Appendix B

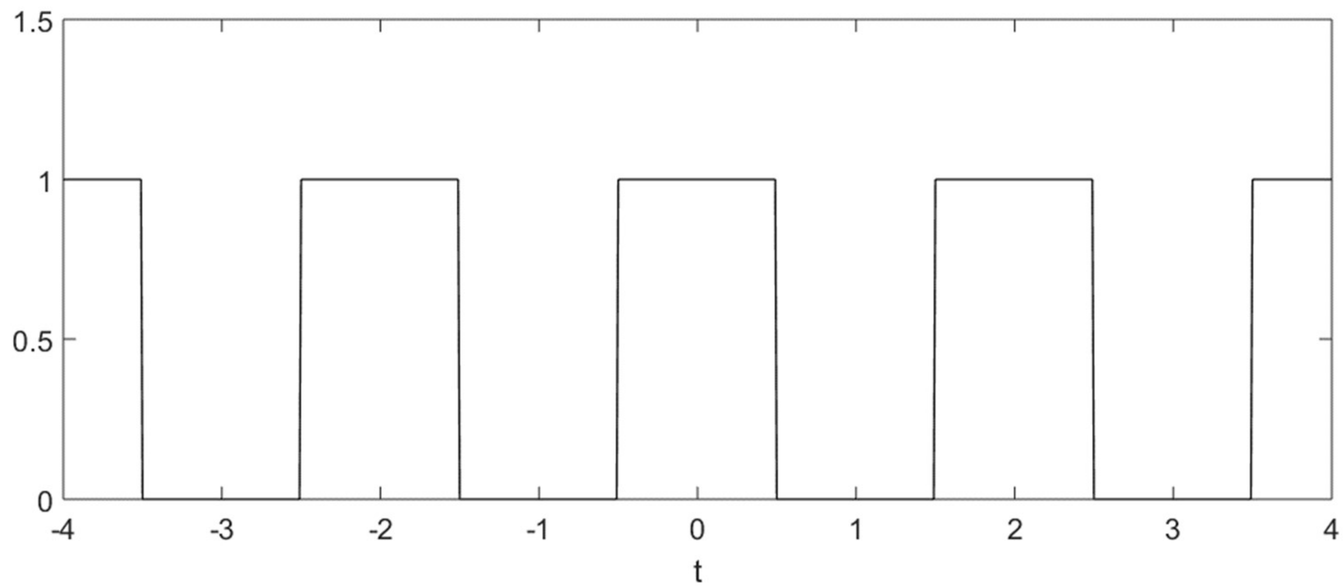
Analogue frequency analysis

Fourier Series, Fourier Transform, Laplace Transform

Fourier series

- A method to split a periodic function into harmonic components
- The harmonic functions are: $\cos(n\omega_0 t)$ and $\sin(n\omega_0 t)$, where n is the harmonic number and $\omega_0 = 2\pi/T_0$ is the fundamental frequency, and T_0 is the period.
- Trigonometric FS: $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$
- The Fourier coefficients of the harmonics are found as correlations between the function, x(t), and the harmonic:
- Symmetric components: $a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,\mathrm{dt}$;
- Antisymmetric comp.: $b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\mathrm{dt}$
- DC-component: $a_0 = \frac{1}{T_0}\int_{T_0} x(t)\mathrm{dt}$ *(average value of x(t))*

Example 5.1



Period: T_0=2 ; ω_0=π

Average value: $$a_0 = \frac{1}{T_0}\int_{-1}^{1} x(t)\mathrm{dt} = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dt = \frac{1}{2}$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,\mathrm{dt} = \frac{2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(n\pi t)\,dt = \frac{1}{n\pi}\,{}_{-\frac{1}{2}}^{\frac{1}{2}}[\sin(n\pi t)]$$

$$= \frac{1}{n\pi}\left[\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right] = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = sinc\left(\frac{n\pi}{2}\right) \;\; ; \;\; a_n = 0\ for\ n\ even$$

