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Theory and Methodology

On some useful properties of the Perron eigenvalue of a positive reciprocal matrix in the context of the analytic hierarchy process

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Abstract: A positive $n \times n$ matrix, $R = (r_{ij})$, is said to be reciprocal if its entries verify $r_{ji} = 1/r_{ij} > 0$ for all $1 \le i,j \le n$. In the context of the analytic hierarchy process, where such matrices arise from the pairwise comparison of $n \ge 2$ decision alternatives on an arbitrary ratio scale, Saaty (in *Journal of Mathematical Psychology*, 1977) proposed to use $\mu = (\lambda_{\max} - n)/(n-1) \ge 0$, a linear transform of the Perron eigenvalue $\lambda_{\max}$ of R, as a measure of the cardinal consistency in an agent's responses and posed the problem of determining how it might vary as a function of the r_{ij}'s. He also suggested that an upper bound could be found for that consistency index when the entries of R are restricted to take their values in a bounded set. Both of these questions are answered here using classical results from linear algebra.

Keywords: Analytic hierarchy process; Decision theory; Pairwise comparisons; Perron–Frobenius Theorem; Reciprocal matrices

1. Introduction

Saaty [20] has introduced a procedure for prioritising decision alternatives based on an $n \times n$ positive matrix, $R = (r_{ij})$, of pairwise comparisons of $n \ge 2$ items expressed on a ratio scale. His method, which has been used extensively in the subject-matter literature, treats $r_{ij} > 0$ as an estimate of the perceived intensity, w_i^*/w_j^*, of an agent's preference in favour of alternative i versus j, where $w^* = (w_1^*, \ldots, w_n^*)^t$ is an underlying column vector of non-negative weights that sum up to one. The judgement matrix, $R = (r_{ij})$, is thus taken to be reciprocal, namely

$$r_{ji} = 1/r_{ij}, \quad 1 \le i,j \le n, \tag{1}$$

on the assumption that the respondent's preference in favour of item i or j is elicited with only one of the pairs (i, j) or (j, i) for $i \ne j$. Accordingly, quantities derived from R may be regarded mathematically as functions of its $\frac{1}{2}n(n-1)$ upper triangular elements, r_{ij}, $i < j$, and this point of view will be strictly adhered to herein.

When the agent's stated preferences are cardinally consistent, the response matrix verifies

$$r_{ij} \times r_{jk} = r_{ik}, \quad 1 \le i,j,k \le n, \tag{2}$$

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and is sometimes said to be supertransitive (see, for instance, Mirkin [15]). The underlying priority vector, w^*, can then be recovered easily by renormalising any column of R, since it is of unit rank. In practice, however, human judgements are frequently incoherent, and a technique is required for extracting preference weights from an arbitrary positive reciprocal response matrix.

In [20–23] and elsewhere, Saaty argues that an appropriate solution to this problem is given by the non-negative, probability-normed column vector $w = (w_1, \ldots, w_n)^t$ that verifies

$$Rw = \lambda_{\max} w, \qquad \sum_{i=1}^{n} w_i = 1,$$

where $\lambda_{\max}$ stands for the largest eigenvalue of R in modulus. The Perron–Frobenius Theorem insures that $\lambda_{\max}$ is real and positive, and that the components of the corresponding right-eigenvector, w, are all of the same sign; the latter vector is thus unique, under the constraints that its components be non-negative and add up to one (see Saaty [20], pp. 240–241, or refer to Theorem 2.1, p. 30, in Varga [28], among others, for a statement of the Perron–Frobenius Theorem). In addition, the agent's underlying priority vector will be recovered, meaning that $w = w^*$, whenever the judgement matrix is supertransitive. This proposal has gained widespread acceptance as a valuable decision-making tool in the scientific literature of the past fifteen years, despite some criticisms. For expositions of the theoretical foundations of the analytic hierarchy process and original applications in a variety of subject-matter areas, the reader is referred to Golden, Wasil and Harker [6], Harker [7], Saaty [20–23], Saaty and Alexander [25], Saaty and Vargas [26, 27], as well as to a recent special issue of the *European Journal of Operational Research* devoted to this topic (Vargas and Whittaker [29]). Criticisms of the method relating essentially to its right–left asymmetry (Johnson, Beine and Wang [12]) and the legitimacy of rank reversal (Belton and Gear [1]) have generated lively debates, in which Dyer [4], Harker and Vargas [10], and Saaty [24] himself have been the most recent protagonists.

