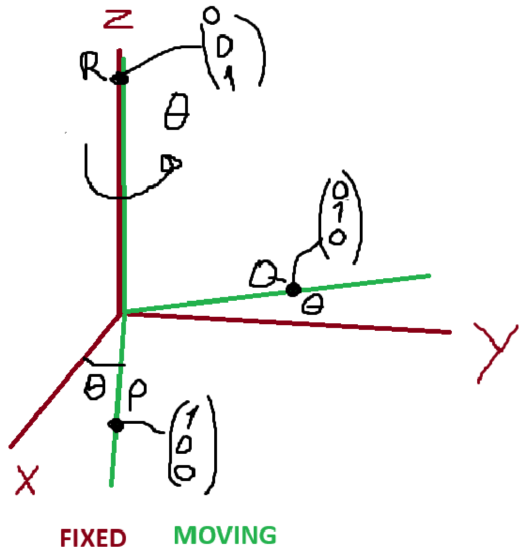
Robot Kinematics

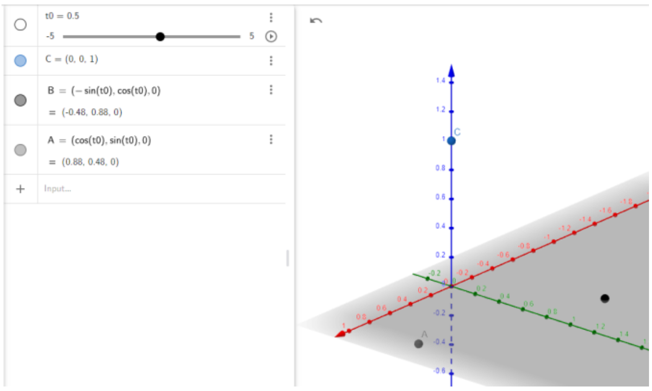
Rotation Matrices

Understanding frames and rotation matrices are crucial for deep understanding of kinematics. Frames are used everywhere in kinematics and is a helpful tool because it enables us to kinematic calculations in matrix form.

Source: https://www.youtube.com/watch?v=ZNcdYDGP-PA

Z-axis





In this example, the axes have been rotated around the Z axis by θ degrees. We want to find point P's position vector in the fixed frame. We can see that in P's moving frame, it is at location (1, 0, 0). The location of P in the fixed frame is called X. The location of P in the moving frame is called x.

$$X = A * x$$

Where A is a rotation matrix. This means that X is a rotation matrix multiplied with x. If the moving frame was moved backwards to the fixed frame, we can see that it's position vector would be correct. We can see from GeoGebra that P's location in the fixed frame is $(\cos(\theta)\,, \sin(\theta)\,, 0)$, therefore:

$$\begin{matrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{matrix} = A * x = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} * \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{bmatrix} * \begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix} = \begin{bmatrix} a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31} \\ a_{21} * b_{11} + a_{22} * b_{21} + a_{32} * b_{31} \\ a_{31} * b_{11} + a_{32} * b_{21} + a_{33} * b_{31} \end{bmatrix}$$

Because of this multiplication we know that $a_{11} * b_{11}$ must be equal to cos (θ). The rest is impossible to know. This means X is solved.

$$X = \begin{matrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{matrix}$$

For point Q. X is the location of Q in the fixed frame. Q's location in the moving frame is x. A is the rotation matrix.

$$X = A * x$$

$$A * x = \begin{bmatrix} \cos(\theta) & ? & ? \\ \sin(\theta) & ? & ? \\ 0 & ? & ? \end{bmatrix} * \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

We can see from GeoGebra that $Q = (-\sin(\theta), \cos(\theta), 0)$, therefore:

$$\begin{matrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{matrix} = A * x = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & ? \\ \sin(\theta) & \cos(\theta) & ? \\ 0 & 0 & ? \end{bmatrix} * \begin{Bmatrix} 0 \\ 1 \\ 0 \end{Bmatrix}$$

For point R. X is R's location in fixed frame. R's location in moving frame is x.

