CHAPTER 5

THE Z-TRANSFORM

CHAPTER OUTLINE

5.1 DEFINITION

The *z-transform* is a very important tool in describing and analyzing digital systems. It also offers techniques for digital filter design and frequency analysis of digital signals. We begin with the definition of z-transform.

The z-transform of a causal sequence $x(n)$, designated by $X(z)$ or $Z(x(n))$, is defined as

$$\begin{aligned} X(z) &= Z(x(n)) = \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + \cdots \end{aligned} \tag{5.1}$$

where z is the complex variable. Here, the summation taken from $n=0$ to $n=\infty$ is in accordance with the fact that for most situations, the digital signal $x(n)$ is a causal sequence, that is, $x(n)=0$ for $n<0$. Thus, the definition in Eq. (5.1) is referred to as *one-sided z-transform* or *a unilateral transform*. In Eq. (5.1), all the values of z that make the summation to exist form a *region of convergence* in the z-transform domain, while all other values of z outside the region of convergence will cause the summation to diverge. The region of convergence is defined based on the particular sequence $x(n)$ being applied. Note that we deal with the unilateral z-transform first, and hence when performing an inverse z-transform (which we shall study later), we are restricted to the causal sequence. Next, we will briefly study two-sided z-transform for the noncausal sequence. Now let us study the following typical examples.

Digital Signal Processing. https://doi.org/10.1016/B978-0-12-815071-9.00005-1

EXAMPLE 5.1

Given the sequence

$$x(n) = u(n),$$

Find the z-transform of $x(n)$.

Solution:

From the definition of Eq. (5.1), the z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} u(n)z^{-n} = \sum_{n=0}^{\infty} \left(z^{-1}\right)^n = 1 + \left(z^{-1}\right) + \left(z^{-1}\right)^2 + \cdots.$$

This is an infinite geometric series that converges to

$$X(z) = \frac{z}{z-1}$$

with a condition $|z^{-1}| < 1$. Note that for an infinite geometric series, we have $1 + r + r^2 + \cdots = \frac{1}{1-r}$ when $|r| < 1$. The region of convergence for all values of z is given as $|z| > 1$.

EXAMPLE 5.2

Considering the exponential sequence

$$x(n) = a^n u(n),$$

Find the z-transform of the sequence $x(n)$.

Solution:

From the definition of the z-transform in Eq. (5.1), it follows that

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + \left(az^{-1}\right) + \left(az^{-1}\right)^2 + \cdots.$$

Since this is a geometric series which will converge for $|az^{-1}| < 1$, it is further expressed as

$$X(z) = \frac{z}{z-a}, \text{ for } |z| > |a|.$$

The z-transforms for common sequences are summarized in Table 5.1.

Example 5.3 illustrates the method of finding the z-transform using Table 5.1.

EXAMPLE 5.3

Find the z-transform for each of the following sequences:

(a) $x(n) = 10u(n)$

(b) $x(n) = 10\sin(0.25\pi n)u(n)$

(c) $x(n) = (0.5)^n u(n)$

(d) $x(n) = (0.5)^n \sin(0.25\pi n)u(n)$

(e) $x(n) = e^{-0.1n}\cos(0.25\pi n)u(n)$

Solution:

(a) From Line 3 in Table 5.1, we get

$$X(z) = Z(10u(n)) = \frac{10z}{z-1}.$$

(b) Line 9 in Table 5.1 leads to

$$\begin{aligned} X(z) &= 10Z(\sin(0.2\pi n)u(n)) \\ &= \frac{10\sin(0.25\pi)z}{z^2 - 2z\cos(0.25\pi) + 1} = \frac{7.07z}{z^2 - 1.414z + 1}. \end{aligned}$$

(c) From Line 6 in Table 5.1, we yield

$$X(z) = Z((0.5)^n u(n)) = \frac{z}{z-0.5}.$$

(d) From Line 11 in Table 5.1, it follows that

$$\begin{aligned} X(z) = Z((0.5)^n \sin(0.25\pi n)u(n)) &= \frac{0.5 \times \sin(0.25\pi)z}{z^2 - 2\times 0.5\cos(0.25\pi)z + 0.5^2} \\ &= \frac{0.3536z}{z^2 - 0.7071z + 0.25}. \end{aligned}$$

(e) From Line 14 in Table 5.1, it follows that

$$\begin{aligned} X(z) = Z(e^{-0.1n}\cos(0.25\pi n)u(n)) &= \frac{z(z - e^{-0.1}\cos(0.25\pi))}{z^2 - 2e^{-0.1}\cos(0.25\pi)z + e^{-0.2}} \\ &= \frac{z(z-0.6397)}{z^2 - 1.2794z + 0.8187}. \end{aligned}$$

Table 5.1 Table of *z*-Transform Pairs

Line No.	$x(n), n \geq 0$	*z*-Transform $X(z)$	Region of Convergence
1	$x(n)$	$\sum_{n=0}^{\infty} x(n)z^{-n}$	
2	$\delta(n)$	1	$\|z\| > 0$
3	$au(n)$	$\frac{az}{z-1}$	$\|z\| > 1$
4	$nu(n)$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
5	$n^2u(n)$	$\frac{z(z+1)}{(z-1)^3}$	$\|z\| > 1$
6	$a^n u(n)$	$\frac{z}{z-a}$	$\|z\| > \|a\|$
7	$e^{-na}u(n)$	$\frac{z}{(z-e^{-a})}$	$\|z\| > e^{-a}$
8	$na^n u(n)$	$\frac{az}{(z-a)^2}$	$\|z\| > \|a\|$
9	$\sin(an)u(n)$	$\frac{z\sin(a)}{z^2 - 2z\cos(a) + 1}$	$\|z\| > 1$

Continued

Table 5.1 Table of z-Transform Pairs—cont'd

Line No.	$x(n)$, $n \geq 0$	z-Transform $X(z)$	Region of Convergence
10	$\cos(an)u(n)$	$\frac{z[z-\cos(a)]}{z^2-2z\cos(a)+1}$	$\lvert z\rvert > 1$
11	$a^n\sin(bn)u(n)$	$\frac{[a\sin(b)]z}{z^2-[2a\cos(b)]z+a^2}$	$\lvert z\rvert > \lvert a\rvert$
12	$a^n\cos(bn)u(n)$	$\frac{z[z-a\cos(b)]}{z^2-[2a\cos(b)]z+a^2}$	$\lvert z\rvert > \lvert a\rvert$
13	$e^{-an}\sin(bn)u(n)$	$\frac{[e^{-a}\sin(b)]z}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$
14	$e^{-an}\cos(bn)u(n)$	$\frac{z[z-e^{-a}\cos(b)]}{z^2-[2e^{-a}\cos(b)]z+e^{-2a}}$	$\lvert z\rvert > e^{-a}$
15	$2\lvert A\rvert\lvert P\rvert^n\cos(n\theta+\phi)u(n)$ where P and A are complex constants defined by $P=\lvert P\rvert\angle\theta$, $A=\lvert A\rvert\angle\phi$	$\frac{Az}{z-P}+\frac{A^*z}{z-P^*}$	

5.2 PROPERTIES OF THE Z-TRANSFORM

In this section, we study some important properties of the z-transform. These properties are widely used in deriving the z-transfer functions of difference equations and solving the system output responses of linear digital systems with constant system coefficients, which will be discussed in Chapter 6.

Linearity: The z-transform is a linear transformation, which implies

$$Z(ax_1(n)+bx_2(n)) = aZ(x_1(n))+bZ(x_2(n)), \tag{5.2}$$

where $x_1(n)$ and $x_2(n)$ denote the sampled sequences, while a and b are the arbitrary constants.

