Appendix B

Fourier Series, Fourier Transform, Laplace Transform

1

Fourier series

- A method to split a periodic function into harmonic components
- The harmonic functions are: $\cos(n\omega_0 t)$ and $\sin(n\omega_0 t)$, where n is the harmonic number and $\omega_0 = 2\pi/T$ is the fundamental frequency, and T is the period.
- Trigonometric FS: $x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$
- The Fourier coefficients of the harmonics are found as correlations between the function, x(t), and the harmonic:
- Symmetric components: $a_n = \frac{2}{T_0} \int_{T_0} x(t) \cos(n\omega_0 t)\, \mathrm{dt}$;
- Antisymmetric comp.: $b_n = \frac{2}{T_0} \int_{T_0} x(t) \sin(n\omega_0 t)\mathrm{dt}$
- DC-component: $a_0 = \frac{1}{T_0} \int_{T_0} x(t)\mathrm{dt}$ *(average value of x(t))*

2

Example 5.1
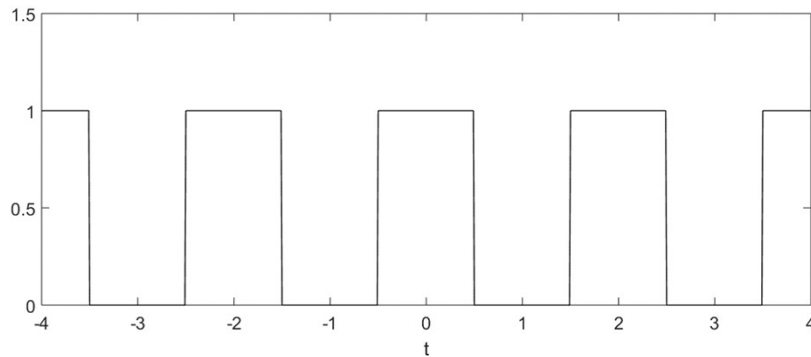
Period: $T_0=2$; $\omega_0=\pi$

Average value: $\quad a_0 = \frac{1}{T_0}\int_{-1}^{1} x(t)\mathrm{dt} = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} dt = \frac{1}{2}$

3

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos(n\omega_0 t)\,\mathrm{dt} = \frac{2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(n\pi t)\,dt = \frac{1}{n\pi}\Big[\sin(n\pi t)\Big]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{1}{n\pi}\Big[\sin\Big(\frac{n\pi}{2}\Big) - \sin\Big(-\frac{n\pi}{2}\Big)\Big] = \frac{2}{n\pi}\sin\Big(\frac{n\pi}{2}\Big) = sinc\Big(\frac{n\pi}{2}\Big) \ ; \ a_n = 0 \ for \ n \ even$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin(n\omega_0 t)\,\mathrm{dt} = \frac{2}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(n\pi t)\,dt = \frac{1}{n\pi}\Big[-\cos(n\pi t)\Big]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{-1}{n\pi}\Big[\cos\Big(\frac{n\pi}{2}\Big) - \cos\Big(-\frac{n\pi}{2}\Big)\Big] = 0 \ ; \ \text{All } b_n = 0 \ because \ x(t) \ is \ symmetric$$

The Fourier series is:
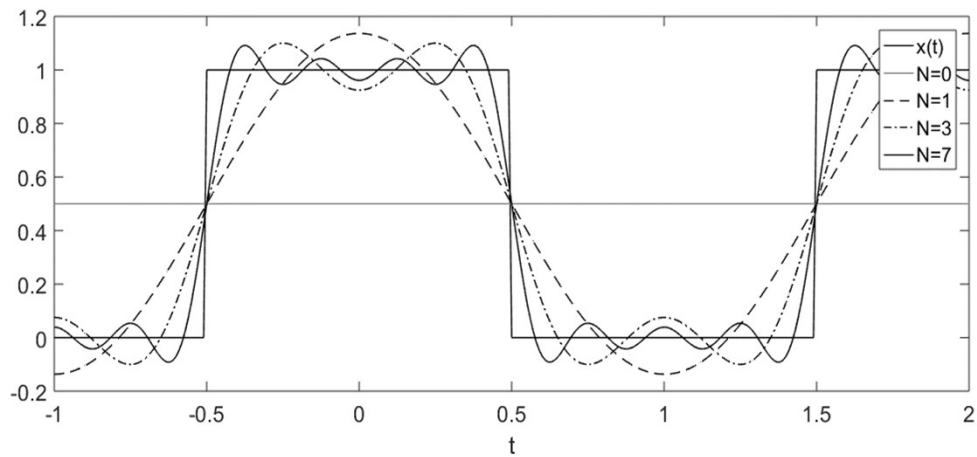
$$x(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin\Big(\frac{n\pi}{2}\Big)\cos(n\pi t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t)\dots$$

4

Summing up results in x(t) for n=1,2,3,…N

Summing up the harmonics, results in x(t) as N approaches infinity.