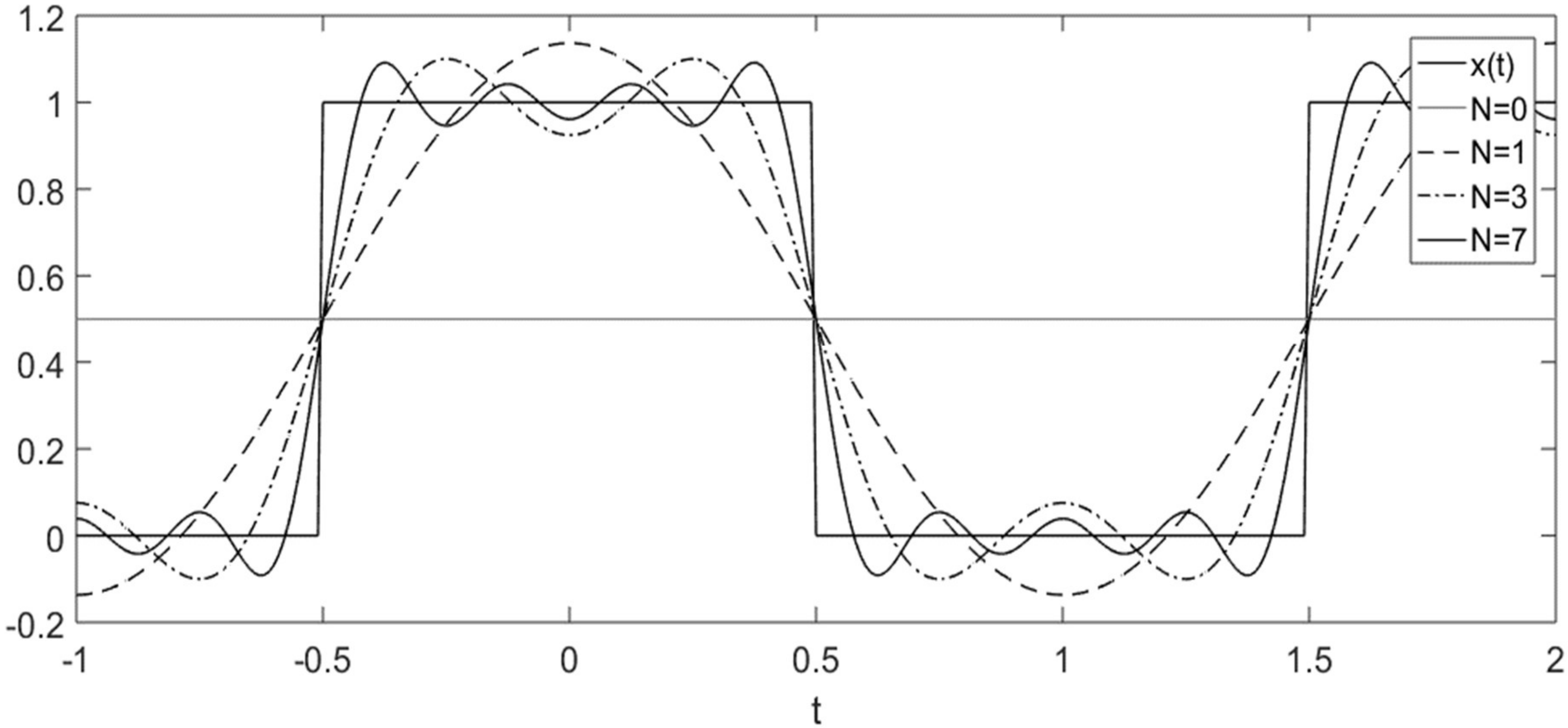
$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,\mathrm{dt} = \frac{2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(n\pi t)\,dt = \frac{1}{n\pi}\,{}_{-\frac{1}{2}}^{\frac{1}{2}}[-\cos(n\pi t)]$$

$$= \frac{-1}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - \cos\left(-\frac{n\pi}{2}\right)\right] = 0 \;\; ; \;\; \text{All } b_n = 0\ because\ x(t)\ is\ symmetric$$

The Fourier series is:

$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(\,n\pi t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t) \ldots$$

Summing up results in x(t) for n=1,2,3,…N



Summing up the harmonics, results in x(t) as N approaches infinity.

