Topics of the week Compute with matrices:

1. matrix-vector multiplication, column space, row space, rank;
2. perform matrix multiplication, including matrix- vector, vector-matrix, scalar and outer product, distributivity, associativity.

Basis of a vector space is a maximal set of linearly independent vectors.
Equivalent definition: $\mathbf{e}_1, \dots, \mathbf{e}_n$ s.t. any $\mathbf{v} \in V$ can be *uniquely* represented as

$$\mathbf{v} = v_1\mathbf{e}_1 + \dots + v_n\mathbf{e}_n.$$

The numbers $(v_1, \dots, v_n)$ are called **coordinates** of $\mathbf{v}$ in $\mathbf{e}_1, \dots, \mathbf{e}_n$.
Equivalent definition: A minimal $\mathbf{e}_1, \dots, \mathbf{e}_n$ s.t. any $\mathbf{v} \in V$ has unique coordinates.

Linear transformation between vector spaces V and W is a function $f : V \to W$, such that

$$\begin{aligned} \mathbf{u} + \mathbf{v} &\mapsto f(\mathbf{u}) + f(\mathbf{v}), \\ k\mathbf{v} &\mapsto kf(\mathbf{v}). \end{aligned}$$

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be the basis of V and $\mathbf{g}_1, \dots, \mathbf{g}_m$ be the basis of W. Then,

$$f(\mathbf{v}) = v_1 f(\mathbf{e}_1) + \dots + v_n f(\mathbf{e}_n).$$

So, f is fully defined by $f(\mathbf{e}_1), \dots, f(\mathbf{e}_n)$:

$$f(\mathbf{e}_k) = f_{1k}\mathbf{g}_1 + \dots + f_{mk}\mathbf{g}_m.$$

From this, we fully describe the linear map as

$$\begin{aligned} f(a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n) = &(a_1 f_{11} + \dots + a_n f_{1n})\mathbf{g}_1 + \\ &(a_1 f_{21} + \dots + a_n f_{2n})\mathbf{g}_2 + \\ &(a_1 f_{31} + \dots + a_n f_{3n})\mathbf{g}_3 + \\ &\dots + \\ &(a_1 f_{m1} + \dots + a_n f_{mn})\mathbf{g}_m \end{aligned}$$

This rewrites as

$$\begin{cases} b_1 = a_1 f_{11} + a_2 f_{12} + \dots + a_n f_{1n}, \\ b_2 = a_1 f_{21} + a_2 f_{22} + \dots + a_n f_{2n}, \\ \dots, \\ b_m = a_1 f_{m1} + a_2 f_{m2} + \dots + a_n f_{mn}. \end{cases}$$

Conventionally, this is written in matrix form as $b = Fa$:

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

Matrix product is defined in a way that AB corresponds to the linear map $\mathbf{x} \mapsto A(B(\mathbf{x}))$:
The columns of A are $A(\mathbf{e}_1), \dots, A(\mathbf{e}_n) \implies$ the columns of AB are $A(B(\mathbf{e}_1)), \dots, A(B(\mathbf{e}_n))$.

Interpretations of matrix product

1. AB corresponds to applying A to each column of B;
2. AB corresponds to applying B to each row of A;
3. $(AB)_{ij}$ is the dot product of the i-th row of A and the j-th column of B.

Dimension of a vector space is the size of its basis. All bases have the same size.

Row and column spaces of a matrix are spans of their rows/columns.

Matrix rank is the dimension of its row/column space.

Lemma 2.21 Let A be an $m \times n$ matrix. The following statements are equivalent:

1. $\operatorname{rank} A \leq k$;
2. There are vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$ and $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$ such that $A = \sum\limits_{i=1}^{k} \mathbf{v}_i \mathbf{w}_i^\top$.
3. $A = BC$, where $B \in \mathbb{R}^{m \times k}$ and $C \in \mathbb{R}^{k \times n}$ (rank decomposition).

In-class exercises

1. Let $m \in \mathbb{N}_{\geq 2}$ be arbitrary and consider the $m \times m$ matrix
$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$
with $a_{ij} = i + j$ for all $i, j \in \{1, 2, \dots, m\}$. Determine the rank of A.
2. Show that $(AB)^\top = B^\top A^\top$.
3. A **conjugate** of A is a matrix A^* such that $\mathbf{x} \cdot A\mathbf{y} = A^*\mathbf{x} \cdot \mathbf{y}$. Find A^*.
4. A matrix is **orthogonal** if it preserves distances. When is a matrix orthogonal?

Only the first two exercises were actually covered in the class.