ACIT4710 Appendix B

5

5

Fourier series as amplitude and phase

We use the trigonometric identity:

$$a\cos(\omega t) + b\sin(\omega t) = C\cos(\omega t + \theta) ; C = \sqrt{a^2 + b^2} ; \theta = \tan^{-1}\left(\frac{-b}{a}\right)$$

We apply this to all the harmonics and get:

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

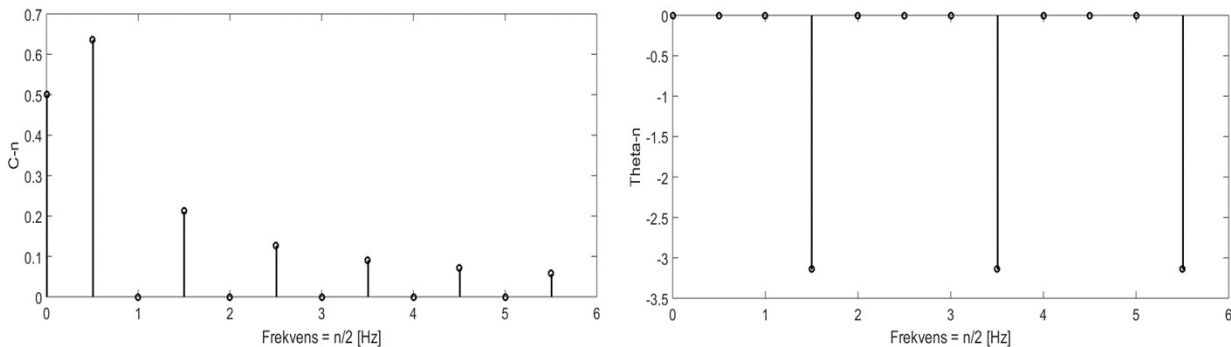$$= C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

Her er: $C_0 = a_0 ; C_n = \sqrt{a_n^2 + b_n^2} ; \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

ACIT4710 Appendix B

6

6

3

Frequency spectrum as amplitude and phase :

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) - \frac{2}{3\pi}\cos(3\pi t) + \frac{2}{5\pi}\cos(5\pi t) - \frac{2}{7\pi}\cos(7\pi t) + \cdots$$

$$= \frac{1}{2} + \frac{2}{\pi}\cos(\pi t) + \frac{2}{3\pi}\cos(3\pi t - \pi) + \frac{2}{5\pi}\cos(5\pi t) + \frac{2}{7\pi}\cos(7\pi t - \pi) + \cdots$$

ACIT4710 Appendix B
7

7

Exponential Fourier series

We now use: $\cos(\omega t) = \frac{1}{2}[\exp(j\omega t) + \exp(-j\omega t)]$

We apply this to all harmonics and get:

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$= D_0 + \sum_{n=1}^{\infty} D_n \exp(jn\omega_0 t) + D_{-n}\exp(-jn\omega_0 t) = \sum_{n=-\infty}^{\infty} D_n \exp(jn\omega_0 t)$$

We have introduced negative n and thereby also negative frequencies.

We see that: $D_0 = C_0$; $D_{-n} = D_n^*$; $|D_n| = |D_{-n}| = \frac{C_n}{2}$; $\angle D_n = \theta_n$; $\angle D_{-n} = -\theta_n$

The amplitude is symmetric, and the phase is antisymmetric.