Fourier series as amplitude and phase

We use the trigonometric identity:

$$\mathrm{a}\cos(\omega t) + \mathrm{b}\sin(\omega t) = \mathrm{C}\cos(\omega t + \theta)\,; \mathrm{C} = \sqrt{\mathrm{a}^2 + \mathrm{b}^2}\,;\ \theta = \tan^{-1}\left(\frac{-\mathrm{b}}{\mathrm{a}}\right)$$

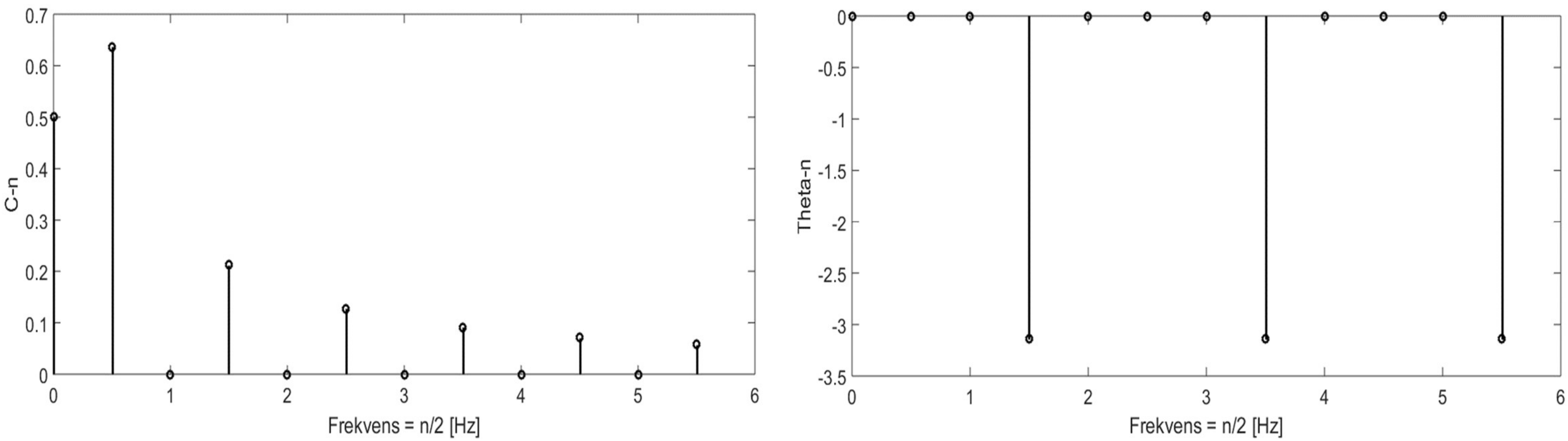
We apply this to all the harmonics and get:

$$\mathrm{x}(t) = \mathrm{a}_0 + \sum_{n=1}^{\infty} \mathrm{a}_n \cos(n\omega_0 t) + \mathrm{b}_n \sin(n\omega_0 t)$$

$$= \mathrm{C}_0 + \sum_{n=1}^{\infty} \mathrm{C}_n \cos(n\omega_0 t + \theta_n)$$

$$\text{Her er:}\quad \mathrm{C}_0 = \mathrm{a}_0\,;\ \mathrm{C}_n = \sqrt{{\mathrm{a}_n}^2 + {\mathrm{b}_n}^2}\,;\ \theta_n = \tan^{-1}\left(\frac{-\mathrm{b}_n}{\mathrm{a}_n}\right)$$

Frequency spectrum as amplitude and phase：



$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t) - \frac{2}{7\pi}\cos(7\pi t) + \cdots$$
$$= \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) + \frac{2}{3\pi}\cos(3\pi t - \pi) + \frac{2}{5\pi}\cos(5\pi t) + \frac{2}{7\pi}\cos(7\pi t - \pi) + \cdots$$

