^{i+1,D}}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_TD^F_{i+1}(T)/X_0D^F_{i+1}(0)}{D^D_{i+1}(T)/D^D_{i+1}(0)}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{X_TD^F_{i+1}(T)/D^D_{i+1}(T)}{X_0D^F_{i+1}(0)/D^D_{i+1}(0)}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(T)\cdot\frac{F_T}{F_0}\right]$

$= \mathbb{E}^{i+1,D}\left[L_i^D(0)e^{-\frac{\sigma_i^2(T)}{2}+\sigma_iW_T^{i+1}}e^{-\frac{\sigma_X^2T}{2}+\sigma_XW_T^D}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2T}{2}}e^{-\frac{\sigma_X^2T}{2}}\mathbb{E}^{i+1,D}\left[e^{\sigma_iW_T^{i+1}+\sigma_XW_T^D}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2T}{2}-\frac{\sigma_X^2T}{2}}\mathbb{E}^{i+1,D}\left[e^{\sigma_iZ_T^1+\sigma_X(\rho Z_T^1+\sqrt{1-\rho^2}Z_T^2)}\right]$

$= L_i^D(0)e^{-\frac{\sigma_i^2T}{2}-\frac{\sigma_X^2T}{2}}\mathbb{E}^{i+1,D}\left[e^{(\sigma_i+\rho\sigma_X)Z_T^1+\sigma_X\sqrt{1-\rho^2}Z_T^2}\right]$

$= L_i^D(0)\exp\left[T\left\{\cancel{-\frac{\sigma_i^2}{2}}\cancel{-\frac{\sigma_X^2}{2}}\cancel{+\frac{1}{2}\sigma_i^2}+\boxed{\sigma_i\rho\sigma_X}\cancel{+\frac{1}{2}\rho^2\sigma_X^2}\cancel{+\frac{1}{2}\sigma_X^2}\cancel{-\frac{1}{2}\sigma_X^2\rho^2}\right\}\right]$

$= L_i^D(0)\exp(\sigma_i\sigma_X\rho T)$

(Q) 3 correlated brownian motion W_t^f, W_t^g, W_t^h where $dW_t^fdW_t^g = \rho_{fg}dt$, $dW_t^gdW_t^h = \rho_{gh}dt$, $dW_t^fdW_t^h = \rho_{fh}dt$. Calculate cholesky decomposition

(A) $dW_t^f = dZ_t^1$, $dW_t^g = \rho_{fg}dZ_t^1 + \rho_{fh}dZ_t^2$

$dW_t^h = \rho_{fh}dZ_t^1 + \frac{\rho_{gh}-\rho_{fg}\rho_{fh}}{\sqrt{1-\rho_{fh}^2}}dZ_t^2$

$+\sqrt{1-(\rho_{fh})^2-\left[\frac{\rho_{gh}-\rho_{fg}\rho_{fh}}{\sqrt{1-\rho_{fh}^2}}\right]^2}\,dZ_t^3$

(Q) Contract pays $\Delta_i\sqrt{L_i(T)}$ at Time T_{i+1} derive valuation formula using LIBOR market model

(A) $dL_i(t) = \sigma_iL_i(t)\,dW_t^{i+1}$ where $L_i(t) = L_i(t;T_i,T_{i+1})$ and W_t^{i+1} is brownian motion under risk neutral measure associated with numeraire

$D_{i+1}(t) = D(t,T_{i+1})$

$L_i(T) = L_i(0)e^{-\frac{\sigma_i^2T}{2}+\sigma_iW_T^{i+1}}$

$\frac{V_0}{D_{i+1}(0)} = \mathbb{E}^{i+1}\left[\frac{V_T}{D_{i+1}(T)}\right]$

$V_0 = D(0,T_{i+1})\mathbb{E}^{i+1}[\Delta_i\sqrt{L_i(T)}]$

$= D(0,T_{i+1})\Delta_i\mathbb{E}^{i+1}\left[L_i(0)^{\frac{1}{2}}e^{-\frac{\sigma_i^2T}{4}+\frac{\sigma_i}{2}W_T^{i+1}}\right]$

$= D(0,T_{i+1})\Delta_iL_i(0)^{\frac{1}{2}}e^{-\frac{\sigma^2T}{4}+\frac{\sigma^2T}{8}}$

$= D(0,T_{i+1})\Delta_iL_i(0)^{\frac{1}{2}}e^{-\frac{\sigma^2T}{8}}$

Q) Use LIBOR market model using LIBOR mkt model for contract with payoff

$\begin{cases}\$1 & \text{if } K_1\le L_i(T)\le K_2\\ 0 & \text{otherwise}\end{cases}$

(A) $K_1\le L_i(0)e^{-\frac{\sigma^2T}{2}+\sigma_i\sqrt{T}x}\le K_2$

$x_L^* = \frac{\ln(\frac{K_1}{L_i(0)})+\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\le x$

$x\le\frac{\ln(\frac{K_2}{L_i(0)})+\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}} = x_U^*$

$V_0 = D(0,T_{i+1})\mathbb{E}^{i+1}[\mathbb{1}_{K_1\le L_i(T)\le K_2}]$

$= D(0,T_{i+1})\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\mathbb{1}_{K_1\le L_i(T)\le K_2}e^{-\frac{x^2}{2}}dx$

$= D(0,T_{i+1})\frac{1}{\sqrt{2\pi}}\int_{x_L^*}^{x_U^*}e^{-\frac{x^2}{2}}dx$

$= D(0,T_{i+1})\left[\Phi\left(-\frac{\log\frac{L_i(0)}{K_2}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right) - \Phi\left(-\frac{\log\frac{L_i(0)}{K_1}-\frac{\sigma_i^2T}{2}}{\sigma_i\sqrt{T}}\right)\right]$

Affine interest rate model can be written as $D(t,T) = \exp[A(t,T) - r_tB(t,T)]$ for some deterministic A(t,T), B(t,T) and T. This implies that $R(t,T) = \frac{1}{T-t}(-A(t,T) + r_tB(t,T))$

Q) Swap mkt model. forward swap rate $dS_{n,N}(t) = \sigma_{n,N}S_{n,N}(t)\,dW^{n+1,N}$

A floating-leg-or-nothing digital option pays $P_{n+1,N}(T)S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K}$ on maturity T, where $P_{n+1,N}$ is PVBP. derive valuation formula

(A) For risk neutral measure $Q^{n+1,N}$ associated with numeraire asset PVBP that is $P_{n+1,N}(t) = \sum_{i=n+1}^{N}\Delta_{i-1}D_i(t)$

$V_T = P_{n+1,N}(T)S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K}$

$= P_{n+1,N}(T)\frac{D(T,T_n)-D(T,T_N)}{P_{n+1,N}(T)}\mathbb{1}_{S_{n,N}(T)>K}$

$= [D(T,T_n)-D(T,T_N)]\mathbb{1}_{S_{n,N}(T)>K}$

$= \sum_{i=n+1}^{N}\Delta_{i-1}D_i(T)L_{i-1}(T)\mathbb{1}_{S_{n,N}(T)>K}$

$\frac{V_0}{P_{n+1,N}(0)} = \mathbb{E}^{n+1,N}\left[\frac{V_T}{P_{n+1,N}(T)}\right]$

$= \mathbb{E}^{n+1,N}\left[\frac{P_{n+1,N}(T)S_{n,N}(T)\mathbb{1}_{S_{n,N}(T)>K}}{P_{n+1,N}(T)}\right]$

float payoff $= S_{n,N}(T) > K$ and zero otherwise

$V_0 = P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{x^*}^\infty\left[S_{n,N}(0)e^{-\frac{\sigma_{n,N}^2T}{2}+\sigma_{n,N}\sqrt{T}x}e^{-\frac{x^2}{2}}dx\right]$ $-\frac{[(\sigma_{n,N}(T)-x)^2]}{2}$

$V_0 = P_{n+1,N}(0)S_{n,N}(0)\Phi(-x^*+\sigma_{n,N}\sqrt{T})$

$= P_{n+1,N}(0)S_{n,N}(0)\Phi\left[\frac{\log\frac{S_{n,N}(0)}{K}+\frac{\sigma_{n,N}^2T}{2}}{\sigma_{n,N}\sqrt{T}}\right]$

(Q) contract pays $S_{n,N}(T)$ on maturity T. explain why we cannot value this contract directly using Swap Market model without applying convexity correction

(A) $\frac{V_0}{D(0,T)} = \mathbb{E}^T\left[\frac{V_T}{D(T,T)}\right]$

$V_0 = D(0,T)\mathbb{E}^T[S_{n,N}(T)]$

$= D(0,T)\mathbb{E}^T\left[S_{n,N}(0)e^{-\frac{\sigma_{n,N}^2T}{2}+\sigma_{n,N}W^{n+1,N}(T)}\right]$

$W^{n+1,N}(T)$ is not a standard brownian motion under Q^T, we cannot evaluate without convexity correction

Term structure model

$E_t^*[e^{-\int_t^Tr_u\,du}] = D(t,T) = e^{-R(t,T)(T-t)}$

example: Let initial 3m rate = 5%

continuous compounding rate (R)

discount rate (D)

