W0: Vectors and linear combinations

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19.09.2025

Group info

Group: 24
Location: CHN G 46
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Language: English
Focus group[1]: Yes
Website: https://ti.inf.ethz.ch/ew/courses/LA25
Please don't hesitate to interrupt at any moment if have any questions!

[1] More thorough and detailed explanation of basics

Introductions

Let's get to know each other!

Intro questions

- Your name?
- A bit about yourself?
- What do you expect from a focus group?

Course info

- Exam structure:
    - **Calculations**: Solve standard compute tasks (answer-only);
    - **Proofs**: Need to justify anything that wasn't in *lectures*;
    - **Multi-choice questions**: Pick the only correct option;
- Weekly bonus tasks: up to 0.25 extra points to grade;
- Hand-ins: One exercise per week, get feedback;

Scalars

Recall "standard" sets of numbers:

- $\mathbb{N}$: **Natural** numbers;
- $\mathbb{Z}$: **Integer** numbers;
- $\mathbb{Q}$: **Rational** numbers;
- $\mathbb{R}$: **Real** numbers;

Represent *quantity*, typically called the **scalars**[2].

[2] from "scale"

Tuples

$\mathbb{R}^m$ is the set of **tuples**: $\mathbb{R}^m = \{(x_1, \ldots, x_m) : x_i \in \mathbb{R}\}$[3]:

- $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ – points of a 2D **plane**;
- $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$ – points of a 3D **space**;

The **power** symbol corresponds to the **Cartesian product**:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

We may use *indices* to refer to the *components* of a tuple.

[3] $x_i \in \mathbb{R}$ means that x_i is an *element* of the *set* $\mathbb{R}$

Vectors

Real vectors[4] are elements of a set V, called **vector space**, with:

- **Addition**: $\mathbf{u} + \mathbf{v} \in V$ for $\mathbf{u}, \mathbf{v} \in V$;
- **Scaling**: $k\mathbf{v} \in V$ for $k \in \mathbb{R}$ and $\mathbf{v} \in V$;

Addition and scaling are collectively called **linear operations**[5].
In most general case, addition and scaling are **axiomatic**[6].
In our course, we will mostly work with $V = \mathbb{R}^m$.

[4] in our course, denoted by lowercase **bold** letters
[5] hence, they are the subject of *linear* algebra
[6] defined by their properties

Column vectors

For $V = \mathbb{R}^m$, our course uses columns to represent vectors:

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix},$$

and the addition and scaling are **component-wise**:

- $(\mathbf{u} + \mathbf{v})_i = u_i + v_i$;
- $(k\mathbf{v})_i = kv_i$;

This is geometry-motivated for $m = 2$ and $m = 3$.

Core properties

Vector addition properties[7]

- Associativity: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$;
- Commutativity: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$;
- Zero vector: $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$;
- Negatives: $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$;

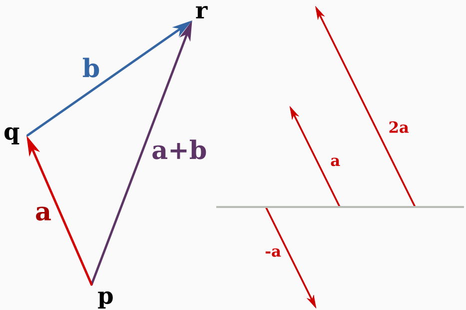
Scaling properties

- Compatibility with multiplication of scalars: $\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v}$;
- Unit scaling: $1\mathbf{v} = \mathbf{v}$;
- Distributivity with vector addition: $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$;
- Distributivity with scalar addition: $(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v}$;

[7] in abstract algebra sense, vector space is a commutative group

Vector geometry

In geometry, vectors have *direction* and *magnitude*[8]:



Addition adds vector direction, scaling changes magnitude

[8] but **not** origin!