Exponential Fourier series

We now use: $\cos(\omega t) = \frac{1}{2}[\exp(j\omega t) + \exp(-j\omega t)]$

We apply this to all harmonics and get:

$$x(t) = \mathrm{C_0} + \sum_{\mathrm{n=1}}^{\infty} \mathrm{C_n} \cos(\mathrm{n\omega_0 t} + \theta_n)$$

$$= D_0 + \sum_{n=1}^{\infty} D_n \exp(\mathrm{jn\omega_0 t}) + D_{-n}\exp(-\mathrm{jn\omega_0 t}) = \sum_{n=-\infty}^{\infty} D_n \exp(jn\omega_0 t)$$

We have introduced negative n and thereby also negative frequencies.

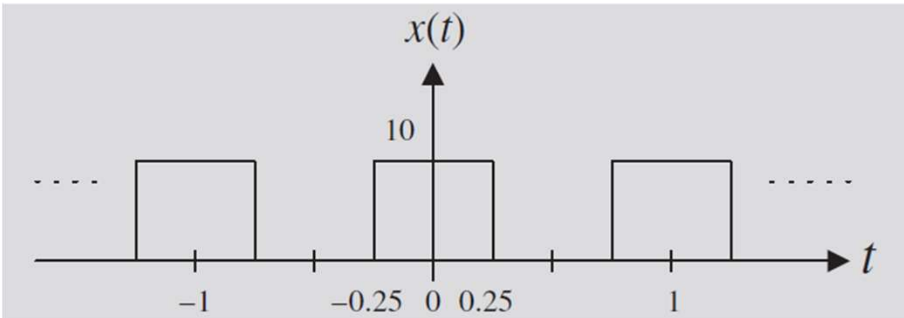
We see that: $D_0 = C_0$; $D_{-n} = D_n^*$; $|D_n| = |D_{-n}| = \frac{C_n}{2}$; $\angle D_n = \theta_n$; $\angle D_{-n} = -\theta_n$

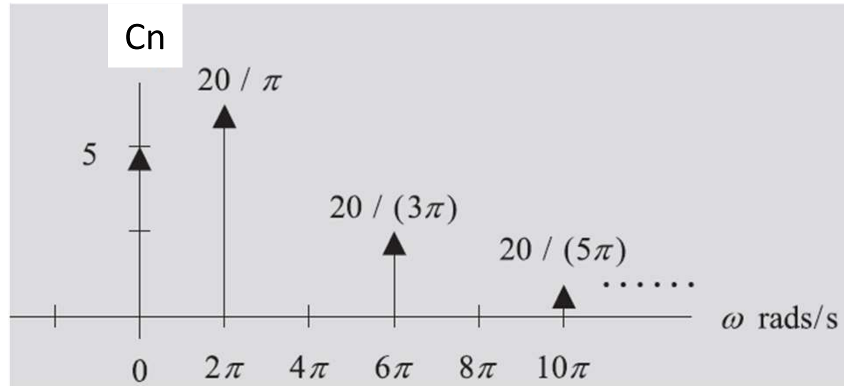
The amplitude is symmetric, and the phase is antisymmetric.

The D’s are therefore pare wise complex conjugate for n and –n.

This is necessary for x(t) to be real.

