COMP 550
Algorithms and Analysis
Spring 2020
Final Exam.
Monday, May 4, 2020
Open Book - Open Notes
Don't forget to write your name or ID and pledge on the exam sheet.
This exam has five pages.

For many of the questions, you will need to select one of the following asymptotic growth rates using the letters A,B,C,D,E, or F:
A) Constant ($\Theta(1)$) B) $\Theta(\log n)$ C) $\Theta(\sqrt{n})$ D) $\Theta(n)$ E) $\Theta(n \log n)$ F) $\Theta(n^2)$
Recall that $f(n) = \Theta(g(n))$ means that f and g asymptotically grow at the same rate.

1. (10 points) Solve the recurrence $T(n) = 4T(n/2) + \Theta(n)$. Fill in the blank with the letter A,B,C,D,E, or F indicating the asymptotic growth rate of $T(n)$ as a function of n. _F_

2. (4 points) What is the asymptotic worst case time bound for quicksort? Fill in the blank with A,B,C,D,E, or F as appropriate. _F_

3. (4 points) What is the asymptotic expected time bound for quicksort? Fill in the blank with A,B,C,D,E, or F as appropriate. _E_

4. (6 points) Give an asymptotic estimate for the sum $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \ldots + \frac{1}{2n}$ as a function of n. Fill in the blank with A,B,C,D,E, or F as appropriate. _B_

5. (4 points) How long does it take to build a max-heap of n elements as a function of n? What is the asymptotic growth rate of this time as a function of n? Fill in the blank with A,B,C,D,E, or F as appropriate. _D_

6. (10 points) Suppose you insert the elements from the set $\{3, 6, 7, 10, 15, 20, 30\}$ into an initially empty binary search tree, to produce a perfectly balanced tree. List the four elements that will end up at the leaves. _3, 7, 15, 30_

7. (10 points)

What is a good method to sort n elements in the range $\{0, 1, \ldots, n-1\}$ using $O(n)$ storage? Consider the following methods:
a) Merge sort b) Radix sort c) Quick sort d) Bucket sort e) Counting sort

7a) Fill in the blank with the letter of the correct choice. ___e___

7b) Fill in the blank with A,B,C,D,E, or F indicating the asymptotic time bound of this method as a function of n. ___D___

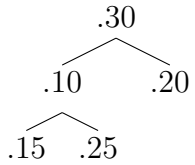
What is a good method to sort $\sqrt{n}$ elements in the range $\{0, 1, \ldots, n\}$ using $O(n)$ storage? Consider the following methods: a) Merge sort b) Radix sort c) Quick sort d) Bucket sort e) Counting sort

7c). Fill in the blank with the letter of the correct choice. ___b___

7d) Fill in the blank with A,B,C,D,E, or F indicating the asymptotic time bound of this method as a function of n. Remember that the total number of elements is much smaller than n. ___C___

8. (10 points) For hashing by the multiplication method, which of the following values for A is best: (a) 0.125 (b) .618033988 (c) .700 (d) .555555555555555
_____B_____

9. (10 points) Consider a binary search tree where the probabilities of searching for each key are as indicated:



What is the expected number of nodes visited in a random successful search operation on this tree? _____2.1_____

10. (10 points) Suppose the symbols a,b,c,d,e,f,g,h are used with the probabilities 1/56, 1/56, 1/28, 1/14, 1/7, 5/28, 1/4 and 2/7, respectively. Construct an optimal prefix code (a Huffman code) for these symbols. Fill in the blanks with the number of bits in the code word for each symbol.

a ___6___ b ___6___ c ___5___ d ___4___ e ___3___ f ___2___ g ___2___ h ___2___

A:111110 B:111111 C:11110 D:1110 E:110 F:00 G:01 H:10

11. (6 points) Suppose that a directed graph contains the following edges. Find the strongly connected components. $\{(h,i),(i,j),(j,k),(k,h),(l,m),(m,n),(n,p),(p,l),(f,i),(c,e),(j,b),(k,l),(a,b),(b,c),(c,a),(d,e),(e,f),(f,g),(g,d)\}$

a) How many vertices are there in the component having the smallest number of vertices? __4__

b) How many vertices are there in the component having the second smallest number of vertices? __11__

12. (10 points) For the parts of this question, assume (x,y) is an edge and the values $d[x], f[x], d[y]$, and $f[y]$ obtained from depth-first search satisfy the given inequalities. $d[x]$ tells when the edge is discovered and $f[x]$ tells when it is finished. For each part, state whether the edge (x,y) is a tree edge (t), forward edge (f), backward edge (b), or cross edge (c) or whether the listed values are impossible (i). If it is not possible to say what kind of edge (x,y) is, give one of the possible kinds of edges it could be. Fill in the blanks with the appropriate letters t,f,b,c, or i.