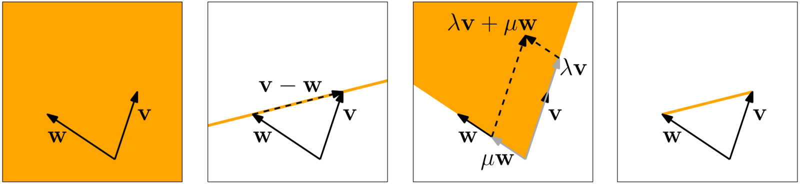
Linear combinations

A **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ with coeffs. $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$:

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n \in V$$

Some α_i may be zero. **Trivial** combination: $\alpha_1 = \cdots = \alpha_n = 0$.

Special combinations



(i) linear (ii) affine (iii) conic (iv) convex

- Affine: $\alpha_1 + \cdots + \alpha_n = 1$;
- Conic: $\alpha_1, \ldots, \alpha_n \geq 0$;
- Convex: $\alpha_1 + \cdots + \alpha_n = 1$ *and* $\alpha_1, \ldots, \alpha_n \geq 0$.

We can rewrite $\alpha_1 + \cdots + \alpha_n = 1$ as $\alpha_1 = 1 - \alpha_2 - \cdots - \alpha_n$:

$$\mathbf{v}_1 + \alpha_2(\mathbf{v}_2 - \mathbf{v}_1) + \cdots + \alpha_n(\mathbf{v}_n - \mathbf{v}_1) \in V$$

Thus, affine combinations are linear combinations with “shifted” origin.

Triangle combinations

Statement

Sketch sets of all possible linear, affine, conic and convex combinations of 3 distinct points on a plane. Consider all cases.

Triangle combinations

Linear: The whole plane or a line (if collinear with 0).
Affine: The whole plane or a line (if collinear).
Conic: The area between "tangent" rays or the whole plane (if 0 is inside).
Convex: The triangle spanned on the 3 points.

The perfect long drink (a)

Statement

You have drinks G and T. You have two drinks:

1. 15 ml of G and 85 ml of T;
2. 35 ml of G and 65 ml of T;

Can we mix a perfect drink (23 ml of G and 77 ml of T)?

The perfect long drink (a)

Use x of first drink and y of second drink:

$$x \begin{bmatrix} 15 \\ 85 \end{bmatrix} + y \begin{bmatrix} 35 \\ 65 \end{bmatrix} = \begin{bmatrix} 23 \\ 77 \end{bmatrix} \iff \begin{cases} 15x + 35y = 23 \\ 85x + 65y = 77 \end{cases}$$

Add equations:

$$100(x+y) = 100 \iff x + y = 1$$

Substitute $x = 1 - y$ into first equation:

$$15(1-y) + 35y = 23 \iff 20y = 8 \iff y = \frac{2}{5}$$

From $x = 1 - y$, we get $x = \frac{3}{5}$. It's feasible, because $0 \leq x, y \leq 1$.

The perfect long drink (b)

Statement

Consider the set of all 100 ml drinks:

$$D = \left\{ \begin{bmatrix} g \\ t \end{bmatrix} \in \mathbb{R}^2 : g, t \geq 0 \text{ and } g + t = 100 \right\}$$

For $\mathbf{v} = \begin{bmatrix} 15 \\ 85 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 35 \\ 65 \end{bmatrix}$, which 100 ml drinks can be mixed?

The perfect long drink (b)

Any mix can be represented as $x\mathbf{v} + y\mathbf{w}$, where $x, y \geq 0$.
Since initial drinks are 100 ml, final drink volume is $x \cdot 100 + y \cdot 100$.
We need it to still be 100, hence $(x + y) \cdot 100 = 100 \iff x + y = 1$.
On the other hand, it is also sufficient for the drink to be in D, thus

$$\hat{D} = \{x\mathbf{v} + y\mathbf{w} : x, y \geq 0 \text{ and } x + y = 1\}$$

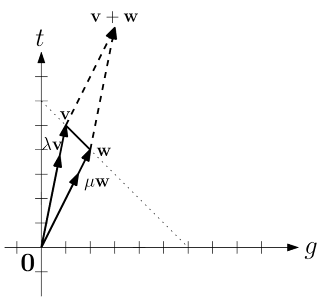
$\hat{D}$ are convex combinations of $\mathbf{v}$ and $\mathbf{w}$, aka the segment between them.

