Topics of the week

1. explain the CR decomposition;
2. linear transformations, visualizing linear transformations in 2d, properties of linear transformations, matrix representation of linear transformations;
3. systems of linear equations, systems of linear equations with unique solutions.

Actions on rows/columns AB for $A \in \mathbb{R}^{n\times m}$ and $B \in \mathbb{R}^{m\times k}$ is interpreted as:

- Applying matrix A to all columns of B as $c \mapsto Ac$;
- Applying matrix B to all rows of A as $r \mapsto rB$;
- $(AB)_i$ is the linear combination of the rows $B_1, \dots, B_m$ with the coefficients from A_i;
- $(AB)^j$ is the linear combination of the columns $A^1, \dots, A^k$ with the coefficients from B^j.

Here, lower indices denote rows, and upper indices denote columns.

Rank decomposition Let A be an $m \times n$ matrix. The following statements are equivalent:

1. $\operatorname{rank} A \le k$;
2. There are vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^m$ and $\mathbf{w}_1, \dots, \mathbf{w}_k \in \mathbb{R}^n$ such that $A = \mathbf{v}_1\mathbf{w}_1^\top + \dots + \mathbf{v}_k\mathbf{w}_k^\top$.
3. $A = CR$, where $C \in \mathbb{R}^{m\times k}$ and $R \in \mathbb{R}^{k\times n}$.

When $\operatorname{rank} A = k$, $A = CR$ is called a rank decomposition of A, and

- The columns of C form a basis in the column space of A;
- The columns of R are the coordinates of the columns of A in the basis of the columns of C;
- The rows of R form a basis in the row space of A;
- The rows of C are the coordinates of the rows of A in the basis of the rows of R.

Systems of linear equations Can be represented in matrix form. If the solution is unique, it is found by applying transforms on the rows until you get a system with the unit matrix: $(A|b) \mapsto (I|b')$.

$$\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + \dots + a_{2n}x_n = b_2, \\ \dots, \\ a_{n1}x_1 + \dots + a_{nn}x_n = b_n. \end{cases} \iff Ax = b \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{cases} x_1 = b'_1, \\ x_2 = b'_2, \\ \dots, \\ x_n = b'_n. \end{cases} \iff A^{-1}Ax = A^{-1}b \iff \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix}$$

In-class exercises

1. Consider $T : \mathbb{R}^n \to \mathbb{R}$ for $n > 0$ defined as $T(x) = \sum\limits_{k=1}^{n} kx_k$. Prove that T is a linear transformation.
2. Consider $T : \mathbb{R}^n \to \mathbb{R}$ for $n \ge 2$ defined as $T(x) = \sum\limits_{k=1}^{n} (x_k)^k$. Is $T(x)$ a linear transformation?