$D(0,3m) = e^{-0.05\times0.25} = e^{-R(0,3m)(0.25)}$

$R(0,3m) = 0.05$ $D(t,T) = e^{-R(t,T)(T-t)}$

$D(0,6m) = e^{-(0.05+0.051)(0.25)} = e^{-R(0,6m)(0.5)}$

$\therefore R(0,6m) = 0.0505$

example: short rate model $dr_t = \mu dt$

integrate both sides $r_t = r_0 + \mu t$

integrate again $\int_t^Tr_u\,du = r_0(T-t) + \frac{1}{2}\mu(T^2-t^2)$

$= r_t(T-t) + \frac{1}{2}\mu(T-t)^2$

Let discount factor $= e^{-R(t,T)(T-t)}$

$D(t,T) = E_t^*\left[e^{-\int_t^Tr_u\,du}\right] = e^{-r_t(T-t)-\frac{1}{2}\mu(T-t)^2}$

$\therefore R(t,T) = -\frac{1}{T-t}\ln[D(t,T)] = \frac{1}{2}\mu(T-t) + r_t$

example: $dr_t = \sigma dW_t^*$ (short rate model)

integrate SDE → $r_t = r_0 + \sigma W_t^*$

where $r_t\sim N(r_0,\sigma^2t)$

integrate again → $\int_t^Tr_u\,du$

$= r_0(T-t) + \sigma\int_t^TW_u^*\,du$

$= \int_0^TW_u\,du = \int_0^T(T-u)\,dW_u$

$\mu = 0$
$\sigma^2 = \frac{T^3}{3}$

$E[\int_t^Tr_u\,du] = r_t(T-t)$

$V[\int_t^Tr_u\,du] = V\left[\sigma\int_t^T(W_u^*-W_t^*)\,du\right]$

$= \frac{\sigma^2(T-t)^3}{3}$

Reconstruct discount factor

$D(t,T) = E_t^*\left[e^{-\int_t^Tr_u\,du}\right]$

$E[e^{\theta X}] = e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2}$

$D(t,T) = e^{-R(t,T)(T-t)} = e^{-r_t(T-t)+\frac{\sigma^2}{6}(T-t)^3}$

$\therefore R(t,T) = -\frac{1}{T-t}\log D(t,T) = r_t - \frac{\sigma^2}{6}(T-t)^2$

Vasicek model for short rate

$dr_t = k(\theta-r_t)dt + \sigma dW_t^*$ θ long run mean

k mean reversion coefficient; see pg2 for full steps

Ito's $f(r_t,t) = r_te^{kt}$

$r_t = \underbrace{r_0e^{-kt} + \theta(1-e^{-kt})}_{E[r_t]} + \underbrace{\sigma\int_0^te^{k(u-t)}dW_u^*}$ $V[r_t] = \frac{\sigma^2}{2k}(1-e^{-2kt})$

reconstruct discount factor $D(t,T) = E[e^{-\int_t^Tr_u\,du}]$

$D(t,T) = e^{-R(t,T)(T-t)} = e^{A(t,T)-B(t,T)r_t}$

$R(t,T) = \frac{1}{T-t}[-A(t,T) + B(t,T)r_t]$

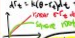
where $B(t,T) = \frac{1}{k}[1-e^{-k(T-t)}]$

$A(t,T) = \frac{[B(t,T)-(T-t)](k^2\theta-\frac{1}{2}\sigma^2)}{k^2} - \frac{\sigma^2B(t,T)^2}{4k}$

Cox-Ingersoll-Ross model

r_t is chi-square distributed

$dr_t = k(\theta-r_t)dt + \sigma\sqrt{r_t}\,dW_t^*$



No-arbitrage short rate affine model

Ho-Lee: $dr_t = \theta(t)dt + \sigma dW_t^*$

choose θ to match the observed spot curve

integrate both sides

$r_t = r_0 + \int_0^t\theta(s)\,ds + \int_0^t\sigma\,dW_s^*$

Integrate again

$\int_0^Tr_u\,du = \int_0^Tr_0\,du + \int_0^T\int_0^u\theta(s)\,ds\,du + \int_0^T\int_0^u\sigma\,dW_s^*\,du$

$= r_0T + \int_0^T\theta(s)[T-s]\,ds + \int_0^T\sigma[T-s]\,dW_s^*$

$= E[\int_0^Tr_u\,du]$ $V[\int_0^Tr_u\,du]$

$= \int_0^T\sigma^2(T-s)^2\,ds = \frac{1}{3}\sigma^2T^3$

Reconstruct zero-coupon discount factor

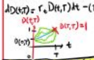
$D(0,T) = E[e^{-\int_0^Tr_u\,du}]$

$= \exp[-r_0T - \int_0^T\theta(s)(T-s)\,ds + \frac{1}{6}\sigma^2T^3]$

This proves that Ho-Lee is an affine model i.e. $D(t,T) = e^{A(t,T)-r_tB(t,T)}$ constant, linear function

apply Ito's formula to find SDE

$dD(t,T) = r_tD(t,T)dt - (T-t)\sigma D(t,T)\,dW_t^*$



(Q) Ho-Lee model, 2 period binomial tree every step, rate can move up or down by 0.5%, risk-neutral probabilities of up or down move are both 0.5. $D(0,1y) = 0.9656$, $D(0,2y) = 0.9229$, $D(0,3y) = 0.8903$. Find no-arbitrage value of θ_0 and θ_1

(A) $D(0,1y) = e^{-R(0,1y)} = 0.9656 \Rightarrow R(0,1y) = 3.5\%$

$r=3.5\% \to r=4\%+\theta_0$, $r=3\%+\theta_0$ → $r=4.5\%+\theta_0+\theta_1$, $r=3.5\%+\theta_0+\theta_1$, $r=2.5\%+\theta_0+\theta_1$

$D(0,2y) = D(0,1y)[0.5e^{-0.04-\theta_0} + 0.5e^{-0.03-\theta_0}]$

$0.9229 = 0.9656[0.5e^{-0.04} + 0.5e^{-0.03}]e^{-\theta_0}$

$\theta_0 = 0.0078$

$D_u(1y,3y) = e^{-0.04-\theta_0}\begin{bmatrix}0.5e^{-0.045-\theta_0-\theta_1}\\+0.5e^{-0.035-\theta_0-\theta_1}\end{bmatrix} = 0.90393e^{-\theta_1}$

$D_d(1y,3y) = e^{-0.03-\theta_0}\begin{bmatrix}0.5e^{-0.035-\theta_0-\theta_1}\\+0.5e^{-0.025-\theta_0-\theta_1}\end{bmatrix} = 0.92168e^{-\theta_1}$

$D(0,3y) = D(0,1y)[0.5\times0.90393e^{-\theta_1} + 0.5\times0.92168e^{-\theta_1}]$

$0.8903 = 0.9656[0.5\times0.90393 + 0.5\times0.92168]e^{-\theta_1}$

$\theta_1 = -0.01032$

Hull White trinomial model

$r=12\%$ → $r=14\%$, $r=12\%$, $r=10\%$

time 0 time 1

if payoff at next period is $100(r-0.11)^+$ where $\Delta t = 1$

Then the PV of payoff at time 0 is $e^{-0.12\times\Delta t}[0.25\times3 + 0.5\times1 + 0.25\times0]$

$= 1.11$

Given $P_u = 0.25$, $P_m = 0.5$, $P_d = 0.25$

Vasicek short rate model — Question

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

Apply Ito's Lemma to $f(r,t) = e^{kt}r_t$

Solution: $r_t = r_0e^{-kt} + \theta(1-e^{-kt}) + \sigma\int_0^t e^{k(u-t)}dW_u^*$ — $D(t,T) = \exp[A(t,T) - r_tB(t,T)]$

integrate both sides from 0 to T:

$\int_0^T r_tdt = \int_0^T r_0e^{-kt}dt + \int_0^T\theta(1-e^{-kt})dt + \int_0^T\int_0^t\sigma e^{k(u-t)}dW_u^*dt$

$= \int_0^T\int_u^T\sigma e^{k(u-t)}dt\,dW_u^*$

$= \int_0^T[-\frac{\sigma}{k}e^{k(u-t)}]_u^TdW_u^*$

$= \frac{\sigma}{k}\int_0^T(1-e^{k(u-T)})dW_u^*$

$\therefore \int_0^T r_tdt = \int_0^T r_0e^{-kt} + \int_0^T\theta(1-e^{-kt})dt + \frac{\sigma}{k}\int_0^T(1-e^{k(u-T)})dW_u^*$

$E^*[\int_0^T r_tdt] = \int_0^T r_0e^{-kt}dt + \int_0^T\theta(1-e^{-kt})dt = \frac{r_0}{k}(1-e^{-kT}) + \theta T - \frac{\theta}{k}(1-e^{-kT})$

$V[\int_0^T r_tdt] = V[\frac{\sigma}{k}\int_0^T(1-e^{k(u-T)})dW_u^*]$

$= \frac{\sigma^2}{k^2}\int_0^T(1-e^{k(u-T)})^2du$

$= \frac{\sigma^2}{k^2}\int_0^T[1-2e^{k(u-T)} + e^{2k(u-T)}]du$

$= \frac{\sigma^2}{k^2}[T - \frac2k(1-e^{-kT}) + \frac{1}{2k}(1-e^{-2kT})]$

$\therefore D(0,T) = E^*[\exp(-\int_0^T r_tdt)] = \exp\{E^*[\int_0^T r_tdt] + \frac12V[\int_0^T r_tdt]\}$

(Q) Ho Lee interest rate model

$dr_t = \theta(t)dt + \sigma dW_t^*$

show that $\theta(t) = \frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2t$