EXAMPLE 5.4

Find the z-transform of the sequence defined by

$$x(n) = u(n) - (0.5)^n u(n).$$

Solution:

Applying the linearity of the z-transform discussed above, we have

$$X(z) = Z(x(n)) = Z(u(n)) - Z(0.5^n(n)).$$

Using Table 5.1 yields

$$Z(u(n)) = \frac{z}{z-1}$$

and

$$Z(0.5^n u(n)) = \frac{z}{z-0.5}$$

Substituting these results in $X(z)$ leads to the final solution

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}.$$

Shift theorem: Given $X(z)$, the z-transform of a sequence $x(n)$, the z-transform of $x(n-m)$, the time-shifted sequence, is given by

$$Z(x(n-m)) = z^{-m}X(z), m \geq 0. \tag{5.3}$$

Note that $m \geq 0$, then $x(n-m)$ is obtained by right shifting $x(n)$ by m samples. Since the shift theorem plays a very important role in developing the transfer function from a difference equation, we verify the shift theorem for the causal sequence. Note that the shift theorem also works for the noncausal sequence.

Verification: Applying the z-transform to the shifted causal signal $x(n-m)$ leads to

$$\begin{aligned} Z(x(n-m)) &= \sum_{n=0}^{\infty} x(n-m)z^{-n} \\ &= x(-m)z^{-0} + \cdots + x(-1)z^{-(m-1)} + x(0)z^{-m} + x(1)z^{-m-1} + \cdots. \end{aligned}$$

Since $x(n)$ is assumed to be a causal sequence, this means that

$$x(-m) = x(-m+1) = \cdots = x(-1) = 0.$$

Then we achieve

$$Z(x(n-m)) = x(0)z^{-m} + x(1)z^{-m-1} + x(2)z^{-m-2} + \cdots. \tag{5.4}$$

Factoring z^{-m} from Eq. (5.4) and applying the definition of z-transform of $X(z)$, we get

$$Z(x(n-m)) = z^{-m}\left(x(0) + x(1)z^{-1} + x(2)z^{-2} + \ldots\right) = z^{-m}X(z).$$

EXAMPLE 5.5

Determine the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \times u(n-5),$$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n < 5$.

Solution:

We first use the shift theorem to have

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z[(0.5)^n u(n)].$$

Using Table 5.1 leads to

$$Y(z) = z^{-5} \times \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}.$$

Convolution: Given two sequences $x_1(n)$ and $x_2(n)$, their convolution can be determined as follows:

$$x(n) = x_1(n)*x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k), \tag{5.5}$$

where * designates the linear convolution. In z-transform domain, we have

$$X(z) = X_1(z)X_2(z) \tag{5.6}$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

EXAMPLE 5.6

Verify Eq. (5.6) using causal sequences $x_1(n)$ and $x_2(n)$.

Solution:

Taking the z-transform of Eq. (5.5) leads to

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_1(n-k)x_2(k)z^{-n}.$$

This expression can be further modified to

$$X(z) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} x_2(k)z^{-k}x_1(n-k)z^{-(n-k)}.$$

Now interchanging the order of the previous summation gives

$$X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k}\sum_{n=0}^{\infty} x_1(n-k)z^{-(n-k)}.$$

Now, let $m = n - k$:

$$X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k}\sum_{m=-k}^{\infty} x_1(m)z^{-m} = \sum_{k=0}^{\infty} x_2(k)z^{-k}\left(x_1(-k)z^{k} + \cdots x_1(-1)z + \sum_{m=0}^{\infty} x_1(m)z^{-m}\right).$$

Since $x_1(m)$ is a causal sequence, we have

$$X(z) = \sum_{k=0}^{\infty} x_2(k)z^{-k}\sum_{m=0}^{\infty} x_1(m)z^{-m}.$$

By the definition of Eq. (5.1), it follows that

$$X(z) = X_2(z)X_1(z) = X_1(z)X_2(z).$$

EXAMPLE 5.7

Given two sequences,

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$

$$x_2(n) = 2\delta(n) - \delta(n-1),$$

(a) Find the z-transform of the convolution:

$$X(z) = Z(x_1(n) * x_2(n)).$$

(b) Determine the convolution sum using the z-transform:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(k)x_2(n-k).$$

Solution:

(a) Applying z-transform for $x_1(n)$ and $x_2(n)$, respectively, it follows that

$$X_1(z) = 3 + 2z^{-1}$$

$$X_2(z) = 2 - z^{-1}.$$

Using the convolution property, we have

$$\begin{aligned} X(z) = X_1(z)X_2(z) &= (3 + 2z^{-1})(2 - z^{-1}) \\ &= 6 + z^{-1} - 2z^{-2}. \end{aligned}$$

(b) Applying the inverse z-transform and using the shift theorem and Line 1 of Table 5.1 leads to

$$x(n) = Z^{-1}\left(6 + z^{-1} - 2z^{-2}\right) = 6\delta(n) + \delta(n-1) - 2\delta(n-2).$$

Initial value theorem. Given the z-transfer function $X(z)$, then initial value of the causal sequence $x(n)$ can be determined by

$$x(0) = \lim_{z\to\infty} X(z). \tag{5.7}$$

Proof:

According to the definition of z-transform,

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots.$$

Taking a limit for $z \to \infty$ yields the initial theorem.

Final value theorem: Given the z-transfer function $X(z)$, then final value of the causal sequence can be determined by

$$x(\infty) = \lim_{z\to 1}[(z-1)X(z)]. \tag{5.8}$$

Proof:

Let us define

$$\tilde{X}(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(N)z^{-N} \tag{5.9}$$

$$\hat{\tilde{X}}(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots + x(N)z^{-N} + x(N+1)z^{-(N+1)} \tag{5.10}$$

Multiplying Eq. (5.10) by z leads to

$$z\hat{\tilde{X}}(z) = x(0)z + x(1)z^0 + x(2)z^{-1} + \cdots + x(N)z^{-N+1} + x(N+1)z^{-N}. \tag{5.11}$$

Then, on subtracting Eq. (5.9) from (5.11), we obtain

$$z\hat{\tilde{X}}(z) - \tilde{X}(z) = x(0)(z-1) + x(1)(z-1)/z + \cdots + x(N)(z-1)/z^N + x(N+1)z^{-N}. \tag{5.12}$$

Considering $N \to \infty$, then $\tilde{X}(z) = \hat{\tilde{X}}(z) = X(z)$. Let $z \to 1$. Hence, we yield the final theorem as

$$\lim_{z\to 1}[zX(z) - X(z)] = \lim_{\substack{N\to\infty \\ z\to 1}} \left[x(0)(z-1) + x(1)(z-1)/z + \cdots + x(N)(z-1)/z^N + x(N+1)z^{-N}\right] = x(\infty).$$

EXAMPLE 5.8

Determine the initial and final values for each of the following z-transform functions:

(a) $X(z) = \dfrac{z}{z-0.5}$

(b) $X(z) = \dfrac{z^2}{(z-1)(z-0.5)}$

Solution:

(a) $x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \dfrac{z}{z-0.5} = 1$

$$x(\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}\frac{(z-1)z}{z-0.5} = 0$$

(b) $x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} \dfrac{z^2}{(z-1)(z-0.5)} = 1$

$$x(\infty) = \lim_{z\to 1}(z-1)X(z) = \lim_{z\to 1}\frac{(z-1)z^2}{(z-1)(z-0.5)} = 2$$

Table 5.2 Properties of z-Transform

Property	Time Domain	z-Transform
Linearity	$ax_1(n)+bx_2(n)$	$aZ(x_1(n))+bZ(x_2(n))$
Shift theorem	$x(n-m)$	$z^{-m}X(z)$
Linear convolution	$x_1(n)*x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k)$	$X_1(z)X_2(z)$
Initial value theorem	$x(0) = \lim_{z\to\infty} X(z)$	$X(z)$
Final value theorem	$x(\infty) = \lim_{z\to 1}[(z-1)X(z)]$	$X(z)$

The properties of the z-transform discussed in this section are listed in Table 5.2.