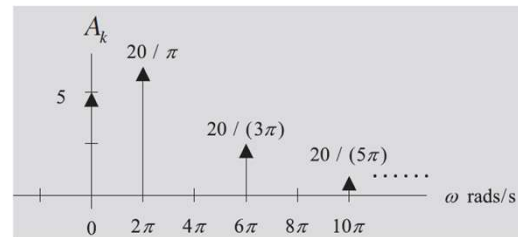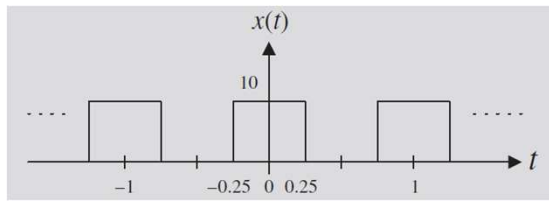The D's are therefore pare wise complex conjugate for n and –n.

This is necessary for x(t) to be real.

ACIT4710 Appendix B
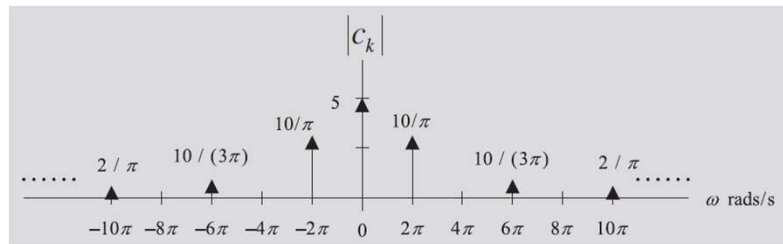8

8

4

Example B1 with complex notation

a(0)=5, A(0)=5, c(0)=5

a(k)=10sinc(πk/2)
b(k)=0

A(k)=|a(k)|

|c(k)|=|c(-k)|=A(k)/2

ACIT4710 Appendix B   9

9

Table B.1 Fourier Series Expansions for Some Common Waveform Signals in the Sine-Cosine Form

Time domain signal $x(t)$	Fourier series expansion

Positive square wave

$x(t) = \dfrac{A}{2} + \dfrac{2A}{\pi}\left(\sin\omega_0 t + \dfrac{1}{3}\sin 3\omega_0 t + \dfrac{1}{5}\sin 5\omega_0 t + \dfrac{1}{7}\sin 7\omega_0 t + \cdots\right)$

Square wave

$x(t) = \dfrac{4A}{\pi}\left(\cos\omega_0 t - \dfrac{1}{3}\cos 3\omega_0 t + \dfrac{1}{5}\cos 5\omega_0 t - \dfrac{1}{7}\cos 7\omega_0 t + \cdots\right)$

Triangular wave

$x(t) = \dfrac{8A}{\pi^2}\left(\cos\omega_0 t + \dfrac{1}{9}\cos 3\omega_0 t + \dfrac{1}{25}\cos 5\omega_0 t + \dfrac{1}{49}\cos 7\omega_0 t + \cdots\right)$

Sawtooth wave

$x(t) = \dfrac{2A}{\pi}\left(\sin\omega_0 t - \dfrac{1}{2}\sin 2\omega_0 t + \dfrac{1}{3}\sin 3\omega_0 t - \dfrac{1}{4}\sin 4\omega_0 t + \cdots\right)$

Rectangular wave (Pulse train)

$x(t) = Ad + 2Ad\left(\dfrac{\sin\pi d}{\pi d}\right)\cos\omega_0 t + 2Ad\left(\dfrac{\sin 2\pi d}{2\pi d}\right)\cos 2\omega_0 t + 2Ad\left(\dfrac{\sin 3\pi d}{3\pi d}\right)\cos 3\omega_0 t + \cdots$

Duty cycle $= d = \dfrac{\tau}{T_0}$

Ideal impulse train

$x(t) = \dfrac{1}{T_0} + \dfrac{2}{T_0}\left(\cos\omega_0 t + \cos 2\omega_0 t + \cos 3\omega_0 t + \cos 4\omega_0 t + \cdots\right)$