(A) integrate both sides

$r_t = r_0 + \int_0^t\theta(s)ds + \int_0^t\sigma dW_s^*$

$\int_0^t r_udu = \int_0^t r_0du + \int_0^t\int_0^u\theta(s)ds\,dt + \int_0^t\int_0^u\sigma dW_s^*du$

$= r_0t + \int_0^t\theta(s)(t-s)ds + \int_0^t\sigma(t-s)dW_s^*$

$E[\int_0^t r_udu] = r_0(t-0) + \int_0^t\theta(s)(t-s)ds$

$V[\int_0^t r_udu] = \int_0^t\sigma^2(t-s)^2dt = \frac13\sigma^2(t-0)^3$ — $\exp[A(t,T) - r_tB(t,T)]$

$D(0,t) = E[\exp\{-\int_0^t r_udu\}] = \exp[\underbrace{-r_0t - \int_0^t\theta(s)(t-s)ds}_{E} + \underbrace{\frac16\sigma^2t^3}_{\frac12V}]$

$\log D(0,t) = -r_0t - \int_0^t\theta(s)(t-s)ds + \frac16\sigma^2t^3$

$\frac{\partial}{\partial t}\log D(0,t) = -r_0 - \int_0^t\theta(s)ds + \frac12\sigma^2t^2$

$\frac{\partial^2}{\partial t^2}\log D(0,t) = -\theta(t) + \sigma^2t$

$\theta(t) = -\frac{\partial^2}{\partial t^2}\log D(0,t) + \sigma^2t$

Instantaneous forward rate and discount factor

$F(t;T,T+\delta)$ is forward interest rate defined at time t for the period between T and T+δ where t < T and δ > 0. No arbitrage gives us

$D(t,T) = D(t,T+\delta)e^{\delta F(t;T,T+\delta)}$

$F(t;T,T+\delta) = -\frac{\log D(t,T+\delta) - \log D(t,T)}{\delta}$

Instantaneous forward rate $f(t,T)$ as limit of $\delta\to0$

$f(t,T) = \lim_{\delta\to0}F(t;T,T+\delta) = -\frac{\partial\log D(t,T)}{\partial T}$

Integrate from t to T

$\int_t^T f(t,u)du = -\log D(t,T)$

$\log D(t,T) = \exp[-\int_t^T f(t,u)du] = E^*[\exp(-\int_t^T r_udu)]$

Ho Lee Model: change to T-forward Measure

Since $\int_0^T\theta(s)(T-s)ds = -\log D(0,T) - r_0T + \frac16\sigma^2T^3$

Express the integrated rate as

$\int_0^T r_udu = -\log D(0,T) + \frac16\sigma^2T^3 + \int_0^T\sigma(T-s)dW_s^*$

Ho-Lee interest rate process under forward measure

$Q^*: dr_t = \theta(t) + \sigma dW_t^*$

$Q^T: dr_t = [\theta(t) - \sigma^2(T-t)]dt + \sigma dW_t^T$

$\frac{\partial}{\partial T}\log D(t,T) = -r_t - \int_t^T\theta(s)ds + \frac12\sigma^2(T-t)^2$

$-f(t,T) = -r_t - \int_t^T\theta(s)ds + \frac12\sigma^2(T-t)^2$

$f(t,T) = r_t + \int_t^T\theta(s)ds - \frac12\sigma^2(T-t)^2$

$df(t,T) = \sigma dW_t^T \to \exp[\int_0^T f(0,u)du]$

$\theta(T) = -\frac{\partial^2}{\partial T^2}\log D(0,T) + \sigma^2T = \frac{\partial f(0,T)}{\partial T} + \sigma^2T$

$\therefore D(t,T) = \frac{D(0,T)}{D(0,t)}\exp[-r_t(T-t) + (T-t)f(0,t) - \frac{\sigma^2t}{2}(T-t)^2]$

Ho-Lee Model: zero coupon bond option price under Q^*

$dD(t,T) = r_tD(t,T)dt - (T-t)\sigma D(t,T)dW_t^*$

Solution

$D(t,T) = D(0,T)\exp[\int_0^t r_udu - \frac{\sigma^2}{2}\int_0^t(T-u)^2du - \sigma\int_0^t(T-u)dW_u^*]$

$\int_0^t r_udu = -\log D(0,t) + \frac16\sigma^2t^3 + \sigma\int_0^t(t-u)dW_u^*$

$\int_0^t r_udu = -\log D(0,t) + \frac{\sigma^2}{2}\int_0^t(t-u)^2du + \sigma\int_0^t(t-u)dW_u^*$

examples $dS_t = rS_tdt + \sigma S_tdW_t^*$

$S_T = S_0\exp[(r-\frac{\sigma^2}{2})T + \sigma W_T^*]$

$dS_t = r_tS_tdt + \sigma_tS_tdW_t^*$

$S_T = S_0\exp[\int_0^T(r_u - \frac{\sigma_u^2}{2})du + \int_0^T\sigma_udW_u^*]$

$D(t,T) = \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2}\int_0^t[(T-u)^2 - (t-u)^2]du - \sigma\int_0^t[(T-u)-(t-u)]dW_u^*]$

$= dW_u^T - \sigma(t-u)du$

$= \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2}\int_0^t((T-u)-(t-u))^2du - \sigma\int_0^t[(T-u)-(t-u)]dW_u^T]$

$= \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2}(T-t)^2t - \sigma(T-t)W_t^T]$

(Q) zero coupon bond option with option expiry t, bond maturity T is given by $[D(t,T)-K]^+$. Value it

(A) $V_0 = E[\frac{B_0(D(t,T)-K)^+}{B_t}]$

$= E^T[(D(t,T)-K)^+\cdot\frac{D(t,t)/D(0,t)}{B_t/B_0}]\cdot D(0,t)$ — =1

$= D(0,t)E^T[(D(t,T)-K)^+]$

$= D(0,T)\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} + \frac12\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt t}]$

$- D(0,t)K\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} - \frac12\sigma^2(T-t)^2t}{\sigma(T-t)\sqrt t}]$

and let $\sigma_{HL} = \sigma(T-t)\sqrt t$ which is the Ho-Lee integrated volatility

Vasicek model — integrated short rate

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

$\int_0^t r_udu = \int_0^t r_0e^{-ku}du + \int_0^t\theta(1-e^{-ku})du + \frac{\sigma}{k}\int_0^t(1-e^{k(u-t)})dW_u^*$

Integrated short rate

$E^*[\int_0^t r_udu] = \int_0^t r_0e^{-ku}du + \int_0^t\theta(1-e^{-ku})du = \frac{r_0}{k}(1-e^{-kt}) + \theta t - \frac{\theta}{k}(1-e^{-kt})$

Let $B(t,T) = \frac{1-e^{-k(T-t)}}{k}$

$E^*[\int_0^t r_udu] = r_0B(0,t) + \theta t - \theta B(0,t)$

$V[\int_0^t r_udu] = V[\frac{\sigma}{k}\int_0^t(1-e^{k(u-t)})dW_u^*]$

$= \frac{\sigma^2}{k^2}\int_0^t(1-e^{k(u-t)})^2du$ — Ito's Isometry

$= \frac{\sigma^2}{k^2}\int_0^t(1-2e^{k(u-t)} + e^{2k(u-t)})du$

$= \frac{\sigma^2}{k^2}[t - \frac2k(1-e^{-kt}) + \frac{1}{2k}(1-e^{-2kt})]$

$= \frac{\sigma^2}{2k^3}[2kt - 4 + 4e^{-kt} + 1 - e^{-2kt}]$

$= \frac{\sigma^2}{2k^3}[2kt - 2 + 2e^{-kt} - (1-2e^{-kt} + e^{-2kt})]$

$= -\frac{\sigma^2}{2k^3}[-2kt + 2(1-e^{-kt}) + (1-e^{-kt})^2]$

$= \frac{\sigma^2}{2k^2}2t - \frac{\sigma^2}{2k^2}2B(0,t) - \frac{\sigma^2}{2k}B(0,t)^2$

Reconstructing discount factor

$D(0,t) = E^*[e^{-\int_0^t r_udu}] = e^{A(0,t) - r_0B(0,t)}$

where $A(0,t) = \exp\{(\theta - \frac{\sigma^2}{2k^2})[B(0,t) - t] - \frac{\sigma^2}{4k}B(0,t)^2\}$

$B(0,t) = \frac{1-e^{-kt}}{k}$

$D(t,T) = E^*[e^{-\int_t^T r_udu}] = e^{A(t,T) - r_tB(t,T)}$

$A(t,T) = \exp\{(\theta - \frac{\sigma^2}{2k^2})[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2\}$

$B(t,T) = \frac{1-e^{-k(T-t)}}{k}$

Vasicek model: Instantaneous forward process

$f(t,T) = [k\theta - \frac{\sigma^2}{2}B(t,T)]B(t,T) + r_te^{-k(T-t)}$

SDE $df(t,T) = \sigma e^{-k(T-t)}dW_t^T$

Discount factor follows the SDE

$dD(t,T) = r_tD(t,T)dt - \frac{\sigma}{k}(1-e^{k(t-T)})D(t,T)dW_t^*$

$= r_tD(t,T)dt - \sigma B(t,T)D(t,T)dW_t^*$

zero coupon bond follows lognormal process and is solved by

$D(t,T) = D(0,T)\exp[\int_0^t r_udu - \frac{\sigma^2}{2k^2}\int_0^t(1-e^{k(u-T)})^2du - \frac{\sigma}{k}\int_0^t(1-e^{k(u-T)})dW_u^*]$

