Calculus 3
Fall 2020
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Assignment 3

1. Done

2. $\vec{r}(t) = e^{t}\hat{i} + 4t\hat{j} + 2e^{-t}\hat{k}$

a) Compute $\vec{r}'(t)$ and $\vec{r}''(t)$

$$\vec{r}'(t) = \frac{d}{dt}\left(e^{t}\hat{i} + 4t\hat{j} + 2e^{-t}\hat{k}\right)$$

$$= e^{t}\hat{i} + 4\hat{j} + (-2)e^{-t}\hat{k}$$

$$\boxed{r'(t) = \langle e^{t}, 4, -2e^{-t} \rangle}$$

$$\vec{r}''(t) = \frac{d}{dt} r'(t)$$

$$\boxed{\vec{r}''(t) = \langle e^{t}, 0, 2e^{-t} \rangle}$$

b) Find the curvature K.

②

$$K(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3}$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle e^t, 4, -2e^{-t} \rangle \times \langle e^t, 0, 2e^{-t} \rangle$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^t & 4 & -2e^{-t} \\ e^t & 0 & 2e^{-t} \end{vmatrix} = \begin{vmatrix} 4 & -2e^{-t} \\ 0 & 2e^{-t} \end{vmatrix} \hat{i} - \begin{vmatrix} e^t & -2e^{-t} \\ e^t & 2e^{-t} \end{vmatrix} \hat{j}$$

$$+ \begin{vmatrix} e^t & 4 \\ e^t & 0 \end{vmatrix} \hat{k}$$

$$= 8e^{-t}\hat{i} - (2+2)\hat{j} + 4e^t \cdot \hat{k}$$

$$= \langle 8e^{-t}, -4, 4e^t \rangle$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{64e^{-2t} + 16 + 16e^{2t}}$$

$$\|\vec{r}'(t)\|^3 = \sqrt{(e^t)^2 + 16 + 4e^{-2t}}^{\,3}$$

$$= \sqrt{e^{2t} + 16 + 4e^{-2t}}$$

$$K(t) = \frac{4\sqrt{4e^{-2t} + 1 + e^{2t}}}{\sqrt{e^{2t} + 16 + 4e^{-2t}}^{\,3}}$$

@ $(1, 0, 2)$ $\quad e^t = 1 \Rightarrow t = 0$

$$K(0) = \frac{4\sqrt{4+1+1}}{\sqrt{1+16+4}^{\,3}}$$

2.c) Cont'd

$$K(0) = \frac{4\sqrt{6}}{\sqrt{21}^3}$$

$$= \frac{4\sqrt{3}\cdot\sqrt{2}}{21\sqrt{21}} = \frac{4\sqrt{3}\cdot\sqrt{2}}{21\sqrt{3}\cdot\sqrt{7}}$$

$$= \frac{4\sqrt{2}}{21\sqrt{7}}$$

$$\boxed{K(0) = \frac{4\sqrt{14}}{147}}$$

3) $\vec{r}(t) = \langle t, -\frac{1}{3}\ln(\cos 3t)\rangle \quad 0 < t < \pi/6$

a. expression of $\hat{T}(t)$

$$\hat{T}(t) = \frac{\vec{r}\,'(t)}{\|r'(t)\|}$$

$$r' = \frac{\langle 1, -\frac{1}{3}(-3\sin 3t) \times \frac{1}{\cos 3t}\rangle}{\sqrt{1^2 + \tan^2 3t}}$$

$$= \frac{\langle 1, \tan 3t\rangle}{\sqrt{\sec^2 3t}}$$

$$= \left\langle \frac{1}{\sec 3t}, \frac{\tan 3t}{\sec 3t}\right\rangle \quad \text{b/c } 0 < t < \pi/6,\ \sec 3t > 0$$

$$\boxed{\hat{T}(t) = \langle \cos 3t, \sin 3t \rangle}$$

b. Compute unit normal ~~tangen~~ $\hat{N}$

$$\hat{N}(t) = \frac{\hat{T}'(t)}{\|\hat{T}'(t)\|}$$

$$\hat{T}'(t) = \langle -3\sin 3t, 3\cos 3t \rangle$$

$$\hat{T}'(t) = \sqrt{9\sin^2 3t + 9\cos^2 3t}$$

$$= 3$$

$$\hat{N}(t) = \frac{\langle -3\sin 3t, 3\cos 3t \rangle}{3}$$

$$\boxed{\hat{N}(t) = \langle -\sin 3t, \cos 3t \rangle}$$

c) Show that $\hat{N}(t) \perp \hat{T}(t)$

$$\hat{N}(t) \cdot \hat{T}(t) = \langle -\sin 3t, \cos 3t \rangle \cdot \langle \cos 3t, \sin 3t \rangle$$

$$= -\sin 3t \cos 3t + \sin 3t \cos 3t$$

$\hat{N}(t) \cdot T(t) = 0$, so $\hat{T}(t) \perp \hat{N}(t)$

3) cont'd*

d) Expression for $K(t)$

$$K(t) = \frac{\|\hat{T}'(t)\|}{\|\vec{r}'(t)\|}$$

$$= \frac{\| \langle -3\sin 3t, 3\cos 3t \rangle \|}{\| \langle 1, \tan 3t \rangle \|}$$

