Topics of the week Compute with matrices:

1. do elimination and back substitution on square systems of linear equations, explain when and why this works or fails;
2. define the inverse of a matrix, compute inverses of 2×2 matrices, characterize when the inverse exists (The Inverse Theorem), invert a product of matrices and the transpose of a matrix.

CR decomposition $A = CR$, where C are *the* independent columns of A.

Elementary operations

- $\boxed{A_i \leftarrow A_i - cA_j}$ for $j \neq i$ and $c \in \mathbb{R}$;
- $\boxed{A_i \leftarrow cA_i}$ for any $c \neq 0$;
- $\boxed{A_i, A_j \leftarrow A_j, A_i}$ for any $i \neq j$;

Can be represented by multiplying with a matrix on the left (see week 3).

Systems of linear equations $Ax = b$. Solve by applying row operations to A and b until $A = I$.

Inverse matrix A^{-1} s.t. $AA^{-1} = A^{-1}A = I$. Unique if exists. How to find?

- Solve $A(A^{-1}e_k) = e_k$ for all e_k to find the k-th column of A^{-1}.
- Do elementary row operations on I and A simultaneously: $(A, I) \mapsto (I, A^{-1})$.

Determinant The volume of a hypercube spanned on $(v_1, \dots, v_n)$. Properties:

1. $V(e_1, \dots, e_n) = 1$.
2. $V(a + b, \dots) = V(a, \dots) + V(b, \dots)$.
3. $V(\alpha v, \dots) = \alpha V(\dots)$.
4. $V(a, b, \dots) = -V(b, a, \dots)$.

Inverse of 2×2 Try to derive Cramer's rule.

In-class exercises

1. Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree at most 2, i.e. $p(x) = ax^2 + bx + c$.
   Find a, b and c s.t. $p(-1) = 0$, $p(0) = 2$ and $p(1) = 2$.

$$\begin{pmatrix} (-1)^2 & -1 & 1 \\ 0^2 & 0 & 1 \\ 1^2 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

2. Assume that $Ax = b$ doesn't have an exact solution. Find x that minimizes $\|Ax - b\|$.