Topics of the week

1. explain the concept of a vector space;
2. give examples that are not $\mathbb{R}^m$;
3. define and identify subspaces;
4. explain when vectors span a subspace / form a basis of it;
5. prove that every basis has the same number of vectors;
6. define the dimension of a vector space;
7. find a basis for a given vector space / subspace;

Real vector space is a set V with $+ : V \times V \to V$ and $\cdot : \mathbb{R} \times V \to V$, such that

1. V is a commutative group over $+$;
2. Compatibility of scalar multiplication: $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$;
3. Scalar identity: $1\mathbf{x} = \mathbf{x}$;
4. Distributivities: $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ and $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.

Examples: Polynomials, functions, matrices, linear recurrences, linear ODE solutions.

Hamel basis of a vector space is a maximal set of linearly independent vectors.
Equivalent definition: $\mathbf{e}_1, \dots, \mathbf{e}_n$ s.t. any $\mathbf{v} \in V$ can be *uniquely* represented as

$$\mathbf{v} = v_1\mathbf{e}_1 + \dots + v_n\mathbf{e}_n.$$

The numbers $(v_1, \dots, v_n)$ are called **coordinates** of $\mathbf{v}$ in $\mathbf{e}_1, \dots, \mathbf{e}_n$.
Equivalent definition: A minimal $\mathbf{e}_1, \dots, \mathbf{e}_n$ s.t. any $\mathbf{v} \in V$ has unique coordinates.
Note: In the case of infinite bases, only finite linear combinations are considered.

Dimension of a vector space is the size of its basis. All bases have the same size.

Subspace is a non-empty subset of V, closed under linear operations.

In-class exercises

1. Prove that $H = \{\mathbf{v} \cdot \mathbf{d} = 0 : \mathbf{v} \in \mathbb{R}^m\}$ is a subspace of $\mathbb{R}^m$.
2. Prove that dimension of H is $m - 1$.
3. Consider the vector space V of real-valued functions on $[0, 1]$.
   Prove that $U = \{f \in V : f(x) = f(1 - x)\}$ is a subspace of V.