

**Steinitz exchange lemma** Let  $V$  be a vector space, and  $L, S \subset V$  are finite such that

1. Elements of  $L$  are linearly independent;
2. The span of  $S$  is  $V$ .

Then,  $|L| \leq |S|$  and there is a set  $S' \subset S$  such that  $|S'| = |S| - |L|$  and the span of  $L \cup S'$  is  $V$ .

**Proof** Consider the sequence  $T = S_1, \dots, S_{|S|}$ . We do the following for each of  $L_1, \dots, L_{|L|}$ :

1. Append  $L_i$  to the beginning of  $T$ ;
2. Remove the last vector of  $T$  that is a linear combination of preceding vectors.

Neither of the steps changes the span of  $T$  (Lemma 1.23). Therefore,  $L_i \in \text{span } T$ , and adding  $L_i$  to  $T$  creates a linear dependence in  $T$ . This ensures that  $T$  has a dependent vector (Lemma 1.19), and it is not a vector of  $L$ , because they are only preceded by other vectors of  $L$ .

In the end,  $T$  will consist of  $L$  and  $S' \subset S$  such that  $|S'| = |S| - |L|$ , and  $\text{span } L \cup S' = \text{span } T = V$ .