$$= \frac{\sqrt{9\sin^2 3t + 9\cos^2 3t}}{\sqrt{1 + \tan^2 3t}}$$

$$= \frac{3}{\sec 3t}$$

$$K(t) = 3\cos 3t \qquad t \in [0, \pi/6]$$

e) Re-computing $K(t)$

$$K(t) = \frac{\|\vec{r}' \times r''(t)\|}{\|r''(t)\|}$$

$r'(t) = \langle 1, \tan 3t, 0 \rangle$
$r''(t) = \langle 0, 3\sec^2 3t, 0 \rangle$

$$\vec{r}(t) \times r''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & \tan 3t & 0 \\ 0 & 3\sec^2 3t & 0 \end{vmatrix} = 3\sec^2 3t \, \hat{k}$$

$$\|\vec{r}'(t)\times\vec{r}''(t)\| = |3\sec^2 3t| = 3\sec^2 3t$$

$$\|r'(t)\|^3 = \sqrt{1+\tan^2 3t}^{\,3} = \sqrt{\sec^2 3t}^{\,3}$$

$$= (\sec 3t)^3$$

$$K(t) = \frac{3\sec^2 3t}{\sec^3 3t} = \frac{3}{\sec 3t}$$

$$K(t) = 3\cos 3t$$ ✓ Checked!

4) a. $\vec{r}(t) = \langle t, f(t), 0\rangle$

a) Show that $K(x) = \dfrac{|f''(x)|}{[1+[f'(x)]^2]^{3/2}}$

$$K(t) = \frac{\vec{r}'(t)\times\vec{r}''(t)}{\|r'(t)\|^3}$$

$\vec{r}'(t) = \langle 1, f'(t), 0\rangle$ $\quad \vec{r}''(t) = \langle 0, f''(t), 0\rangle$

$$\vec{r}'(t)\times\vec{r}''(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(t) & 0 \\ 0 & f''(t) & 0 \end{vmatrix} = (f''(t)-0)\hat{i} - \hat{j}(0-0) + \hat{k}(f''(t)-0)$$

$$= f''(t)\hat{k}$$

$$\|\vec{r}'(t)\times\vec{r}''(t)\| = |f''(t)|$$

$$\vec{r}'(t) = \langle 1, f'(t), 0 \rangle$$

$$\|\vec{r}'(t)\|^3 = \sqrt{1 + [f'(t)]^2}^{\,3}$$

$$= \left(1 + [f'(t)]^2\right)^{3/2}$$

Since $y = f(t)$ and $x = t$,
$y = f(x)$

$$\boxed{K(x) = \frac{|f''(x)|}{\left[1 + [f'(x)]^2\right]^{3/2}}}$$

b) K for $y = \sin 2x$

$$K(x) = \frac{|f''(x)|}{[1 + f'(x)^2]^3} = \frac{|-4\sin 2x|}{(1 + 4\cos^2 2x)^{3/2}}$$

for $x = \frac{\pi}{4}$ $\quad K = \dfrac{|-4\sin\frac{\pi}{2}|}{\left(1 + 4(\cos\frac{\pi}{2})^2\right)^{3/2}}$

$$= \frac{4}{1}$$

$$\boxed{K = 4}$$

c) Equation for the circle

$$r = \frac{1}{k} \Rightarrow r = \frac{1}{4}$$

Let's find the center of the circle.

$$\hat{T}(t) = \frac{\cancel{2\cos 2\theta}\ \frac{d}{dt}(\langle t, \sin 2t, 0\rangle)}{\|\langle t, \sin 2t, 0\rangle\|}$$

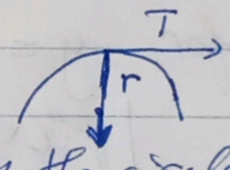
$$= \frac{\langle 1, 2\cos 2t, 0\rangle}{\sqrt{1+4\cos^2(2t)}}$$

$$@\ t = \pi/4,\ \hat{T} = \frac{\langle 1, 0, 0\rangle}{1}$$

$$\hat{T} = \hat{i}$$

$$\Rightarrow \hat{N} = \frac{\frac{d}{dt}(\hat{i})}{\|\hat{T}\|}$$

$$\vec{N} = \hat{j}$$



Since we know that diameter of the circle goes through $(\pi/4, 1)$ and is vertical ($\parallel$ to $\hat{j}$) and that $r = 1/4$

we can conclude that $x_0 = \pi/4$, $y = 1 - 1/4 = 3/4$

$$\boxed{\text{Equation: } \left(x - \frac{\pi}{4}\right)^2 + \left(y - \frac{3}{4}\right)^2 = \frac{1}{16}}$$

