Topics of the week

1. Spectral theorem: eigenvalues and eigenvectors of symmetric matrices
2. Rayleigh quotients and their connection to eigenvalues of symmetric matrices
3. Positive definite matrices, positive semidefinite matrices, Gram matrices
4. Cholesky decomposition
5. Singular value decomposition (SVD), derivation of singular value decomposition, connection to eigenvalue decomposition of A^TA and AA^T, compact form of singular value decomposition
6. Singular values, left singular vectors, right singular vectors.

Invariant subspace of $A : V \to V$ is a subspace $U \subset V$ s.t. $u \in U \implies A(u) \in U$.
Note: Any eigensubspace is invariant.
Note: If U is an invariant subspace of A, then $U^\perp$ is an invariant subspace of $A^\top$:

$$y \in U^\perp \iff \forall x \in U : x \cdot y = 0 \implies \forall x \in U : Ax \cdot y = 0 \iff \forall x \in U : x \cdot A^\top y = 0 \iff A^\top y \in U^\perp$$

In particular, when $A = A^\top$, then $U^\perp$ is an invariant subspace of A.

Reduction of A on its invariant subspace U is a transformation $A' : U \to U$, defined as $A'u = Au$.
Note: If $V = U_1 + U_2$ and $U_1 \cap U_2$, then there is a basis, in which

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 and A_2 are reductions on U_1 and U_2 correspondingly.

Spectral theorem If $A = A^\top$, then A has a real orthonormal eigenbasis.
Proof: Any eigenvalue of A is real. Assume $Av = \lambda v$, then

$$\lambda \|v\|^2 = Av \cdot v = v \cdot Av = \bar{\lambda}\|v\|^2$$

Now, consider eigenvector v and $U = \{kv : k \in \mathbb{R}\}$, then $U^\perp$ is invariant subspace of A. It means that we can consider the reduction of A on $U^\perp$ and inductively find a basis of eigenvectors of A that will all be orthogonal to v.
Note: It means that $A = V\Lambda V^\top = \sum\limits_{i=1}^{n} \lambda_i v_i v_i^\top$, where V has eigenvectors as the columns.

Bilinear form is a function $A(x, y)$ that is linear in x and y.
Note: Any bilinear form can be represented in the coordinate form $A(x, y) = x^\top Ay$.

Quadratic form is $B(x) = A(x, x)$, where $A(x, y)$ is a bilinear form s.t. $A(x, y) = A(y, x)$.
Note: Any quadratic form can be represented as $A(x) = x^\top Ax$, where $A = A^\top$.

Change of basis In matrices, $A \mapsto X^{-1}AY$, in bilinear forms $A \mapsto X^\top AY$.
Note: In particular $A \mapsto V^\top AV$ for quadratic forms. Thus, only for orthonormal change of bases, A changes in the same way as a quadratic form, and as a linear transformation.

Rayleigh quotient Let $A = A^\top$, then the Rayleigh quotient of $x \neq 0$ is

$$R(x) = \frac{x^\top A x}{x^\top x}$$

For Rayleigh quotient, it holds that $R(v_{\min}) = \lambda_{\min}$ and $R(v_{\max}) = \lambda_{\max}$.
Proof: Consider an orthonormal basis in which $A = \Lambda$:

$$R(x) = \frac{\sum \lambda_i x_i^2}{\sum x_i^2} \in [\lambda_{\min}, \lambda_{\max}]$$

Positive (semi)definite matrices A symmetric matrix A is:

1. Positive definite if $x^\top A x > 0$ for all $x \neq 0$;
2. Positive semidefinite if $x^\top A x \geq 0$ for all x;

Equivalently, all eigenvalues are positive/non-negative.

Gram matrix of $v_1, \dots, v_n$ is $V^\top V$, where $V = (v_1, \dots, v_n)$.
Note: Always positive semidefinite, and definite if linearly independent.

Cholesky decomposition A is positive semidefinite $\iff$ $A = C^\top C$, where C is upper triangular.

$$A = (V\sqrt{\Lambda})(V\sqrt{\Lambda})^\top = (QR)^\top (QR) = R^\top R$$

Take $C = R$ from the QR decomposition.

Singular value decomposition is $A = U\Sigma V^\top = \sum\limits_{i=1}^{n} \sigma_i u_i v_i^\top$, where U and V are orthogonal, and Σ is diagonal. Columns of U and V are left (right) singular vectors, and elements of Σ are singular values.

1. U are the eigenvectors of $AA^\top$ and V are the eigenvectors of $A^\top A$;
2. Singular values are the square roots of eigenvalues of $A^\top A$ or $AA^\top$ (they're the same).

In-class exercises

1. Let $S^\top = -S$. Prove that $-S^2$ is symmetric and positive definite.
2. Let A be a matrix of rank 1. Find its singular values.
3. Let $B_{ij} = 1$. Prove that $A = I + B$ is invertible.