Topics of the week

1. define the nullspace of a matrix;
2. compute a basis for the nullspace of a matrix;
3. solve Ax=b by elimination to REF, read off all solutions, count the number of solutions;
4. define the four fundamental subspaces of a matrix: column space, row space, nullspace, left nullspace;
5. compute their dimensions, depending on shape and rank of the matrix;
6. define orthogonal complement and orthogonal subspaces;
7. prove that nullspace and row space of a matrix are orthogonal;
8. argue about dimensions of two orthogonal subspaces;

Nullspace of $A \in \mathbb{R}^{n\times m}$ is the set $\ker A = \{\mathbf{x} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{0}\} = A^{-1}(\mathbf{0})$.

Nullspace basis in row-echelon form:

1. Classify the variables into "free" and "leading":
   (a) Free variables: Can be assigned any value from $\mathbb{R}$;
   (b) Leading variables: Fully determined from the free variables.
2. Form the basis by setting exactly one free variable to 1, and others to 0, and setting up the leading variables accordingly.

   We have $\dim\ker A = |I|$ and $\dim \operatorname{im} A = \operatorname{rank} A = m - |I|$.

Unorthodox approach: Reduce $(A^\top|I)$ to REF (suboptimal when $m \gg n$).

Solve $Ax = b$ In general, any solution x can be represented as $x = x_0 + y$, where

- x_0 is *any* specific solution to $Ax_0 = b$.
- y is a certain solution to $Ay = 0$.

If at least one solution exists!

In-class exercises Consider a system:

$$\begin{pmatrix} -1 & 2 & 5 & -2 \\ -3 & 3 & 12 & -3 \\ 1 & -14 & -7 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -6 \\ -15 \\ 8 \end{pmatrix}$$

1. Determine the set of solutions $\mathcal{L} = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$.
2. Write down a basis for $\mathbf{N}(A)$ and $\mathbf{C}(A)$.
3. What are the dimensions of $\mathbf{N}(A)$, $\mathbf{C}(A)$, $\mathbf{N}(A^\top)$ and $\mathbf{C}(A^\top)$?
4. Find a basis of $\mathbf{C}(A^\top)$.