5.3 INVERSE Z-TRANSFORM

The z-transform of a sequence $x(n)$ and the inverse z-transform of a function $X(z)$ are defined as, respectively

$$X(z) = Z(x(n)) \tag{5.13}$$

$$x(n) = Z^{-1}(X(z)), \tag{5.14}$$

where $Z(\,)$ is the z-transform operator, while $Z^{-1}(\,)$ is the inverse z-transform operator.

The inverse of the z-transform may be obtained by at least three methods:

1. Partial fraction expansion and look-up table (z-transform function must be rational).
2. Power series expansion.
3. Inversion Formula Method.

The first method is widely used, and it is assumed that the reader is well familiar with the partial fraction expansion method in learning Laplace transform. Therefore, we concentrate on the first method in this book. We will also outline the power series expansion and inversion formula methods. The detail of these two methods can also be found in the textbook by Oppenheim and Shafer (1975).

5.3.1 PARTIAL FRACTION EXPANSION AND LOOK-UP TABLE

For simple z-transform functions, we can directly find the inverse z-transform using Table 5.1. The key idea of the partial fraction expansion is that if $X(z)$ is a proper rational function of z, we can expand it to a sum of the first-order factors or higher-order factors using the partial fraction expansion that could be inverted by inspecting the z-transform table. The inverse z-transform using the z-transform table is first illustrated via the following example.

EXAMPLE 5.9

Find the inverse z-transform for each of the following functions:

(a) $X(z)=2+\dfrac{4z}{z-1}-\dfrac{z}{z-0.5}$

(b) $X(z)=\dfrac{5z}{(z-1)^2}-\dfrac{2z}{(z-0.5)^2}$

(c) $X(z)=\dfrac{10z}{z^2-z+1}$

(d) $X(z)=\dfrac{z^{-4}}{z-1}+z^{-6}+\dfrac{z^{-3}}{z+0.5}$

Solution:

(a) $x(n)=2Z^{-1}(1)+4Z^{-1}\left(\dfrac{z}{z-1}\right)-Z^{-1}\left(\dfrac{z}{z-0.5}\right)$.

From Table 5.1, we have

$$x(n)=2\delta(n)+4u(n)-(0.5)^n u(n).$$

(b) $x(n)=Z^{-1}\left(\dfrac{5z}{(z-1)^2}\right)-Z^{-1}\left(\dfrac{2z}{(z-0.5)^2}\right)=5Z^{-1}\left(\dfrac{z}{(z-1)^2}\right)-\dfrac{2}{0.5}Z^{-1}\left(\dfrac{0.5z}{(z-0.5)^2}\right)$.

Then $x(n)=5nu(n)-4n(0.5)^n u(n)$.

(c) Since $X(z)=\dfrac{10z}{z^2-z+1}=\left(\dfrac{10}{\sin(a)}\right)\dfrac{\sin(a)z}{z^2-2z\cos(a)+1}$,

By coefficient matching, we have

$$-2\cos(a)=-1.$$

Hence, $\cos(a)=0.5$, and $a=60°$. Substituting $a=60°$ into the sine function leads to

$$\sin(a)=\sin(60°)=0.866.$$

Finally, we have

$$x(n)=\frac{10}{\sin(a)}Z^{-1}\left(\frac{\sin(a)z}{z^2-2z\cos(a)+1}\right)=\frac{10}{0.866}\sin(n\times 60°)=11.547\sin(n\times 60°).$$

(d) Since $x(n)=Z^{-1}\left(z^{-5}\dfrac{z}{z-1}\right)+Z^{-1}\left(z^{-6}\times 1\right)+Z^{-1}\left(z^{-4}\dfrac{z}{z+0.5}\right)$,

Table 5.3 Partial Fraction(s) and Formulas for Constant(s)

Partial fraction with the first-order real pole:
$\dfrac{R}{z-p}, \ R=(z-p)\dfrac{X(z)}{z}\Big\|_{z=p}$
Partial fraction with the first-order complex poles:
$\dfrac{Az}{(z-P)}+\dfrac{A^{*}z}{(z-P^{*})}, \ A=(z-P)\dfrac{X(z)}{z}\Big\|_{z=P}$
$P^{*}=$complex conjugate of P,
$A^{*}=$complex conjugate of A
Partial fraction with mth-order real poles:
$\dfrac{R_m}{(z-p)}+\dfrac{R_{m-1}}{(z-p)^2}+\cdots+\dfrac{R_1}{(z-p)^m}, \ R_k=\dfrac{1}{(k-1)!}\dfrac{d^{k-1}}{dz^{k-1}}\left((z-p)^m\dfrac{X(z)}{z}\right)\Big\|_{z=p}$

Using Table 5.1 and the shift property, we get

$$x(n)=u(n-5)+\delta(n-6)+(-0.5)^{n-4}u(n-4).$$

Now, we are ready to deal with the inverse z-transform using the partial fraction expansion and look-up table. The general procedure is as follows:

(1) Eliminate the negative powers of z for the z-transform function $X(z)$.
(2) Determine the rational function $X(z)/z$ (assuming it is proper), and apply the partial fraction expansion to the determined rational function $X(z)/z$ using the formula in Table 5.3.
(3) Multiply the expanded function $X(z)/z$ by z on both sides of the equation to obtain $X(z)$.
(4) Apply the inverse z-transform using Table 5.1.

The partial fraction format and the formulas for calculating the constants are listed in Table 5.3.

Example 5.10 considers the situation of the z-transform function having first-order poles.

EXAMPLE 5.10

Find the inverse of the following z-transform:

$$X(z)=\frac{1}{(1-z^{-1})(1-0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z)=\frac{z^2}{z^2(1-z^{-1})(1-0.5z^{-1})}$$
$$=\frac{z^2}{(z-1)(z-0.5)}.$$

Dividing both sides by z leads to

$$\frac{X(z)}{z}=\frac{z}{(z-1)(z-0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

Then A and B are constants found using the formula in Table 5.3, that is,

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying both sides by z gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

Using Table 5.1 of the z-transform pairs, it follows that

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Tabulating this solution in terms of integer values of n, we obtain the results in Table 5.4.

Table 5.4 Determined Sequence in Example 5.10

n	0	1	2	3	4	…	∞
$x(n)$	1.0	1.5	1.75	1.875	1.9375	…	2.0

The following example considers the case where $X(z)$ has first-order complex poles.

EXAMPLE 5.11

Find $y(n)$ if $Y(z) = \dfrac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$.

Solution:

Dividing $Y(z)$ by z, we have

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}.$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}.$$

Continued

EXAMPLE 5.11—CONT'D

We first find B:

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4.$$

Note that A and A^* are a complex conjugate pair. We determine A as follows:

$$A = (z-0.5-j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$= \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}.$$

Using the polar form, we get

$$A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$$

$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

Assume that a first-order complex pole has form
$P = 0.5+0.5j = |P|\angle\theta = 0.707\angle 45^\circ$ and $P^* = |P|\angle -\theta = 0.707\angle -45^\circ$.
We have

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}$$

Applying the inverse z-transform from Line 15 in Table 5.1 leads to

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right).$$

Using the previous formula, the inversion and subsequent simplification yield

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n\cos(n\theta+\phi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n\cos\left(45^\circ n - 161.57^\circ\right)u(n). \end{aligned}$$

The situation dealing with the real repeated poles is presented below.

