Topics of the week

- 1. Orthogonal vectors, Orthonormal vectors, Orthonormal bases;
- 2. Orthogonal Matrices. Orthogonal matrices preserve norm and inner-product;
- 3. Projections with orthonormal bases;
- 4. Build an orthonormal basis with Gram-Schmidt (and show correctness of Gram-Schmidt)
- 5. QR decomposition. Projections and least squares with QR decomposition.

Orthogonal vectors means pairwise orthogonal.

Let e_1, \ldots, e_n be orthogonal and $v = x_1 e_1 + \cdots + x_n e_n$, then $e_i \cdot v = x_i (e_i \cdot e_i) \iff x_i = \frac{v \cdot e_i}{e_i \cdot e_i}$.

Orthonormal vectors means orthogonal + each vector has a unit length.

Let e_1, \ldots, e_n be orthonormal and $v = x_1 e_1 + \cdots + x_n e_n$, then $x_i = e_i \cdot v$.

Orthonormal basis is a basis consisting of orthonormal vectors.

Gram matrix For v_1, \ldots, v_n :

$$\Gamma(v_1, \dots, v_n) = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \dots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \dots & v_n \cdot v_n \end{bmatrix} = A^{\top} A$$

The columns of A are orthogonal $\iff A^{\top}A$ is diagonal.

The columns of A are orthonormal $\iff A^{\top}A = I$.

Interpretation of projection $A^{\top}b$ are scalar products of b with all basis vectors of C(A). Then, $(A^{\top}A)^{-1}$ recovers a vector of C(A) from its dot products with the basis of C(A). $(A^{\top}A)^{-1}$ allows to consider coordinate system in dot products with basis vectors, rather than in their linear combinations.

Dual basis is the basis of C(A) formed of the columns of $A(A^{T}A)^{-1}$.

Orthogonal matrix is a matrix that preserves distances: $||Ax - Ay|| = ||x - y|| \iff A^{\top}A = I$. In other words, matrix is orthogonal \iff it maps e_1, \ldots, e_n into orthonormal basis.

Projection with orthonormal basis Assume $A^{\top}A = I$, then the projection of y on Ax is $AA^{\top}y$.

Gram-Schmidt process construct an orthonormal basis in C(A) by subtracting projections.

QR decomposition A = QR, where Q is the result of Gram-Schmidt process, and $R = Q^{T}A$. **Note**: $AM = Q \implies A = QM^{-1} \implies R = M^{-1}$.

Project with QR decomposition $\operatorname{proj}_{C(A)}(b) = QQ^{\top}b$.

In-class exercises Consider the invertible matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

- 1. Apply the Gram-Schmidt process to the columns of A.
- 2. Write down a QR-decomposition of A.
- 3. Apply the Gram-Schmidt process to the columns of B.
- 4. Is it always true that the Gram-Schmidt process on the columns of an upper triangular $n \times n$ matrix with non-zero diagonal entries yields the canonical basis e_1, \ldots, e_n ? Provide a proof or counterexample.