Sub in $\int_0^t r_udu$ ← zero coupon bond options

$= \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2k^2}\int_0^t\{(1-e^{k(u-T)})^2 - (1-e^{k(u-t)})^2\}du - \frac{\sigma}{k}\int_0^t\{(1-e^{k(u-T)}) - (1-e^{k(u-t)})\}dW_u^*]$

Girsanov theorem to change measure from Q^* to Q^T

$dW_t^T = dW_t^* + \sigma\frac{1-e^{-k(T-t)}}{k}dt = dW_t^* + \sigma B(t,T)dt$

Q^T brownian motion

$= \frac{D(0,T)}{D(0,t)}\exp[-\frac{\sigma^2}{2k^2}\int_0^t(e^{k(u-t)} - e^{k(u-T)})^2du - \frac{\sigma}{k}\int_0^t(e^{k(u-t)} - e^{k(u-T)})dW_u^T]$

Price of call option on zero-coupon bond with strike K and maturity T is

$= D(0,T)\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} + \frac12\sigma_v^2}{\sigma_v}] - D(0,t)K\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} - \frac12\sigma_v^2}{\sigma_v}]$

where the Vasicek integrated volatility is $\sigma_v = \sigma\sqrt{\frac{1-e^{-2kt}}{2k}}\frac{1-e^{-k(T-t)}}{k} = \sigma\sqrt{\frac{1-e^{-2kt}}{2k}}B(t,T)$

Radon Nikodym derivative of measure Q^T wrt Q^*

$\frac{dQ^T}{dQ} = \frac{\exp[-\int_0^T r_udu]}{D(0,T)}$ ← changing to T-forward measure

$= \exp[-\log D(0,T) - \int_0^T r_udu]$

$= \exp[-\frac{\sigma^2}{2k^2}\int_0^T(1-e^{k(u-T)})^2du - \frac{\sigma}{k}\int_0^T(1-e^{k(u-T)})dW_u^*]$

W_t^T is martingale under Q^T

$dW_t^T = dW_t^* + \frac{\sigma}{k}(1-e^{k(t-T)})dt = dW_t^* + \sigma B(t,T)dt$

under Q^T, short rate model is

$dr_t = [k\theta - \sigma^2B(t,T) - kr_t]dt + \sigma dW_t^T$

Hull-White model: DF reconstruction

SDE $dr_t = k[\theta(t) - r_t]dt + \sigma dW_t^*$

$r_t = r_0e^{-kt} + k\int_0^t\theta(s)e^{k(s-t)}ds + \sigma\int_0^te^{k(s-t)}dW_s^*$

more generally

$r_T = r_te^{-k(T-t)} + k\int_t^T\theta(s)e^{-k(T-s)}ds + \sigma\int_t^Te^{-k(T-s)}dW_s^*$

Hull-White model = Vasicek model + θ(t)

Integrate r_t to obtain integrated short rate. The discount factor maturing at time T can be constructed as

$D(t,T) = \exp(A(t,T) - r_tB(t,T))$

where A(t,T)

$= \exp[(0 - \frac{\sigma^2}{2k^2})[B(t,T) - T + t] - \frac{\sigma^2}{4k}B(t,T)^2]\exp[-k\int_t^T\theta(s)B(s,T)ds]$

and $B(t,T) = \frac{1-e^{-k(T-t)}}{k}$ — Vasicek's B(t,T)

Instantaneous forward rate

$f(t,T) = k\int_t^T\theta(s)e^{-k(T-s)}ds - \frac{\sigma^2}{2}B(t,T)^2 + r_te^{-k(T-t)}$

To fit Hull White model to initial term structure

$\theta(t) = f(0,t) + \frac1k\frac{\partial f(0,t)}{\partial t} + \frac{\sigma^2}{2k^2}(1-e^{-2kt})$

Discount factor reconstructed

$D(t,T) = \frac{D(0,T)}{D(0,t)}\exp[-r_tB(t,T) + B(t,T)f(0,t) - \frac{\sigma^2}{4k}(1-e^{-2kt})B(t,T)^2]$

Zero-coupon bond valuation formula

$V_0 = E^*[\frac{B_0[D(t,T)-K]^+}{B_t}]$

$= D(0,t)E^T[(D(t,T)-K)^+]$

$= D(0,T)\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} + \frac12\sigma_{IHW}^2}{\sigma_{IHW}}] - D(0,t)K\Phi[\frac{\log\frac{D(0,T)}{D(0,t)K} - \frac12\sigma_{IHW}^2}{\sigma_{IHW}}]$

where σ_{IHW} is integrated Hull-White volatility given $\sigma_{IHW} = \sigma\sqrt{\frac{1-e^{-2kt}}{2k}}\frac{1-e^{-k(T-t)}}{k}$

Zero-coupon bond dynamics

Let D(t,T) be value at time t of a zero-coupon bond paying 1 at time T and let r_t denote spot interest rate at time t, then zero-coupon bond dynamics is given by

$dD(t,T) = r_tD(t,T)dt + \sigma(t,T)D(t,T)dW_t^*$

apply Ito's lemma

$D(t,T) = D(0,T)\exp[\int_0^t(r_u - \frac12\sigma^2(u,T))du + \int_0^t\sigma(u,T)dW_u^*]$

Let $B_t = \exp(\int_0^t r_udu)$ denote money-market paying interest r_t. Radon-Nikodym derivative is

$\underset{\text{forward measure}}{\frac{dQ^T}{dQ^*}} = \frac{D(t,T)/D(0,T)}{\underset{\text{risk neutral measure}}{B_t/B_0}} = \exp[-\int_0^t\frac12\sigma^2(u,T)du + \int_0^t\sigma(u,T)dW_u^*]$

Girsanov theorem $dW_t^T = dW_t^* - \sigma(t,T)dt$

$dW_t^T = dW_t^* - \sigma(t,T)dt$ → if choose σ(t,T)

If let $\sigma(t,T) = -\sigma(T-t)$ give rise to Ho-Lee model; $\sigma(t,T) = -\sigma\frac{1-e^{-k(T-t)}}{k}$ give rise to Hull-White model

Two-factor Hull White model

At part all models in this course assume 1-dimensional Brownian motion as single source of randomness. This assumes that zero rate R(t,T) across all maturities T will be perfectly correlated, term structure always move up or down in full synchronicity, term structure cannot invert, term structure cannot change curvature. A 2-factor model using 2 Brownian motions for short rate model allows term structure to change shape (or invert)

$$\begin{cases}dr_t = k(\theta(t) + X_t - r_t)dt + \sigma_rdW_t^*\\ dX_t = -\gamma X_tdt + \sigma_xdZ_t^*\\ dW_t^*dZ_t^* = \rho dt\end{cases}$$

In long run, short rate r_t revert to mean of θ(t) + X_t. θ(t) is used to match initial term structure. X_t is second source of randomness by adjusting ρ, we can let make term structure change its shape, or invert.

(Q) SDE short rate r_t

$dr_t = \mu dt + \sigma dW_t^*$ under Q^*. Let D(t,T) denote price of zero coupon bond maturing at time T. show that

$D(t,T) = E^*[\exp(-\int_t^T r_udu)]$

(A) $dr_t = \mu dt + \sigma dW_t^*$

$r_s - r_t = \mu(s-t) + \sigma(W_s^* - W_t^*)$ — integrate $\int_t^T$

$\int_t^T r_udu = r_t(T-t) + \mu\int_t^T(u-t)du + \sigma\int_t^T(W_u^* - W_t^*)du$

$= r_t(T-t) + \mu[\frac{u^2}{2} - tu]_t^T + \sigma\int_t^T\int_t^u dW_s\,du$

$= r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \sigma\int_t^T\int_s^T du\,dW_s$

$= r_t(T-t) + \frac{\mu}{2}(T-t)^2 + \sigma\int_t^T(T-s)dW_s$

$E[\int_t^T r_udu] = r_t(T-t) + \frac{\mu}{2}(T-t)^2$

$V[\int_t^T r_udu] = V[\sigma\int_t^T(T-s)dW_s] = \sigma^2\int_t^T(T-s)^2ds = \frac{\sigma^2}{3}(T-t)^3$

$-\int_t^T r_udu \sim N(-r_t(T-t) - \frac{\mu}{2}(T-t)^2, [\frac{\sigma^2}{3}(T-t)^3])$

$D(t,T) = E^*[e^{-\int_t^T r_udu}] = \exp[-r_t(T-t) - \frac{\mu}{2}(T-t)^2 + \frac16(T-t)^3]$

(B) zero rate $D(t,T) = \exp[-R(t,T)(T-t)]$

$R(t,T) = r_t + \frac{\mu}{2}(T-t) - \frac{\sigma^2}{6}(T-t)^2$ ← hedge [illegible]

(Q) S_t = stock price at time t. Price a contract that pays $\log(\frac{\alpha S_T}{\beta})$ at T, where α and β are constants.