$$X = A * x$$

$$A * x = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & ? \\ \sin(\theta) & \cos(\theta) & ? \\ 0 & 0 & ? \end{bmatrix} * \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

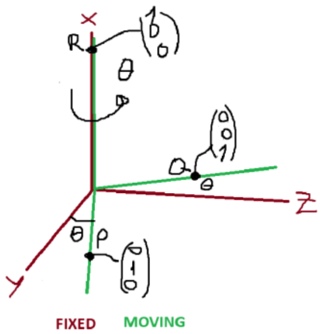
Looking at geogebra we know that

$$\begin{matrix} 0 \\ 0 \\ 1 \end{matrix} = A * x = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} * \begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}$$

We know have the rotation matrix A to retrieve any point between the frames.

X-axis

A moving frame rotated around the X axis is also possible to calculate.



The same technique applies. Point R in the moving frame is x. Point R in the fixed frame is X. A is the rotation matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{bmatrix} * \begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix} = \begin{bmatrix} a_{11} * b_{11} + a_{12} * b_{21} + a_{13} * b_{31} \\ a_{21} * b_{11} + a_{22} * b_{21} + a_{32} * b_{31} \\ a_{31} * b_{11} + a_{32} * b_{21} + a_{33} * b_{31} \end{bmatrix}$$

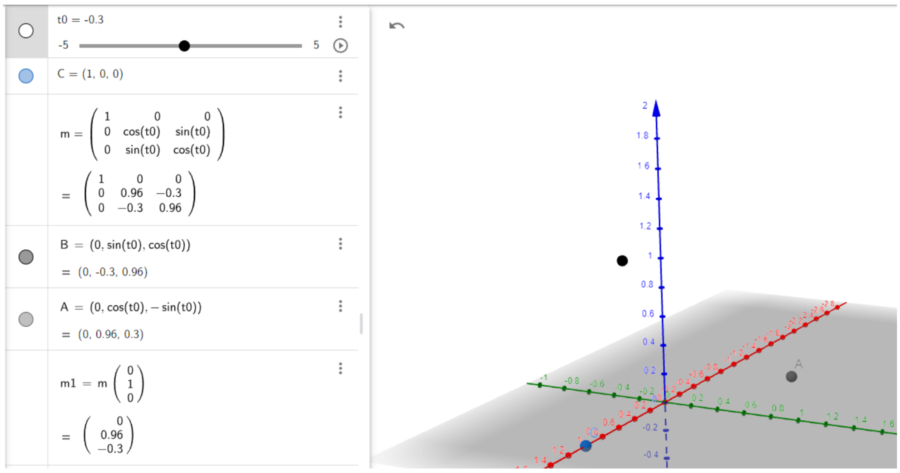
Point R. X = r_f. x=r_m. A=rot_m

$$X = \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} = A * x = \begin{bmatrix} 1 & ? & ? \\ 0 & ? & ? \\ 0 & ? & ? \end{bmatrix} * \begin{matrix} 1 \\ 0 \\ 0 \end{matrix}$$

Point Q. X = q_f. x=q_m. A = rot_m

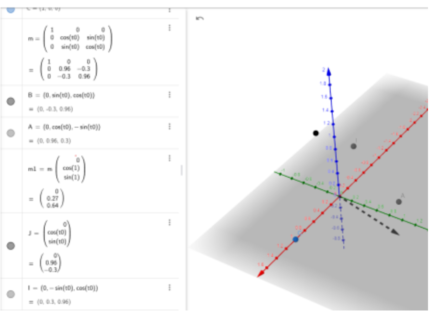
$$X = \begin{matrix} 0 \\ -\sin(\theta) \\ \cos(\theta) \end{matrix} = A * x = \begin{bmatrix} 1 & ? & 0 \\ 0 & ? & -\sin(\theta) \\ 0 & ? & \cos(\theta) \end{bmatrix} * \begin{matrix} 0 \\ 0 \\ 1 \end{matrix}$$

