APPENDIX

NORMALIZED BUTTERWORTH AND CHEBYSHEV FUNCTIONS

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C.1 NORMALIZED BUTTERWORTH FUNCTION

The normalized Butterworth squared magnitude function is given by

$$|P_n(\omega)|^2 = \frac{1}{1+\varepsilon^2(\omega)^{2n}}, \tag{C.1}$$

where n is the order and ε is the specified ripple on filter passband. The specified ripple in dB is expressed as $\varepsilon_{dB} = 10 \cdot \log_{10}(1+\varepsilon^2)$ dB.

To develop the transfer function $P_n(s)$, we first let $s = j\omega$ and then substitute $\omega^2 = -s^2$ into Eq. (C.1) to obtain

$$P_n(s)P_n(-s) = \frac{1}{1+\varepsilon^2(-s^2)^n}. \tag{C.2}$$

Eq. (C.2) has $2n$ poles, and $P_n(s)$ has n poles on the left-hand half plane (LHHP) on the s-plane, while $P_n(-s)$ has n poles on the right-hand half plane (RHHP) on the s-plane. Solving for poles leads to

$$(-1)^n s^{2n} = -1/\varepsilon^2. \tag{C.3}$$

If n is an odd number, Eq. (C.3) becomes

$$s^{2n} = 1/\varepsilon^2$$

and the corresponding poles are solved as

$$p_k = \varepsilon^{-1/n} e^{j\frac{2\pi k}{2n}} = \varepsilon^{-1/n}[\cos(2\pi k/2n) + j\sin(2\pi k/2n)], \tag{C.4}$$

where $k = 0, 1, \cdots, 2n-1$. Thus in the phasor form, we have

$$r = \varepsilon^{-1/n}, \text{ and } \theta_k = 2\pi k/(2n) \text{ for } k = 0, 1, \cdots, 2n-1. \tag{C.5}$$

When n is an even number, it follows that

$$s^{2n} = -1/\varepsilon^2$$

$$p_k = \varepsilon^{-1/n} e^{j\frac{2\pi k+\pi}{2n}} = \varepsilon^{-1/n}[\cos((2\pi k+\pi)/2n) + j\sin((2\pi k+\pi)/2n)], \tag{C.6}$$

where $k = 0, 1, \cdots, 2n-1$. Similarly, the phasor form is given by

$$r = \varepsilon^{-1/n}, \text{ and } \theta_k = (2\pi k + \pi)/(2n) \text{ for } k = 0, 1, \cdots, 2n-1. \tag{C.7}$$

When n is an odd number, we can identify the poles on the LHHP as

$$\begin{aligned} p_k &= -r, k = 0 \text{ and} \\ p_k &= -r\cos(\theta_k) + jr\sin(\theta_k), k = 1, \cdots, (n-1)/2 \end{aligned}. \tag{C.8}$$

Using complex conjugate pairs, we have

$$p_k^* = -r\cos(\theta_k) - jr\sin(\theta_k).$$

Note that

$$(s - p_k)(s - p_k^*) = s^2 + (2r\cos(\theta_k))s + r^2,$$

and a factor from the real pole $(s + r)$, it follows that

$$P_n(s) = \frac{K}{(s+r)\prod_{k=1}^{(n-1)/2}(s^2 + (2r\cos(\theta_k))s + r^2)} \tag{C.9}$$

and $\theta_k = 2\pi k/(2n)$ for $k = 1, \cdots, (n-1)/2$.

Setting $P_n(0) = 1$ for the unit passband gain leads to

$$K = r^n = 1/\varepsilon.$$

When n is an even number, we can identify the poles on the LHHP as

$$p_k = -r\cos(\theta_k) + jr\sin(\theta_k), k = 0, 1, \cdots, n/2 - 1. \tag{C.10}$$

Using complex conjugate pairs, we have

$$p_k^* = -r\cos(\theta_k) - jr\sin(\theta_k).$$

The transfer function is given by

$$\begin{aligned} P_n(s) &= \frac{K}{\prod_{k=0}^{n/2-1}\left(s^2 + (2r\cos(\theta_k))s + r^2\right)} \\ \theta_k &= (2\pi k + \pi)/(2n) \text{ for } k = 0, 1, \cdots, n/2 - 1 \end{aligned}. \tag{C.11}$$

Setting $P_n(0) = 1$ for the unit passband gain, we have

$$K = r^n = 1/\varepsilon.$$

Let us examine the following examples.