EXAMPLE 5.12

Find $x(n)$ if $X(z) = \frac{z^2}{(z-1)(z-0.5)^2}$.

Solution:

Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

where $\quad A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)^2}\bigg|_{z=1} = 4.$

Using the formulas for mth-order real poles in Table 5.3, where $m=2$ and $p=0.5$, to determine B and C yields

$$B=R_2=\frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$
$$=\frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5}=\frac{-1}{(z-1)^2}\bigg|_{z=0.5}=-4$$
$$C=R_1=\frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$
$$=\frac{z}{z-1}\bigg|_{z=0.5}=-1.$$

Then

$$X(z)=\frac{4z}{z-1}+\frac{-4z}{z-0.5}+\frac{-1z}{(z-0.5)^2}$$

The inverse z-transform for each term on the right-hand side of the above equation can be achieved by the result listed in Table 5.1, that is,

$$Z^{-1}\left\{\frac{z}{z-1}\right\}=u(n),$$
$$Z^{-1}\left\{\frac{z}{z-0.5}\right\}=(0.5)^n u(n),$$
$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\}=2n(0.5)^n u(n).$$

From these results, it follows that

$$x(n)=4u(n)-4(0.5)^n u(n)-2n(0.5)^n u(n).$$

5.3.2 PARTIAL FRACTION EXPANSION USING MATLAB

The MATLAB function **residue()** can be applied to perform the partial fraction expansion of a z-transform function $X(z)/z$. The syntax is given as

[R, P, K] = residue(B,A).

Here, B and A are the vectors consisting of coefficients for the numerator and denominator polynomials, $B(z)$ and $A(z)$, respectively. Note that $B(z)$ and $A(z)$ are the polynomials with increasing positive powers of z.

$$\frac{B(z)}{A(z)}=\frac{b_0z^M+b_1z^{M-1}+b_2z^{M-2}+\cdots+b_M}{z^N+a_1z^{N-1}+a_2z^{-2}+\cdots+a_N}. \tag{5.15}$$

The function returns the residues in vector R, corresponding poles in vector P, and polynomial coefficients (if any) in vector K. The expansion format is shown as

$$\frac{B(z)}{A(z)} = \frac{r_1}{z-p_1} + \frac{r_2}{z-p_2} + \cdots + k_0 + k_1 z^{-1} + \cdots.$$

For a pole p_j of multiplicity m, the partial fraction includes the following terms:

$$\frac{B(z)}{A(z)} = \cdots + \frac{r_j}{z-p_j} + \frac{r_{j+1}}{\left(z-p_j\right)^2} + \cdots + \frac{r_{j+m}}{\left(z-p_j\right)^m} + \cdots + k_0 + k_1 z^{-1} + \cdots.$$

EXAMPLE 5.13

Find the partial expansion for each of the following z-transform functions:

(a) $X(z) = \dfrac{1}{(1-z^{-1})(1-0.5z^{-1})}$

(b) $Y(z) = \dfrac{z^2(z+1)}{(z-1)(z^2-z+0.5)}$

(c) $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2}$

Solution:

(a) From MATLAB, we can show the denominator polynomial as

» conv([1 –1], [1 –0.5])
D =
1.0000 –1.5000 0.5000

This leads to

$$X(z) = \frac{1}{(1-z^{-1})(1-0.5z^{-1})} = \frac{1}{1-1.5z^{-1}+0.5^{-2}} = \frac{z^2}{z^2-1.5z+0.5}$$

and $\dfrac{X(z)}{z} = \dfrac{z}{z^2-1.5z+0.5}$.

From MATLAB, we have

» [R,P,K]=residue([1 0], [1 –1.5 0.5])
R =
2
–1
P =
1.0000
0.5000
K =
[]
»

Then the expansion is written as

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5}.$$

(b) From the MATLAB

```
» N = conv([1 0 0], [1 1])
N =
1 1 0 0
» D = conv([1 –1], [1 –1 0.5])
D =
1.0000 –2.0000 1.5000 –0.5000
```

we get

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)} = \frac{z^3+z^2}{z^3-2z^2+1.5z-0.5}$$

and

$$\frac{Y(z)}{z} = \frac{z^2+z}{z^3-2z^2+1.5z-0.5}.$$

Using the MATLAB residue function yields

```
» [R,P,K]=residue([1 1 0], [1 –2 1.5 –0.5])
R =
4.0000
–1.5000 – 0.5000i
–1.5000 + 0.5000i
P =
1.0000
0.5000 + 0.5000i
0.5000 – 0.5000i
K =
[]
»
```

Then the expansion is shown below

$$X(z) = \frac{Bz}{z-p_1} + \frac{Az}{z-p} + \frac{A^*z}{z-p^*},$$

where $B = 4$, $p_1 = 1$,
$A = -1.5 - 0.5j$, $p = 0.5 + 0.5j$,
$A^* = -1.5 + 0.5j$, and $p = 0.5 - 0.5j$.
c. Similarly,

```
» D = conv(conv([1 –1], [1 –0.5]),[1 –0.5])
D =
1.0000 –2.0000 1.2500 –0.2500
```

Continued

EXAMPLE 5.13—CONT'D

then $X(z) = \dfrac{z^2}{(z-1)(z-0.5)^2} = \dfrac{z^2}{z^3 - 2z^2 + 1.25z - 0.25}$ and we yield

$$\frac{X(z)}{z} = \frac{z}{z^3 - 2z^2 + 1.25z - 0.25}.$$

From MATLAB, we obtain

```
» [R,P,K]=residue([1 0], [1 –2 1.25 –0.25])
R =
4.0000
–4.0000
–1.0000
P =
1.0000
0.5000
0.5000
K =
[]
»
```

Using the previous results leads to

$$X(z) = \frac{4z}{z-1} - \frac{4z}{z-0.5} - \frac{z}{(z-0.5)^2}.$$

5.3.3 POWER SERIES METHOD

By using a long division method, a rational function $X(z)$ can be expressed in the form of power series, that is,

$$X(z) = a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots. \tag{5.16}$$

Based on the definition of the unilateral z-transform, we also have

$$X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots. \tag{5.17}$$

It can be seen that there is a one-to-one coefficient match between coefficients a_n and the sequence $x(n)$, that is,

$$x(n) = a_n, \text{ for } n \geq 0 \tag{5.18}$$

As an illustration, let us find the inverse of a given rational z-transform function:

$$X(z) = \frac{z}{z^2 - 2z + 1}$$

Using the long division method, we yield

$$
\begin{array}{r|l}
 & z^{-1}+2z^{-2}+\cdots \\
\hline
z^2-2z+1 & z \\
 & \underline{z-2+z^{-1}} \\
 & \quad 2-z^{-1} \\
 & \quad \underline{2-4z^{-1}+2z^{-2}} \\
 & \qquad 3z^{-1}-2z^{-2} \\
 & \qquad \cdots
\end{array}
$$

This leads to

$$X(z) = 0 + 1z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + \cdots$$

We see that

$a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$, ...

which can be concluded as

$$x(n) = nu(n).$$

5.3.4 INVERSION FORMULA METHOD

Based on the definition of z-transform for a given sequence:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \text{ for } |z| > R, \tag{5.19}$$

where $|z| > R$ specifies the region of convergence. From the complex variable and function analysis, we have

$$x(n) = \frac{1}{2\pi j}\oint_C z^{n-1}X(z)dz, \text{ for } n \geq 0, \tag{5.20}$$

where C is any simple closed curve enclosing $|z| = R$, and $\oint_C$ designates contour integration along C in the counterclockwise direction.

