W2: Matrix-column mul, rank, transpose

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Orthogonality and Linear independence (a)

Statement

For which $s \in \mathbb{R}$ are $\mathbf{v} = \begin{pmatrix} s \\ 3 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 2 \\ s \end{pmatrix}$ orthogonal?

By definition:

$$\mathbf{v} \cdot \mathbf{w} = s \cdot 1 + 3 \cdot 2 + 2 \cdot s = 0 \iff 3s = -6 \iff s = -2$$

Linear combinations of vectors (b)

Statement

For which $t \in \mathbb{R}$ are $\mathbf{u} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ lin. dep.?

$\mathbf{v}$ and $\mathbf{w}$ are indep., so $\mathbf{u} = \lambda \mathbf{v} + \mu \mathbf{w} = \begin{pmatrix} \mu \\ \lambda \\ \lambda + \mu \end{pmatrix} = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix}$.

Can only happen for $t = 0$. Avoid determinants where they aren't needed.

Linear combinations of vectors (c)

Statement

Show that if $\mathbf{v} \cdot \mathbf{w} = 0$ and $\mathbf{v}, \mathbf{w} \neq \mathbf{0}$, then they are lin. indep.
Find lin. indep. $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ s.t. $\mathbf{v} \cdot \mathbf{w} \neq 0$.

$\mathbf{v} \cdot \mathbf{w} = 0 \implies$ lin. indep.

Assume $\lambda \mathbf{v} + \mu \mathbf{w} = \mathbf{0}$, then:

$$\begin{cases} \lambda \|\mathbf{v}\|^2 = (\lambda \mathbf{v} + \mu \mathbf{w}) \cdot \mathbf{v} = \mathbf{0} \cdot \mathbf{v} = 0 \implies \lambda = 0, \\ \mu \|\mathbf{w}\|^2 = (\lambda \mathbf{v} + \mu \mathbf{w}) \cdot \mathbf{w} = \mathbf{0} \cdot \mathbf{w} = 0 \implies \mu = 0 \end{cases}$$

Hence, only trivial lin. comb. produces $\mathbf{0}$.

lin. indep. $\not\Longrightarrow \mathbf{v} \cdot \mathbf{w} = 0$

E.g. $\mathbf{v} = \mathbf{e}_1$ and $\mathbf{w} = \mathbf{e}_1 + \mathbf{e}_2$, with $\mathbf{v} \cdot \mathbf{w} = 1$.

Exchange lemma

Lemma 1.28

$T = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq \mathbb{R}^m$ are lin. indep. $\implies \operatorname{span}(T) = \mathbb{R}^m$.

Let $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ be s.t. $\operatorname{span} S = \mathbb{R}^m$:

- Append vectors of T to the front of S one by one
- New vector is lin. comb. of S, because $\operatorname{span}(S) = \mathbb{R}^m$
- Delete first vector that is a lin. comb. of the ones before it
- Deleted vector could have only been one of $\mathbf{e}_i$
- After m steps, all vectors are replaced

Can be generalized for any vector space V instead of $\mathbb{R}^m$.

Linear transformation

Definition

Linear transformation is $f: V \to W$ that preserves[1] *linear operations*:

$$f(\mathbf{u}+\mathbf{v}) = f(\mathbf{u}) + f(\mathbf{v})$$
$$f(k\mathbf{v}) = kf(\mathbf{v})$$

Similar to dot products, $f(\mathbf{v})$ is fully defined by $f(\mathbf{e}_1), \dots, f(\mathbf{e}_n)$:

$$f(\mathbf{v}) = f(v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n) = v_1 f(\mathbf{e}_1) + \cdots + v_n f(\mathbf{e}_n)$$

[1] In abstract algebra words, it is a **homomorphism** between vector spaces

Matrix

Definition

An $m \times n$ **matrix** is a table of numbers with m rows and n columns. The set of all $m \times n$ matrices is denoted $\mathbb{R}^{m\times n}$.

Matrices represent linear transformations:

$$f(\mathbf{v}) = \begin{bmatrix} | & & | \\ f(\mathbf{e}_1) & \dots & f(\mathbf{e}_n) \\ | & & | \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1 f(\mathbf{e}_1) + \cdots + v_n f(\mathbf{e}_n)$$

And linear systems of equations:

$$\begin{cases} \mathbf{u}_1 \cdot \mathbf{v} = b_1 \\ \dots \\ \mathbf{u}_m \cdot \mathbf{v} = b_m \end{cases} \iff \begin{bmatrix} - & \mathbf{u}_1^\top & - \\ & \vdots & \\ - & \mathbf{u}_m^\top & - \end{bmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

Matrix-vector multiplication

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & 0 & 3 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \iff (A\mathbf{x})_i = \sum_j A_{ij} x_j$$

Lin. comb. of columns

$$A\mathbf{x} = 1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} 3 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