Example B1 with complex notation





a(0)=5, C(0)=5, D(0)=5

a(n)=10sinc(πn/2)
b(n)=0

C(n)=|a(n)|

|D(n)|=|D(-n)|=C(n)/2

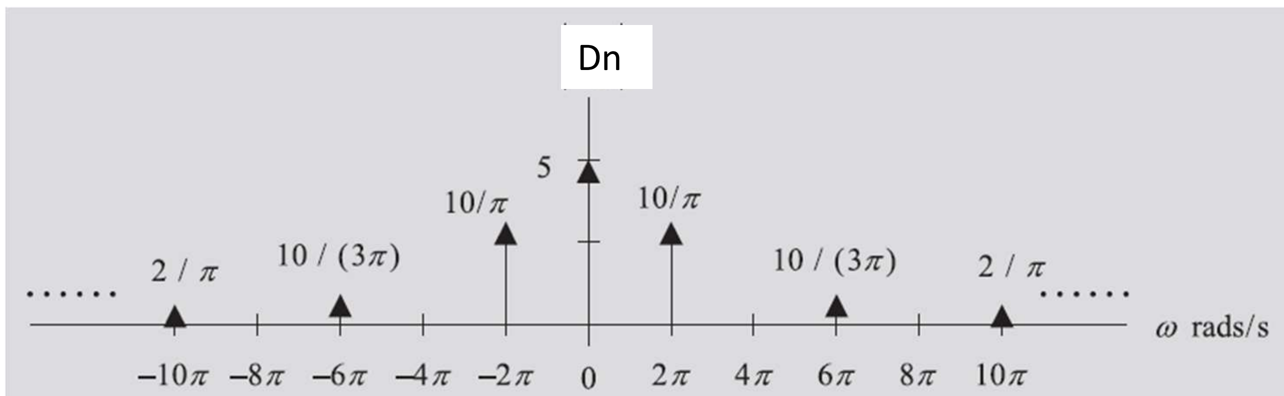
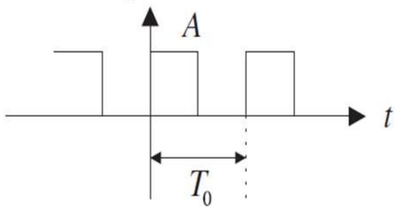
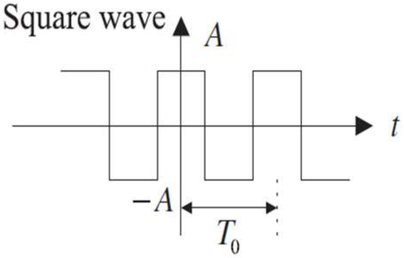
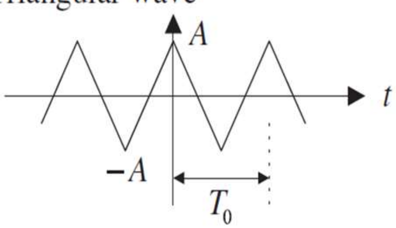
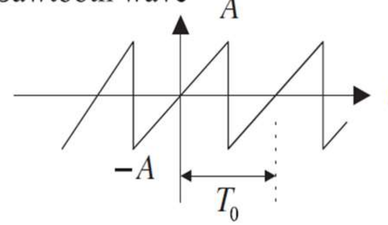
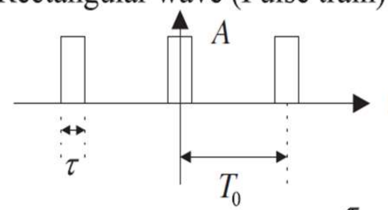
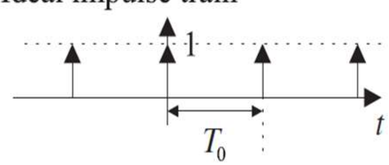


Table B.1 Fourier Series Expansions for Some Common Waveform Signals in the Sine-Cosine Form

<table>
<tr><th>Time domain signal $x(t)$</th><th>Fourier series expansion</th></tr>
<tr><td>Positive square wave

</td><td>$$x(t)=\frac{A}{2}+\frac{2A}{\pi}\left(\sin\omega_0 t+\frac{1}{3}\sin 3\omega_0 t+\frac{1}{5}\sin 5\omega_0 t+\frac{1}{7}\sin 7\omega_0 t+\cdots\right)$$</td></tr>
<tr><td>Square wave

</td><td>$$x(t)=\frac{4A}{\pi}\left(\cos\omega_0 t-\frac{1}{3}\cos 3\omega_0 t+\frac{1}{5}\cos 5\omega_0 t-\frac{1}{7}\cos 7\omega_0 t+\cdots\right)$$</td></tr>
<tr><td>Triangular wave

