W1: Scalar products, lengths and angles

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Linear combinations of vectors (a)

Statement

Prove that every $\mathbf{v} \in \mathbb{R}^2$ is a lincomb of $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Need to find λ and μ s.t. $\mathbf{v} = \lambda\mathbf{u} + \mu\mathbf{w}$.
Solving the system yields $\lambda = \frac{v_1+v_2}{2}$ and $\mu = \frac{v_2-v_1}{2}$.
Beware of arithmetic mistakes!

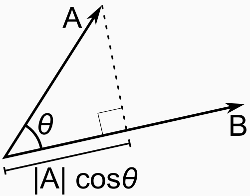
Linear combinations of vectors (b)

Statement

For $\mathbf{v} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, find $\mathbf{u} \in \mathbb{R}^3$ that isn't their lincomb.

$\lambda \mathbf{v} + \mu \mathbf{w} = \mathbf{u}$ means that $\lambda = u_1$, $\mu = u_2$ and $\lambda + \mu = u_3$. We can use any vector s.t. $u_1 + u_2 \neq u_3$ as an example, e.g. $(1, 1, 1)$.
Note: If problem asks for example, provide it, not just show it exists.

Geometric definitions



Definitions

- **Length** $\|\mathbf{v}\|$ of $\mathbf{v}$ is its “magnitude”.
- **Vector projection** of $\mathbf{u}$ on $\mathbf{v}$ is $t\mathbf{v}$ s.t. $\|\mathbf{u} - t\mathbf{v}\| \to \min.$
- **Scalar product** $\mathbf{u} \cdot \mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is $\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$.
- Vectors are **orthogonal** if $\theta = 90°$.

Core properties

Normed space

- Positive-definite: $\|\mathbf{v}\| \geq 0$ and $\|\mathbf{v}\| = 0 \iff \mathbf{v} = \mathbf{0}$
- Homogeneity: $\|\alpha\mathbf{v}\| = |\alpha| \cdot \|\mathbf{v}\|$
- Triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Inner product space

- Commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
- Bilinear: $(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{w} = \alpha(\mathbf{u} \cdot \mathbf{w}) + \beta(\mathbf{v} \cdot \mathbf{w})$
- Positive-definite: $\mathbf{v} \cdot \mathbf{v} \geq 0$ and $\mathbf{v} \cdot \mathbf{v} = 0 \iff \mathbf{v} = \mathbf{0}$

In $\mathbb{R}^n$, there are standard norms and scalar products.

Unit vectors

A **unit vector** is a vector of length 1. The **standard unit vectors** of $\mathbb{R}^n$:

$$\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Any $\mathbf{v} \in \mathbb{R}^n$ is a lincomb of the standard vectors with its coordinates:

$$\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n$$

We may use $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ for $n \leq 3$.

Algebraic definition

It is natural to assume $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ and $\mathbf{e}_i \cdot \mathbf{e}_i = 1$. From this:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (u_1\mathbf{e}_1 + \cdots + u_n\mathbf{e}_n) \cdot (v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) \\ &= \sum_{i=1}^{n} u_i v_i (\mathbf{e}_i \cdot \mathbf{e}_i) + \sum_{i \neq j} u_i v_j (\mathbf{e}_i \cdot \mathbf{e}_j) \end{aligned}$$

Definition

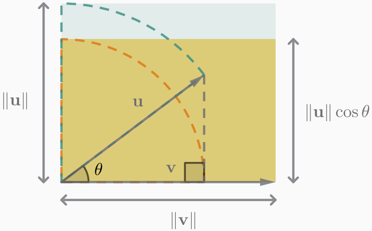
- The **scalar product** of $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is the component-wise sum:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

- The **length** of $\mathbf{v} \in \mathbb{R}^n$ is $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$
- $\mathbf{u}$ and $\mathbf{v}$ are **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$

$\mathbf{v}$ can be decomposed in *length* $\|\mathbf{v}\|$ and (unit) *direction vector* $\frac{\mathbf{v}}{\|\mathbf{v}\|}$.

Cauchy-Schwarz inequality

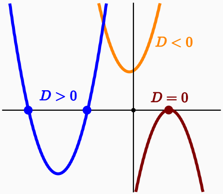


$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$$

Allows to define the **angle** between vectors:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{\mathbf{u}}{\|\mathbf{u}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|}$$

Inequality proof



Let $f(t) = \|\mathbf{u} - t\mathbf{v}\|^2$. It may have at most one zero (when $\mathbf{u} = t\mathbf{v}$), but

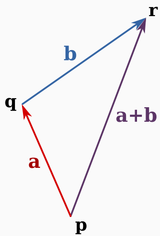
$$f(t) = (\mathbf{u} - t\mathbf{v}) \cdot (\mathbf{u} - t\mathbf{v}) = \|\mathbf{u}\|^2 - 2t(\mathbf{u} \cdot \mathbf{v}) + t^2\|\mathbf{v}\|^2,$$

so it is quadratic in t. Its number of zeros is determined by the sign of

$$D = 4(\mathbf{u} \cdot \mathbf{v})^2 - 4\|\mathbf{u}\|^2 \cdot \|\mathbf{v}\|^2 \leq 0$$

Moreover, $D = 0 \iff |\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \iff \mathbf{u} = t\mathbf{v}$.

