## Topics of the week

- 1. explain the CR decomposition;
- 2. linear transformations, visualizing linear transformations in 2d, properties of linear transformations, matrix representation of linear transformations;
- 3. systems of linear equations, systems of linear equations with unique solutions.

Actions on rows/columns AB for  $A \in \mathbb{R}^{n \times m}$  and  $B \in \mathbb{R}^{m \times k}$  is interpreted as:

- Applying matrix A to all columns of B as  $c \mapsto Ac$ ;
- Applying matrix B to all rows of A as  $r \mapsto rB$ ;
- $(AB)_i$  is the linear combination of the rows  $B_1, \ldots, B_m$  with the coefficients from  $A_i$ ;
- $(AB)^j$  is the linear combination of the columns  $A^1, \ldots, A^k$  with the coefficients from  $B^j$ .

Here, lower indices denote rows, and upper indices denote columns.

**Rank decomposition** Let A be an  $m \times n$  matrix. The following statements are equivalent:

- 1. rank  $A \leq k$ ;
- 2. There are vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$  and  $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$  such that  $A = \mathbf{v}_1 \mathbf{w}_1^\top + \dots + \mathbf{v}_k \mathbf{w}_k^\top$ .
- 3. A = CR, where  $C \in \mathbb{R}^{m \times k}$  and  $R \in \mathbb{R}^{k \times n}$ .

When rank A = k, A = CR is called a rank decomposition of A, and

- The columns of C form a basis in the column space of A;
- The columns of R are the coordinates of the columns of A in the basis of the columns of C;
- The rows of R form a basis in the row space of A;
- The rows of C are the coordinates of the rows of A in the basis of the rows of R.

**Systems of linear equations** Can be represented in matrix form. If the solution is unique, it is found by applying transforms on the rows until you get a system with the unit matrix:  $(A|b) \mapsto (I|b')$ .

$$\begin{cases} a_{11}x_{1} + \dots + a_{1n}x_{n} = b_{1}, \\ a_{21}x_{1} + \dots + a_{2n}x_{n} = b_{2}, \\ \dots, \\ a_{n1}x_{1} + \dots + a_{nn}x_{n} = b_{n}. \end{cases} \iff Ax = b \iff \begin{cases} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{cases} \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{n} \end{pmatrix} = \begin{pmatrix} b_{1} \\ b_{2} \\ \vdots \\ b_{n} \end{pmatrix}$$

$$\begin{cases} x_{1} = b'_{1}, \\ x_{2} = b'_{2}, \\ \dots, \\ x_{n} = b'_{n}. \end{cases} \iff A^{-1}Ax = A^{-1}b \iff \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_{1} \\ x_{2} \\ \vdots \\ x_{n} \end{pmatrix} = \begin{pmatrix} b'_{1} \\ b'_{2} \\ \vdots \\ b'_{n} \end{pmatrix}$$

## In-class exercises

- 1. Consider  $T: \mathbb{R}^n \to \mathbb{R}$  for n > 0 defined as  $T(x) = \sum_{k=1}^n kx_k$ . Prove that T is a linear transformation.
- 2. Consider  $T: \mathbb{R}^n \to \mathbb{R}$  for  $n \geq 2$  defined as  $T(x) = \sum_{k=1}^n (x_k)^k$ . Is T(x) a linear transformation?