CHAPTER

DIGITAL SIGNAL PROCESSING SYSTEMS, BASIC FILTERING TYPES, AND DIGITAL FILTER REALIZATIONS

6

CHAPTER OUTLINE

6.1 DIFFERENCE EQUATION AND DIGITAL FILTERING

In this chapter, we begin with developing the filtering concept of digital signal processing (DSP) systems. With the knowledge acquired in Chapter 5, z-transform, we will learn how to describe and analyze linear time-invariant systems. We will also become familiar with digital filtering types and their realization structures.

A DSP system (digital filter) is described in Fig. 6.1.

Let $x(n)$ and $y(n)$ be a DSP system's input and output, respectively. We can express the relationship between the input and the output of a DSP system by the following *difference equation*

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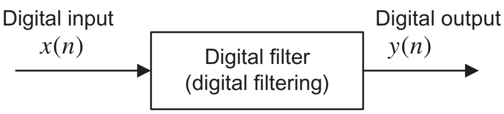


FIG. 6.1

DSP system with input and output.

$$\begin{aligned} y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ - a_1 y(n-1) - \cdots - a_N y(n-N) \end{aligned}, \tag{6.1}$$

where b_i, $0 \le i \le M$ and a_j, $1 \le j \le N$ represent the coefficients of the system and n is the time index. Eq. (6.1) can also be written as

$$y(n) = \sum_{i=0}^{M} b_i x(n-i) - \sum_{j=1}^{N} a_j y(n-j). \tag{6.2}$$

From Eqs. (6.1) and (6.2), we observe that the DSP system output is the weighted summation of the current input value $x(n)$ and its past values: $x(n-1)$, …, $x(n-M)$, and past output sequence: $y(n-1)$, …, $y(n-N)$. The system can be verified as linear, time invariant, and causal. If the initial conditions are specified, we can compute system output (time response) $y(n)$ recursively. This process is referred to as *digital filtering*. We will illustrate filtering operations by Examples 6.1 and 6.2.

EXAMPLE 6.1

Compute the system output

$$y(n) = 0.5y(n-2) + x(n-1)$$

for the first four samples using the following initial conditions:

(a) Initial conditions: $y(-2) = 1$, $y(-1) = 0$, $x(-1) = -1$, and input $x(n) = (0.5)^n u(n)$.

(b) Zero initial conditions: $y(-2) = 0$, $y(-1) = 0$, $x(-1) = 0$, and input $x(n) = (0.5)^n u(n)$.

Solution:

According to Eq. (6.1), we identify the system coefficients as.
$N = 2$, $M = 1$, $a_1 = 0$, $a_2 = -0.5$, $b_0 = 0$, and $b_1 = 1$.

(a) Setting $n = 0$, and using initial conditions, we obtain the input and output as

$$x(0) = (0.5)^0 u(0) = 1$$

$$y(0) = 0.5y(-2) + x(-1) = 0.5 \cdot 1 + (-1) = -0.5.$$

Setting $n = 1$, and using the initial condition $y(-1) = 0$, we achieve

$$x(1) = (0.5)^1 u(1) = 0.5$$

$$y(1) = 0.5y(-1) + x(0) = 0.5 \cdot 0 + 1 = 1.0.$$

Similarly, using the past output $y(0) = -0.5$, we get

$$x(2) = (0.5)^2 u(2) = 0.25$$

$$y(2) = 0.5y(0) + x(1) = 0.5 \cdot (-0.5) + 0.5 = 0.25$$

and with $y(1) = 1.0$, we yield

$$x(3) = (0.5)^3 u(3) = 0.125$$

$$y(3) = 0.5y(1) + x(2) = 0.5 \cdot 1 + 0.25 = 0.75$$

$$\dots$$

Clearly, $y(n)$ could be recursively computed for $n > 3$.

(b) Setting $n = 0$, we obtain

$$x(0) = (0.5)^0 u(0) = 1$$

$$y(0) = 0.5y(-2) + x(-1) = 0 \cdot 1 + 0 = 0$$

Setting $n = 1$, we achieve

$$x(1) = (0.5)^1 u(1) = 0.5$$

$$y(1) = 0.5y(-1) + x(0) = 0 \cdot 0 + 1 = 1.$$

Similarly, with the past output $y(0) = 0$, we determine

$$x(2) = (0.5)^2 u(2) = 0.25$$

$$y(2) = 0.5y(0) + x(1) = 0.5 \cdot 0 + 0.5 = 0.5$$

and with $y(1) = 1$, we obtain

$$x(3) = (0.5)^3 u(3) = 0.125$$

$$y(3) = 0.5y(1) + x(2) = 0.5 \cdot 1 + 0.25 = 0.75$$

$$\dots$$

Clearly, $y(n)$ could be recursively computed for $n > 3$.