While the eigenvalue procedure is not entirely uncontroversial and several competing prioritization techniques are now in existence (see Blankmeyer [2] for a recent review), a unique feature of this solution is the related quantity $\mu = (\lambda_{\max} - n)/(n-1)$ which, according to Saaty [20–23], constitutes a useful measure of the degree of cardinal consistency in an agent's responses. In particular, it can be seen from (1) that $\lambda \geq n$ and hence that $\mu \geq 0$, with equality if and only if the entries of the judgement matrix meet requirement (2). It also follows from the Perron–Frobenius Theorem that $\lambda_{\max}$ (and hence μ) would become larger if a single entry, r_{ij}, of the response matrix were increased, but Saaty [20] correctly points out that this may not hold true for reciprocal matrices, since r_{ij} and r_{ji} must be perturbed simultaneously in order to preserve condition (1).

To provide intuitive support for his consistency index, Saaty [20] suggestively rewrites the entries of R as $r_{ij} = (w_i^*/w_j^*)\varepsilon_{ij}$ in terms of a given vector of priorities, $w^* = (w_1^*, \ldots, w_n^*)^t$. Using the fact that $\varepsilon_{ji} = 1/\varepsilon_{ij} > 0$ for all $1 \leq i,j \leq n$, he then expresses $\lambda_{\max}$, the corresponding Perron root of that judgement matrix, in the form

$$\lambda_{\max} = 1 + \sum_{i<j} \frac{\varepsilon_{ij} + 1/\varepsilon_{ij}}{n}, \tag{3}$$

under the assumption that $Rw^* = \lambda_{\max} w^*$. Although it is clear that perturbing an ε_{ij} (and ε_{ji} accordingly) would invalidate (3) if the assumption that w^* is the principal eigenvector of the modified response matrix did not remain valid, this representation suggests that $\lambda_{\max}$ (and hence μ) is likely to be small or large, depending on ε_{ij} being near or far from unity, respectively. This heuristic argument also made it possible for Saaty to surmise an interpretation of μ in terms of root mean square error, which he then proceeded to test empirically via computer simulations. To the best of the present authors' knowledge, however, the question of determining how $\lambda_{\max}$ or μ varies as a function of the (upper triangular) entries of a reciprocal matrix has been left unanswered to this date.

The main purpose of the present note is to address this issue. Drawing upon classical tools from linear algebra, it will be shown in Section 2 that if the entries of an $n \times n$ positive reciprocal matrix, $R = (r_{ij})$, are expressed as

$$r_{ij} = \exp(a_{ij}), \quad 1 \leq i,j \leq n, \tag{4}$$

then the largest eigenvalue, $\lambda_{\max}$, of R is logarithmically convex in a_{ij} for $i<j$ when all the other entries of that response matrix (except $a_{ji} = -a_{ij}$) are held constant. Actually, this turns out to be an immediate consequence of a rather specialized result from linear algebra obtained by Kingman [13] which, applied to the current context, implies in particular that $\lambda_{\max}$ is either increasing, decreasing or U-shaped as a function of any single upper triangular entry of R. This is in accordance with one's expectation as to the way in which a cardinal consistency measure should behave. In Section 3, this fact will be used, in conjunction with a classical spectrum localisation inequality due to Parker [18], to derive a sharp upper bound on $\lambda_{\max}$ for situations, frequently encountered in practice, where an agent's responses are coded on a bounded scale. More specifically, it will be seen that if all the entries of an $n \times n$ positive reciprocal matrix, R, are assumed to take their values in the closed interval $[1/S, S]$ for some constant $S \geq 1$, the Perron eigenvalue $\lambda_{\max}$ of R must then lie between n and $1 + \frac{1}{2}(n-1)(S + 1/S)$, as might have been conjectured from an empirical observation due to Lusk [14]. Interestingly, the response matrices that reach this upper bound are connected with what graph theorists call regular tournaments.