(a) $d[x] < f[x] < d[y] < f[y]$ _____i_____
(b) $d[y] < f[y] < d[x] < f[x]$ _____c_____
(c) $d[x] < d[y] < f[y] < f[x]$ _____f_____
(d) $d[y] < d[x] < f[x] < f[y]$ _____b_____
(e) $d[x] < d[y] < f[x] < f[y]$ _____i_____

13. (10 points) Consider the following undirected graph G with (x,y,n) indicating an edge from x to y having weight n.

$\{(a,h,3),(a,j,1),(b,f,3),(b,g,4),(c,f,1),(c,g,2),(c,h,3),(c,j,2),(d,g,6),(d,h,5),(e,f,4),(e,j,6)\}$.

What is the cost of a minimum spanning tree for G? _____21_____

14. (6 points) For the following questions, among the methods presented in class, give the best method to use to solve the single-source shortest paths problem on the kind of graph specified. Choices: A. The Bellman-Ford algorithm B. Dijkstra's algorithm C. An algorithm that is simpler and faster

than Dijkstra's algorithm on the specified kind of graphs.

(a) For graphs with no negative weight edges. B
(b) For directed acyclic graphs C
(c) For graphs with no negative weight cycles. A

15. (8 points) For each problem, choose all elements F of the set $\{\Theta, O, o, \Omega, \omega\}$ such that the statement $f(x) = F(g(x))$ is a correct statement of the asymptotic relationship between f and g. Thus if $f(x) = \Omega(g(x))$ and $f(x) = \Theta(g(x))$ and $f(x) = O(g(x))$ are the only three valid asymptotic relationships between f and g, choose Ω, Θ, and O. Hint: $f(x) = \omega(g(x))$ means f grows faster than g, in the limit. $f(x) = o(g(x))$ means f grows slower than g, in the limit. Θ means they grow the same, roughly speaking.

a). $f(x) = 2^x + 3x^2, g(x) = 3^x + 2x + 1$. O, o
b). $f(x) = x^2 + 6, g(x) = 5x^2 - 4$. O, Θ, Ω
c). $f(x) = 2x^2 + 1, g(x) = 3x - 5$. Ω, ω
d). $f(x) = 4x + 2, g(x) = 2\log^2 x + 1$. Ω, ω

16. (8 points) A fair *die* when tossed will give each of the possible values with equal probability. The plural of die is *dice*. A *5-sided die* has the numbers 1,2,3,4,5 on its sides and each one can come up with equal probability,

a). Suppose two fair 5-sided dice are tossed. What is the probability that the sum of their values will equal 6? 5/25
b). What is the expected value for a single toss of a fair 5-sided die? 3
c). What is the expected value for the sum of three tosses of a fair 5-sided die? 9
d). Suppose three fair 5-sided dice are tossed. What is the probability that all three dice will produce equal values? 1/25

17. (5 points) EXTRA CREDIT: Let $f(n)$ be the number of permutations of $2n$ elements in which even numbered elements map to even numbered element and osdd numbered elements map to odd numbered elements. Thus if $n = 2$ then the permutation g with $g(1) = 3$, $g(3) = 1$, $g(2) = 4$, and $g(4) = 2$ is an example of such a permutation.

How many such permutations are there, as a function of n?

a) $n!$
b) $(2n)!$
c) $(n!)^2$
d) $(2n)!^2$
e) $\binom{n}{2}$
f) $\binom{2n}{2}$
g) $\binom{2n}{n}$

Fill in the blank with the letter of the correct choice. __c__

18. (5 points) EXTRA CREDIT: Compute $\sum_{j=0}^{\infty} \frac{j}{2^j}$. Fill in the blank with an integer or a fraction. __2__