Proof: Applying the integral $\frac{1}{2\pi j}\oint_C z^{n-1}(.)dz$ with a close path of C on both sides of Eq. (5.20), we obtain

$$\frac{1}{2\pi j}\oint_C z^{n-1}X(z)dz = \sum_{m=0}^{\infty} x(m)\frac{1}{2\pi j}\oint_C z^{n-m}\frac{dz}{z}. \tag{5.21}$$

Let C be a circle of radius $\overline{R}$, where $\overline{R} > R$. Note that

$z = \overline{R}e^{j\theta}$ and $dz = j\overline{R}e^{j\theta}d\theta = jzd\theta$

The integral on the right side of Eq. (5.21) becomes

$$\frac{1}{2\pi j}\oint_C z^{n-m}X(z)\frac{dz}{z} = \frac{\overline{R}^{(n-m)}}{2\pi}\int_0^{2\pi} e^{j(n-m)\theta}d\theta = \begin{cases} 1 & n = m \\ 0 & \text{otherwise} \end{cases}. \tag{5.22}$$

Thus, we have

$$\frac{1}{2\pi j}\oint_C z^{n-1}X(z)dz = x(n).$$

The inversion integral can be evaluated through Cauchy's residue theorem, which is an important subject in the area of complex variables and function analysis. The key result is given by the following residue theorem.

Residue theorem: Given $X(z), |z| > R$, the inverse z-transform can be evaluated by

$$\begin{aligned} x(n) &= \frac{1}{2\pi j}\oint_C z^{n-1}X(z)dz \quad \text{for } n \geq 0 \\ &= \text{Sum of the residues } z^{n-1}X(z) \text{ corresponding to the poles of } z^{n-1}X(z) \\ &\quad \text{that reside inside a simple closed curve C enclosing } |z| = R. \end{aligned} \tag{5.23}$$

If the function $z^{n-1}X(z)$ has a simple pole at $z = p$, the residue is evaluated by

$$R_{z=p} = (z-p)z^{n-1}X(z)\big|_{z=p}. \tag{5.24}$$

For a pole with the order of m at $z = p$, the residue can be determined by

$$R_{z=p} = \frac{d^{m-1}}{dz^{m-1}}\left[\frac{(z-p)^m}{(m-1)!}z^{n-1}X(z)\right]\Bigg|_{z=a}, \tag{5.25}$$

where p is the pole value and n is the order of pole.

EXAMPLE 5.14

Given the following z-transfer function, determine $x(n)$ using the inversion formula method.

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}$$

Solution:

Since $z^{n-1}X(z) = \dfrac{z^{n+1}}{(z-1)(z-0.5)^2}$

has two poles, the sequence can be written as two residues, that is,

$$x(n) = (R_{z=1} + R_{z=0.5})u(n)$$

where

$$R_{z=1} = (z-1)\frac{z^{n+1}}{(z-1)(z-0.5)^2}\Bigg|_{z=1} = 4$$

$$\begin{aligned} R_{z=0.5} &= \frac{d^{2-1}}{dz^{2-1}}\left[\frac{(z-0.5)^2}{(2-1)!}\frac{z^{n+1}}{(z-1)(z-0.5)^2}\right]\Bigg|_{z=0.5} \\ &= \frac{d}{dz}\left[\frac{z^{n+1}}{(z-1)}\right]\Bigg|_{z=0.5} = \left[\frac{(n+1)z^n(z-1) - z^{n+1}}{(z-1)^2}\right]\Bigg|_{z=0.5} \\ &= -2(n+1)(0.5)^n - 4(0.5)^{n+1} = -2n(0.5)^n - 4(0.5)^n \end{aligned}$$

Thus, we have

$$x(n) = (R_{z=1} + R_{z=0.5})u(n) = 4u(n) - 2n(0.5)^n u(n) - 4(0.5)^n u(n)$$

EXAMPLE 5.15

Given the following z-transfer function, determine $x(n)$ using the inversion formula method.

$$X(z) = \frac{1}{z^2 - z + 0.5}$$

Solution:

$z^{n-1}X(z)$ has three poles for $n = 0$ and two poles for $n > 0$, that is,

$$z^{n-1}X(z) = \frac{1}{z(z^2 - z + 0.5)} \text{ for } n = 0$$

$$z^{n-1}X(z) = \frac{z^{n-1}}{(z^2 - z + 0.5)} \text{ for } n > 0$$

We need to apply the inversion formula for two different cases.

For $n = 0$

$x(0) = R_{z=0} + R_{z=0.5+j0.5} + R_{z=0.5-j0.5}$

$$R_{z=0} = z\frac{1}{z(z^2 - z + 0.5)} = 2$$

$$R_{z=0.5+j0.5} = (z - 0.5 - j0.5)\left.\frac{1}{z(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\right|_{z=0.5+j0.5}$$

$$= \frac{1}{-0.5 + j0.5}$$

$$R_{z=0.5-j0.5} = (z - 0.5 + j0.5)\left.\frac{1}{z(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\right|_{z=0.5-j0.5}$$

$$= \frac{1}{-0.5 - j0.5}$$

$$x(0) = 2 + \frac{1}{-0.5 + j0.5} + \frac{1}{-0.5 - j0.5} = 0$$

For $n > 0$

$x(n) = R_{z=0.5+j0.5} + R_{z=0.5-j0.5}$

$$R_{z=0.5+j0.5} = (z - 0.5 - j0.5)\left.\frac{z^{n-1}}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\right|_{z=0.5+j0.5}$$

$$= \frac{(0.5 + j0.5)^{n-1}}{j} = \frac{(0.5 + j0.5)^n}{-0.5 + j0.5}$$

$$R_{z=0.5-j0.5} = (z - 0.5 + j0.5)\left.\frac{z^{n-1}}{(z - 0.5 - j0.5)(z - 0.5 + j0.5)}\right|_{z=0.5-j0.5}$$

$$= \frac{(0.5 - j0.5)^{n-1}}{-j} = \frac{(0.5 - j0.5)^n}{-0.5 - j0.5}$$

Continued

EXAMPLE 5.15—CONT'D

For $n > 0$

$$\begin{aligned}
x(n) &= R_{z=0.5+j0.5} + R_{z=0.5-j0.5} \\
&= \frac{\left(0.707e^{\pi/4}\right)^n}{0.707e^{3\pi/4}} + \frac{\left(0.707e^{-\pi/4}\right)^n}{0.707e^{-3\pi/4}} \\
&= (0.707)^{n-1}e^{\pi n/4 - 3\pi/4} + (0.707)^{n-1}e^{-(\pi n/4 - 3\pi/4)} \\
&= 2(0.707)^{n-1}\cos(\pi n/4 - 3\pi/4)
\end{aligned}$$

Finally,

$$x(n) = \begin{cases} 0 & n = 0 \\ 2.282(0.707)^n \cos(\pi n/4 - 3\pi/4) & n > 0 \end{cases}.$$

5.4 SOLUTION OF DIFFERENCE EQUATIONS USING THE Z-TRANSFORM

To solve a difference equation with initial conditions, we have to deal with time-shifted sequences such as $y(n-1), y(n-2), \ldots, y(n-m)$, and so on. Let us examine the z-transform of these terms. Using the definition of the z-transform, we have

$$\begin{aligned}
Z(y(n-1)) &= \sum_{n=0}^{\infty} y(n-1)z^{-n} \\
&= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \cdots \\
&= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots\right).
\end{aligned}$$