The perfect long drink (c)

Statement

For $\mathbf{v} = \begin{bmatrix} 15 \\ 85 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 35 \\ 65 \end{bmatrix}$, which (any) drinks can be mixed?

The perfect long drink (c)



Since the drink no longer needs to be 100 ml, we get rid of $x + y = 1$. But, it should be $x, y \leq 1$, as we can't use more than 100% of a drink.

$$\bar{D} = \{x\mathbf{v} + y\mathbf{w} : 0 \leq x, y \leq 1\}$$

$\bar{D}$ is a parallelogram spanned on $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{v} + \mathbf{w}$.

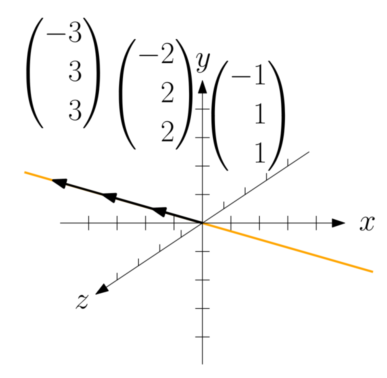
Geometry of linear combinations (a)

Statement

Sketch the following set in $\mathbb{R}^3$:

$$\left\{ \lambda_1 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$

Geometry of linear combinations (a)



lets use $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to denote vectors. Note that $\mathbf{b} = 2\mathbf{a}$ and $\mathbf{c} = 3\mathbf{a}$. We can safely remove vectors that are linear combinations of others. Thus, the set is simply $\lambda_1 \mathbf{a}$, aka the line going through $\mathbf{0}$ and $\mathbf{a}$.

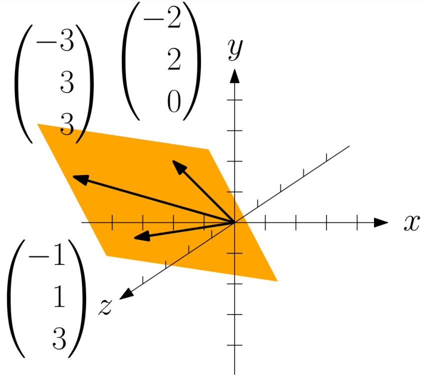
Geometry of linear combinations (b)

Statement

Sketch the following set in $\mathbb{R}^3$:

$$\left\{ \lambda_1 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \right\}$$

Geometry of linear combinations (b)



Note that $\mathbf{c} = \mathbf{a} + \mathbf{b}$, so we can "ignore" $\mathbf{c}$.
Thus, the set is $\lambda_1\mathbf{a} + \lambda_2\mathbf{b}$, a plane passing through $\mathbf{0}$, $\mathbf{a}$ and $\mathbf{b}$.

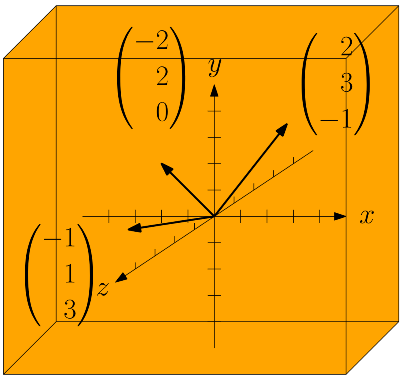
Geometry of linear combinations (c)

Statement

Sketch the following set in $\mathbb{R}^3$:

$$\left\{\lambda_1 \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + \lambda_3 \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} : \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\right\}$$

Geometry of linear combinations (c)



Vectors point in three independent directions.
The set of all linear combinations covers the whole $\mathbb{R}^3$.