5) $\vec{r}(t) = 3t\,\hat{i} + (\cos 4t)\,\hat{j} + (\sin 4t)\,\hat{k}$

a) Find $\hat{T}\left(\frac{\pi}{8}\right)$

$$\hat{T} = \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} = \frac{\langle 3, -4\sin 4t, 4\cos 4t \rangle}{\sqrt{9 + 16(\sin^2 4t + \cos^2 4t)}}$$

$$= \left\langle \frac{3}{5}, -\frac{4}{5}\sin 4t, \frac{4}{5}\cos 4t \right\rangle$$

@ $t = \frac{\pi}{8}$ $\boxed{\hat{T}(\pi/8) = \left\langle \frac{3}{5}, -\frac{4}{5}, 0 \right\rangle}$

b) Find $\hat{N}(\pi/8)$

$$\hat{N}(t) = \frac{\cancel{d}\,\hat{T}'(t)}{\left\|\frac{d\hat{T}}{dt}\right\|} = \frac{\langle 0, -\frac{16}{5}\cos 4t, -\frac{16}{5}\sin 4t \rangle}{\sqrt{\left(\frac{16}{5}\right)^2(\cos^2 4t + \sin^2 4t)}}$$

$$= \frac{\langle 0, -\frac{16}{5}\cos 4t, -\frac{16}{5}\sin 4t \rangle}{16/5}$$

$$= \langle 0, -\cos 4t, -\sin 4t \rangle$$

$\boxed{\hat{N}(\pi/8) = \langle 0, 0, -1 \rangle}$

c) $\hat{B}$ · computed

$$\hat{B} = \hat{T} \times \hat{N}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3/5 & -4/5 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \hat{i}\left(\frac{4}{5}-0\right) - \hat{j}\left(0+\frac{3}{5}\right) + \hat{k}(0-0)$$

$$= \cancel{-\frac{3}{5}\hat{j}} \; \frac{4}{5}\hat{i} - \frac{3}{5}\hat{j}$$

$$\boxed{\hat{B} = \left\langle \frac{4}{5}, -\frac{3}{5}, 0 \right\rangle}$$

6) $\vec{r}(t) = \langle t^2, t^3 \rangle$

a)

$$\vec{v}(t) = \langle 2t, 3t^2 \rangle$$

$$\vec{a}(t) = \langle 2, 6t \rangle$$

$$\frac{ds}{dt} = \|\vec{v}\| = \sqrt{(2t)^2 + (3t^2)^2}$$

$$\text{Speed} = t\sqrt{4+9t^2}$$

b) $$a_T = \frac{d^2s}{dt^2} = \frac{d}{dt}\left(t\sqrt{4+9t^2}\right)$$

$$= 1\cdot\sqrt{4+9t^2} + t\cdot\frac{18t}{2\sqrt{4+9t^2}}$$

$$= \sqrt{4+9t^2} + \frac{9t^2}{\sqrt{4+9t^2}}$$

$$a_T = \frac{18t^2+4}{\sqrt{4+9t^2}}$$

$$\hat{T}(t) = \frac{\vec{v}(t)}{\vec{v}(t)}$$

$$= \frac{\langle 2t, 3t^2 \rangle}{t\sqrt{4+9t^2}}$$

$$\hat{T}(t) = \left\langle \frac{2}{\sqrt{4+9t^2}}, \frac{3t}{\sqrt{4+9t^2}} \right\rangle$$

d) $$a_T = \vec{a} \cdot \hat{T}$$

$$= \langle 2, 6t \rangle \cdot \left\langle \frac{2}{\sqrt{4+9t^2}}, \frac{3t}{\sqrt{4+9t^2}} \right\rangle$$

$$a_T = \frac{4+18t^2}{\sqrt{4+9t^2}}$$

e) $$\vec{a}(1) = \langle 2, 6 \rangle \qquad \hat{T}(1) = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$$a_T(1) = \frac{4+18}{\sqrt{13}}$$

$$a_T(1) = \frac{22\sqrt{13}}{13}$$

f)

$$a_N = \sqrt{\|\vec{a}(1)\|^2 - [a_T(1)]^2}$$

$$= \sqrt{(4+36) - \frac{22^2}{13}}$$

$$= \sqrt{\frac{40\times 13}{13} - \frac{22^2}{13}}$$

$$= \frac{\sqrt{36}}{\sqrt{13}}$$

$$\boxed{a_N(1) = \frac{6\sqrt{13}}{13}}$$

g) Find $\hat{N}(1)$

$$\hat{T}(1) = \left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle$$

$\Rightarrow \hat{N} = \left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$ since $\hat{N}$ is point toward the left. (x-component negative)

h) $\vec{a}(1) = a_T(1)\,\hat{T}(1) + a_N(1)\,\vec{N}$

$$= \frac{22\sqrt{13}}{13}\left\langle \frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}} \right\rangle + \frac{6\sqrt{13}}{13}\left\langle -\frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

$$= \left\langle \frac{44}{13}, \frac{6}{13} \right\rangle + \left\langle \frac{-18}{13}, \frac{12}{13} \right\rangle$$

$\boxed{\vec{a}(1) = \langle 2, 6 \rangle}$ verified