(a) Value contract using Black Scholes model $dS_t = rS_tdt + \sigma S_tdW_t^*$ under Q^*

(A) $S_T = S_0\exp((r - \frac{\sigma^2}{2})T + \sigma W_T^*)$

$\log(\frac{\alpha S_T}{\beta}) = \log(\frac{\alpha}{\beta}) + \log(S_T) = \log(\frac{\alpha}{\beta}) + \log(S_0) + (r - \frac{\sigma^2}{2})T + \sigma W_T^*$

$V_0 = e^{-rT}[\log(\frac{\alpha S_T}{\beta})] = e^{-rT}E^*[\log(\frac{\alpha S_T}{\beta})] = e^{-rT}[\log(\frac{\alpha}{\beta}) + \log(S_0) + (r - \frac{\sigma^2}{2})T]$

(b) value contract using static replication starting with $\int_0^\infty g(K)dK$ where $g(K) = \log(\frac{\alpha K}{\beta})$

(A) $h(K) = \log(\frac{\alpha K}{\beta})$

$h'(K) = \frac1K$ $h''(K) = -\frac{1}{K^2}$

choose $F = S_0\exp(rT)$ as expansion point, the Breeden-Litzenberger formula is by

$V_0 = e^{-rT}h(F) + \int_0^F h''(K)P(K)dK + \int_F^\infty h''(K)C(K)dK$

$= e^{-rT}\log(\frac{\alpha S_0e^{rT}}{\beta}) - \int_0^F\frac{P(K)}{K^2}dK - \int_F^\infty\frac{C(K)}{K^2}dK$

$= e^{-rT}[\log(\frac{\alpha}{\beta}) + \log(S_0) + rT] - \int_0^F\frac{P(K)}{K^2}dK - \int_F^\infty\frac{C(K)}{K^2}dK$

(Q) State LIBOR mkt model for forward LIBOR rate $L_i(t)$

(A) $dL_i(t) = \sigma_iL_i(t)dW_t^{i+1}$ $L_i(t) = \frac{1}{\Delta_i}\frac{D_i(t) - D_{i+1}(t)}{D_{i+1}(t)}$

(Q) What numeraire for expectation of $L_i(t)$ to be martingale?

(A) D_{i+1} zero coupon bond maturing at T_{i+1}

(Q) Contract pays $100 if LIBOR rate L_i is above 5% on fixing date T_i. Payment is made at time T_{i+1}. Value it

$V_0 = D_{i+1}(0)E^{i+1}[100\cdot\mathbb{1}_{L_i(T)>5\%}]$

(A) $V_0 = D_{i+1}(0)\frac{100}{\sqrt{2\pi}}\int_{x^*}^\infty e^{-\frac{x^2}{2}}dx$

$= D_{i+1}(0)100\Phi[\frac{\log\frac{L_i(0)}{5\%} - \frac{\sigma_i^2T}{2}}{\sigma_i\sqrt T}]$

(Q) Value a contract paying $L_i(T)^2$ observed at time T_i and paid at T_{i+1}

(Q) Value it using LIBOR market model

(A) $dL_i(t) = \sigma_iL_i(t)dW_t^{i+1}$

$L_i(T) = L_i(0)\exp(-\frac{\sigma_i^2T}{2} + \sigma_iW_T^{i+1})$

$V_0 = D(0,T_{i+1})E^{i+1}[L_i(T)^2]$

$= D(0,T_{i+1})E^{i+1}[L_i(0)^2\exp(-\sigma_i^2T + 2\sigma_iW_T^{i+1})]$

$= D(0,T_{i+1})L_i(0)^2e^{-\sigma_i^2T}E^{i+1}[e^{2\sigma_i\sqrt T\varepsilon^{i+1}}]$

MGF $E[e^{\theta X}] = e^{\theta\mu + \frac12\sigma^2\theta^2}$

$= D(0,T_{i+1})L_i(0)^2e^{-\sigma_i^2T}e^{\frac12(4\sigma_i^2T)}$

$= D(0,T_{i+1})L_i(0)^2e^{\sigma_i^2T}$

(Q) From the static replication portfolio using Breeden-Litzenberger approach, if we are able to obtain liquid caplet $V^C(K)$ and floorlet $V^F(K)$ prices across a wide range of strikes

(A) choose expansion point at the forward LIBOR rate $F = L_i(0)$ we obtain

$V_0 = D(0,T)L_i(0)^2 + \int_0^{L_i(0)}2V^F(K)dK + \int_{L_i(0)}^\infty 2V^C(K)dK$

FX rate process

FX rate process in an arbitrage-free market, FX product in domestic and foreign denomination is equal. Let FX rate follow GBM (geometric brownian motion)

$dB_t^F = r^FB_t^Fdt$

$dX_t = \mu X_tdt + \sigma X_tdW_t$

$dB_t^D = r^DB_t^Ddt$ — domestic (D) money mkt account; risk-free rate r^D of B^D

From domestic view, there are 2 traded assets B_t^D and $B_t^FX_t$ (foreign money mkt account in domestic currency). Apply Ito's lemma to $Y_t = f(B_t^F,X_t) = B_t^FX_t$, we obtain

$dY_t = (r^F + \mu)Y_tdt + \sigma Y_tdW_t$

in domestic mkt, use domestic money market account as numeraire, define $Z_t = g(B_t^D,Y_t) = \frac{Y_t}{B_t^D} = \frac{B_t^FX_t}{B_t^D}$, apply Ito's lemma again

$dZ_t = (r^F - r^D + \mu)Z_tdt + \sigma Z_tdW_t$

Apply Girsanov's theorem with $K = \frac{r^F - r^D + \mu}{\sigma}$ to obtain equivalent martingale measure Q^D, under the process $Z_t = \frac{B_t^FX_t}{B_t^D}$ is a martingale. Under Q^D measure, exchange rate process is

$dX_t = (r^D - r^F)X_tdt + \sigma X_tdW_t^D$

Foreign market perspective, we have 2 assets B_t^F and $\frac{B_t^D}{X_t}$. Using B_t^F as numeraire, we can derive the process $\frac{B_t^D}{X_tB_t^F}$ as follows

$d(\frac{B_t^D}{X_tB_t^F}) = (r^D - r^F - \mu - \sigma^2)(\frac{B_t^D}{X_tB_t^F}) - \sigma(\frac{B_t^D}{X_tB_t^F})dW_t$

Apply Girsanov theorem with $K = -\frac{r^D - r^F - \mu}{\sigma} - \sigma$. We obtain unique equivalent martingale measure Q^F. Under this measure, exchange rate $\frac{1}{X_t}$ follows $d\frac{1}{X_t} = (r^F - r^D)\frac{1}{X_t}dt - \sigma\frac{1}{X_t}dW_t^F$

(Q) Ho-Lee interest rate model

$dr_t = \theta(t)dt + \sigma dW_t^*$ where W_t^* is brownian motion under risk-neutral measure Q^*. Derive distribution, mean and variance of integral $\int_t^T r_udu$

(A) $r_s = r_t + \int_t^s\theta(u)du + \int_t^s\sigma dW_u^*$

$\int_t^T r_sds = r_t(T-t) + \int_t^T\int_t^s\theta(u)du\,ds + \int_t^T\int_t^s\sigma dW_u^*ds$

$= r_t(T-t) + \int_t^T\int_u^T\theta(u)ds\,du + \int_t^T\int_u^T\sigma ds\,dW_u^*$

$= r_t(T-t) + \int_t^T\theta(u)(T-u)du + \int_t^T\sigma(T-u)dW_u^*$

$E^*[\int_t^T r_sds] = r_t(T-t) + \int_t^T\theta(u)(T-u)du$

$V[\int_t^T r_sds] = \frac{\sigma^2(T-t)^3}{3}$ — $E[\int_0^T r_udu] = r_0T + \int_0^T \dots du$, $V[\int_0^T r_udu] = \frac{\sigma^2T^3}{3}$

$D(t,T) = E^*[\exp(-\int_t^T r_udu)] = \exp[-r_t(T-t) - \int_t^T\theta(u)(T-u)du + \frac{\sigma^2(T-t)^3}{6}]$

$D(t,T) = E^*[\exp(-\int_t^T r_udu)] = \exp[A(t,T) - r_tB(t,T)]$

where $A(t,T) = -\int_t^T\theta(u)(T-u)du + \frac{\sigma^2(T-t)^3}{6}$ and $B(t,T) = T-t$

(Q) Vasicek interest rate model

$dr_t = k(\theta - r_t)dt + \sigma dW_t^*$

Explain how this model produce mean reversion

(A) $E^*[dr_t] = k(\theta - r_t)dt$ so if short rate is lower than long run mean (i.e. θ > r_t), the drift is positive, and dr_t is expected to drift upward, while if short rate is higher than the long run mean (i.e. θ < r_t), the drift term is negative, and dr_t is expected to drift downward. When θ = r_t, the drift is zero, and so dr_t is expected to remain at the same time. K is a positive multiplier and controls the mean reversion speed.