Point P. X = p_f. x=p_m. A = rot_m



$$X = \begin{matrix} 0 \\ \cos(\theta) \\ \sin(\theta) \end{matrix} = A * x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} * \begin{matrix} 0 \\ 1 \\ 0 \end{matrix}$$

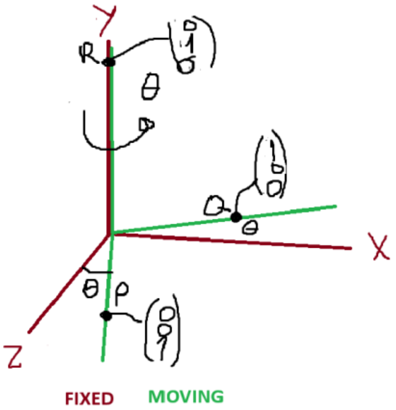
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{bmatrix} * \begin{matrix} 0 \\ 1 \\ b_{31} \end{matrix} = \begin{bmatrix} a_{12} * b_{21} \\ a_{22} * b_{21} \\ a_{32} * b_{21} \end{bmatrix}$$



We now have the rotation matrix for rotations around the X axis:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$$

Y-axis



Point P. X = p_f. x=p_m. A = rot_m

$$\begin{matrix}\sin(\theta)\\ 0\\ \cos(\theta)\end{matrix} = A * x = A * \begin{matrix}0\\ 0\\ 1\end{matrix}$$

$$X = \begin{matrix}\sin(\theta)\\ 0\\ \cos(\theta)\end{matrix} = A * x = \begin{bmatrix}? & ? & \sin(\theta)\\ ? & ? & 0\\ ? & ? & \cos(\theta)\end{bmatrix} * \begin{matrix}0\\ 0\\ 1\end{matrix}$$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{33} & a_{33}\end{bmatrix} * \begin{matrix}b_{11}\\ b_{21}\\ b_{31}\end{matrix} = \begin{bmatrix}a_{13} * b_{31}\\ a_{32} * b_{31}\\ a_{33} * b_{31}\end{bmatrix}$$

Point Q X=q_f. x=q_m. A=rot_m.

$$\begin{matrix}\cos(\theta)\\ 0\\ -\sin(\theta)\end{matrix} = A * x = A * \begin{matrix}1\\ 0\\ 0\end{matrix}$$

$$X = \begin{matrix}\cos(\theta)\\ 0\\ -\sin(\theta)\end{matrix} = A * x = \begin{bmatrix}\cos(\theta) & ? & \sin(\theta)\\ 0 & ? & 0\\ -\sin(\theta) & ? & \cos(\theta)\end{bmatrix} * \begin{matrix}0\\ 0\\ 1\end{matrix}$$

$$\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{33} & a_{33}\end{bmatrix} * \begin{matrix}b_{11}\\ b_{21}\\ b_{31}\end{matrix} = \begin{bmatrix}a_{11} * b_{11}\\ a_{21} * b_{11}\\ a_{31} * b_{11}\end{bmatrix}$$

Point R X=r_f. x=r_m. A=rot_m.

$$X = \begin{matrix}0\\ 1\\ 0\end{matrix} = A * x = \begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{bmatrix} * \begin{matrix}0\\ 1\\ 0\end{matrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{33} & a_{33} \end{bmatrix} * \begin{matrix} b_{11} \\ b_{21} \\ b_{31} \end{matrix} = \begin{bmatrix} a_{12} * b_{21} \\ a_{22} * b_{21} \\ a_{32} * b_{21} \end{bmatrix}$$

We now have the completed transformation matric for rotations around the Y-axis.

$$\begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$$

Transformation Matrix

The transformation matrix is defined by

$$T = \begin{bmatrix} A & P \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & P_{11} \\ A_{21} & A_{22} & A_{23} & P_{21} \\ A_{31} & A_{32} & A_{33} & P_{31} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Where $A = A_{x\theta}A_{y\theta}A_{z\theta}$ $where\ A_{n\theta}\ is\ the\ rotation\ matrix\ around\ n - axis\ rotated\ by\ \theta.$ P is the position vector or translation matrix defined as $\{P_x, P_y, P_z\}$. The first joint of the robot arm is T_0, the second T_1 etc. So the end effector will be $T = T_0 * T_1 * \ldots * T_N$ where N denotes the amount of joints.

