Topics of the week

1. Characteristic polynomial, algebraic multiplicity, finding eigenvalues and eigenvectors, properties of eigenvalues and eigenvectors;
2. Linear independence of eigenvectors corresponding to distinct eigenvalues;
3. Determinant, trace, and connection to eigenvalues;
4. Eigenvalues and eigenvectors of rotations and other linear transformations;
5. Eigenvalues and eigenvectors of orthogonal matrices;
6. Eigenvalues and eigenvectors of diagonal matrices;
7. Eigenvalues and eigenvectors of projection matrices;
8. Repeated eigenvalues and geometric multiplicity;
9. Linear independence of eigenvectors, complete sets of real eigenvectors;
10. Change of basis, diagonalization, diagonalizable matrices;
11. Similar matrices, eigenvalues of similar matrices.

Change of basis What happens to a matrix when we switch from $e_1, \dots, e_n$ to $b_1, \dots, b_n$?
Consider $v = x$ on the basis $E = (e_1, \dots, e_n)$. We want to find y such that $v = By$, i.e. $y = B^{-1}x$.
Then Ax becomes $B(B^{-1}ABy)$, that is, $A \mapsto B^{-1}AB$.

Similar matrices are matrices A and B s.t. $A = P^{-1}BP$.

Eigenbasis Some basis changes turn A into a nice form, e.g. $A \mapsto \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.
A basis, in which A is diagonal is called an eigenbasis of A.

Eigenvector Any $v \neq 0$ s.t. $Av = \lambda v$ for some λ, called **eigenvalue**.
Correspondingly, eigenbasis is any basis formed by eigenvectors.
Criterion: λ is an eigenvalue $\iff (A - \lambda I)v = 0 \iff \det(A - \lambda I) = 0$.

Characteristic polynomial of the matrix A is $p(\lambda) = \det(\lambda I - A)$.
Characteristic polynomial doesn't change when the basis is changed!
Each coefficient stays invariant under the change of the basis, in particular:

1. $[\lambda^0]p(\lambda) = (-1)^n \lambda_1 \dots \lambda_n = (-1)^n \det A$, the **determinant** of A;
2. $[\lambda^{n-1}]p(\lambda) = -(\lambda_1 + \dots + \lambda_n) = -(A_{11} + \dots + A_{nn}) = -\operatorname{tr} A$, the **trace** of A.

Every matrix has at least one (complex) eigenvalue and eigenvector.

Linear independence Eigenvector is not a linear combination of eigenvectors with other eigenvalues.

$$\begin{aligned} v_{n+1} = \alpha_1 v_1 + \dots + \alpha_n v_n &\implies \lambda v_{n+1} = \alpha_1 \lambda_1 v_1 + \dots + \alpha_n \lambda_n v_n \\ &\implies \alpha_1(\lambda_1 - \lambda)v_1 + \dots + \alpha_n(\lambda_n - \lambda)v_n = 0 \end{aligned}$$

Corollary: All eigenvalues distinct $\implies$ there is an eigenbasis.

Eigenvalues of rotations Q is orthogonal $\implies |\lambda| = 1$.
Because $\|v\| = \|Qv\| = \|\lambda v\| = |\lambda| \cdot \|v\|$.

Eigenvalues of projections Let $P^2 = P$, then $\lambda \in \{0, 1\}$.
The basis of $\operatorname{im} P$ has eigenvalues 1, and the basis of $\ker P^\top$ has eigenvalues 0.
Note: Also true in oblique projections because eigenvalues of P^k are λ^k.

Algebraic multiplicity is the multiplicity of $\lambda - \lambda_i$ in $p(\lambda)$.

Geometric multiplicity is $\dim \ker(A - \lambda I)$.
There is an egeinbasis $\iff$ algebraic and geometric multiplicities are the same for all eigenvalues.

Diagonalization With an eigenbasis, we can factorize $A = P \Lambda P^{-1}$, where $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.
Note: Columns of P are the eigenvectors.

In-class exercises Let $A, B \in \mathbb{R}^{n \times n}$.

1. Let M be such that $M \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$, find all eigenvalues and eigenvectors of M;
2. Construct a matrix with eigenvalues $0, 1, 2$;
3. Construct a non-diagonal matrix with eigenvalues $0, 1, 2$;
4. Prove that AB and BA have the same eigenvalues;
5. Assume that B is invertible and AB has an eigenbasis, prove that BA has an eigenbasis;
6. Assume that both A and B are invertible, prove that AB has an eigenbasis $\iff$ BA has an eigenbasis;
7. Find A and B such that BA has an eigenbasis, but AB doesn't.