(Q)

( ) Consider function $e^{kt}r_t$, the total derivative is given by $d(e^{kt}r_t) = ke^{kt}r_tdt + e^{kt}dr_t = k\theta e^{kt}dt + \sigma e^{kt}dW_t$

integrating both sides from 0 to t, we can obtain a solution to the SDE

$\int_0^td(e^{ku}r_u) = ke^{kt}r_tdt + e^{kt}dr_t = k\theta e^{kt}dt + \sigma e^{kt}dW_t$

Integrate both sides 0 to t, we obtain a solution to the SDE

$\int_0^td(e^{ku}r_u) = \int_0^tk\theta e^{ku}du + \int_0^t\sigma e^{ku}dW_u$

$r_t = r_0e^{-kt} + \theta(1-e^{-kt}) + \sigma\int_0^te^{k(u-t)}dW_u$

Taking expectation on both sides gives us the mean $E(r_t) = r_0e^{-kt} + \theta(1-e^{-kt})$

$V(r_t) = E[(\sigma\int_0^te^{k(u-t)}dW_u)^2] = E[\sigma^2\int_0^te^{2k(u-t)}du] = \frac{\sigma^2}{2k}(1-e^{-2kt})$

The distribution of r_t is $r_t \sim N[r_0e^{-kt} + \theta(1-e^{-kt}), \frac{\sigma^2}{2k}(1-e^{-2kt})]$

$\lim_{t\to\infty}E^*[r_t] = \lim_{t\to\infty}(r_0e^{-kt} + \theta(1-e^{-kt})) = \theta$

$\lim_{t\to\infty}V[r_t] = \lim_{t\to\infty}(\frac{\sigma^2}{2k}(1-e^{-2kt})) = \frac{\sigma^2}{2k}$

(Q) IRS mkt

6m LIBOR 2%   IRS pays semi-annual and overnight interest rate is f = 0.8%
1y IRS 2.2%
2y IRS 2.5%

Calc value of forward LIBOR rate L(1.5,2) if swap mkt is collateralised & uncollateralised

express forward LIBOR using interpolated discount factors

$\underset{\text{day count fraction } 0.5}{\Delta}(2.5\%)[D_{OIS}(0,0.5) + D_{OIS}(0,1) + D_{OIS}(0,1.5) + \underset{\text{OIS discount factor}}{D_{OIS}(0,2)}]$

$= D_{OIS}(0,0.5)\Delta L(0,0.5) + D_{OIS}(0,1)\Delta L(0.5,1) + D_{OIS}(0,1.5)\Delta L(1,1.5) + D_{OIS}(0,2)\Delta L(1.5,2)$

use LIBOR discount factors to calc what 1.5y par swap rate → LIBOR discount factor

$IRS_{1.5y} = \frac{1 - \tilde D(0,1.5)}{\Delta[\tilde D(0,0.5) + \tilde D(0,1) + \tilde D(0,1.5)]} = 2.3\%$

$2.3\%\,\Delta(D_0(0,0.5) + D_0(0,1) + D_0(0,1.5)) = D_0(0,0.5)\Delta L(0.5) + D_0(0,1)\Delta L(0.5,1) + D_0(0,1.5)\Delta L(1,1.5)$

(Q) IRS mkt

6m LIBOR 2%   IRS pays semi-annual and overnight interest rate is f = 0.8%
1y IRS 2.2%
2y IRS 2.5%

(A) Calc value of forward LIBOR rate L(1.5,2) if swap mkt is collateralised & uncollateralised

$D_{OIS}(0,0.5) = \frac{1}{(1+\frac{0.8\%}{360})^{180}} = 0.9960937$

$D_{OIS}(0,1) = 0.992032$ $D_{OIS}(0,1.5) = 0.988072$

$D_{OIS}(0,2) = 0.984127$

Δ = 0.5 = day count fraction, $\tilde D$ (LIBOR discount factor), D_{OIS} (OIS discount factor), L (forward LIBOR rate)

uncollateralised

$\tilde D(0,0.5) = \frac{1}{1+\Delta L(0,0.5)} = \frac{1}{1+0.5(2\%)} = 0.990099$ — 6m LIBOR 2%

1y IRS = 2.2% (given) = $\frac{1-\tilde D(0,1)}{\Delta[\tilde D(0,0.5) + \tilde D(0,1)]}$

$2.2\% = \frac{1-\tilde D(0,1)}{0.5[0.990099 + \tilde D(0,1)]}$

$0.01089109 + 0.011\tilde D(0,1) = 1 - \tilde D(0,1)$

$\tilde D(0,1) = 0.978347$

2y IRS = 2.5% (given) = $\frac{1-\tilde D(0,2)}{\Delta[\tilde D(0,0.5) + \tilde D(0,1) + \tilde D(0,1.5) + \tilde D(0,2)]}$

$2.5\% = \frac{1-\tilde D(0,2)}{\Delta[\tilde D(0,0.5) + \frac32\{\tilde D(0,1) + \tilde D(0,2)\}]}$

$2.5\% = \frac{1-\tilde D(0,2)}{0.5[0.990099 + \frac32\{0.978347 + \tilde D(0,2)\}]}$

$0.01237624 + 0.018344 + 0.01875\tilde D(0,2) = 1 - \tilde D(0,2)$

$\tilde D(0,2) = 0.95144$

$\tilde D(0,1.5) = \frac12[\tilde D(0,2) + \tilde D(0,1)] = \frac12[0.95144 + 0.978347] = 0.9648935$

$L(1.5,2) = \frac1\Delta\frac{D(0,1.5) - D(0,2)}{D(0,2)} = \frac{1}{0.5}(\frac{0.9648935 - 0.95144}{0.95144})$

$= 0.02828$ (Ans)

collateralised

$2.2\%\,\Delta[D_{OIS}(0,0.5) + D_{OIS}(0,1)] = L(0,0.5)D_{OIS}(0,0.5) + L(0.5,1)D_{OIS}(0,1)$ — 1y IRS given

$2.2\%[0.9960937 + 0.992032] = 0.02(0.9960937) + L(0.5,1)(0.992032)$

$\therefore L(0.5,1) = 0.02397702$

1.5y IRS $= \frac{1-\tilde D(0,1.5)}{\Delta[\tilde D(0,0.5) + \tilde D(0,1) + \tilde D(0,1.5)]}$

$= \frac{1 - 0.9648935}{0.5[0.990099 + 0.978347 + 0.9648935]}$

$= 0.02392851$

1.5y IRS $[D_{OIS}(0,0.5) + D_{OIS}(0,1) + D_{OIS}(0,1.5)] = L(0,0.5)D_{OIS}(0,0.5) + L(0.5,1)D_{OIS}(0,1) + L(1,1.5)D_{OIS}(0,1.5)$

$0.02392851[0.9960937 + 0.992032 + 0.988072] = 0.02(0.9960937) + 0.02397702(0.992032) + L(1,1.5)(0.988072)$

$\therefore L(1,1.5) = 0.02747127$

$2.5\%[D_{OIS}(0,0.5) + D_{OIS}(0,1) + D_{OIS}(0,1.5) + D_{OIS}(0,2)] = L(0,0.5)D_{OIS}(0,0.5) + L(0.5,1)D_{OIS}(0,1) + L(1,1.5)D_{OIS}(0,1.5) + L(1.5,2)D_{OIS}(0,2)$ — 2y IRS given

$2.5\%[0.9960937 + 0.992032 + 0.988072 + 0.984127] = 0.02(0.9960937) + 0.02397702(0.992032) + 0.02747127(0.988072) + L(1.5,2)(0.984127)$

$\therefore L(1.5,2) = 0.0282007$ (Ans)

(b) A contract pays L(1.5,2) accrued over 6m at T = 2y. Calculate PV of this contract if it is collateralised or uncollateralised — should be higher since D_{OIS} is < $\tilde D$

collateralised $L(1.5,2)\times\Delta\times D_{OIS}(0,2)$

$= 2.82007\%(0.5)(0.984127) = 0.0138765$

uncollateralised $L(1.5,2)\times\Delta\times\tilde D(0,2)$

$= 2.828\%(0.5)(0.95144) = 0.013453$

(c) calc OIS discount factor $D_{OIS}(1,2)$ and LIBOR discount factor $\tilde D(1,2)$

$D_{OIS}(1,2) = \frac{D_{OIS}(0,2)}{D_{OIS}(0,1)} = \frac{0.984127}{0.992032} = 0.992035$

$\tilde D(1,2) = \frac{\tilde D(0,2)}{\tilde D(0,1)} = \frac{0.95144}{0.978347} = 0.972497$

---

can interpolate zero rate & discount rate

(a) 1y OIS 1.5% $\frac{1}{D_o(0,t_n)} = (1+\Delta S_o)$
12m LIBOR 2.75%

(A) 1y OIS discount factor $= \frac{1}{1+1\times0.015} = D_o(0,1y) = 0.9852$

12m-LIBOR discount factor for 1y $= \frac{1}{1+1\times0.0275} = 0.9732$

(Q) Find settlement 10 days into a month, 30 days this month, average realized rate 2.156%, futures quotes is 97.75 (= 2.25%)

(A) $2.25 = \frac{10}{30}\times2.156 + \frac{20}{30}E[\sum_{i=k+1}^{30}\frac{f_i}{20}]$

$= \frac{10}{30}\times2.156 + \frac{20}{30}E[\bar f_i]$

$E[\bar f_i] = 2.297$

(Q) State forward value of foreign exchange rate at time T. Suppose that we know the spot exchange rate today, and the domestic and foreign discount bond [$D_d(0,T)$ and $D_f(0,T)$] maturing at time T

(A) If 1 unit of foreign currency is equal to FX × 1 unit of domestic currency

$FX_T = FX_0\frac{D_f(0,T)}{D_d(0,T)}$ $D_f(0,T) = \frac{1}{(1+r_f)^T}$

(Q)