ACIT4710 Appendix B   10

10

The Fourier transform

- The Fourier transform is a generalization of the Fourier series for non-periodic functions.
- The Fourier transform results in the frequency spectrum of x(t):
- $X(\omega) = \int_{-\infty}^{\infty} x(t) \exp(-j\omega t)\, dt$
- We get back to the time function, x(t), by the inverse transform:
- $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp(j\omega t)\, d\omega$
- We use the Fourier transform to analyze the frequency content of the signal.
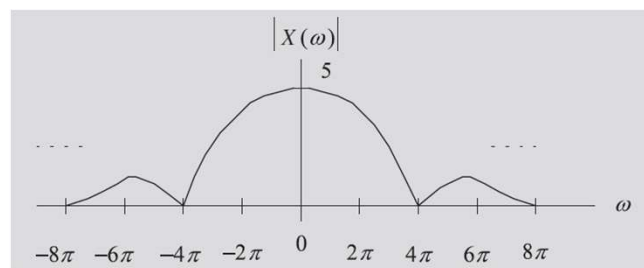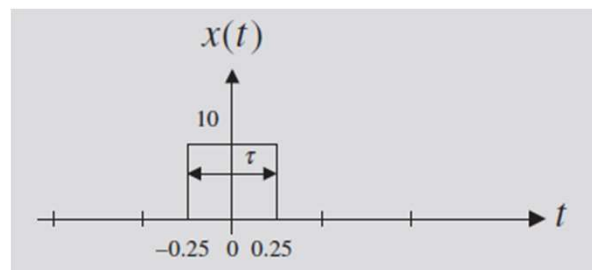
ACIT4710 Appendix B   11

11

Example B5

x(t) = Π(t/τ)

X(ω) = τ sinc(ωτ/2)
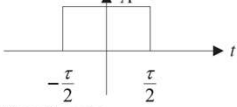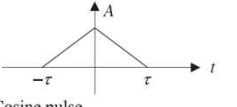
ACIT4710 Appendix B   12

12

Table B.3 Fourier Transform for Some Common Signals

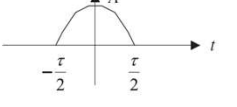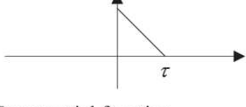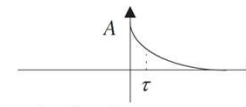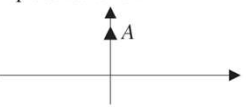| Time domain signal $x(t)$	Fourier spectrum $X(f)$
Rectangular pulse	$X(f) = A\tau \frac{\sin \pi f \tau}{\pi f \tau}$
Triangular pulse	$X(f) = A\tau \left(\frac{\sin \pi f \tau}{\pi f \tau}\right)^2$
Cosine pulse	$X(f) = \frac{2A\tau}{\pi}\left(\frac{\cos \pi f \tau}{1 - 4f^2\tau^2}\right)$
Sawtooth pulse	$X(f) = \frac{jA}{2\pi f}\left(\frac{\sin \pi f \tau}{\pi f \tau}e^{-j\pi f \tau} - 1\right)$
Exponential function $\alpha = \frac{1}{\tau}$	$X(f) = \frac{A}{\alpha + j2\pi f}$
Impulse function	$X(f) = A$

Real $x(t) \to$ Symmetric $|X(\omega)|$ and antisymmetric $\angle X(\omega)$
Symmetric $x(t) \to$ Real $X(\omega)$

ACIT4710 Appendix B   13

13

Table B.4 Properties of Fourier Transform

Line	Time Function	Fourier Transform
1	$\alpha x_1(t) + \beta x_2(t)$	$\alpha X_1(f) + \beta X_2(f)$
2	$\frac{dx(t)}{dt}$	$j2\pi f X(f)$
3	$\int_{-\infty}^{t} x(t)dt$	$\frac{X(f)}{j2\pi f}$
4	$x(t - \tau)$	$e^{-j2\pi f \tau}X(f)$
5	$e^{j2\pi f_0 t}x(t)$	$X(f - f_0)$
6	$x(at)$	$\frac{1}{a}X\left(\frac{f}{a}\right)$

ACIT4710 Appendix B   14

14

The Laplace Transform

- The Laplace transform was developed to solve linear differential equations.
- Definition: $X(s) = \int_0^\infty x(t)\exp(-st)\,dt$
- This is the unilateral transform, and we need starting conditions.
- The Laplace variable is complex: $s = \sigma + j\omega$.
- We use the Laplace transform to analyze systems.
- An important property of the LT is the following: $\mathcal{L}\left\{\dfrac{dx(t)}{dt}\right\} = sX(s)$
- A time derivative results in a multiplication with s.