$$T_0\ is\ the\ fixed\ frame\ in\ this\ case. With\ zero\ rotation\ and\ 0\ displacement, T_0\ is\ defined\ as:$$

$$T_0 = \begin{bmatrix} Z_{\theta_1} & P_1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} Z_{11\theta_1} & Z_{12\theta_1} & Z_{13\theta_1} & 0 \\ Z_{21\theta_1} & Z_{22\theta_1} & Z_{23\theta_1} & 0 \\ Z_{31\theta_1} & Z_{32\theta_1} & Z_{33\theta_1} & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$where\ Z_\theta\ is\ the\ rotation\ matrix\ for\ the\ Z - axis\ rotated\ by\ \theta\ radians. d_1\ is\ the\ height\ offset\ up\ to\ the\ joint.$$

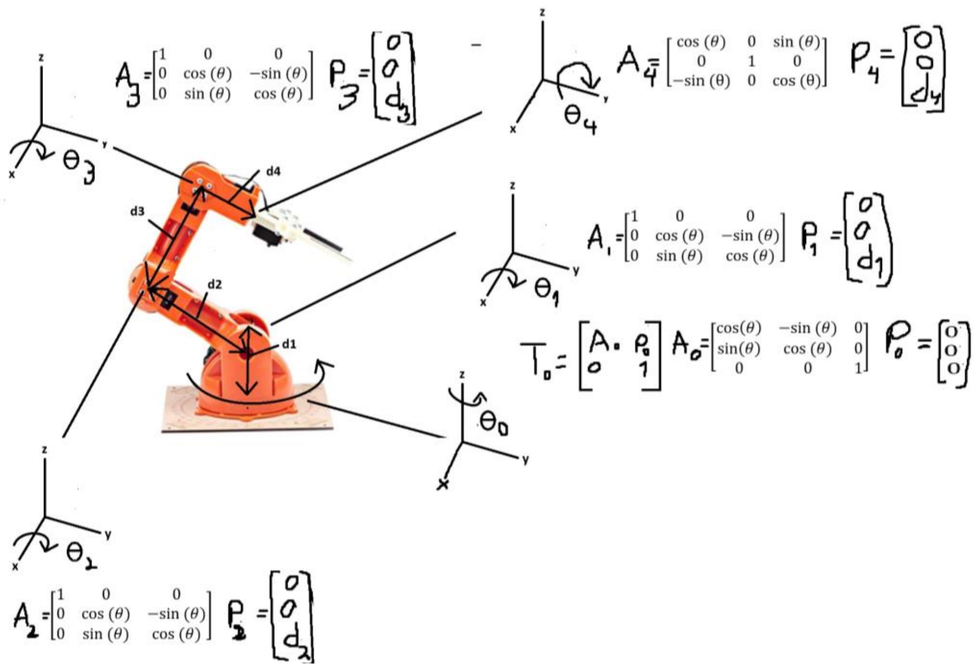
The following joints follows the same, however the rotation matrix and displacement matrices change. The rotation matrix may be anyone of X, Y or Z, and combinations of them. The displacement vector defined by the dimensions of the bone its attached to. For a robot with 5 joints, each rotated a different amount with different displacements looks like this:

$$T = T_0T_1T_2T_3T_4 = \begin{bmatrix} Z_{\theta_1} & P_1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z_{\theta_2} & P_2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z_{\theta_3} & P_3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z_{\theta_4} & P_4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} Z_{\theta_5} & P_5 \\ 0 & 1 \end{bmatrix}$$

This is also written as

$$H = H_{01}H_{12}H_{23}H_{34}H_{45}$$

We can now assign rotation matrices and displacements for each joint of the robot.