It holds that

$$Z(y(n-1)) = y(-1) + z^{-1}Y(z). \tag{5.26}$$

Similarly, we can have

$$\begin{aligned}
Z(y(n-2)) &= \sum_{n=0}^{\infty} y(n-2)z^{-n} \\
&= y(-2) + y(-1)z^{-1} + y(0)z^{-2} + y(1)z^{-3} + \cdots \\
&= y(-2) + y(-1)z^{-1} + z^{-2}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \cdots\right)
\end{aligned}$$

$$Z(y(n-2)) = y(-2) + y(-1)z^{-1} + z^{-2}Y(z), \tag{5.27}$$

$$\cdots$$

$$Z(y(n-m)) = y(-m) + y(-m+1)z^{-1} + \cdots + y(-1)z^{-(m-1)} + z^{-m}Y(z), \tag{5.28}$$

where $y(-m)$, $y(-m+1)$, …, $y(-1)$ are the initial conditions. If all initial conditions are considered to be zero, that is,

$$y(-m) = y(-m+1) = \cdots = y(-1) = 0, \tag{5.29}$$

then Eq. (5.28) becomes

$$Z(y(n-m)) = z^{-m}Y(z), \tag{5.30}$$

which is the same as the shift theorem in Eq. (5.3).

The following two examples serve as illustrations of applying the z-transform to find the solutions of the difference equations. The procedure is as follows:

1. Apply z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve the difference equation in z-transform domain.
4. Find the solution in time domain by applying the inverse z-transform.

EXAMPLE 5.16

A digital signal processing (DSP) system is described by a difference equation

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n).$$

Determine the solution when the initial condition is given by $y(-1) = 1$.

Solution:

Applying the z-transform on both sides of the difference equation and using Eq. (5.28), we have

$$Y(z) - 0.5\left(y(-1) + z^{-1}Y(z)\right) = 5Z(0.2^n u(n)).$$

Substituting the initial condition and $Z(0.2^n u(n)) = Z(0.2^n u(n)) = z/(z-0.2)$, we achieve

$$Y(z) - 0.5\left(1 + z^{-1}Y(z)\right) = 5z/(z-0.2).$$

Simplification leads to

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2).$$

Factoring out $Y(z)$ and combining the right-hand side of the equation, it follows that

$$Y(z)\left(1 - 0.5z^{-1}\right) = (5.5z - 0.1)/(z-0.2).$$

Then we obtain

$$Y(z) = \frac{(5.5z - 0.1)}{(1 - 0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{5.5z - 0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2},$$

where $$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5 \times 0.5 - 0.1}{0.5 - 0.2} = 8.8333,$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5 \times 0.2 - 0.1}{0.2 - 0.5} = -3.3333.$$

Thus $$Y(z) = \frac{8.8333z}{(z-0.5)} + \frac{-3.3333z}{(z-0.2)},$$

which gives the solution as

$$y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n).$$

EXAMPLE 5.17

A relaxed (zero initial conditions) DSP system is described by a difference equation

$$y(n) + 0.1y(n-1) - 0.2y(n-2) = x(n) + x(n-1).$$

(a) Determine the impulse response $y(n)$ due to the impulse sequence $x(n) = \delta(n)$.

(b) Determine the system response $y(n)$ due to the unit step function excitation, where $u(n) = 1$ for $n \geq 0$.

Solution:

(a) Applying the z-transform on both sides of the difference equation and using Eq. (5.3) or Eq. (5.30), we yield

$$Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}.$$

Factoring out $Y(z)$ on the left-hand side and substituting $X(z) = Z(\delta(n)) = 1$ on the right-hand side of the above equation achieves

$$Y(z)\left(1 + 0.1z^{-1} - 0.2z^{-2}\right) = 1\left(1 + z^{-1}\right).$$

Then $Y(z)$ can be expressed as

$$Y(z) = \frac{1 + z^{-1}}{1 + 0.1z^{-1} - 0.2z^{-2}}.$$

To obtain the impulse response, which is the inverse z-transform of the transfer function, we multiply the numerator and denominator by z^2.

Thus

$$Y(z) = \frac{z^2 + z}{z^2 + 0.1z - 0.2} = \frac{z(z+1)}{(z-0.4)(z+0.5)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.4)(z+0.5)} = \frac{A}{z-0.4} + \frac{B}{z+0.5},$$

where

$$A = (z-0.4)\frac{Y(z)}{z}\bigg|_{z=0.4} = \frac{z+1}{z+0.5}\bigg|_{z=0.4} = \frac{0.4+1}{0.4+0.5} = 1.5556$$

$$B = (z+0.5)\frac{Y(z)}{z}\bigg|_{z=-0.5} = \frac{z+1}{z-0.4}\bigg|_{z=-0.5} = \frac{-0.5+1}{-0.5-0.4} = -0.5556.$$

Thus

$$Y(z) = \frac{1.5556z}{(z-0.4)} + \frac{-0.5556z}{(z+0.5)}$$

which gives the impulse response:

$$y(n) = 1.5556(0.4)^n u(n) - 0.5556(-0.5)^n u(n).$$

(b) To obtain the response due to a unit step function, the input sequence is set to be

$$x(n) = u(n)$$

and the corresponding z-transform is given by

$$X(z) = \frac{z}{z-1},$$

and note that

$$Y(z) + 0.1Y(z)z^{-1} - 0.2Y(z)z^{-2} = X(z) + X(z)z^{-1}.$$

Then the z-transform of the output sequence $y(n)$ can be yielded as

$$Y(z) = \left(\frac{z}{z-1}\right)\left(\frac{1+z^{-1}}{1+0.1z^{-1}-0.2z^{-2}}\right) = \frac{z^2(z+1)}{(z-1)(z-0.4)(z+0.5)}.$$

Using the partial fraction expansion method as before gives

$$Y(z) = \frac{2.2222z}{z-1} + \frac{-1.0370z}{z-0.4} + \frac{-0.1852z}{z+0.5},$$

and the system response is found by using Table 5.1:

$$y(n) = 2.2222u(n) - 1.0370(0.4)^n u(n) - 0.1852(-0.5)^n u(n).$$

Since $x(n) = u(n) = \sum_{k=0}^{n} \delta(k)$, we expect $y_b(n) = \sum_{k=0}^{n} y_a(k)$ based on linearity, where $y_a(k)$ is the response from (a) while $y_b(n)$ the response from (b). The verification is shown below: For $n \geq 0$

$$\begin{aligned} y_b(n) &= \sum_{k=0}^{n}\left[1.5556(0.4)^k - 0.5556(-0.5)^k\right] = \sum_{k=0}^{n} 1.5556(0.4)^k - \sum_{k=0}^{n} 0.5556(-0.5)^k \\ &= 1.5556\frac{1-(0.4)^{n+1}}{1-0.4} - 0.5556\frac{1-(-0.5)^{n+1}}{1-(-0.5)} \\ &= 2.2222 - 1.0370(0.4)^n - 0.1852(-0.5)^n. \end{aligned}$$

5.5 TWO-SIDED Z-TRANSFORM

It is practical to study z-transform for a causal sequence, that is, $x(n) = 0$ for $n < 0$. However, from a theoretical point of view, we can briefly investigate a two-sized z-transform, which is applied for a noncausal sequence, $x(n) \neq 0$ for $n < 0$, that is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}. \tag{5.31}$$

Note that Eq. (5.31) converges everywhere outside a circle with a radius R_1 if $x(n) = 0$ for $n < 0$; and on the other hand, Eq. (5.31) converges everywhere inside a circle with a radius R_2 if $x(n) = 0$ for $n \geq 0$. Thus, region of convergence for Eq. (5.31) becomes an annular region , that is, $R_1 < |z| < R_2$.

