Topics of the week

1. Orthogonal vectors, Orthonormal vectors, Orthonormal bases;
2. Orthogonal Matrices. Orthogonal matrices preserve norm and inner-product;
3. Projections with orthonormal bases;
4. Build an orthonormal basis with Gram-Schmidt (and show correctness of Gram-Schmidt)
5. QR decomposition. Projections and least squares with QR decomposition.

Orthogonal vectors means pairwise orthogonal.
Let $e_1, \dots, e_n$ be orthogonal and $v = x_1 e_1 + \dots + x_n e_n$, then $e_i \cdot v = x_i(e_i \cdot e_i) \iff x_i = \frac{v \cdot e_i}{e_i \cdot e_i}$.

Orthonormal vectors means orthogonal + each vector has a unit length.
Let $e_1, \dots, e_n$ be orthonormal and $v = x_1 e_1 + \dots + x_n e_n$, then $x_i = e_i \cdot v$.

Orthonormal basis is a basis consisting of orthonormal vectors.

Gram matrix For $v_1, \dots, v_n$:

$$\Gamma(v_1, \dots, v_n) = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \dots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \dots & v_n \cdot v_n \end{bmatrix} = A^\top A$$

The columns of A are orthogonal $\iff$ $A^\top A$ is diagonal.
The columns of A are orthonormal $\iff$ $A^\top A = I$.

Interpretation of projection $A^\top b$ are scalar products of b with all basis vectors of $C(A)$. Then, $(A^\top A)^{-1}$ recovers a vector of $C(A)$ from its dot products with the basis of $C(A)$. $(A^\top A)^{-1}$ allows to consider coordinate system in dot products with basis vectors, rather than in their linear combinations.

Dual basis is the basis of $C(A)$ formed of the columns of $A(A^\top A)^{-1}$.

Orthogonal matrix is a matrix that preserves distances: $\|Ax - Ay\| = \|x - y\| \iff A^\top A = I$.
In other words, matrix is orthogonal $\iff$ it maps $e_1, \dots, e_n$ into orthonormal basis.

Projection with orthonormal basis Assume $A^\top A = I$, then the projection of y on Ax is $AA^\top y$.

Gram-Schmidt process construct an orthonormal basis in $C(A)$ by subtracting projections.

QR decomposition $A = QR$, where Q is the result of Gram-Schmidt process, and $R = Q^\top A$.
Note: $AM = Q \implies A = QM^{-1} \implies R = M^{-1}$.

Project with QR decomposition $\text{proj}_{C(A)}(b) = QQ^\top b$.

In-class exercises Consider the invertible matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

1. Apply the Gram-Schmidt process to the columns of A.
2. Write down a QR-decomposition of A.
3. Apply the Gram-Schmidt process to the columns of B.
4. Is it always true that the Gram-Schmidt process on the columns of an upper triangular $n \times n$ matrix with non-zero diagonal entries yields the canonical basis $e_1, \ldots, e_n$? Provide a proof or counterexample.