Topics of the week

1. Scalar product, length, cosine formula;
2. Cauchy-Schwarz inequality, triangle inequality, perpendicular vectors;
3. Define linear independence of vectors in three different ways;
4. Work with the span of vectors.

Real vector space is a set V with $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$, such that

1. V is a commutative group over $+$;
2. Compatibility of scalar multiplication: $\alpha(\beta\vec{x}) = (\alpha\beta)\vec{x}$;
3. Scalar identity: $1\vec{x} = \vec{x}$;
4. Distributivities: $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ and $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$.

Consequence: $0\vec{x} = \vec{0}$.

Inner product is a map $\cdot : V \times V \to \mathbb{R}$, such that

1. Symmetry: $\vec{x} \cdot \vec{y} = \vec{y} \cdot \vec{x}$;
2. Linearity: $(a\vec{x} + b\vec{y}) \cdot \vec{z} = a(\vec{x} \cdot \vec{z}) + b(\vec{y} \cdot \vec{z})$;
3. Positive-definite: $\vec{x} \cdot \vec{x} \geq 0$ and $\vec{x} \cdot \vec{x} = 0 \iff \vec{x} = 0$.

The standard scalar (dot) product of $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ is $a_1b_1 + \dots + a_nb_n$.

Norm is a map $\| \cdot \| : V \to \mathbb{R}$, such that

1. Positive-definite: $\|\vec{x}\| \geq 0$ and $\|\vec{x}\| = 0 \iff \vec{x} = \vec{0}$;
2. Absolute homogeneity: $\|\lambda\vec{x}\| = |\lambda| \cdot \|\vec{x}\|$;
3. Triangle inequality: $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$.

Distance between two vectors: $\|\vec{x} - \vec{y}\|$. In inner product spaces, $\|\vec{x}\| = \sqrt{\vec{x} \cdot \vec{x}}$.

Cauchy-Schwarz inequality For any $\vec{x}, \vec{y} \in V$, we have

$$|\vec{x} \cdot \vec{y}| \leq \|\vec{x}\| \cdot \|\vec{y}\|$$

Proof: Analyze $f(t) = \|\vec{x} - t\vec{y}\|^2$.
What does it mean geometrically?

Angle between vectors Due to Cauchy-Schwarz inequality, we can define θ from

$$\vec{x} \cdot \vec{y} = \|x\|\|y\| \cos\theta$$

Triangle inequality For any $\vec{x}, \vec{y} \in V$, we have

$$\|x + y\| \leq \|x\| + \|y\|$$

In-class exercises

1. Let $\vec{0} \in \mathbb{R}^m$ denote the vector whose entries are all zero. We say that a set L is a line in $\mathbb{R}^m$ if and only if there exists $\vec{w} \in \mathbb{R}^m$ with $\vec{w} \neq 0$ such that $L = \{\lambda\vec{w} : \lambda \in \mathbb{R}\}$. Let now L be a line in $\mathbb{R}^m$ and let $\vec{u}$ be an arbitrary nonzero element of L. Prove $L = \{\lambda\vec{u} : \lambda \in \mathbb{R}\}$.
2. For two lines L_1 and L_2 in $\mathbb{R}^m$, prove that we have $L_1 \cap L_2 = \{\vec{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.
3. There are three points $A, B, C \in \mathbb{R}^n$. You need to go from A to B without getting closer to C than r. What is the length of the shortest path that satisfies this?
4. Show that the following operations do not change the span:
   (a) Multiplying any vector with a non-zero scalar;
   (b) Adding one vector to another;
5. We have a set of vectors $\vec{v}_1, \dots, \vec{v}_n$. How to find $\lambda_1, \dots, \lambda_n$ such that

$$\lambda_1\vec{v}_1 + \dots + \lambda_n\vec{v}_n = \vec{v}$$

   for a given $\vec{v}$? Assume that $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$. What if it does not hold?
6. Given a set $\vec{v}_1, \dots, \vec{v}_n$, find $\vec{u}_1, \dots, \vec{u}_n$ that is orthogonal and has the same span.

Only first two exercises were actually covered.