2. Log-convexity of $\lambda_{\max}$ as a function of $\log(r_{ij})$ for $i<j$

Let us momentarily abstract ourselves from the context of Saaty's analytic hierarchy process and consider more generally a class of $n \times n$ positive matrices M_θ, indexed by a real parameter θ, whose entries would be of the form

$$m_{ij}(\theta) = \exp\{f_{ij}(\theta)\}, \quad 1 \leq i,j \leq n, \tag{5}$$

with arbitrary real-valued, convex functions $f_{ij}(\theta)$. For every θ, let also $\lambda_{\max}(\theta)$ stand for the largest eigenvalue of M_θ in modulus, which must be real and positive in view of the Perron–Frobenius Theorem. With these hypotheses and notations, one is then in a position to state the following key result, originally due to Kingman [13], but rediscovered independently by the present authors, who are grateful to L. Elsner and V. Mehrmann of the Universität Bielefeld (Germany) for bringing the relevant reference to their attention.

Proposition 1 (Kingman). *The mapping $\theta \to \lambda_{\max}(\theta)$ is logarithmically convex (and hence convex) on the real line.*

A short proof of this remarkable, though somewhat obscure, result is included below for convenience. The argument, which is surprisingly simple and elegant, is essentially that which was initially supplied by Kingman [13].

Proof. For arbitrary integer $k \geq 1$, let C_k denote the set of k-dimensional vectors $c = (c_1, \ldots, c_k)$ in $\{1, \ldots, n\}^k$. By definition, the trace of $M_\theta^k = M_\theta \times \cdots \times M_\theta$ (k times) is given by

$$\operatorname{trace}(M_\theta^k) = \sum_{c \in C_k} \exp\{f_{c_1,c_2}(\theta) + f_{c_2,c_3}(\theta) + \cdots + f_{c_{k-1},c_k}(\theta) + f_{c_k,c_1}(\theta)\}.$$

It is thus strictly positive and must be a convex function of θ, since the f_{ij}'s are convex by hypothesis and finite sums or exponentials of convex functions are themselves convex. Likewise, $\exp(\beta\theta) \times \operatorname{trace}(M_\theta^k)$ is convex in θ for all real β and integer $k \geq 1$, which implies that trace (M_θ^k) is logarithmically convex in θ by the theorem on p. 90 in Vladimirov [30].

Now, since M_θ is primitive, it is well known that

$$\lambda_{\max}(\theta) = \lim_{k \to \infty} \{\operatorname{trace}(M_\theta^k)\}^{1/k}$$

for all θ (see, for example, Exercise 6, p. 44 in Varga [28]). Consequently,

$$\log\{\lambda_{\max}(\theta)\} = \lim_{k \to \infty} \log\{\operatorname{trace}(M_\theta^k)\}/k$$

is the limit of a sequence of convex functions, and hence must be convex by Theorem 10.8, p. 90 in Rockafellar [19]. □

In view of this result, it is plain that the mapping $\log(r_{ij}) \to \log[\lambda_{\max}\{\log(r_{ij})\}]$ is convex for fixed $i<j$ when all other entries of R (except $r_{ji} = 1/r_{ij}$) are held fixed. Indeed, identifying (4) and (5) and treating $\theta = a_{ij} = \log(r_{ij})$ for given $i<j$ as the variable in the statement of Proposition 1, the functions $f_{ij}(\theta) = \theta$ and $f_{ji}(\theta) = -\theta$ are immediately seen to be linear in θ, while all

the other f_{ij}'s are constant with respect to that variable. This yields the following

