

**Topics of the week**

1. derive and explain the LU factorization from elimination;
2. compute the REF and RREF of a given  $m \times n$  matrix  $A$ , explain why it equals  $R$  in  $A = CR$ ;

**LUP factorization** Do Gaussian elimination and reduce  $A$  to upper-triangular form  $U$ :

$$M_n \dots M_1 A = U \iff A = (M_1^{-1} \dots M_n^{-1})U$$

To recover  $L$ , keep a matrix  $B = I$ . Whenever applying  $M_i$  to  $A$ , apply  $M_i^{-\top}$  to  $B$ . In the end,

$$B = M_n^{-\top} \dots M_1^{-\top} \iff B^\top = M_1^{-1} \dots M_n^{-1}$$

So, we get the decomposition  $A = B^\top U$ . If  $B$  is not lower-triangular, its rows can be sorted such that  $L = PB^\top$  is lower-triangular. Thus,  $PA = LU$ .

**REF and RREF** Gauss-Jordan elimination turns a matrix into the row-echelon form:

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$

Reduced row-echelon form: Remove zero rows. Other conventions:

1. REF: Not necessary to keep pivots 1 and elements above them 0;
2. RREF: What we call REF in this course.

**In-class exercises**

1. Elimination, back substitution, LU factorization:

(a) Compute an LU factorization of the matrix

$$A = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix}$$

(b) For the factorization above, solve  $Ly = b$  with

$$b = \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix}$$

(c) For the  $y$  from the previous exercise, solve  $Ux = y$ .

(d) Prove that  $Ax = b$ .

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix}$$

with  $CR$ -decomposition

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} = CR$$

Let  $R_0$  be the matrix in row echelon form obtained by performing Gauss-Jordan elimination on  $A$ . Determine  $R_0$  by performing the elimination and verify Theorem 3.24 by checking that  $R$  corresponds to the non-zero rows of  $R_0$ .