Instrument	Quote
FX_0	1.5
FX_T	1.45
$D_f(0,T)$	0.95

what is the discount bond value $D_d(0,T)$ implied from the market

(A) $1.45 = 1.5\frac{0.95}{D_d(0,T)}$

$D_d(0,T) = 0.982759$

(Q) 4 coupon bonds with annual payment from the same issuer

Maturity	Coupon	Price
1y	4.5	101.5
2y	5	102
2y	5.25	?
3y	5.5	102.5

(A) $101.5 = \frac{104.5}{1+r_1} \to r_1 = 0.029557$

$102 = \frac{5}{1+r_1} + \frac{105}{(1+r_2)^2}$ — 0.029557

$97.14854 = \frac{105}{(1+r_2)^2} \to r_2 = 0.0296512$

Price of bond 3 $= \frac{5.25}{1+r_1} + \frac{105.25}{(1+r_2)^2}$

$= 102.4741$

(Q) S_t is forward swap rate at time t. Suppose a CMS product has this payoff on maturity T

$$g(S_T) = \begin{cases}0 & S_T < K_1\\ S_T - K_1 & K_1 \le S_T \le K_2\\ K_2 - K_1 & S_T > K_2\end{cases}$$

start with $\int_0^\infty h(K)\frac{\partial^2V^{pay}(K)}{\partial K^2}dK$

where $h(K) = \frac{g(K)}{IRR(K)}$, derive static replication formula for this payoff

(A) $\int_0^\infty h(K)\frac{\partial^2V^{pay}(K)}{\partial K^2}dK$

$= \int_{K_1}^{K_2}h_1(K)\frac{\partial^2V^p(K)}{\partial K^2}dK + \int_{K_2}^\infty h_2(K)\frac{\partial^2V^p(K)}{\partial K^2}dK$

$= [h_1(K)\frac{\partial V^p(K)}{\partial K}]_{K_1}^{K_2} - \int_{K_1}^{K_2}h_1'(K)\frac{\partial V^p(K)}{\partial K}dK$

$+ [h_2(K)\frac{\partial V^p(K)}{\partial K}]_{K_2}^\infty - \int_{K_2}^\infty h_2'(K)\frac{\partial V^p(K)}{\partial K}dK$

$= h_1(K_2)\frac{\partial V^p(K_2)}{\partial K} - h_1(K_1)\frac{\partial V^p(K_1)}{\partial K} - [h_1'(K)V^p(K)]_{K_1}^{K_2}$

$+ \int_{K_1}^{K_2}h_1''(K)V^p(K)dK + \underbrace{h_2(\infty)\frac{\partial V^p(\infty)}{\partial K}}_{=0}$

$- h_2(K_2)\frac{\partial V^p(K_2)}{\partial K} - [h_2'(K)V^p(K)]_{K_2}^\infty$

$+ \int_{K_2}^\infty h_2''(K)V^p(K)dK$

$= h_1(K_2)\frac{\partial V^p(K_2)}{\partial K} - h_1(K_1)\frac{\partial V^p(K_1)}{\partial K} - h_1'(K_2)V^p(K_2)$

$+ h_1'(K_1)V^p(K_1) + \int_{K_1}^{K_2}h_1''(K)V^p(K)dK$

$- h_2(K_2)\frac{\partial V^p(K_2)}{\partial K} - h_2'(K_2)V^p(K_2)$

$+ \int_{K_2}^\infty h_2''(K)V^p(K)dK$

where $h_1(K) = \frac{K-K_1}{IRR(K)}$

$h_1'(K) = \frac{IRR(K) - (K-K_1)IRR'(K)}{IRR(K)^2}$

$h_1''(K) = \frac{-IRR''(K)(K-K_1) - 2IRR'(K)}{IRR(K)^2} + \frac{2IRR'(K)^2(K-K_1)}{IRR(K)^3}$

and $h_2(K) = \frac{K_2-K_1}{IRR(K)}$ $h_2'(K) = -\frac{(K_2-K_1)IRR'(K)}{IRR(K)^2}$

$h_2''(K) = \frac{-IRR''(K)(K_2-K_1)}{IRR(K)^2} + \frac{2IRR'(K)^2(K_2-K_1)}{IRR(K)^3}$

(Q) Swap rate SDE

$dS_{n,N}(t) = \sigma_{n,N}S_{n,N}(t)dW^{n+1,N}$ derive a valuation formula for payer swaption in Black normal model

$V_{n,N}^{pay}(0) = P_{n+1,N}E^{n+1,N}[(S_{n,N}(T) - K)^+]$

(A) $\frac{V_{n,N}(0)}{P_{n+1,N}(0)} = E^{n+1,N}[\frac{V_{n,N}(T)}{P_{n+1,N}(T)}] = E^{n+1,N}[\frac{P_{n+1,N}(T)[S_{n,N}(T)-K]^+}{P_{n+1,N}(T)}]$

normal dist $= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^\infty xe^{-\frac{(x-\mu)^2}{2\sigma}}dx$

$V_{n,N}(0) = P_{n+1,N}(0)E^{n+1,N}[\{S_{n,N}(T) - K\}^+]$

$= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\{S_{n,N}(T) - K\}^+e^{-\frac{x^2}{2}}dx$

$dS_t = \sigma S_0dW \to$ integrate both sides $S_T - S_0 = \sigma S_0\int_0^TdW_t$

$\to S_T = S_0 + \sigma S_0W_T = S_0 + \sigma S_0\sqrt Tx$

$= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty\{S_{n,N}(0)[1 + \sigma_{n,N}\sqrt Tx] - K\}^+e^{-\frac{x^2}{2}}dx$

$= P_{n+1,N}(0)\frac{1}{\sqrt{2\pi}}[\int_{x^*}^\infty(S_{n,N}(0) - K)e^{-\frac{x^2}{2}}dx + \int_{x^*}^\infty S_{n,N}(0)\sigma_{n,N}\sqrt Txe^{-\frac{x^2}{2}}dx]$

$= P_{n+1,N}(0)[\{S_{n,N}(0) - K\}\Phi(-x^*) + S_{n,N}(0)\sigma_{n,N}\sqrt T\frac{1}{\sqrt{2\pi}}\int_{x^*}^\infty e^{-\frac{x^2}{2}}dx]$

let $S_{n,N}(T) > K$ assume stock price follows BS model which assumes stock return follow lognormal dist

$S_0e^{(\mu - \frac{\sigma^2}{2})T + \sigma W_T^*} > K \to (\mu - \frac{\sigma^2}{2})T + \sigma\sqrt Tx^* > \ln(\frac{K}{S_0})$

$x^* > \frac{\ln(\frac{K}{S_0}) - (\mu - \frac{\sigma^2}{2})T}{\sigma\sqrt T}$

Sub $u = \frac{x^2}{2}$ $\frac{du}{dx} = x$ $dx = \frac1xdu$

$\frac{1}{\sqrt{2\pi}}\int_{x^*}^\infty xe^{-u}(\frac1x)du = \frac{1}{\sqrt{2\pi}}\int_{x^*}^\infty e^{-u}du = \frac{1}{\sqrt{2\pi}}[e^{-u}]_{x^*}^\infty$

$= -\frac{1}{\sqrt{2\pi}}[e^{-\frac{x^2}{2}}]_{x^*}^\infty = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^{*2}}{2}}$

$V_{n,N}(0) = P_{n+1,N}[\{S_{n,N}(0) - K\}\Phi(-x^*) + S_{n,N}(0)\sigma_{n,N}\sqrt T\frac{1}{\sqrt{2\pi}}e^{-\frac{x^{*2}}{2}}]$ = pdf = $\phi(x^*)$

$= P_{n+1,N}(0)[\{S_{n,N}(0) - K\}\Phi(-x^*) + S_{n,N}(0)\sigma_{n,N}\sqrt T\phi(x^*)]$

(Q) displaced diffusion LIBOR $dL_i(t) = \sigma_i[\beta L_i(t) + (1-\beta)L_i(0)]dW_t^{i+1}$ under risk-neutral measure Q^{i+1} associated with zero coupon bond $D_{i+1}(t)$

calc $E^{i+1}[L_i(T_i)]$ and $E^{i+1}[(L_i(T_i) - K)^+]$

(A) $f(L) = \ln[\beta L_i(t) + (1-\beta)L_i(0)]$ — univariate Ito's lemma $y = f(X)$, $dy = f'(X)dX + \frac12f''(X)(dX)^2$

$f'(L) = \frac{\beta}{\beta L_i(t) + (1-\beta)L_i(0)}$

$f''(L) = -\frac{\beta^2}{\{\beta L_i(t) + (1-\beta)L_i(0)\}^2}$

$f(L) = \frac{\beta}{\beta L_i(t) + (1-\beta)L_i(0)}dL_i(t) + \frac{\beta^2}{2[\beta L_i(t) + (1-\beta)L_i(0)]^2}(dL_i(t))^2$

sub in $dL_i(t)$

$f(L) = \beta\sigma_idW_t^{i+1} + \frac{\beta^2\sigma_i^2}{2}dt$ integrate both sides 0 to T

$\ln[\frac{\beta L_i(T) + (1-\beta)L_i(0)}{\beta L_i(0) + (1-\beta)L_i(0)}] = \beta\sigma_iW_T^{i+1} + \frac{\beta^2\sigma_i^2}{2}T$

$L_i(T) = \frac{L_i(0)}{\beta}e^{\frac{\beta^2\sigma_i^2}{2}T + \beta\sigma_iW_T^{i+1}} - \frac{1-\beta}{\beta}L_i(0)$

$E[L_i(T_m)] = E^{i+1}[\frac{L_i(0)}{\beta}e^{-\frac12\beta^2\sigma_i^2T_m + \beta\sigma_iW_{T_m}^{i+1}} - \frac{1-\beta}{\beta}L_i(0)]$

