

**Topics of the week**

1. Orthogonal vectors, Orthonormal vectors, Orthonormal bases;
2. Orthogonal Matrices. Orthogonal matrices preserve norm and inner-product;
3. Projections with orthonormal bases;
4. Build an orthonormal basis with Gram-Schmidt (and show correctness of Gram-Schmidt)
5. QR decomposition. Projections and least squares with QR decomposition.

**Orthogonal vectors** means pairwise orthogonal.

Let  $e_1, \dots, e_n$  be orthogonal and  $v = x_1 e_1 + \dots + x_n e_n$ , then  $e_i \cdot v = x_i(e_i \cdot e_i) \iff x_i = \frac{v \cdot e_i}{e_i \cdot e_i}$ .

**Orthonormal vectors** means orthogonal + each vector has a unit length.

Let  $e_1, \dots, e_n$  be orthonormal and  $v = x_1 e_1 + \dots + x_n e_n$ , then  $x_i = e_i \cdot v$ .

**Orthonormal basis** is a basis consisting of orthonormal vectors.

**Gram matrix** For  $v_1, \dots, v_n$ :

$$\Gamma(v_1, \dots, v_n) = \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 & \dots & v_1 \cdot v_n \\ v_2 \cdot v_1 & v_2 \cdot v_2 & \dots & v_2 \cdot v_n \\ \vdots & \vdots & \ddots & \vdots \\ v_n \cdot v_1 & v_n \cdot v_2 & \dots & v_n \cdot v_n \end{bmatrix} = A^\top A$$

The columns of  $A$  are orthogonal  $\iff A^\top A$  is diagonal.

The columns of  $A$  are orthonormal  $\iff A^\top A = I$ .

**Interpretation of projection**  $A^\top b$  are scalar products of  $b$  with all basis vectors of  $C(A)$ . Then,  $(A^\top A)^{-1}$  recovers a vector of  $C(A)$  from its dot products with the basis of  $C(A)$ .  $(A^\top A)^{-1}$  allows to consider coordinate system in dot products with basis vectors, rather than in their linear combinations.

**Dual basis** is the basis of  $C(A)$  formed of the columns of  $A(A^\top A)^{-1}$ .

**Orthogonal matrix** is a matrix that preserves distances:  $\|Ax - Ay\| = \|x - y\| \iff A^\top A = I$ .

In other words, matrix is orthogonal  $\iff$  it maps  $e_1, \dots, e_n$  into orthonormal basis.

**Projection with orthonormal basis** Assume  $A^\top A = I$ , then the projection of  $y$  on  $Ax$  is  $AA^\top y$ .

**Gram-Schmidt process** construct an orthonormal basis in  $C(A)$  by subtracting projections.

**QR decomposition**  $A = QR$ , where  $Q$  is the result of Gram-Schmidt process, and  $R = Q^\top A$ .

Note:  $AM = Q \implies A = QM^{-1} \implies R = M^{-1}$ .

**Project with QR decomposition**  $\text{proj}_{C(A)}(b) = QQ^\top b$ .

**In-class exercises** Consider the invertible matrices

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

1. Apply the Gram-Schmidt process to the columns of  $A$ .
2. Write down a  $QR$ -decomposition of  $A$ .
3. Apply the Gram-Schmidt process to the columns of  $B$ .
4. Is it always true that the Gram-Schmidt process on the columns of an upper triangular  $n \times n$  matrix with non-zero diagonal entries yields the canonical basis  $e_1, \dots, e_n$ ? Provide a proof or counterexample.