Corollary. *Let $R = (r_{ij})$ be an $n \times n$ positive reciprocal matrix and let $\lambda_{\max}$ represent its largest eigenvalue in modulus, which is known to be real and positive from the Perron–Frobenius Theorem. Suppose that all entries of R are held constant except r_{ij} and $r_{ji} = 1/r_{ij}$ for fixed $i \neq j$. The quantities $\log(\lambda_{\max})$ and $\lambda_{\max}$ are then convex as functions of $\log(r_{ij})$.*

For completeness, it should be mentioned here that Harker [8] had already derived an expression for the second partial derivative of $\lambda_{\max}$ as a function of r_{ij} for arbitrary $i < j$ in a positive reciprocal matrix, but that the convexity or log-convexity of $\lambda_{\max}$ was not apparent from it. As a further consequence of Proposition 1, one can note that $\lambda_{\max}$ and its scaled relative, $\mu = (\lambda_{\max} - n)/(n-1)$, must be either increasing, decreasing or U-shaped as a function of any single upper triangular entry of their corresponding response matrix. This fact will be used below to find an achievable upper bound on the value of Saaty's cardinal consistency index, μ, in situations where a respondent's preferences in favour of one of the two items in each pair of decision alternatives is expressed on a bounded scale.

3. A sharp bound on $\lambda_{\max}$ under the assumption $1/S \leq r_{ij} \leq S$

In applications of the analytic hierarchy process, the agent's responses are often restricted to a bounded measurement scale. In fact, it has even been argued by some that a bounded response scale is always sufficient for comparing pairs of items (see, for instance, Harker and Vargas [9]). Based on experience and a number of psychological considerations, Saaty himself advocates, in [20–23] and elsewhere, the use of a linear, 1 to 9 integer scale, and his interpretation of its various intensity levels has been put to advantage in a large number of case studies. Another possibility would be to express preferences on the multiplicative scale

$$1,\ \alpha,\ \alpha^2, \ldots, \alpha^m,$$

where $\alpha > 0$ would represent the smallest detectable ratio of weights in a respondent's judgements and α^m, with integer $m \geq 2$, would be the largest elicited ratio for the problem at hand. This alternative response scale has recently been proposed by Holder [11], in order to overcome incompleteness and semantic problems which he uncovered in relation with Saaty's recommendation.

When preferences between a fixed number of items are expressed on a bounded scale, so that

$$1/S \leq r_{ij} \leq S, \quad 1 \leq i, j \leq n,$$

is known to hold true for some real $S \geq 1$ specified in advance, it is clear that there must be a limit both to the size of $\lambda_{\max}$ and μ. In his pioneering work, Saaty [20] had already noted that if r_{i+} represents the i-th row sum of a positive reciprocal matrix R, then

$$n \leq \lambda_{\max} \leq \max(r_{1+}, \ldots, r_{n+}) \leq nS,$$

which implies that $0 \leq \mu \leq (S-1)n/(n-1)$. In an early application of the analytic hierarchy process, however, Lusk [14] had further observed empirically that, at least for odd values of n, $\lambda_{\max}$ seemed to reach a maximum at $1 + \frac{1}{2}(n-1)(S + 1/S) \leq nS$ when the response matrix was of the form

$$R = \begin{pmatrix} 1 & S & 1/S & S & 1/S & \cdots \\ 1/S & 1 & S & 1/S & S & \\ S & 1/S & 1 & S & 1/S & \\ \vdots & & & & & \ddots \end{pmatrix}.$$

