MOM PS Estimator under Strong Monotonicity

April 30, 2021

[NOTE: I updated this taking **out** references to *estimated* principal scores, cuz I realized the math was wrong. I'm following the propensity score literature and ignoring estimation error, for the moment]

Say members of the control group of an RCT have no access to the treatment, but among the treatment group there are compliers and non-compliers—i.e. one-sided non-compliance. (Of course, this could be, e.g., compliance in two different ways, etc.) Then there are two principal strata S, never-takers and compliers. Say the exclusion restriction doesn't hold.

Feller et al. [2017] and Ding and Lu [2017] and others discuss estimating principal effects in one-sided noncompliance (which they call "strong monotonicity") under the assumption of principal ignorability, viz. $Y_C \perp\!\!\!\perp S|\boldsymbol{X}$. This has two problems, to my mind:

1. Sometimes principal ignorability is not plausible
2. If you assume the principal ignorability for Y_T ("strong" principal ignorability) then, as Feller et al. [2017] points out, the principal effects conditional on covariates are equal to the ITT conditional on covariates—the stratum plays no role. But sometimes the role of the strata is what we're trying to estimate!

The approach here takes a different tack, based on a different assumption which is maybe even less plausible—but which, I think, can be relaxed.

1 Setup

Estimating $\mathbb{E}[Y_T|S]$ is straightforward, so for the remainder just consider the "control" group, $Z = 0$, with n (control) subjects and drop the C subscript. The control group is a mixture of compliers and never-takers.

For subject i, $i = 1, \ldots, n$, we have:

- Y_i observed outcome
- $S_i \in \{0, 1\}$ unobserved stratum
- $e(\boldsymbol{X_i}) = Pr(S_i = 1|\boldsymbol{X}_i)$ take as given (like, estimated from

Let $\mu_0 = \mathbb{E}[Y|S = 0]$ and $\mu_1 = \mathbb{E}[Y|S = 1]$. The goal is to estimate μ_0 and μ_1.

Assumption:

$$\mathbb{E}[Y_i|\boldsymbol{X}_i, S_i] = \mathbb{E}[Y_i|S_i] = \mu_0 \text{ or } \mu_1 \tag{1}$$

i.e. Y is mean-independent of $\boldsymbol{X}$ conditional on S. This is problematic—why should S contain all information about Y?

Note, it is kinda related to "principal ignorability" of Feller et al. [2017] and Ding and Lu [2017], $Y \perp\!\!\!\perp S|\boldsymbol{X}_i$ but, obviously, different too.

2 M-Estimator

As a preliminary, note that

$$\begin{aligned}
\mathbb{E}[Y|e(\boldsymbol{X})] &= \mathbb{E}\left\{\mathbb{E}[Y|e(\boldsymbol{X}), S]|e(\boldsymbol{X})\right\} \\
&= \mathbb{E}\left\{\mathbb{E}[Y|S]|e(\boldsymbol{X})\right\} \text{ by (1)} \\
&= \mathbb{E}[\mu_1 S + \mu_0(1 - S)|e(\boldsymbol{X})] \\
&= \mu_1 e(\boldsymbol{X}) + \mu_0(1 - e(\boldsymbol{X}))
\end{aligned}$$

Then we have

$$\mathbb{E}[Y] = \mathbb{E}\mathbb{E}[Y|e(\boldsymbol{X})] = \mu_1\mathbb{E}e(\boldsymbol{X}) + \mu_0(1 - \mathbb{E}e(\boldsymbol{X}))$$

Next we have

$$\begin{aligned}
\mathbb{E}[Ye(\boldsymbol{X})] &= \mathbb{E}\left\{\mathbb{E}[Ye(\boldsymbol{X})|e(\boldsymbol{X})]\right\} \\
&= \mathbb{E}\left\{e(\boldsymbol{X})\mathbb{E}[Y|e(\boldsymbol{X})]\right\} \\
&= \mathbb{E}\left\{e(\boldsymbol{X})\left[\mu_1 e(\boldsymbol{X}) + \mu_0(1 - e(\boldsymbol{X}))\right]\right\} \\
&= \mu_1\mathbb{E}[e(\boldsymbol{X})^2] + \mu_0\left(\mathbb{E}[e(\boldsymbol{X})] - \mathbb{E}[e(\boldsymbol{X})^2]\right)
\end{aligned}$$

That gives us four parameters:

$$\theta_1 = \mathbb{E}[e(\boldsymbol{X})]$$
$$\theta_2 = \mathbb{E}[e(\boldsymbol{X})^2]$$
$$\theta_3 = \mu_0$$
$$\theta_4 = \mu_1$$

and four estimating equations:

$$\sum_i e(\boldsymbol{X_i}) - \theta_1 = 0$$
$$\sum_i e(\boldsymbol{X_i})^2 - \theta_2 = 0$$
$$\sum_i Y_i - \theta_4\theta_1 - \theta_3(1-\theta_1) = 0$$
$$\sum_i Y_i e(\boldsymbol{X_i}) - \theta_4\theta_2 - \theta_3(\theta_1 - \theta_2) = 0$$

3 Relaxing (1) with regression

There's no particular reason to assume (1) and it seems like it would typically be pretty implausible.

But say you believe the model

$$Y_i = \mu_1 S_i + \mu_0(1 - S_i) + \boldsymbol{x}_i'\boldsymbol{\beta} + \epsilon_i \tag{2}$$

Then instead of (1) we could assume something like

$$\mathbb{E}[Y_i - \boldsymbol{x}_i'\boldsymbol{\beta}|\boldsymbol{X}_i, S_i] = \mathbb{E}[Y_i - \boldsymbol{x}_i'\boldsymbol{\beta}|S_i]$$

Under model (2), we have new estimating equations.

$$\mathbb{E}[Y] = \mathbb{E}\left[\mu_1 e(\boldsymbol{X}) + \mu_0(1 - e(\boldsymbol{X})) + \boldsymbol{X\beta}\right]$$
$$\mathbb{E}[Ye(\boldsymbol{X})] = \mathbb{E}\left[e(\boldsymbol{X})\left(\mu_1 e(\boldsymbol{X}) + \mu_0(1 - e(\boldsymbol{X})) + \boldsymbol{X\beta}\right)\right]$$
$$\mathbb{E}[\boldsymbol{X}Y]\mathbb{E}\left[\boldsymbol{X}\left(\mu_1 e(\boldsymbol{X}) + \mu_0(1 - e(\boldsymbol{X})) + \boldsymbol{X\beta}\right)\right]$$

in other words, no interaction between $\boldsymbol{X}$ and S in (1). Then, if you had an estimate for β, you could just substitute $Y_i - \boldsymbol{x}_i'\boldsymbol{\beta}$ for Y_i in the estimates. Alternatively, you could use a stacked estimating equation approach and estimate it all together.

References

Avi Feller, Fabrizia Mealli, and Luke Miratrix. Principal score methods: Assumptions, extensions, and practical considerations. *Journal of Educational and Behavioral Statistics*, 42(6):726–758, 2017.

P. Ding and J. Lu. Principal stratification analysis using principal scores. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 79:757–777, 2017.