Topics of the week Compute with matrices:

1. derive and explain the LU factorization from elimination;
2. compute the REF and RREF of a given $m \times n$ matrix A, explain why it equals R in $A = CR$;

LUP factorization Do Gaussian elimination and reduce A to upper-triangular form U:

$$M_n \dots M_1 A = U \iff A = (M_1^{-1} \dots M_n^{-1})U$$

To recover L, keep a matrix $B = I$. Whenever applying M_i to A, apply $M_i^{-\top}$ to B. In the end,

$$B = M_n^{-\top} \dots M_1^{-\top} \iff B^\top = M_1^{-1} \dots M_n^{-1}$$

So, we get the decomposition $A = B^\top U$. If B is not lower-triangular, its rows can be sorted such that $L = PB^\top$ is lower-triangular. Thus, $PA = LU$.

REF and RREF Gauss-Jordan elimination turns a matrix into the row-echelon form:

$$\begin{bmatrix} 1 & 0 & a_1 & 0 & b_1 \\ 0 & 1 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & 1 & b_3 \end{bmatrix}$$

Reduced row-echelon form: Remove zero rows. Other conventions:

1. REF: Not necessary to keep pivots 1 and elements above them 0;
2. RREF: What we call REF in this course.

In-class exercises

1. Elimination, back substitution, LU factorization:
   (a) Compute an LU factorization of the matrix
   $$A = \begin{bmatrix} 2 & -12 & 6 \\ 1 & -4 & 1 \\ 2 & -11 & -5 \end{bmatrix}$$
   (b) For the factorization above, solve $Ly = b$ with
   $$b = \begin{bmatrix} 4 \\ 4 \\ 25 \end{bmatrix}$$
   (c) For the y from the previous exercise, solve $Ux = y$.
   (d) Prove that $Ax = b$.
2. Consider the matrix
   $$A = \begin{bmatrix} 1 & 2 & 0 & 3 \\ 2 & 4 & 1 & 2 \\ 3 & 6 & 2 & 5 \end{bmatrix}$$
   with CR-decomposition
   $$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix} = CR$$
   Let R_0 be the matrix in row echelon form obtained by performing Gauss-Jordan elimination on A. Determine R_0 by performing the elimination and verify Theorem 3.24 by checking that R corresponds to the non-zero rows of R_0.