We can now write out the forward kinematics for this robot accurately. We don't know the dimensions of the robot yet, therefore we are using variables instead. Our rotation matrices are defined as:

$$X(\theta) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos(\theta) & -\sin(\theta)\\ 0 & \sin(\theta) & \cos(\theta)\end{bmatrix}$$

$$Y(\theta) = \begin{bmatrix}\cos(\theta) & 0 & \sin(\theta)\\ 0 & 1 & 0\\ -\sin(\theta) & 0 & \cos(\theta)\end{bmatrix}$$

$$Z(\theta) = \begin{bmatrix}\cos(\theta) & -\sin(\theta) & 0\\ \sin(\theta) & \cos(\theta) & 0\\ 0 & 0 & 1\end{bmatrix}$$

Where X is the rotation matrix for a joint rotating around the X-axis. Y matrix for Y-axis rotation. Z matrix for Z-axis rotation.

Starting with the first joint:

$$T_0 = \begin{bmatrix}Z(\theta_0) & P_0\\ 0 & 1\end{bmatrix} \qquad P_0 = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Where T_0 is the transformation matrix of the first bone of the arm. We see from the sketch rotates around its Z-axis; therefore, we choose the Z-rotation matrix. θ_0 is joint rotation angle. P_0 is the displacement vector, this is the base of the robot and therefore set to 0,0,0. If we had another origo, not at the base of the robot, this could account for it.

The next two joints:

$$T_1 = \begin{bmatrix} X(\theta_1) & P_1 \\ 0 & 1 \end{bmatrix} \quad P_1 = \begin{bmatrix} 0 \\ 0 \\ d_1 \end{bmatrix}$$

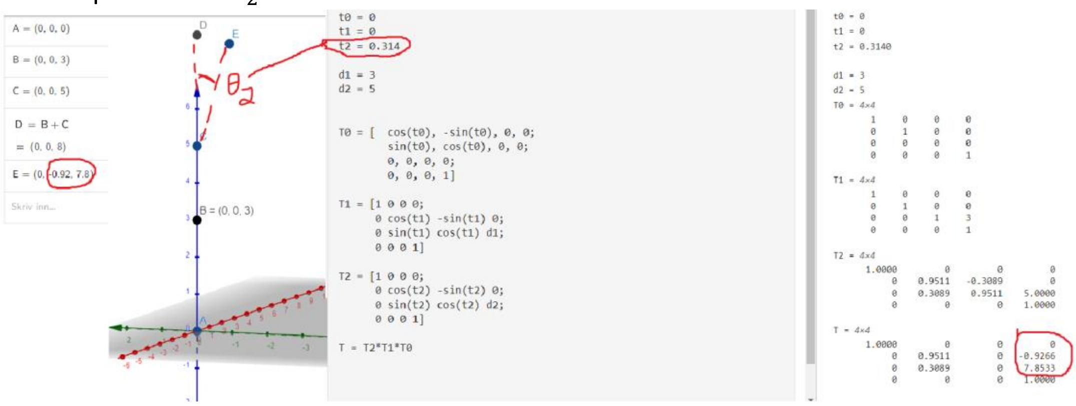
$$T_2 = \begin{bmatrix} X(\theta_2) & P_2 \\ 0 & 1 \end{bmatrix} \quad P_2 = \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}$$