ACIT4710 Appendix B 15

15

Table B.5 Laplace Transform Table

Line	Time Function $x(t)$	Laplace Transform $X(s) = L(x(t))$
1	$\delta(t)$	1
2	1 or $u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$e^{-at}u(t)$	$\frac{1}{s+a}$
5	$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
6	$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
7	$\sin(\omega t+\theta)u(t)$	$\frac{s\sin(\theta)+\omega\cos(\theta)}{s^2+\omega^2}$
8	$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
9	$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
10	$\left(A\cos(\omega t)+\frac{B-aA}{\omega}\sin(\omega t)\right)e^{-at}u(t)$	$\frac{As+B}{(s+a)^2+\omega^2}$
11a	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
11b	$\frac{1}{(n-1)!}t^{n-1}u(t)$	$\frac{1}{s^n}$
12a	$e^{-at}t^n u(t)$	$\frac{n!}{(s+a)^{n+1}}$
12b	$\frac{1}{(n-1)!}e^{-at}t^{n-1}u(t)$	$\frac{1}{(s+a)^n}$
13	$(2\text{Real}(A)\cos(\omega t)-2\text{Imag}(A)\sin(\omega t))e^{-at}u(t)$	$\frac{A}{s+a-j\omega}+\frac{A^*}{s+a+j\omega}$
14	$\frac{dx(t)}{dt}$	$sX(s)-x(0^-)$
15	$\int_0^t x(t)dt$	$\frac{X(s)}{s}$
16	$x(t-a)u(t-a)$	$e^{-as}X(s)$
17	$e^{-at}x(t)u(t)$	$X(s+a)$

ACIT4710 Appendix B 16

16

Solving differential equations with Laplace

- Any linear differential equation with constant coefficients with all starting conditions being zero, can be transformed to:
- $A(s)Y(s) = B(s)X(s)$, where Y(s) and X(s) are the Laplace transforms of the output signal and the input signal, respectively.
- A(s) is the characteristic polynomial and B(s) the driving polynomial.
- Poles: A(s)=0 ; Zeros: B(s)=0.
- The transfer function of the system is defined as the output over the input: $H(s) = \frac{Y(s)}{X(s)} = \frac{B(s)}{A(s)}$
- And we get: $Y(s) = H(s)X(s) \quad ; \quad y(t) = \mathcal{L}^{-1}\{Y(s)\}$

ACIT4710 Appendix B   17

17

The impulse response of a system

- The impulse response, h(t), of a system is defined as the response of the system when the input is an impulse, δ(t).
- The impulse function is defined as a time function that is infinitely short and with infinite amplitude, but so that the area under the curve is 1. It is also called the Dirac's delta function.
- We get: $X(s) = \mathcal{L}\{\delta(t)\} = 1$
- $h(t) = y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{H(s)X(s)\} = \mathcal{L}^{-1}\{H(s)\}$
- And therefore: $H(s) = \mathcal{L}\{h(t)\}$

ACIT4710 Appendix B   18

18

Frequency response

- We can get to the Fourier transform from the Laplace transform by setting: s = jω.
- This requires that the system is asymptotically stable and causal.
- The systems frequency response is defined as: $H(j\omega) = H(s = j\omega)$
- H(jω) is a complex function and therefore split into an amplitude response and a phase response:

$$H(j\omega) = |H(j\omega)|\exp(j\angle H(j\omega)$$

$$|H(j\omega)| = \sqrt{(Re[H(j\omega)])^2 + (Im[H(j\omega)])^2} \; ; \angle H(j\omega) = tan^{-1}\left(\frac{Im[H(j\omega)]}{Re[H(j\omega)]}\right)$$

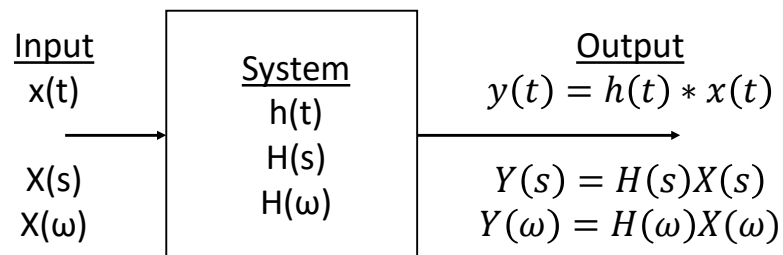- Finally, the frequency response is the Fourier transform of the impulse response: $H(\omega) = \mathcal{F}\{h(t)\}$

19

Summary of analog signal processing

Input
x(t)

X(s)
X(ω)

System
h(t)
H(s)
H(ω)

Output
$y(t) = h(t) * x(t)$

$Y(s) = H(s)X(s)$
$Y(\omega) = H(\omega)X(\omega)$

20