This would suggest that for fixed $n \geq 2$, the consistency index μ actually takes its values between 0 and $\frac{1}{2}(S + 1/S) - 1$. It will now be shown that this upper bound on μ is indeed the best possible and that it is attained, for odd n, by all response matrices, such as Lusk's, on whose rows the values S and $1/S$ occur exactly $\frac{1}{2}(n-1)$ times each. These matrices, which Genest, Lapointe and Drury [5] call 'maximally intransitive', are intimately related with the notion of 'regular tournament' studied by graph theorists (see, for example, Exercise 7, p. 7 in Moon [16]). Genest, Lapointe and Drury's terminology is justified in that matrices of this sort exhibit the largest possible number of 'circular triads', that is, triples (i, j, k) of decision alternatives subject to an ordinal inconsistency of the type $r_{ij} > 1$, $r_{jk} > 1$

and $r_{ki} > 1$. For a more rigorous definition of that concept and a simple formula for computing the number of circular triads from the row sums of the so-called incidence matrix of a round-robin tournament, the reader is referred to Section 2.1 in the book by David [3].

Proposition 2. *Let R be an $n \times n$ positive reciprocal matrix with entries $1/S \leq r_{ij} \leq S$, $1 \leq i,j \leq n$, for some $S \geq 1$, and let $\lambda_{\max}$ denote the largest eigenvalue of R in modulus, which is known to be real and positive from the Perron–Frobenius Theorem. Then $n \leq \lambda_{\max} \leq 1 + \frac{1}{2}(n-1)(S+1/S)$, the lower and upper bound being reached if and only if R is supertransitive or maximally intransitive, respectively.*

Proof. That $\lambda_{\max} \geq n$ with equality only if the respondent's preferences are cardinally consistent was first established by Saaty [20]. Furthermore, in view of Proposition 1 and the ensuing corollary, the search for an upper bound on $\lambda_{\max}$ can clearly be limited to reciprocal matrices with off-diagonal entries equal to S or $1/S$. If n_i represents the number of occurrences of S on the i-th row of such a matrix, R, and if r_{i+} and r_{+i} respectively stand for its i-th row and i-th column sum, one has

$$r_{i+} = 1 + n_i S + (n-1-n_i)/S,$$

$$r_{+i} = 1 + (n-1-n_i)S + n_i/S,$$

and hence

$$r_{i+} + r_{+i} = 2 + (n-1)(S+1/S), \quad 1 \leq i \leq n.$$

The conclusion then follows at once from a classical inequality due to Parker [18], which asserts that

$$\lambda_{\max} \leq \max \tfrac{1}{2}(r_{1+} + r_{+1}, \ldots, r_{n+} + r_{+n}). \qquad (6)$$

Since R is irreducible, it also follows from Satz VII, p. 212 in Ostrowski [17] that equality in (6) can only occur if $r_{i+} = r_{+i}$ for all $1 \leq i \leq n$, that is, when R is maximally intransitive. □

The bound $\mu \leq \frac{1}{2}(S+1/S) - 1$ intuited by Lusk [14] for preferences expressed on a linear, 1 to S integer scale, trivially obtains from Proposition 2. A similar bound holds with $S = \alpha^m$ if the scale propounded by Holder [11] is used instead.

References

[1] Belton, V., and Gear, T., "On a short-coming of Saaty's method of analytic hierarchies", *Omega* 11 (1983) 228–230.

[2] Blankmeyer, E., "Approaches to consistency adjustment", *Journal of Optimization Theory and Applications* 54 (1987) 479–488.

[3] David, H.A., *The Method of Paired Comparisons*, second edition, revised, Oxford University Press, New York, 1988.

[4] Dyer, J.S., "Remarks on the analytic hierarchy process", *Management Science* 36 (1990) 249–258.

[5] Genest, C., Lapoint, F., and Drury, S.W., "On a proposal of Jensen for the analysis of ordinal pairwise preferences using Saaty's eigenvector sealing method", *Journal of Mathematical Psychology*, in press.