Scalar products with rows

$$A\mathbf{x} = \begin{pmatrix} \mathbf{u}_1 \cdot \mathbf{x} \\ \mathbf{u}_2 \cdot \mathbf{x} \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 0 \cdot 0 \\ 2 \cdot 1 + 1 \cdot 2 + 0 \cdot 3 + 3 \cdot 0 \end{pmatrix} = \begin{pmatrix} 14 \\ 4 \end{pmatrix}$$

Matrix transformation

Definition

For $A \in \mathbb{R}^{m\times n}$, its **matrix transformation** is the function

$$\begin{aligned} T_A : &\mathbb{R}^n \to \mathbb{R}^m \\ T_A(\mathbf{x}) &= A\mathbf{x} \end{aligned}$$

$T_A(\mathbf{x})_i = \sum_j A_{ij}x_j \implies$ matrix transformations are linear:

$$\begin{aligned} T_A(\mathbf{x}+\mathbf{y})_i = && \sum_j A_{ij}(x_j+y_j) = && T_A(\mathbf{x})_i + T_A(\mathbf{y})_i \\ T_A(k\mathbf{x})_i = && \sum_j A_{ij}kx_j = && kT_A(\mathbf{x})_i \end{aligned}$$

Column space

Definition

The **column space** of $A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix}$ is $\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_n) \subseteq \mathbb{R}^m$.

Equivalently, $\mathbf{C}(A) = \{A\mathbf{x} : \mathbf{x} \in V\}$, also called the **image** of T_A.

Definition

The column $\mathbf{v}_j$ is **independent** if it's not a lin. comb. of $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.
The **column rank** of A is the number of independent columns.

Intuitively, it is the *dimension* of $\mathbf{C}(A)$ (to be defined later).

Transpose

Definition

The **transpose** of $A \in \mathbb{R}^{m \times n}$ is $A^\top \in \mathbb{R}^{n \times m}$ s.t. its columns are the corresponding rows of A, and vice versa.

$$A = \begin{bmatrix} | & & | \\ \mathbf{v}_1 & \dots & \mathbf{v}_n \\ | & & | \end{bmatrix} \iff A^\top = \begin{bmatrix} - & \mathbf{v}_1^\top & - \\ & \vdots & \\ - & \mathbf{v}_n^\top & - \end{bmatrix}$$

In index notation, $A_{ij} = A^\top_{ji}$. “Pure” definition:

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^\top \mathbf{y}$$

Note: Because $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\top \mathbf{y}$ and $(A\mathbf{x})^\top = \mathbf{x}^\top A^\top$.

Row space

Definition

The **row space** of $A \in \mathbb{R}^{m \times n}$ is $\mathbf{R}(A) = \mathbf{C}(A^\top) \subseteq \mathbb{R}^n$.

Definition

The **row rank** of A is the number of independent rows.

Row and column ranks are equal (proven later), hence we write $\operatorname{rank} A$.

Definition

The **nullspace** of A is $\mathbf{N}(A) = \{\mathbf{x} \in V : A\mathbf{x} = \mathbf{0}\}$.

Also called the **kernel** of T_A.

Parallel lines

Statement

Show that parallel lines stay parallel under T_A for any A.

Solution

First, we need to formalize the statement.

Lines

A **line** through $\mathbf{x}$ with direction $\mathbf{d} \neq \mathbf{0}$ is $L = \{\mathbf{x} + t\mathbf{d} : t \in \mathbb{R}\}$.

Parallel lines

L_1 and L_2 are parallel if $\mathbf{d}_1 = \lambda \mathbf{d}_2$.

Then, we look at the line *image*:

$$T_A(L) = \{A\mathbf{x} + tA\mathbf{d} : t \in \mathbb{R}\} = \{\mathbf{x}' + t\mathbf{d}' : t \in \mathbb{R}\},$$

where $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{d}' = A\mathbf{d}$. For two lines, we have:

$$\mathbf{d}_1 = \lambda \mathbf{d}_2 \implies \mathbf{d}_1' = A\mathbf{d}_1 = \lambda A\mathbf{d}_2 = \lambda \mathbf{d}_2'$$

Rank of a matrix (a)

Statement

Let $m \geq 2$. Consider

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix},$$

s.t. $a_{ij} = i + j$. Find A_m for $m \in \{2, 3, 4\}$.

Solution

- $A_2 = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$
- $A_3 = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$
- $A_4 = \begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

Rank of a matrix (b)

Statement

Let $m \geq 2$. Consider

$$A_m = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mm} \end{bmatrix},$$

s.t. $a_{ij} = i + j$. Find $\operatorname{rank} A_m$.

Solution

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be the columns of A_m. Then, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, so the rank is at least 2. At the same time

$$\mathbf{v}_j - \mathbf{v}_{j-1} = \mathbf{v}_2 - \mathbf{v}_1 \implies \mathbf{v}_j = (j-1)\mathbf{v}_2 - (j-2)\mathbf{v}_1$$

Above, we expand $\mathbf{v}_{j-1}$ up to $\mathbf{v}_2$, or use induction.