The inversion formula according to Cauchy's residue theorem for the two-sized z-transform is listed below

$$x(n) = \frac{1}{2\pi j}\oint_C z^{n-1}X(z)dz \ \text{ for } |n| < \infty$$
$$= \begin{cases} \text{Sum of the residues } z^{n-1}X(z) \text{ corresponding to the poles of } z^{n-1}X(z) \\ \text{that reside inside a simple closed curve C enclosing } |z| = R_1 & n \geq 0 \\ -\text{Sum of the residues } z^{n-1}X(z) \text{ corresponding to the poles of } z^{n-1}X(z) \\ \text{that reside outside a simple closed curve C enclosing } |z| = R_2 & n < 0 \end{cases}. \tag{5.32}$$

Let us examine the following examples.

EXAMPLE 5.18

Given the noncausal sequence defined below

$$x(n) = \begin{cases} (-2)^n & n < 0 \\ (0.5)^n & n \geq 0 \end{cases}$$

Determine $X(z)$.

Solution:

Appling Eq. (5.31), it follows that

$$X(z) = \sum_{n=-\infty}^{-1}\left(-2z^{-1}\right)^n + \sum_{n=0}^{\infty}\left(0.5z^{-1}\right)^2 = X_1(z) + X_2(z)$$

where

$$X_1(z) = \sum_{n=-\infty}^{-1}\left(-2z^{-1}\right)^n = \sum_{n=0}^{\infty}\left[(-2)^{-1}z\right]^n - 1 = \frac{1}{1-(-2)^{-1}z} - 1,$$

for $|(-2)^{-1}z| < 1$.

That is,

$$X_1(z) = \frac{1}{1-(-2)^{-1}z} - 1 = \frac{-0.5z}{1+0.5z} \text{ for } |z| < 2$$

and

$$X_2(z) = \sum_{n=0}^{\infty}\left(0.5z^{-1}\right)^2 = \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5} \text{ for } |z| > 0.5.$$

Finally, we achieve

$$X(z) = \frac{-0.5z}{1+0.5z} + \frac{z}{z-0.5} = \frac{1.25z}{(1+0.5z)(z-0.5,)} \text{ for } 0.5 < |z| < 2.$$

EXAMPLE 5.19

Given

$$X(z) = \frac{-0.5z}{(z-0.5)(z-1)} \text{ for } 0.5 < |z| < 1,$$

Determine $x(n)$.

Solution:

We first form

$$z^{n-1}X(z) = \frac{-0.5z^n}{(z-0.5)(z-1)}.$$

A closed path C is chosen and plotted in Fig. 5.1 for $n \geq 0$ and $n < 0$ with poles marked correspondingly.

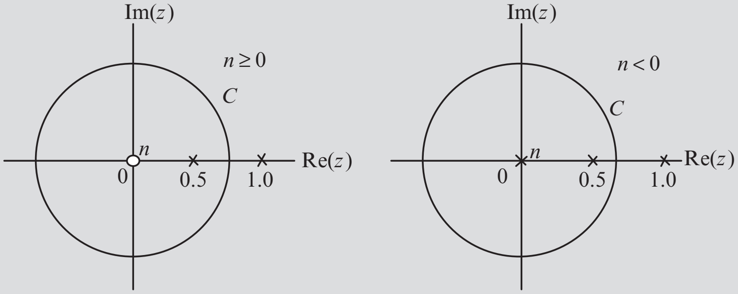

FIG. 5.1

Plots of the poles $z^{n-1}X(z)$ in Example 5.19 for $n \geq 0$ and $n < 0$, respectively.

For $n \geq 0$, C encloses only pole $z = 0.5$. Applying Eq. (5.32), it follows that

$$x(n) = (z-0.5)\frac{-0.5z^n}{(z-0.5)(z-1)}\bigg|_{z=0.5} = (0.5)^n.$$

For $n < 0$, there is only one pole $z = 1$ outside C. The pole at $z = 0.5$ and nth-order pole at $z = 0$ are inside C. Applying Eq. (5.32) leads to

$$x(n) = -(z-1)\frac{-0.5z^n}{(z-0.5)(z-1)}\bigg|_{z=1} = 1.$$

The combined result is given below:

$$x(n) = \begin{cases} 1 & n < 0 \\ (0.5)^n & n \geq 0 \end{cases}.$$

5.6 SUMMARY

1. The one-sided (unilateral) z-transform was defined, which can be used to transform the causal sequence to the z-transform domain.
2. The look-up table of the z-transform determines the z-transform for a simple causal sequence, or the causal sequence from a simple z-transform function.
3. The important properties of the z-transform, such as linearity, shift theorem, convolution, and initial and final value theorems were introduced. The shift theorem can be used to solve a difference equation. The z-transform of a digital convolution of two digital sequences is equal to the product of their z-transforms.

4. The method of finding the inverse of the z-transform, such as the partial fraction expansion, inverses the complicated z-transform function, which can have first-order real poles, multiple-order real poles, and first-order complex poles assuming that the z-transform function is proper. The MATLAB tool for the partial fraction expansion method was introduced. In addition, the power series and inversion formula methods were also described.
5. The z-transform can be applied to solve linear difference equations with nonzero initial conditions and zero initial conditions.
6. Two-sided z-transform was briefly introduced.

5.7 PROBLEMS

5.1 Find the z-transform for each of the following sequences:
- **(a)** $x(n) = 4u(n)$
- **(b)** $x(n) = (-0.7)^n u(n)$
- **(c)** $x(n) = 4e^{-2n}u(n)$
- **(d)** $x(n) = 4(0.8)^n \cos(0.1\pi n)u(n)$
- **(e)** $x(n) = 4e^{-3n}\sin(0.1\pi n)u(n)$.

5.2 Using the properties of the z-transform, find the z-transform for each of the following sequences:
- **(a)** $x(n) = u(n) + (0.5)^n u(n)$
- **(b)** $x(n) = e^{-3(n-4)}\cos(0.1\pi(n-4))u(n-4)$, where $u(n-4) = 1$ for $n \geq 4$ while $u(n-4) = 0$ for $n < 4$.

5.3 Find the z-transform for each of the following sequences:
- **(a)** $x(n) = 3u(n-4)$
- **(b)** $x(n) = 2(-0.5)^n u(n)$
- **(c)** $x(n) = 5e^{-2(n-3)}u(n-3)$
- **(d)** $x(n) = 6(0.6)^n \cos(0.2\pi n)u(n)$
- **(e)** $x(n) = 4e^{-3(n-1)}\sin(0.2\pi(n-1))u(n-1)$.

5.4 Using the properties of the z-transform, find the z-transform for each of the following sequences:
- **(a)** $x(n) = -2u(n) - (0.75)^n u(n)$
- **(b)** $x(n) = e^{-2(n-3)}\sin(0.2\pi(n-3))u(n-3)$, where $u(n-3) = 1$ for $n \geq 3$ while $u(n-3) = 0$ for $n < 3$.