[6] Golden, B., Wasil, E., and Harker, P.T. (eds), *The Analytic Hierarchy Process: Applications and Studies*, Springer-Verlag, New York, 1989.

[7] Harker, P.T. (guest ed.), "The analytic hierarchy process", *Socio-Economic Planning Sciences* 20 (1986) 6.

[8] Harker, P.T., "Derivatives of the Perron root of a positive reciprocal matrix: With application to the analytic hierarchy process", *Applied Mathematics and Computations* 22 (1987) 217–232.

[9] Harker, P.T., and Vargas, L.G., "The theory of ratio scale estimation: Saaty's analytic hierarchy process", *Management Science* 33 (1987) 1383–1403.

[10] Harker, P.T., and Vargas, L.G., "Reply to 'Remarks on the analytic hierachy process' by J.S. Dyer", *Management Science* 36 (1990) 269–273.

[11] Holder, R.D., "Some comments on the analytic hierarchy process", *Journal of the Operational Research Society* 41 (1990) 1073–1076.

[12] Johnson, C.R., Beine, W.B., and Wang, T.J., "Right-left asymmetry in an eigenvector ranking procedure", *Journal of Mathematical Psychology* 19 (1979) 61–64.

[13] Kingman, J.F.C., "A convexity property of positive matrices", *The Quarterly Journal of Mathematics. Oxford. Second Series* 12 (1961) 283–284.

[14] Lusk, E.J., "Analysis of hospital capital decision alternatives: A priority assignment model", *Journal of the Operational Research Society* 30 (1979) 439–448.

[15] Mirkin, B.G., *Group Choice*, Winston, Washington, DC, 1979.

[16] Moon, J.W., *Topics on Tournaments*, Holt, Rinehart & Winston, New York, 1968.

[17] Ostrowski, A.M., "Über das Nichtverschwinden einer Klasse von Determinanten und die Lokalisierung der charakteristischen Wurzeln von Matrizen", *Compositio Mathematica* 9 (1951) 209–226.

[18] Parker, W.V., "The characteristic roots of a matrix", *Duke Mathematical Journal* 3 (1937) 484–487.

[19] Rockafellar, R.T., *Convex Analysis*, Princeton University Press, Princeton, NJ, 1970.

[20] Saaty, T.L., "A scaling method for priorities in hierarchical structures", *Journal of Mathematical Psychology* 15 (1977) 234–281.

[21] Saaty, T.L., *The Analytic Hierarchy Process*, McGraw-Hill, New York, 1980.

[22] Saaty, T.L., *Decision Making for Leaders*, Lifetime Learning Publications, Belmont, CA, 1982.
[23] Saaty, T.L., "Axiomatic foundation of the analytic hierarchy process", *Management Science* 32 (1986) 841–855.
[24] Saaty, T.L., "An exposition of the AHP in reply to the paper 'Remarks on the analytic hierarchy process' ", *Management Science* 36 (1990) 259–268.
[25] Saaty, T.L., and Alexander, J.M., *Conflict Resolution: The Analytic Hierarchy Approach*, Praeger, New York, 1989.
[26] Saaty, T.L., and Vargas, L.G., *The Logic of Priorities*, Kluwer-Nijhoff, Boston, MA, 1982.
[27] Saaty, T.L., and Vargas, L.G. (guest eds), "The analytic hierarchy process: Theoretic developments and some applications", *Mathematical Modelling* 9 (1987) 3–5.
[28] Varga, R.S., *Matrix Iterative Analysis*, Prentice-Hall, Englewood Cliffs, NJ, 1962.
[29] Vargas, L.G., and Whittaker, R.W. (guest eds), "Decision making by the analytic hierarchy process: Theory and applications", *European Journal of Operational Research* 48 (1990) 1.
[30] Vladimirov, V.S., *Methods of the Theory of Functions of Many Complex Variables*, MIT Press, Cambridge, MA, 1966.