Steinitz exchange lemma Let V be a vector space, and $L, S \subset V$ are finite such that

1. Elements of L are linearly independent;
2. The span of S is V.

Then, $|L| \leq |S|$ and there is a set $S' \subset S$ such that $|S'| = |S| - |L|$ and the span of $L \cup S'$ is V.

Proof Consider the sequence $T = S_1, \ldots, S_{|S|}$. We do the following for each of $L_1, \ldots, L_{|L|}$:

1. Append L_i to the beginning of T;
2. Remove the last vector of T that is a linear combination of preceding vectors.

Neither of the steps changes the span of T (Lemma 1.23). Therefore, $L_i \in \operatorname{span} T$, and adding L_i to T creates a linear dependence in T. This ensures that T has a dependent vector (Lemma 1.19), and it is not a vector of L, because they are only preceded by other vectors of L.

In the end, T will consist of L and $S' \subset S$ such that $|S'| = |S| - |L|$, and $\operatorname{span} L \cup S' = \operatorname{span} T = V$.