5.5 Given two sequences

$$x_1(n) = 5\delta(n) - 2\delta(n-2) \text{ and } x_2(n) = 3\delta(n-3),$$

- **(a)** Determine the z-transform of convolution of the two sequences using the convolution property of z-transform

$$X(z) = X_1(z)X_2(z);$$

- **(b)** Determine the convolution by the inverse z-transform from the result in (a)

$$x(n) = Z^{-1}(X_1(z)X_2(z)).$$

5.6 Using Table 5.1 and z-transform properties, find the inverse z-transform for each of the following functions:

(a) $X(z) = 4 - \frac{10z}{z-1} - \frac{z}{z+0.5}$

(b) $X(z) = \frac{-5z}{(z-1)} + \frac{10z}{(z-1)^2} + \frac{2z}{(z-0.8)^2}$

(c) $X(z) = \frac{z}{z^2+1.2z+1}$

(d) $X(z) = \frac{4z^{-4}}{z-1} + \frac{z^{-1}}{(z-1)^2} + z^{-8} + \frac{z^{-5}}{z-0.5}$

5.7 Given two sequences

$$x_1(n) = -2\delta(n) + 5\delta(n-2) \text{ and } x_2(n) = 4\delta(n-4),$$

(a) Determine the z-transform of convolution of the two sequences using the convolution property of z-transform

$$X(z) = X_1(z)X_2(z);$$

(b) Determine convolution by the inverse z-transform from the result in (a)

$$x(n) = Z^{-1}(X_1(z)X_2(z)).$$

5.8 Using Table 5.1 and z-transform properties, find the inverse z-transform for each of the following functions:

(a) $X(z) = 5 - \frac{7z}{z+1} - \frac{3z}{z-0.5};$

(b) $X(z) = \frac{-3z}{(z-0.5)} + \frac{8z}{(z-0.8)} + \frac{2z}{(z-0.8)^2}$

(c) $X(z) = \frac{3z}{z^2+1.414z+1}$

(d) $X(z) = \frac{5z^{-5}}{z-1} - \frac{z^{-2}}{(z-1)^2} + z^{-10} + \frac{z^{-3}}{z-0.75}$

5.9 Using the partial fraction expansion method, find the inverse of the following z-transforms:

(a) $X(z) = \frac{1}{z^2-0.3z-0.04}$

(b) $X(z) = \frac{z}{(z-0.2)(z+0.4)}$

(c) $X(z) = \frac{z}{(z+0.2)(z^2-z+0.5)}$

(d) $X(z) = \frac{z(z+0.5)}{(z-0.1)^2(z-0.6)}$

5.10 A system is described by the difference equation

$$y(n) + 0.5y(n-1) = 2(0.8)^n u(n).$$

Determine the solution when the initial condition is $y(-1) = 2$.

5.11 Using the partial fraction expansion method, find the inverse of the following z-transforms:

(a) $X(z) = \dfrac{1}{z^2 + 0.5z + 0.06}$

(b) $X(z) = \dfrac{z}{(z+0.3)(z-0.5)}$

(c) $X(z) = \dfrac{5z}{(z-0.75)(z^2 - z + 0.5)}$

(d) $X(z) = \dfrac{2z(z-0.4)}{(z-0.2)^2(z+0.8)}$

5.12 A system is described by the difference equation

$$y(n) + 0.2y(n-1) = 4(0.3)^n u(n).$$

Determine the solution when the initial condition is $y(-1) = 1$.

5.13 A system is described by the difference equation

$$y(n) - 0.5y(n-1) + 0.06y(n-2) = (0.4)^{n-1}u(n-1).$$

Determine the solution when the initial conditions are $y(-1) = 1$, and $y(-2) = 2$.

5.14 Given the following difference equation with the input-output relationship of a certain initially relaxed system (all initial conditions are zero),

$$y(n) - 0.7y(n-1) + 0.1y(n-2) = x(n) + x(n-1),$$

(a) Find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$.
(b) Find the output response of the system when the unit step function $u(n)$ is applied.

5.15 A system is described by the difference equation

$$y(n) - 0.6y(n-1) + 0.08y(n-2) = (0.5)^{n-1}u(n-1).$$

Determine the solution when the initial conditions are $y(-1) = 2$, and $y(-2) = 1$.

5.16 Given the following difference equation with the input-output relationship of a certain initially relaxed system (all initial conditions are zero),

$$y(n) - 0.6y(n-1) + 0.25y(n-2) = x(n) + x(n-1),$$

(a) Find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$.
(b) Find the output response of the system when the unit step function $u(n)$ is applied.

5.17 Given the following difference equation with the input-output relationship of a certain initially relaxed DSP system (all initial conditions are zero),

$$y(n) - 0.4y(n-1) + 0.29y(n-2) = x(n) + 0.5x(n-1),$$

(a) Find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$.
(b) Find the output response of the system when the unit step function $u(n)$ is applied.

5.18 Given the following difference equation with the input-output relationship of a certain initially relaxed DSP system (all initial conditions are zero)

$$y(n) - 0.2y(n-1) + 0.17y(n-2) = x(n) + 0.3x(n-1),$$

(a) Find the impulse response sequence $y(n)$ due to the impulse sequence $\delta(n)$.
(b) Find the output response of the system when the unit step function $u(n)$ is applied.

5.19 Use the initial and final value theorems to find $x(0)$ and $x(\infty)$ for Problem 5.11(a), (b), (d).
5.20 Use power series method to find $x(0)$, $x(1)$, $x(2)$, $x(3)$, $x(4)$ for Problem 5.11(a), (b).
5.21 Use the residue formula to find the inverse of the z-transform for Problem 5.11(a), (b), (d).

Advanced Problems:

5.22 If $y(n) = e^{-an}x(n)$, where $a \geq 0$ and $n \geq 0$, show that

$$Y(z) = X(ze^a).$$

5.23 If $y(n) = nx(n)$, where $a \geq 0$ and $n \geq 0$, show that

$$Y(z) = -z\frac{dX(z)}{dz}.$$

5.24 Given a difference equation

$$y(n) - y(n-1) + y(n-2) = 0, \ n \geq 0$$

with initial conditions $y(-1) = 1$ and $y(-2) = 0$.
(a) Show that

$$Y(z) = \frac{z(z-1)}{(z - e^{j\pi/3})(z - e^{-j\pi/3})};$$

(b) Use the inversion formula method to show that

$$y(n) = \frac{\sin[(n+1)\pi/3] - \sin(n\pi/3)}{\sin(\pi/3)}, \ n \geq 0.$$

5.25 Given a difference equation

$$y(n) = 2\cos(\Omega_0)y(n-1) - y(n-2), \ n \geq 0$$

with initial conditions $y(-1) = -\sin(\Omega_0)$ and $y(-2) = -\sin(2\Omega_0)$, show that

$$y(n) = \sin(\Omega_0 n), n \geq 0.$$

5.26 For the following rational z-transfer function

$$X(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \cdots}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3} + \cdots}$$

and the definition of z-transform: $X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \cdots$, show that

$$x(0) = b_0 \text{ and } x(n) = b_n - \sum_{k=1}^{n} x(n-k)a_k \text{ for } n > 0.$$

5.27 Suppose

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}.$$

Show that

$$\sum_{n=0}^{\infty} x^2(n) = \frac{1}{2\pi j}\oint_C z^{-1}X(z)X\left(z^{-1}\right)dz.$$

5.28 For $z = e^{j\Omega}$, show that

$$\sum_{n=0}^{\infty} x^2(n) = \frac{1}{2\pi}\oint_C \left|X\left(e^{j\Omega}\right)\right|^2 d\Omega.$$

5.29 Given

$$X(z) = \frac{(a-1)}{(1-az^{-1})(z-1)} \text{ and } a < |z| < 1, 0 < a < 1,$$

use the inversion formula to show that

$$x(n) = \begin{cases} 1 & n < 0 \\ a^n & n \geq 0 \end{cases}.$$

5.30 Given

$$X(z) = \frac{(a-b)}{(1-az^{-1})(z-b)} \quad \text{and} \quad a < |z| < b,\ 0 < a < 1 \text{ and } b > 1,$$

use the inversion formula to show that

$$x(n) = \begin{cases} b^n & n < 0 \\ a^n & n \geq 0 \end{cases}.$$