Triangle inequality



$$\|\mathbf{u}+\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$$

The inequality directly follows from the Cauchy-Schwarz inequality:

$$\|\mathbf{u}+\mathbf{v}\|^2 \leq \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2|\mathbf{u}\cdot\mathbf{v}| \leq (\|\mathbf{u}\| + \|\mathbf{v}\|)^2$$

Scalar product from norm

We saw that $\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Can we define scalar product from norms?

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 \pm 2(\mathbf{u} \cdot \mathbf{v}) \iff \mathbf{u} \cdot \mathbf{v} = \frac{\|\mathbf{u} + \mathbf{v}\|^2 - \|\mathbf{u} - \mathbf{v}\|^2}{4}$$

Most norms aren't induced by scalar product, e.g. $\mathbb{R}^2$ with $\| \cdot \|_1$-norm:

$$\mathbf{e}_x \cdot \mathbf{e}_y = \frac{\|(1,1)\|_1^2 - \|(1,-1)\|_1^2}{4} = \frac{2-2}{4} = 0,$$

which is incompatible with the fact that $\mathbf{e}_x \neq \mathbf{e}_y$. **Criterion**:

$$\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 = \frac{\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2}{2}$$

Also called the **parallelogram law**.

Linear independence

Definitions

- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are **linearly independent** if neither is lincomb of others.
- $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Equivalent statements

- $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent
- There is no *non-trivial* combination $\lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n = \mathbf{0}$
- None of the vectors is a linear combination of the *previous ones*
- Any vector in $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is produced by a unique combination

Span properties

- $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}) = \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n)$ for $\mathbf{v} = \lambda_1 \mathbf{v}_1 + \cdots + \lambda_n \mathbf{v}_n$
- $\operatorname{span}(\dots, \lambda_i \mathbf{v}_i, \dots) = \operatorname{span}(\dots, \mathbf{v}_i, \dots)$ for $\lambda_i \neq 0$
- $\operatorname{span}(\dots, \mathbf{v}_i + \mathbf{v}_j, \dots) = \operatorname{span}(\dots, \mathbf{v}_i, \dots)$ for $i \neq j$

Can use to "normalize" a set of vectors without changing their span.

Linear independence

Statement

Show that the following vectors are linearly independent:

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

Solution

The j-th vector has 1 in the j-th component, while vectors before it have 0, so j-th vector can't be a linear combination of the previous ones.

Lines in $\mathbb{R}^m$ (a)

Statement

$L \subseteq \mathbb{R}^m$ is a *line through origin*[1] if there is $\mathbf{w} \neq \mathbf{0}$ s.t. $L = \{\lambda\mathbf{w} : \lambda \in \mathbb{R}\}$. For $\mathbf{u} \in L$ and $\mathbf{u} \neq \mathbf{0}$, prove that $L = \{\lambda\mathbf{u} : \lambda \in \mathbb{R}\}$.

[1] we will just call them lines, unless noted otherwise

Solution

How to prove $A = B$ for sets:

- Show that $A \subseteq B$ and $B \subseteq A$
- Show that $A = C$ and $B = C$ for some C

Let $\mathbf{u} = \mu\mathbf{w}$ and $L' = \{\lambda\mathbf{u} : \lambda \in \mathbb{R}\}$. We need to show $L = L'$.

$L' \subseteq L$

Let $\mathbf{v} = \lambda\mathbf{u}$, then $\mathbf{v} = \lambda(\mu\mathbf{w}) = (\lambda\mu)\mathbf{w}$.

$L \subseteq L'$

Let $\mathbf{v} = \lambda\mathbf{w}$, then $\mathbf{v} = \lambda\left(\frac{1}{\mu}\mathbf{u}\right) = \frac{\lambda}{\mu}\mathbf{u}$, because $\mathbf{w} = \frac{1}{\mu}\mathbf{u}$.

In the second part, $\mu \neq 0$ because it would imply $\mathbf{u} = \mathbf{0}$.

Lines in $\mathbb{R}^m$ (b)

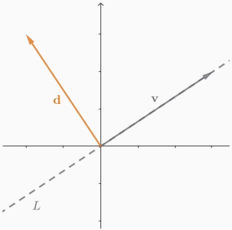
Statement

For lines L_1 and L_2, prove that $L_1 \cap L_2 = \{\mathbf{0}\}$ or $L_1 \cap L_2 = L_1 = L_2$.

