APPENDIX

# WAVELET ANALYSIS AND SYNTHESIS EQUATIONS

# F

## F.1 BASIC PROPERTIES

The inter product of two functions is defined as

$$< x, y > = \int x(t)y(t)dt. \tag{F.1}$$

Two functions are orthogonal if

$$< x(t), x(t-k) > = \begin{cases} A & \text{for } k=0 \\ 0 & \text{for } k \neq 0 \end{cases}. \tag{F.2}$$

Two functions are orthonormal if

$$< x(t), x(t-k) > = \begin{cases} 1 & \text{for } k=0 \\ 0 & \text{for } k \neq 0 \end{cases}. \tag{F.3}$$

The signal energy is defined as

$$E = \int x^2(t)dt. \tag{F.4}$$

Many wavelet families are designed to be orthonormal:

$$E = \int \psi^2(t)dt = 1 \tag{F.5}$$

$$E = \int \psi_{jk}^2(t)dt = \int \left[2^{j/2}\psi\left(2^j t - k\right)\right]^2 dt = \int 2^j \psi^2\left(2^j t - k\right)dt. \tag{F.6}$$

Let $u = 2^j t - k$. Then $du = 2^j dt$.

Eq. (F.5) becomes

$$E = \int 2^j \psi^2(u) 2^{-j} du = 1. \tag{F.7}$$

Both father and mother wavelets are orthonormal at scale $j$:

$$\int \phi_{jk}(t)\phi_{jn}(t)dt = \begin{cases} 1 & k=n \\ 0 & \text{otherwise} \end{cases} \tag{F.8}$$

$$\int \psi_{jk}(t)\psi_{jn}(t)dt = \begin{cases} 1 & k=n \\ 0 & \text{otherwise} \end{cases}. \tag{F.9}$$

## F.2 ANALYSIS EQUATIONS

When a function $f(t)$ is approximated using the scaling functions only at scale $j+1$, it can be expressed as

$$f(t) = \sum_{k=-\infty}^{\infty} c_j(k)2^{j/2}\phi(2^j t - k).$$

Using the inner product,

$$c_j(k) = <f(t), \phi_{jk}(t)> = \int f(t)2^{j/2}\phi(2^j t - k)dt. \tag{F.10}$$

Note that

$$\phi(t) = \sum_{n=-\infty}^{\infty} \sqrt{2}h_0(n)\phi(2t-n). \tag{F.11}$$

Substituting Eq. (F.11) into Eq. (F.10) leads to

$$c_j(k) = <f(t), \phi_{jk}(t)> = \int f(t)2^{j/2} \sum_{n=-\infty}^{\infty} \sqrt{2}h_0(n)\phi\left[2(2^j t - k) - n\right]dt$$

$$c_j(k) = \sum_{n=-\infty}^{\infty} \int f(t)2^{(j+1)/2}h_0(n)\phi\left(2^{(j+1)}t - 2k - n\right)dt.$$

Let $m = n+2k$. Interchange of the summation and integral leads to

$$c_j(k) = \sum_{m=-\infty}^{\infty} \int f(t)2^{(j+1)/2}h_0(m-2k)\phi\left(2^{(j+1)}t - m\right)dt$$

$$c_j(k) = \sum_{m=-\infty}^{\infty} \left(\int f(t)\phi_{(j+1)m}(t)dt\right)h_0(m-2k). \tag{F.12}$$

Using the inner product definition for the DWT coefficient again in (F.12), we achieve

$$c_j(k) = \sum_{m=-\infty}^{\infty} <f(t), \phi_{(j+1)m}(t)> h_0(m-2k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m)h_0(m-2k). \tag{F.13}$$

