

**Topics of the week** Compute with matrices:

1. matrix-vector multiplication, column space, row space, rank;
2. perform matrix multiplication, including matrix- vector, vector-matrix, scalar and outer product, distributivity, associativity.

**Basis** of a vector space is a maximal set of linearly independent vectors.

**Equivalent definition:**  $\mathbf{e}_1, \dots, \mathbf{e}_n$  s.t. any  $\mathbf{v} \in V$  can be *uniquely* represented as

$$\mathbf{v} = v_1 \mathbf{e}_1 + \dots + v_n \mathbf{e}_n.$$

The numbers  $(v_1, \dots, v_n)$  are called **coordinates** of  $\mathbf{v}$  in  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

**Equivalent definition:** A minimal  $\mathbf{e}_1, \dots, \mathbf{e}_n$  s.t. any  $\mathbf{v} \in V$  has unique coordinates.

**Linear transformation** between vector spaces  $V$  and  $W$  is a function  $f : V \rightarrow W$ , such that

$$\begin{aligned} \mathbf{u} + \mathbf{v} &\mapsto f(\mathbf{u}) + f(\mathbf{v}), \\ k\mathbf{v} &\mapsto kf(\mathbf{v}). \end{aligned}$$

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be the basis of  $V$  and  $\mathbf{g}_1, \dots, \mathbf{g}_m$  be the basis of  $W$ . Then,

$$f(\mathbf{v}) = v_1 f(\mathbf{e}_1) + \dots + v_n f(\mathbf{e}_n).$$

So,  $f$  is fully defined by  $f(\mathbf{e}_1), \dots, f(\mathbf{e}_n)$ :

$$f(\mathbf{e}_k) = f_{1k} \mathbf{g}_1 + \dots + f_{mk} \mathbf{g}_m.$$

From this, we fully describe the linear map as

$$\begin{aligned} f(a_1 \mathbf{e}_1 + \dots + a_n \mathbf{e}_n) &= (a_1 f_{11} + \dots + a_n f_{1n}) \mathbf{g}_1 + \\ &\quad (a_1 f_{21} + \dots + a_n f_{2n}) \mathbf{g}_2 + \\ &\quad (a_1 f_{31} + \dots + a_n f_{3n}) \mathbf{g}_3 + \\ &\quad \dots + \\ &\quad (a_1 f_{m1} + \dots + a_n f_{mn}) \mathbf{g}_m \end{aligned}$$

This rewrites as

$$\begin{cases} b_1 = a_1 f_{11} + a_2 f_{12} + \dots + a_n f_{1n}, \\ b_2 = a_1 f_{21} + a_2 f_{22} + \dots + a_n f_{2n}, \\ \dots, \\ b_m = a_1 f_{m1} + a_2 f_{m2} + \dots + a_n f_{mn}. \end{cases}$$

Conventionally, this is written in matrix form as  $b = Fa$ :

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m1} & f_{m2} & \dots & f_{mn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

**Matrix product** is defined in a way that  $AB$  corresponds to the linear map  $\mathbf{x} \mapsto A(B(\mathbf{x}))$ :

The columns of  $A$  are  $A(\mathbf{e}_1), \dots, A(\mathbf{e}_n) \implies$  the columns of  $AB$  are  $A(B(\mathbf{e}_1)), \dots, A(B(\mathbf{e}_n))$ .

### Interpretations of matrix product

1.  $AB$  corresponds to applying  $A$  to each column of  $B$ ;
2.  $AB$  corresponds to applying  $B$  to each row of  $A$ ;
3.  $(AB)_{ij}$  is the dot product of the  $i$ -th row of  $A$  and the  $j$ -th column of  $B$ .

**Dimension** of a vector space is the size of its basis. All bases have the same size.

**Row and column spaces** of a matrix are spans of their rows/columns.

**Matrix rank** is the dimension of its row/column space.

**Lemma 2.21** Let  $A$  be an  $m \times n$  matrix. The following statements are equivalent:

1.  $\text{rank } A \leq k$ ;
2. There are vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$  and  $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$  such that  $A = \sum_{i=1}^k \mathbf{v}_i \mathbf{w}_i^\top$ .
3.  $A = BC$ , where  $B \in \mathbb{R}^{m \times k}$  and  $C \in \mathbb{R}^{k \times n}$  (rank decomposition).

### In-class exercises

1. Let  $m \in \mathbb{N}_{\geq 2}$  be arbitrary and consider the  $m \times m$  matrix

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix}$$

with  $a_{ij} = i + j$  for all  $i, j \in \{1, 2, \dots, m\}$ . Determine the rank of  $A$ .

2. Show that  $(AB)^\top = B^\top A^\top$ .
3. A **conjugate** of  $A$  is a matrix  $A^*$  such that  $\mathbf{x} \cdot A\mathbf{y} = A^* \mathbf{x} \cdot \mathbf{y}$ . Find  $A^*$ .
4. A matrix is **orthogonal** if it preserves distances. When is a matrix orthogonal?

Only the first two exercises were actually covered in the class.