EXAMPLE C.1

Compute the normalized Butterworth transfer function for the following specifications:
Ripple = 3 dB

$n = 2$

Solution:

$$n/2 = 1$$

$$\theta_k = (2\pi \times 0 + \pi)/(2 \times 2) = 0.25\pi$$

$$\varepsilon^2 = 10^{0.1 \times 3} - 1,$$

$$r = 1, \text{ and } K = 1.$$

Applying Eq. (C.11) leads to

$$P_2(s) = \frac{1}{s^2 + 2 \times 1 \times \cos(0.25\pi)s + 1^2} = \frac{1}{s^2 + 1.4141s + 1}.$$

EXAMPLE C.2

Compute the normalized Butterworth transfer function for the following specifications:
Ripple = 3 dB

$n = 3$

Solution:

$$(n-1)/2 = 1$$

$$\varepsilon^2 = 10^{0.1 \times 3} - 1,$$

$$r = 1, \text{ and } K = 1$$

$$\theta_k = (2\pi \times 1)/(2 \times 3) = \pi/3.$$

From Eq. (C.9), we have

$$P_3(s) = \frac{1}{(s+1)\left(s^2 + 2 \times 1 \times \cos(\pi/3)s + 1^2\right)}$$
$$= \frac{1}{(s+1)(s^2 + s + 1)}.$$

For the unfactored form, we can carry out

$$P_3(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}.$$

EXAMPLE C.3

Compute the normalized Butterworth transfer function for the following specifications:
Ripple = 1.5 dB

$n = 3$

Solution:

$$(n-1)/2 = 1$$

$$\varepsilon^2 = 10^{0.1\times1.5} - 1,$$

$$r = 1.1590, \text{ and } K = 1.5569$$

$$\theta_k = (2\pi \times 1)/(2\times 3) = \pi/3.$$

Applying Eq. (C.9), we achieve the normalized Butterworth transfer function as

$$P_3(s) = \frac{1}{(s+1.1590)\left(s^2 + 2\times 1.1590 \times \cos(\pi/3)s + 1.1590^2\right)}$$
$$= \frac{1}{(s+1)(s^2+1.1590s+1.3433)}.$$

For the unfactored form, we can carry out

$$P_3(s) = \frac{1.5569}{s^3 + 2.3180s^2 + 2.6866s + 1.5569}.$$

C.2 NORMALIZED CHEBYSHEV FUNCTION

Similar to analog Butterworth filter design, the transfer function is derived from the normalized Chebyshev function, and the result is usually listed in a table for design reference. The Chebyshev magnitude response function with an order of n and the normalized cutoff frequency $\omega = 1$ radian per second is given by

$$|B_n(\omega)| = \frac{1}{\sqrt{1+\varepsilon^2 C_n^2(\omega)}}, n \geq 1, \tag{C.12}$$

where the function $C_n(\omega)$ is defined as

$$C_n(\omega) = \begin{cases} \cos\left(n\cos^{-1}(\omega)\right) & \omega \leq 1 \\ \cosh\left(n\cosh^{-1}(\omega)\right) & \omega > 1 \end{cases}, \tag{C.13}$$

where ε is the ripple specification on the filter passband. Note that

$$\cosh^{-1}(x) = \ln\left(x + \sqrt{x^2-1}\right). \tag{C.14}$$

To develop the transfer function $B_n(s)$, we let $s = j\omega$ and substitute $\omega^2 = -s^2$ into Eq. (C.12) to obtain