Solution

Note that $\mathbf{0} \in L_1 \cap L_2$ always, because $\lambda = 0$ produces $\mathbf{0}$.
Now, assume there is something besides $\mathbf{0}$, that is, there is $\mathbf{w} \in L_1 \cap L_2$, s.t. $\mathbf{w} \neq \mathbf{0}$. Then, by ex. (a), we get $L_1 = \{\lambda \mathbf{w} : \lambda \in \mathbb{R}\} = L_2$.

Lines in $\mathbb{R}^m$ (c)



Statement

Consider a line $L \subseteq \mathbb{R}^2$. Prove that L is a hyperplane, i.e. there is $\mathbf{d} \neq \mathbf{0}$ s.t. $L = \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v} \cdot \mathbf{d} = 0\}$.

Solution

Let $L = \{\lambda\mathbf{v} : \lambda \in \mathbb{R}\}$ and $L' = \{\mathbf{u} \in \mathbb{R}^2 : \mathbf{u} \cdot \mathbf{d} = 0\}$.
For such $\mathbf{d}$ to exist, it must be that $\mathbf{d} \cdot \lambda\mathbf{v} = 0$. In coordinate form:

$$\lambda(v_x d_x + v_y d_y) = 0$$

One possible solution is $\mathbf{d} = (-v_y, v_x)$.

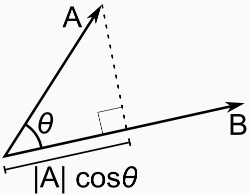
$L \subseteq L'$

We picked $\mathbf{d}$ in a way that $\mathbf{d} \cdot \lambda\mathbf{v} = 0$, so $\lambda\mathbf{v} \in L'$ for any $\lambda \in \mathbb{R}$.

$L' \subseteq L$

Let $\mathbf{d} \cdot \mathbf{u} = 0$, then $-u_x v_y + u_y v_x = 0$.
Without loss of generality, $v_x \neq 0$, then $u_y = \frac{u_x}{v_x} v_y$ and $u_x = \frac{u_x}{v_x} v_x$,
meaning that $\mathbf{u} = \frac{u_x}{v_x}\mathbf{v}$.

Orthogonal projection



Statement

Given $\mathbf{u}, \mathbf{v} \in V$, find t s.t. $\|\mathbf{u} - t\mathbf{v}\| \to \min.$

Solution

Recall Cauchy-Schwarz proof, we had

$$f(t) = \|\mathbf{u} - t\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - 2t(\mathbf{u}\cdot\mathbf{v}) + t^2\|\mathbf{v}\|^2$$

Its minimum is found at $f'(t) = 0$, or:

$$-2(\mathbf{u}\cdot\mathbf{v}) + 2t\|\mathbf{v}\|^2 = 0 \iff \boxed{t = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}}$$

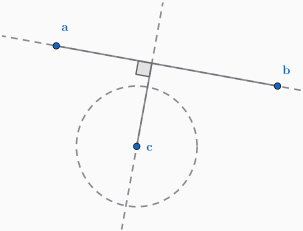
Equivalently, $(\mathbf{u} - t\mathbf{v})\cdot\mathbf{v} = 0$, meaning that $\mathbf{u} - t\mathbf{v}$ is orthogonal to $\mathbf{v}$.

Shortest path with obstacle

Statement

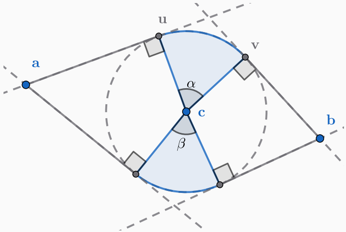
Given points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n$. Find the length of the shortest path from $\mathbf{a}$ to $\mathbf{b}$ that doesn't come to $\mathbf{c}$ closer than r.

Solution



$$\|\mathbf{a} - \mathbf{b}\|$$

Check if $\mathbf{c}$ is far enough with orthogonal projection of $\mathbf{c} - \mathbf{a}$ on $\mathbf{b} - \mathbf{a}$.



$$\|\mathbf{a}-\mathbf{u}\| + r\min(\alpha,\beta) + \|\mathbf{v}-\mathbf{b}\|$$

$$\|\mathbf{a}-\mathbf{u}\| = \sqrt{\|\mathbf{a}-\mathbf{c}\|^2 - r^2} \qquad \|\mathbf{b}-\mathbf{v}\| = \sqrt{\|\mathbf{b}-\mathbf{c}\|^2 - r^2}$$

$$\angle\mathbf{uca} = \arccos\frac{r}{\|\mathbf{a}-\mathbf{c}\|} \qquad \angle\mathbf{bcv} = \arccos\frac{r}{\|\mathbf{b}-\mathbf{c}\|}$$

$$\angle\mathbf{acb} = \arccos\frac{(\mathbf{a}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{c})}{\|\mathbf{a}-\mathbf{c}\|\cdot\|\mathbf{b}-\mathbf{c}\|}$$

$$\min(\alpha,\beta) = \angle\mathbf{acb} - \angle\mathbf{uca} - \angle\mathbf{bcv}$$