</td><td>$$x(t)=\frac{8A}{\pi^2}\left(\cos\omega_0 t+\frac{1}{9}\cos 3\omega_0 t+\frac{1}{25}\cos 5\omega_0 t+\frac{1}{49}\cos 7\omega_0 t+\cdots\right)$$</td></tr>
<tr><td>Sawtooth wave

</td><td>$$x(t)=\frac{2A}{\pi}\left(\sin\omega_0 t-\frac{1}{2}\sin 2\omega_0 t+\frac{1}{3}\sin 3\omega_0 t-\frac{1}{4}\sin 4\omega_0 t+\cdots\right)$$</td></tr>
<tr><td>Rectangular wave (Pulse train)


Duty cycle $= d = \frac{\tau}{T_0}$</td><td>$$x(t)=Ad+2Ad\left(\frac{\sin\pi d}{\pi d}\right)\cos\omega_0 t+2Ad\left(\frac{\sin 2\pi d}{2\pi d}\right)\cos 2\omega_0 t+2Ad\left(\frac{\sin 3\pi d}{3\pi d}\right)\cos 3\omega_0 t+\cdots$$</td></tr>
<tr><td>Ideal impulse train

</td><td>$$x(t)=\frac{1}{T_0}+\frac{2}{T_0}(\cos\omega_0 t+\cos 2\omega_0 t+\cos 3\omega_0 t+\cos 4\omega_0 t+\cdots)$$</td></tr>
</table>

The Fourier transform

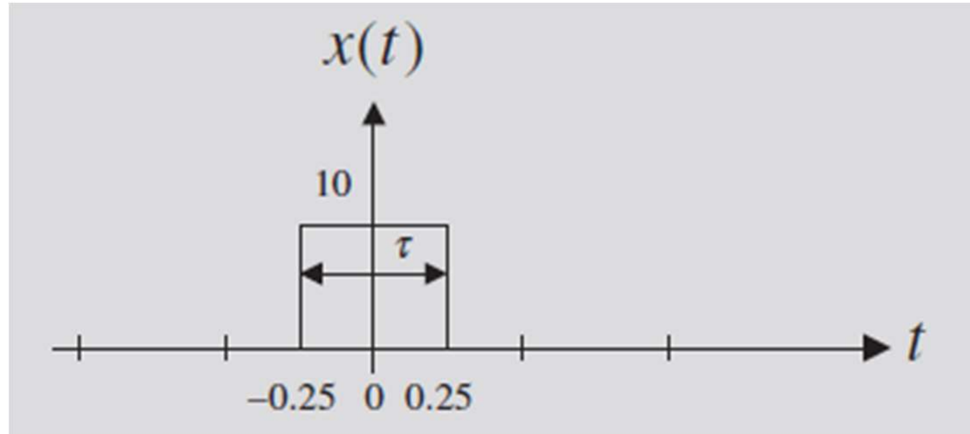
- The Fourier transform is a generalization of the Fourier series for non-periodic functions.
- The Fourier transform results in the frequency spectrum of x(t):
- $X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt$
- We get back to the time function, x(t), by the inverse transform:
- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega$
- We use the Fourier transform to analyze the frequency content of the signal.

Example B5

x(t) = Π(t/τ)

