

**Topics of the week**

1. explain the CR decomposition;
2. linear transformations, visualizing linear transformations in 2d, properties of linear transformations, matrix representation of linear transformations;
3. systems of linear equations, systems of linear equations with unique solutions.

**Actions on rows/columns**  $AB$  for  $A \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{m \times k}$  is interpreted as:

- Applying matrix  $A$  to all columns of  $B$  as  $c \mapsto Ac$ ;
- Applying matrix  $B$  to all rows of  $A$  as  $r \mapsto rB$ ;
- $(AB)_i$  is the linear combination of the rows  $B_1, \dots, B_m$  with the coefficients from  $A_i$ ;
- $(AB)^j$  is the linear combination of the columns  $A^1, \dots, A^k$  with the coefficients from  $B^j$ .

Here, lower indices denote rows, and upper indices denote columns.

**Rank decomposition** Let  $A$  be an  $m \times n$  matrix. The following statements are equivalent:

1.  $\text{rank } A \leq k$ ;
2. There are vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$  and  $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$  such that  $A = \mathbf{v}_1 \mathbf{w}_1^\top + \dots + \mathbf{v}_k \mathbf{w}_k^\top$ .
3.  $A = CR$ , where  $C \in \mathbb{R}^{m \times k}$  and  $R \in \mathbb{R}^{k \times n}$ .

When  $\text{rank } A = k$ ,  $A = CR$  is called a rank decomposition of  $A$ , and

- The columns of  $C$  form a basis in the column space of  $A$ ;
- The columns of  $R$  are the coordinates of the columns of  $A$  in the basis of the columns of  $C$ ;
- The rows of  $R$  form a basis in the row space of  $A$ ;
- The rows of  $C$  are the coordinates of the rows of  $A$  in the basis of the rows of  $R$ .

**Systems of linear equations** Can be represented in matrix form. If the solution is unique, it is found by applying transforms on the rows until you get a system with the unit matrix:  $(A|b) \mapsto (I|b')$ .

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n. \end{cases} \iff Ax = b \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{cases} x_1 = b'_1, \\ x_2 = b'_2, \\ \dots, \\ x_n = b'_n. \end{cases} \iff A^{-1}Ax = A^{-1}b \iff \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$$

**In-class exercises**

1. Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $n > 0$  defined as  $T(x) = \sum_{k=1}^n kx_k$ . Prove that  $T$  is a linear transformation.
2. Consider  $T : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $n \geq 2$  defined as  $T(x) = \sum_{k=1}^n (x_k)^k$ . Is  $T(x)$  a linear transformation?