

## Topics of the week

1. explain the concept of a vector space;
2. give examples that are not  $\mathbb{R}^m$ ;
3. define and identify subspaces;
4. explain when vectors span a subspace / form a basis of it;
5. prove that every basis has the same number of vectors;
6. define the dimension of a vector space;
7. find a basis for a given vector space / subspace;

**Real vector space** is a set  $V$  with  $+: V \times V \rightarrow V$  and  $\cdot: \mathbb{R} \times V \rightarrow V$ , such that

1.  $V$  is a commutative group over  $+$ ;
2. Compatibility of scalar multiplication:  $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$ ;
3. Scalar identity:  $1\mathbf{x} = \mathbf{x}$ ;
4. Distributivities:  $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$  and  $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ .

Examples: Polynomials, functions, matrices, linear recurrences, linear ODE solutions.

**Hamel basis** of a vector space is a maximal set of linearly independent vectors.

**Equivalent definition:**  $\mathbf{e}_1, \dots, \mathbf{e}_n$  s.t. any  $\mathbf{v} \in V$  can be *uniquely* represented as

$$\mathbf{v} = v_1\mathbf{e}_1 + \dots + v_n\mathbf{e}_n.$$

The numbers  $(v_1, \dots, v_n)$  are called **coordinates** of  $\mathbf{v}$  in  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

**Equivalent definition:** A minimal  $\mathbf{e}_1, \dots, \mathbf{e}_n$  s.t. any  $\mathbf{v} \in V$  has unique coordinates.

**Note:** In the case of infinite bases, only finite linear combinations are considered.

**Dimension** of a vector space is the size of its basis. All bases have the same size.

**Subspace** is a non-empty subset of  $V$ , closed under linear operations.

## In-class exercises

1. Prove that  $H = \{\mathbf{v} \cdot \mathbf{d} = 0 : \mathbf{v} \in \mathbb{R}^m\}$  is a subspace of  $\mathbb{R}^m$ .
2. Prove that dimension of  $H$  is  $m - 1$ .
3. Consider the vector space  $V$  of real-valued functions on  $[0, 1]$ .  
Prove that  $U = \{f \in V : f(x) = f(1 - x)\}$  is a subspace of  $V$ .