$$X(\omega) = \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\omega/2}$$

$$= \tau \cdot sinc(\omega\tau/2)$$



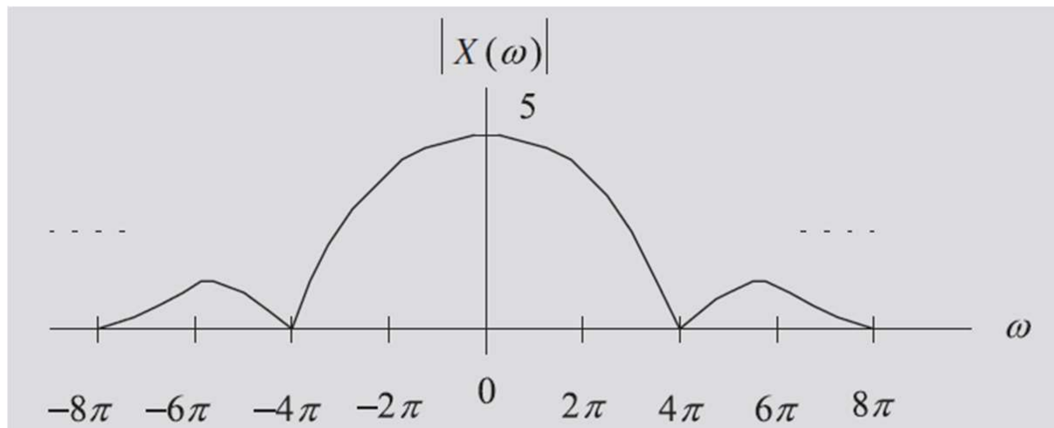
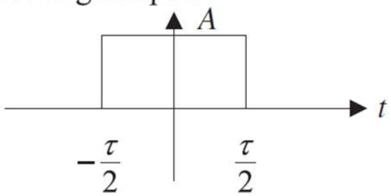
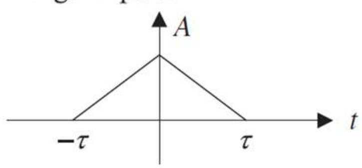
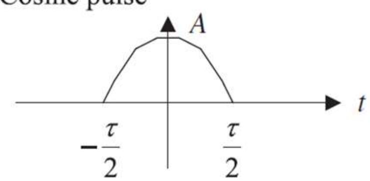
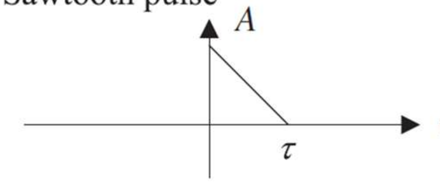
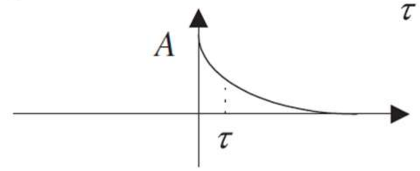
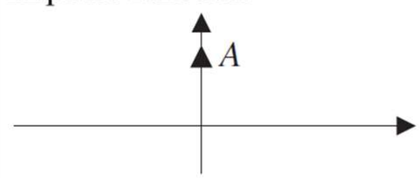


Table B.3 Fourier Transform for Some Common Signals

Time domain signal $x(t)$	Fourier spectrum $X(f)$
Rectangular pulse 	$X(f) = A\tau\frac{\sin \pi f\tau}{\pi f\tau}$
Triangular pulse 	$X(f) = A\tau\left(\frac{\sin \pi f\tau}{\pi f\tau}\right)^2$
Cosine pulse 	$X(f) = \frac{2A\tau}{\pi}\left(\frac{\cos \pi f\tau}{1-4f^2\tau^2}\right)$
Sawtooth pulse 	$X(f) = \frac{jA}{2\pi f}\left(\frac{\sin \pi f\tau}{\pi f\tau}e^{-j\pi f\tau} - 1\right)$
Exponential function $\alpha = \frac{1}{\tau}$ 	$X(f) = \frac{A}{\alpha + j2\pi f}$
Impulse function 	$X(f) = A$

Real x(t) → Symmetric |X(ω)| and antisymmetric ∠X(ω)

Symmetric x(t) → Real X(ω)

Table B.4 Properties of Fourier Transform

Line	Time Function	Fourier Transform
1	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
2	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
3	$\int_{-\infty}^{t} x(t)dt$	$\frac{X(f)}{j2\pi f}$
4	$x(t-\tau)$	$e^{-j2\pi f\tau}X(f)$
5	$e^{j2\pi f_0 t}x(t)$	$X(f-f_0)$
6	$x(at)$	$\frac{1}{a}X\left(\frac{f}{a}\right)$

The Laplace Transform

- The Laplace transform was developed to solve linear differential equations.
- Definition: $X(s) = \int_0^\infty x(t) \exp(-st)\, dt$
- This is the unilateral transform, and we need starting conditions.
- The Laplace variable is complex: $s = \sigma + j\omega$.
- We use the Laplace transform to analyze systems.
- An important property of the LT is the following: $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s)$
- A time derivative results in a multiplication with s.

