Topics of the week Compute with matrices:

1. explain the CR decomposition;
2. linear transformations, visualizing linear transformations in 2d, properties of linear transformations, matrix representation of linear transformations;
3. systems of linear equations, systems of linear equations with unique solutions.

Actions on rows/columns AB for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times k}$ is interpreted as:

1. Applying matrix A to all columns of B;
2. Applying matrix B to all rows of A.

Let A_i denote the i-th row and A^j the j-th column:

1. $(AB)_i$ is the linear combination of rows of $B_1, \dots, B_m$ with coefficients from A_i.
2. $(AB)^j$ is the linear combination of columns of $A^1, \dots, A^k$ with coefficients from B^j.

Rank decomposition Let A be an $m \times n$ matrix. The following statements are equivalent:

1. $\operatorname{rank} A \le k$;
2. There are vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$ and $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$ such that $A = \sum\limits_{i=1}^{k} \mathbf{v}_i \mathbf{w}_i^\top$.
3. $A = CR$, where $C \in \mathbb{R}^{m\times k}$ and $R \in \mathbb{R}^{k\times n}$ (rank decomposition).

Columns of C form a basis in the column space. Rows of R form the basis in the row space. Multiplying R by C on the left defines the way rows of A are represented as linear combinations of the rows of R, similar for C multiplied by R on the right.

Systems of linear equations Consider a system

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n. \end{cases} \iff Ax = b \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{cases} x_1 = b'_1, \\ x_2 = b'_2, \\ \dots, \\ x_n = b'_n. \end{cases} \iff A^{-1}Ax = A^{-1}b \iff \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$$

In-class exercises

1. Consider $T : \mathbb{R}^n \to \mathbb{R}$ for $n > 0$ defined as $T(x) = \sum\limits_{k=1}^{n} kx_k$. Prove that T is a linear transformation.
2. Consider $T : \mathbb{R}^n \to \mathbb{R}$ for $n \ge 2$ defined as $T(x) = \sum\limits_{k=1}^{n} (x_k)^k$. Is $T(x)$ a linear transformation?