Similarly, note that

$$\psi(t) = \sum_{k=-\infty}^{\infty} \sqrt{2}h_1(k)\phi(2t-k).$$

Using the inner product gives

$$d_j(k) = <f(t), \psi_{jk}(t)> = \int f(t)2^{j/2} \sum_{n=-\infty}^{\infty} \sqrt{2}h_1(n)\phi\left[2(2^j t - k) - n\right]dt$$

$$d_j(k) = \sum_{n=-\infty}^{\infty} \int f(t) 2^{(j+1)/2} h_1(n) \phi\left(2^{(j+1)} t - 2k - n\right) dt. \tag{F.14}$$

Let $m = n + 2k$. Interchange of the summation and integral leads to

$$d_j(k) = \sum_{m=-\infty}^{\infty} \int f(t) 2^{(j+1)/2} h_1(m-2k) \phi\left(2^{(j+1)} t - m\right) dt$$

$$d_j(k) = \sum_{m=-\infty}^{\infty} \left( \int f(t) \phi_{(j+1)m}(t) dt \right) h_1(m-2k). \tag{F.15}$$

Finally, applying the inner product definition for the WDT coefficient, we obtain

$$d_j(k) = \sum_{m=-\infty}^{\infty} < f(t), \phi_{(j+1)m}(t) > h_1(m-2k) = \sum_{m=-\infty}^{\infty} c_{j+1}(m) h_1(m-2k). \tag{F.16}$$

## F.3 WAVELET SYNTHESIS EQUATIONS

We begin with

$$f(t) = \sum_{k=-\infty}^{\infty} c_j(k) 2^{j/2} \phi\left(2^j t - k\right) + \sum_{k=-\infty}^{\infty} d_j(k) 2^{j/2} \psi\left(2^j t - k\right).$$

Taking an inner product using the scaling function at scale level $j+1$ gives

$$\begin{aligned} c_{j+1}(k) = &< f(t), \phi_{(j+1)k}(t) > = \sum_{m=-\infty}^{\infty} c_j(m) 2^{j/2} \int \phi\left(2^j t - m\right) \phi_{(j+1)k}(t) dt \\ &+ \sum_{m=-\infty}^{\infty} d_j(m) 2^{j/2} \int \psi\left(2^j t - m\right) \phi_{(j+1)k}(t) dt \end{aligned}$$

$$\begin{aligned} c_{j+1}(k) = &\sum_{m=-\infty}^{\infty} c_j(m) 2^{j/2} \int \sum_{n=-\infty}^{\infty} \sqrt{2} h_0(n) \phi\left(2^{j+1} t - 2m - n\right) \phi_{(j+1)k}(t) dt \\ &+ \sum_{m=-\infty}^{\infty} d_j(m) 2^{j/2} \int \sum_{n=-\infty}^{\infty} \sqrt{2} h_1(n) \phi\left(2^{j+1} t - 2m - n\right) \phi_{(j+1)k}(t) dt. \end{aligned} \tag{F.17}$$

Interchange of the summation and integral, it yields

$$\begin{aligned} c_{j+1}(k) = &\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_j(m) h_0(n) \int 2^{(j+1)/2} \phi\left(2^{j+1} t - 2m - n\right) \phi_{(j+1)k}(t) dt \\ &+ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_j(m) h_1(n) \int 2^{(j+1)/2} \phi\left(2^{j+1} t - 2m - n\right) \phi_{(j+1)k}(t) dt. \end{aligned}$$

Using the inner product, we get

$$\begin{aligned} c_{j+1}(k) = &\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_j(m) h_0(n) < \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) > \\ &+ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} d_j(m) h_1(n) < \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) > . \end{aligned} \quad \text{(F.18)}$$

From the wavelet orthonormal property, we have

$$< \phi_{(j+1)(2m+n)}(t), \phi_{(j+1)k}(t) > = \begin{cases} 1 & n = k - 2m \\ 0 & \text{otherwise} \end{cases}. \quad \text{(F.19)}$$

Substituting Eq. (F.19) into Eq. (F.18), we finally obtain

$$c_{j+1}(k) = \sum_{m=-\infty}^{\infty} c_j(m) h_0(k-2m) + \sum_{m=-\infty}^{\infty} d_j(m) h_1(k-2m). \quad \text{(F.20)}$$