Table B.5 Laplace Transform Table

Line	Time Function $x(t)$	Laplace Transform $X(s) = L(x(t))$
1	$\delta(t)$	1
2	1 or $u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$e^{-at}u(t)$	$\frac{1}{s+a}$
5	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
6	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
7	$\sin(\omega t+\theta)u(t)$	$\frac{s\sin(\theta)+\omega\cos(\theta)}{s^2+\omega^2}$
8	$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
9	$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
10	$\left(A\cos(\omega t)+\frac{B-aA}{\omega}\sin(\omega t)\right)e^{-at}u(t)$	$\frac{As+B}{(s+a)^2+\omega^2}$
11a	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
11b	$\frac{1}{(n-1)!}t^{n-1}u(t)$	$\frac{1}{s^n}$
12a	$e^{-at}t^n u(t)$	$\frac{n!}{(s+a)^{n+1}}$
12b	$\frac{1}{(n-1)!}e^{-at}t^{n-1}u(t)$	$\frac{1}{(s+a)^n}$
13	$(2\text{Real}(A)\cos(\omega t)-2\text{Imag}(A)\sin(\omega t))e^{-\alpha t}u(t)$	$\frac{A}{s+\alpha-j\omega}+\frac{A^*}{s+\alpha+j\omega}$
14	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
15	$\int_0^t x(t)dt$	$\frac{X(s)}{s}$
16	$x(t-a)u(t-a)$	$e^{-as}X(s)$
17	$e^{-at}x(t)u(t)$	$X(s+a)$

Solving differential equations with Laplace

- Any linear differential equation with constant coefficients with all starting conditions being zero, can be transformed to:
- $A(s)Y(s) = B(s)X(s)$, where Y(s) and X(s) are the Laplace transforms of the output signal and the input signal, respectively.
- A(s) is the characteristic polynomial and B(s) the driving polynomial.
- Poles: A(s)=0 ; Zeros: B(s)=0.
- The transfer function of the system is defined as the output over the input: $H(s) = \frac{Y(s)}{X(s)} = \frac{B(s)}{A(s)}$
- And we get: $Y(s) = H(s)X(s) \quad ; \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$

The impulse response of a system

- The impulse response, h(t), of a system is defined as the response of the system when the input is an impulse, δ(t).
- The impulse function is defined as a time function that is infinitely short and with infinite amplitude, but so that the area under the curve is 1. It is also called the Dirac's delta function.
- We get: $X(s) = \mathcal{L}\{\delta(t)\} = 1$
- $h(t) = y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)X(s)\} = \mathcal{L}^{-1}\{H(s)\}$
- And therefore: $H(s) = \mathcal{L}\{h(t)\}$

Frequency response

- We can get to the Fourier transform from the Laplace transform by setting: s = jω.
- This requires that the system is asymptotically stable and causal.
- The systems frequency response is defined as: $H(j\omega) = H(s = j\omega)$
- H(jω) is a complex function and therefore split into an amplitude response and a phase response:

$$H(j\omega) = |H(j\omega)|\exp(j\angle H(j\omega)$$

$$|H(j\omega)| = \sqrt{(Re[H(j\omega)])^2 + (Im[H(j\omega)])^2}\ ; \angle H(j\omega) = tan^{-1}\left(\frac{Im[H(j\omega)]}{Re[H(j\omega)]}\right)$$

- Finally, the frequency response is the Fourier transform of the impulse response: $H(\omega) = \mathcal{F}\{h(t)\}$

Summary of analog signal processing

Input

x(t)

X(s)

X(ω)

System

h(t)

H(s)

H(ω)

Output

$$y(t) = h(t) * x(t)$$

$$Y(s) = H(s)X(s)$$

$$Y(\omega) = H(\omega)X(\omega)$$