$$B_n(s)B_n(-s) = \frac{1}{1+\varepsilon^2 C_n^2(s/j)}. \tag{C.15}$$

The poles can be found from

$$1+\varepsilon^2 C_n^2(s/j)=0$$

or

$$C_n(s/j)=\cos\left(n\cos^{-1}(s/j)\right)=\pm j1/\varepsilon. \tag{C.16}$$

Introduce a complex variable $v=\alpha+j\beta$ such that

$$v=\alpha+j\beta=\cos^{-1}(s/j), \tag{C.17}$$

we can then write

$$s=j\cos(v). \tag{C.18}$$

Substituting Eq. (C.17) into Eq, (C.16) and using trigonometric identities, it follows that

$$\begin{aligned} C_n(s/j) &= \cos(n\cos^{-1}(s/j)) \\ &= \cos(nv)=\cos(n\alpha+jn\beta) \\ &= \cos(n\alpha)\cosh(n\beta)-j\sin(n\alpha)\sinh(n\beta)=\pm j1/\varepsilon. \end{aligned} \tag{C.19}$$

To solve Eq. (C.19), the following conditions must be satisfied:

$$\cos(n\alpha)\cosh(n\beta)=0 \tag{C.20}$$

$$-\sin(n\alpha)\sinh(n\beta)=\pm 1/\varepsilon. \tag{C.21}$$

Since $\cosh(n\beta)\geq 1$ in Eq. (C.20), we must let

$$\cos(n\alpha)=0, \tag{C.22}$$

which therefore leads to

$$\alpha_k=(2k+1)\pi/(2n), k=0,1,2,\cdots,2n-1. \tag{C.23}$$

With Eq. (C.23), we have $\sin(n\alpha_k)=\pm 1$. Then Eq. (C.21) becomes

$$\sinh(n\beta)=1/\varepsilon. \tag{C.24}$$

Solving Eq. (C.24) gives

$$\beta=\sinh^{-1}(1/\varepsilon)/n. \tag{C.25}$$

Again from Eq. (C.18),

$$\begin{aligned} &s=j\cos(v)=j[\cos(\alpha_k)\cosh(\beta)-j\sin(\alpha_k)\sinh(\beta)] \\ &\text{for } k=0,1,\cdots,2n-1 \end{aligned}. \tag{C.26}$$

The poles can be found from Eq. (C.26):

$$\begin{aligned} &p_k=\sin(\alpha_k)\sinh(\beta)+j\cos(\alpha_k)\cosh(\beta) \\ &\text{for } k=0,1,\cdots,2n-1 \end{aligned}. \tag{C.27}$$

Using Eq. (C.27), if n is an odd number, the poles on the left-hand side are solved to be

$$p_k=\sin(\alpha_k)\sinh(\beta)+j\cos(\alpha_k)\cosh(\beta) \text{ for } k=0,1,\cdots,2n-1. \tag{C.28}$$

Using complex conjugate pairs, we have

$$p_k^* = -\sin(\alpha_k)\sinh(\beta) - j\cos(\alpha_k)\cosh(\beta) \tag{C.29}$$

and a real pole

$$p_k = -\sinh(\beta), k = (n-1)/2. \tag{C.30}$$

Note that

$$(s - p_k)(s - p_k^*) = s^2 + b_k s + c_k \tag{C.31}$$

and a factor from the real pole $[s + \sinh(\beta)]$, it follows that

$$B_n(s) = \frac{K}{[s + \sinh(\beta)] \prod_{k=0}^{(n-1)/2-1} (s^2 + b_k s + c_k)}, \tag{C.32}$$

where

$$b_k = 2\sin(\alpha_k)\sinh(\beta) \tag{C.33}$$

$$c_k = [\sin(\alpha_k)\sinh(\beta)]^2 + [\cos(\alpha_k)\cosh(\beta)]^2 \tag{C.34}$$

$$\alpha_k = (2k+1)\pi/(2n) \text{ for } k = 0, 1, \cdots, (n-1)/2 - 1. \tag{C.35}$$

