Topics of the week

1. Determinant and its properties, definition via permutations, connection to matrix inverse, cofactors and the determinant, Cramer's rule
2. Complex numbers, calculations with complex numbers, conversion between Cartesian form and polar form, Euler's formula
3. Fundamental theorem of algebra, roots of polynomials
4. Complex-valued vectors and matrices
5. Eigenvalues and eigenvectors, definition and 2x2 examples

Determinant For a set of vectors $a_1, \dots, a_n$, define the signed volume $V(a_1, \dots, a_n)$:

1. $V(\alpha v, \dots) = \alpha V(v, \dots)$;
2. $V(a + b, \dots) = V(a, \dots) + V(b, \dots)$;
3. $V(a, b, \dots) = -V(b, a, \dots)$;
4. $V(e_1, \dots, e_n) = 1$.

Properties:

1. $\det AB = \det A \det B \iff$ All spatial objects increase their volume by the factor of det;
2. A^{-1} exists $\iff \det A \neq 0 \iff$ columns of A belong to a smaller-dimensional subspace;
3. $\det A = \det A^\top$;

Sign of permutation $\operatorname{sgn} \sigma$ is $(-1)^k$, where k is, equivalently:

1. The parity of the number of inversions in σ;
2. The parity of the number of swaps needed to turn σ into identity;
3. The difference in parity between n and the number of cycles in cycle presentation of σ.

Leibniz formula We can explicitly write out the determinant as

$$\det A = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \prod_{i=1}^{n} A_{i\sigma_i}$$

Cofactor matrix of A is the matrix M s.t. M_{ij} is $(-1)^{i+j}$ times the determinant of the matrix that is obtained from A by removing its i-th row and j-th column.

Laplace expansion also called cofactor expansion:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} M_{ij} = \sum_{i=1}^{n} (-1)^{i+j} A_{ij} M_{ij},$$

where M is the cofactor matrix. Example:

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

Cramer's rule Consider $Ax = b$. We need to find $x_1, \dots, x_n$ s.t. $b = x_1 v_1 + \dots + x_n v_n$. Then,

$$x_i = \frac{V(v_1, \dots, v_{i-1}, b, v_{i+1}, \dots, v_n)}{V(v_1, \dots, v_{i-1}, v_i, v_{i+1}, \dots, v_n)}$$

In other words, $x_i = \frac{\det A_i}{\det A}$, where A_i replaces v_i with b.

Adjugate matrix $\operatorname{adj} A = M^\top$, where M is the co-factor matrix:

1. $A \operatorname{adj} A = \operatorname{adj} A A = I \det A$;
2. If A is invertible, $A^{-1} = \frac{\operatorname{adj} A}{\det A}$.

Determinant in row operations When we do Gaussian elimination,

1. Multiplying a row by α also multiplies the determinant by α;
2. Adding one row, multiplied by α, to another does **not** change the determinant;
3. Swapping two rows multiplies the determinant by -1.

Change of basis What happens to a matrix when we switch from $e_1, \dots, e_n$ to $b_1, \dots, b_n$?
Consider $v = x$ on the basis $E = (e_1, \dots, e_n)$. We want to find y such that $v = By$, i.e. $y = B^{-1}x$.
Then Ax becomes $B(B^{-1}ABy)$, that is, $A \mapsto B^{-1}AB$.

Similar matrices are matrices A and B s.t. $A = P^{-1}BP$.

Eigenbasis Some basis changes turn A into a nice form, e.g. $A \mapsto \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.
A basis, in which A is diagonal is called an eigenbasis of A.

Eigenvector Any $v \neq 0$ s.t. $Av = \lambda v$ for some λ, called **eigenvalue**.
Correspondingly, eigenbasis is any basis formed by eigenvectors.
Criterion: λ is an eigenvalue $\iff (A - \lambda I)v = 0 \iff \det(A - \lambda I) = 0$.

Characteristic polynomial of the matrix A is $p(\lambda) = \det(A - \lambda I)$.
Characteristic polynomial doesn't change when the basis is changed!
Each coefficient stays invariant under the change of the basis, in particular:

1. $[\lambda^0]p(\lambda) = \lambda_1 \dots \lambda_n = \det A$, the determinant of A;
2. $[\lambda^{n-1}]p(\lambda) = -(\lambda_1 + \dots + \lambda_n) = -(A_{11} + \dots + A_{nn}) = -\operatorname{tr} A$, the trace of A.

Complex numbers Are numbers of form $a + bi \in \mathbb{C}$, where $i^2 = -1$.
Algebraically closed field: Every polynomial can be factored in linear factors.

In-class exercises

1. For what values of $a, b, c \in \mathbb{R}$ is the determinant zero?

$$A = \begin{bmatrix} 0 & 1 & 0 & 4 & c \\ a & 5 & 0 & 4 & -1 \\ 2 & 1 & b & -1 & -3 \\ 0 & -2 & 0 & 1 & 0 \\ 0 & -4 & 0 & 3 & 1 \end{bmatrix}$$

2. Find the determinant by performing Gaussian elimination:

$$B = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & 0 \\ -1 & -2 & 2 \end{bmatrix}$$

3. How to find adjugate matrix?

4. When n is odd, any $n \times n$ matrix has a real eigenvalue, why?

5. Example of a matrix that doesn't have eigenbasis.