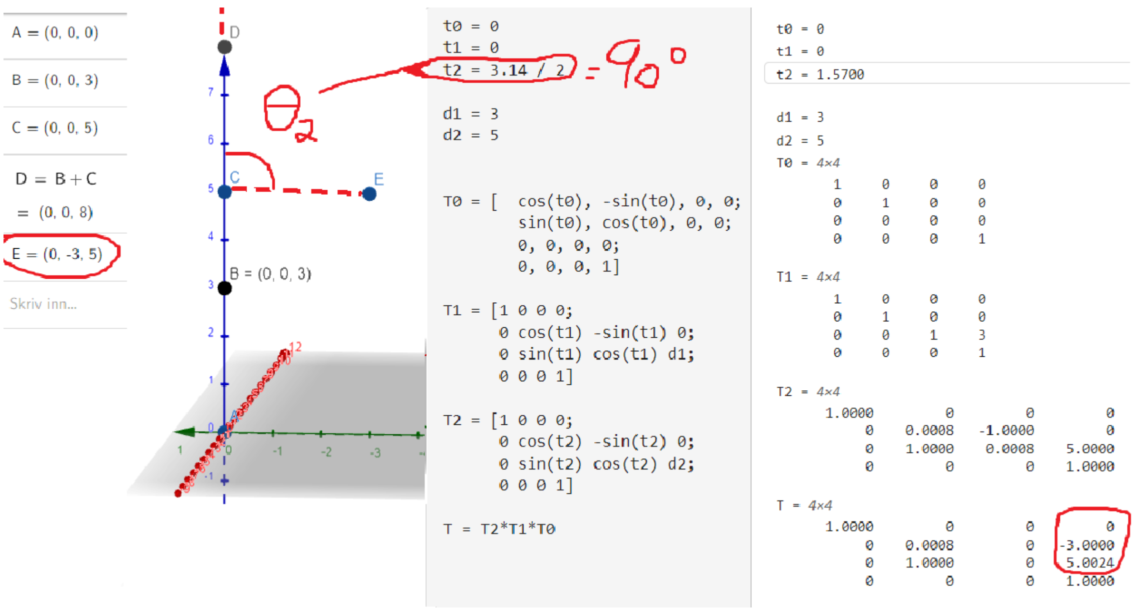
We can now find this point:

We call that point $P_3(x, y, z)$:

$$T = T_0 T_1 T_2 = \begin{bmatrix} R(\theta_{tot}) & P_3 \\ 0 & 1 \end{bmatrix}$$

The following pictures shows visualizations of this operation. The first picture has a low θ_2, the second picture has $\theta_2 = 90°$





We now know that the Forward Kinematics is correct. The exact calculations have been done in matlab and geogebra showing it. Now because we have a formula a position based on angles and lengths, we can reverse it and find angles from a position. Changing out the function for symbols, we can see the formula for the final matrix T.

```
syms c0 s0 c1 s1 c2 s2

T0 = [  c0, -s0, 0, 0;
        s0, c0, 0, 0;
        0, 0, 0, 0;
        0, 0, 0, 1]

T1 = [1 0 0 0;
      0 c1 -s1 0;
      0 s1 c1 d1;
      0 0 0 1]

T2 = [1 0 0 0;
      0 c2 -s2 0;
      0 s2 c2 d2;
      0 0 0 1]
T = T2*T1*T0
```

$$\begin{pmatrix} 0 & c_1 & -s_1 & 0 \\ 0 & s_1 & c_1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T2 =

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_2 & -s_2 & 0 \\ 0 & s_2 & c_2 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

T =

$$\begin{pmatrix} c_0 & -s_0 & 0 & 0 \\ s_0\,(c_1\,c_2 - s_1\,s_2) & c_0\,(c_1\,c_2 - s_1\,s_2) & 0 & -3\,s_2 \\ s_0\,(c_1\,s_2 + c_2\,s_1) & c_0\,(c_1\,s_2 + c_2\,s_1) & 0 & 3\,c_2 + 5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$T = \begin{bmatrix} \cos(\theta_0) & -\sin(\theta_0) & 0 & 0 \\ \sin(\theta_0) * (\cos(\theta_1) * \cos(\theta_2) - \sin(\theta_1) * \sin(\theta_2)) & \cos(\theta_0) * (\cos(\theta_1) * \cos(\theta_2) - \sin(\theta_1) * \sin(\theta_2)) & 0 & -3 * \sin(\theta_2) \\ \sin(\theta_0) * (\cos(\theta_1) * \sin(\theta_2) + \sin(\theta_1) * \cos(\theta_2)) & \cos(\theta_0) * (\cos(\theta_1) * \sin(\theta_2) + \sin(\theta_1) * \cos(\theta_2)) & 0 & 3 * \cos(\theta_2) + 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From this we can figure out the angles required for a given position. T_{11} is $\cos(\theta_0)$, from the previous image we can see