For the unit passband gain and the filter order as an odd number, we set $B_n(0) = 1$. Then

$$K = \sinh(\beta) \prod_{k=0}^{(n-1)/2-1} c_k \tag{C.36}$$

$$\beta = \sinh^{-1}(1/\varepsilon)/n \tag{C.37}$$

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right). \tag{C.38}$$

Following the similar procedure for the even number of n, we have

$$B_n(s) = \frac{K}{\prod_{k=0}^{n/2-1} (s^2 + b_k s + c_k)} \tag{C.39}$$

$$b_k = 2\sin(\alpha_k)\sinh(\beta) \tag{C.40}$$

$$c_k = [\sin(\alpha_k)\sinh(\beta)]^2 + [\cos(\alpha_k)\cosh(\beta)]^2 \tag{C.41}$$

$$\text{where } \alpha_k = (2k+1)\pi/(2n) \text{ for } k = 0, 1, \cdots, n/2 - 1. \tag{C.42}$$

For the unit passband gain and the filter order as an even number, we require that $B_n(0) = 1/\sqrt{1+\varepsilon^2}$, so that the maximum magnitude of the ripple on passband equals 1. Then we have

$$K = \prod_{k=0}^{n/2-1} c_k/\sqrt{1+\varepsilon^2} \tag{C.43}$$

$$\beta = \sinh^{-1}(1/\varepsilon)/n \tag{C.44}$$

$$\sinh^{-1}(x) = \ln\left(x + \sqrt{x^2+1}\right). \tag{C.45}$$

Eqs. (C.32) to (C.45) are applied to compute the normalized Chebyshev transfer function. Now let us look at the following illustrative examples.

EXAMPLE C.4

Compute the normalized Chebyshev transfer function for the following specifications:
Ripple = 0.5 dB

$n = 2$

Solution:

$$n/2 = 1.$$

Applying Eqs. (C.39) to (C.45), we obtain

$$\alpha_0 = (2\times 0+1)\pi/(2\times 2) = 0.25\pi$$

$$\varepsilon^2 = 10^{0.1\times 0.5} - 1 = 0.1220, 1/\varepsilon = 2.8630$$

$$\beta = \sinh^{-1}(2.8630)/n = \ln\left(2.8630 + \sqrt{2.8630^2+1}\right)/2 = 0.8871$$

$$b_0 = 2\sin(0.25\pi)\sinh(0.8871) = 1.4256$$

$$c_0 = [\sin(0.25\pi)\sinh(0.8871)]^2 + [\cos(0.25\pi)\cosh(0.8871)]^2 = 1.5162$$

$$K = 1.5162/\sqrt{1+0.1220} = 1.4314.$$

Finally, the transfer function is derived as

$$B_2(s) = \frac{1..4314}{s^2 + 1.4256s + 1.5162}.$$

EXAMPLE C.5

Compute the normalized Chebyshev transfer function for the following specifications:
Ripple = 1 dB

$n = 3$

Solution:

$$(n-1)/2 = 1.$$

Applying Eqs. (C.32) to (C.38) leads to

$$\alpha_0 = (2\times 0+1)\pi/(2\times 3) = \pi/6$$

$$\varepsilon^2 = 10^{0.1\times 1} - 1 = 0.2589, 1/\varepsilon = 1.9653$$

$$\beta = \sinh^{-1}(1.9653)/n = \ln\left(1.9653 + \sqrt{1.9653^2+1}\right)/3 = 0.4760$$

$$b_0 = 2\sin(\pi/6)\sinh(0.4760) = 0.4942$$

$$c_0 = [\sin(\pi/6)\sinh(0.4760)]^2 + [\cos(\pi/6)\cosh(0.4760)]^2 = 0.9942$$

$$\sinh(\beta) = \sinh(0.4760) = 0.4942$$

$$K = 0.4942 \times 0.9942 = 0.4913.$$

We can conclude the transfer function as

$$B_3(s) = \frac{0.4913}{(s + 0.4942)(s^2 + 0.4942s + 0.9942)}.$$

Finally, the unfactored form is found to be

$$B_3(s) = \frac{0.4913}{s^3 + 0.9883s^2 + 1.2384s + 0.4913}.$$