$= \frac{L_i(0)}{\beta} - \frac{1-\beta}{\beta}L_i(0) = L_i(0)$

(A) $L_i(T_i) > K \Rightarrow \frac{L_i(0)}{\beta}e^{-\frac{\beta^2\sigma_i^2}{2}T_i + \beta\sigma_i\sqrt{T_i}(x)_i^{i+1}} - \frac{1-\beta}{\beta}L_i(0) > K$

$L_i(0)e^{-\frac{\beta^2\sigma_i^2}{2}T_i + \beta\sigma_i\sqrt{T_i}(x)_i^{i+1}} > K\beta + (1-\beta)L_i(0)$

$-\frac{\beta^2\sigma_i^2}{2}T_i + \beta\sigma_i\sqrt{T_i}(x)_i^{i+1} > \ln[\frac{K\beta + (1-\beta)L_i(0)}{L_i(0)}]$

$(x)_i^{i+1} > \frac{\ln[\frac{K\beta}{L_i(0)} + (1-\beta)] + \frac{\beta^2\sigma_i^2}{2}T_i}{\beta\sigma_i\sqrt{T_i}} = (x^*)_i^{i+1}$

let $K' = K + \frac{1-\beta}{\beta}L_i(0)$ → let $\sigma_i' = \sigma_i\beta$

let $L_i(0)' = \frac{L_i(0)}{\beta}$ $x_i^{i+1} > \frac{\log\frac{K'}{L_i(0)'} + \frac{(\sigma_i')^2T}{2}}{\sigma_i'\sqrt T} = x^*$

$E^{i+1}[(L_i(T) - K)^+] = \frac{1}{\sqrt{2\pi}}\int_{x^*}^\infty[\frac{L_i(0)}{\beta}e^{-\frac{\beta^2\sigma_i^2}{2}T + \beta\sigma_i\sqrt{T_i}x} - \frac{1-\beta}{\beta}L_i(0) - K]e^{-\frac{x^2}{2}}dx$

Sub in $\sigma_i' = \sigma_i\beta$, $L_i(0)' = \frac{L_i(0)}{\beta}$, $K' = K + \frac{1-\beta}{\beta}L_i(0)$

$E^{i+1}[(L_i(T_i) - K)^+] = \frac{1}{\sqrt{2\pi}}\int_{x^*}[L_i(0)'e^{-\frac{(\sigma_i')^2}{2}T_i + \sigma_i'\sqrt Tx} - K']e^{-\frac{x^2}{2}}dx$

$= L_i(0)'e^{-\frac{(\sigma_i')^2T}{2}}\int_{x^*}\frac{1}{\sqrt{2\pi}}e^{\sigma_i'\sqrt Tx}e^{-\frac{x^2}{2}}dx - K'\int_{x^*}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$

Complete square $= \frac{2\sigma_i'\sqrt Tx - x^2}{2} = -\frac{(\sigma_i'\sqrt T - x)^2}{2} + \frac{(\sigma_i')^2T}{2}$ $= -K'\Phi(-x^*)$

$= L_i(0)'\int_{x^* - \sigma_i'\sqrt T}^\infty e^{-\frac{x^2}{2}}dx - K'\Phi(-x^*)dx$

$= L_i(0)'\Phi(-x^* + \sigma_i'\sqrt T) - K'\Phi(-x^*)$

where $\sigma_i' = \sigma_i\beta$, $L_i(0)' = \frac{L_i(0)}{\beta}$, $K' = K + \frac{1-\beta}{\beta}L_i(0)$

(Q) calc expectation of receiver swaption payoff maturing at T and strike at K. Show that we cannot evaluate the expectation under Q^*, the risk neutral measure associated with risk-free money market account numeraire $B_t = B_0\exp(\int_0^tr_udu)$, but by changing the measure to $Q^{n+1,N}$, the risk neutral measure associated with the present value for basis point (PVBP) numeraire $P_{n+1,N}(t) = \sum_{i=n+1}^N\Delta_{i-1}D_i(t)$, we can derive an analytical expression for receiver swaption

(A) $V_{n,N}^r = P_{n+1,N}(0)(K - S_{n,N}(T))^+$ $\frac{V_t^r}{B_t} = E[\frac{V_T^r}{B_T}]$

$V_0^r = E^*[\frac{B_0}{B_T}P_{n+1,N}(0)\{K - S_{n,N}(T)\}^+]$ cannot evaluate

cannot evaluate further as $B_T = B_0\exp(\int_0^Tr_udu)$

But if change measure to $Q^{n+1,N}$

$V_0^r = E^{n+1,N}\{\frac{B_0}{B_T}P_{n+1,N}(T)[K - S_{n,N}(T)]^+\frac{dQ^*}{dQ^{n+1,N}}\}$

$= E^{n+1,N}\{\frac{B_0}{B_T}P_{n+1,N}(T)[K - S_{n,N}(T)]^+\frac{B_T/B_0}{P_{n+1,N}(T)/P_{n+1,N}(0)}\}$

$V_0^r = P_{n+1,N}(0)E^{n+1,N}[\{K - S_{n,N}(T)\}^+]$

Sub in h_1, h_1', h_2, h_2'

$\int_0^\infty h(K)\frac{\partial^2V^P(K)}{\partial K^2}dK = \frac{1}{IRR(K_1)}V^P(K_1) - \frac{1}{IRR(K_2)}V^P(K_2) + \int_{K_1}^{K_2}h_1''(K)V^P(K)dK + \int_{K_2}^\infty h_2''(K)V^P(K)dK$

(Q) Spot LIBOR rates, uncollateralised

1m	1.15%
2m	1.5%
3m	1.65%
6m	1.95%
12m	1.80%

calc forward rate for 9x12 FRA

(A) $D(0,6m) = \frac{1}{1+\frac{6}{12}(1.95\%)} = 0.9903$

$D(0,12m) = \frac{1}{1+1.8\%} = 0.982318$

$\therefore L(9m,12m) = \frac{1}{\Delta_{12m}}\frac{D(0,9m) - D(0,12m)}{D(0,12m)} = 1.834\%$

(Q) calc 1y par swap rate for IRS with quarter payment

(A) $S(0,12m) = \frac{1 - D(0,12m)}{\Delta_{3m}[D(0,3m) + D(0,6m) + \dots + D(0,12m)]}$

$\frac{1}{1+\frac{3}{12}(1.65\%)} = 0.99589$ $= 1.7877\%$

(Q) calc continuously compounded zero rates R(0,3m), R(0,6m) and R(0,12m)

(A) $D(0,T) = e^{-R(0,T)\cdot T}$ $R(0,T) = -\frac{\ln D(0,T)}{T}$

$R(0,3m) = -\frac{\ln(0.99589)}{\frac14} = 1.6466\%$

$R(0,6m) = 1.7424\%$ $R(0,9m) = 1.7687\%$

(Q) spot FX rate for USD/SGD is 1.42, and USD 6m LIBOR rate is 1.5%. If 6m forward FX rate for USD/SGD is 1.39, calc implied 6m interest rate (SOR) in SGD

(A) $FX_T = FX_0(\frac{1+r_d}{1+r_f})^T$ 1.39   1.42

$1 + \Delta_{6m}L_{SGD}(0,6m) = \frac{USDSGD(6m)}{USDSGD(0)}[1 + \Delta_{6m}L_{USD}(0,6m)]$

$L_{SGD}(0,6m) = -2.757\%$

(Q) uncollateralised, IRS have semi-annual payment

6m LIBOR	1.5%
1y IRS	1.8%
2y IRS	2.0%
3y IRS	2.05%

calc 1.5y IRS

(A) $L(0,6m) \to D(0,0.5y) = \frac{1}{1+0.5(1.5\%)} = 0.99256$

$1.8\% = \frac{1 - D(0,1y)}{0.5[D(0,0.5y) + D(0,1y)]} \Rightarrow D(0,1y) = 0.98223$

$2\% = \frac{1 - D(0,2)}{0.5[D(0,0.5) + D(0,1) + D(0,1.5) + D(0,2)]}$ $= \frac{D_{01} + D_{02}}{2}$

$D(0,2) = 0.960927$ $D(0,1.5) = 0.97158$

$IRS(0,1.5y) = \frac{1 - D(0,1.5y)}{0.5[D(0,0.5y) + D(0,1y) + D(0,1.5y)]}$

$= 1.92936\%$

(Q) A forward starting swap with 2y tenor start at t = 1y has these cashflows

Time (t)	Pay	Receive
1.5 2.0 2.5 3.0	par swap rate	6m LIBOR

what is the par swap rate for the forward starting swap?

(A) $IRS(1y,3y) = \frac{D(0,1y) - D(0,3y)}{0.5[D_{0,1.5} + D_{0,2} + D_{0,2.5} + D_{0,3}]}$

$IRS(0,3y) = \frac{1 - D(0,3)}{0.5[D_{0,0.5} + D_{0,1} + D_{0,1.5} + 1.5D_{0,2} + 1.5D_{0,3}]} = 2.149\%$

$D_{0,3} = 0.940564$

$D_{0,2.5} = 0.950746$

the forward starting swap has a par swap rate of $S = \frac{D(0,1y) - D(0,3y)}{0.5[D_{0,1.5} + D_{0,2} + D_{0,2.5} + D_{0,3}]} = 2.18\%$

(Q)