EXAMPLE 6.2

Given the DSP system

$$y(n) = 2x(n) - 4x(n-1) - 0.5y(n-1) - y(n-2)$$

with initial conditions $y(-2) = 1$, $y(-1) = 0$, $x(-1) = -1$, and the input $x(n) = (0.8)^n u(n)$, compute the system response $y(n)$ for 20 samples using MATLAB.

Solution:

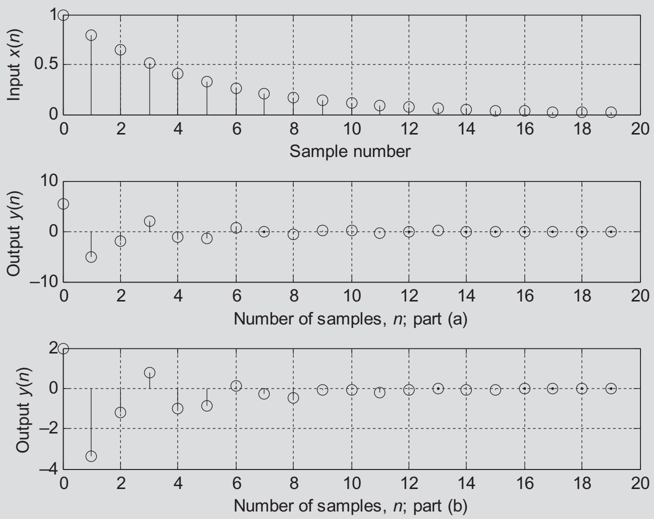
Program 6.1 lists the MATLAB program for computing the system response $y(n)$. The top plot in Fig. 6.2 shows the input sequence. The middle plot displays the filtered output using the initial conditions, and the bottom plot shows the filtered output for zero initial conditions. As we can see, both system outputs are different at the beginning portion, while they approach the same value later.

Program 6.1 MATLAB program for Example 6.2.

```
% Example 6.2
% Compute y(n)=2x(n)-4x(n-1)-0.5y(n-1)-0.5y(n-2)
% Nonzero initial conditions:
% y(-2)=1, y(-1)=0, x(-1) =-1, and x(n)=(0.8)^n*u(n)
%
```

Continued

EXAMPLE 6.2—CONT'D



Part (a): response with initial conditions;
Part (b): response with zero initial conditions.

FIG. 6.2

Plots of the input and system outputs $y(n)$ for Example 6.2. Part (a): response with initial conditions; Part (b): response with zero initial conditions.

```
y = zeros(1,20);      %Set up a vector to store y(n)
y = [ 1 0 y];         %Add initial conditions of y(-2) and y(-1)
n=0:1:19;             %Compute time indexes
x=(0.8).^n;           %Compute 20 input samples of x(n)
x = [ 0 -1 x];        %Add initial conditions of x(-2)=0 and x(-1)=1
for n=1:20
  y(n+2)= 2*x(n+2)-4*x(n+1)-0.5*y(n+1)-0.5*y(n); %Compute 20 outputs of y(n)
end
n=0:1:19;
subplot(3,1,1);stem(n,x(3:22));grid;ylabel('Input x(n)');xlabel('Sample number');
Subplot(3,1,2); stem(n,y(3:22)),grid;
xlabel('Number of samples, n; part (a)'); ylabel('Output y(n)');
y(3:22)         %Output y(n)
%Zero- initial conditions:
% y(-2)=0, y(-1)=0, x(-1) =0, and x(n)=(0.8)^n
%
y = zeros(1,20);             %Set up a vector to store y(n)
y = [ 0 0 y];                %Add zero initial conditions of y(-2) and y(-1)
n=0:1:19;                    %Compute time indexes
x=(0.8).^n;                  %Compute 20 input samples of x(n)
x = [ 0 0 x];                %Add zero initial conditions of x(-2)=0 and x(-1)=0
```

```
for n=1:20
 y(n+2)= 2*x(n+2)-4*x(n+1)-0.5*y(n+1)-0.5*y(n); %Compute 20 outputs of y(n)
end
n=0:1:19
subplot(3,1,3); stem(n,y(3:22)),grid;
xlabel('Number of samples, n; part (b)'); ylabel('Output y(n)');
y(3:22)                                    %Output y(n)
```

MATLAB function **filter(),** developed using a direct-form II realization (which will be discussed in Section 6.6.2), can be used to operate digital filtering, and the syntax is

$$\mathbf{Zi} = \mathbf{filtic}(\mathbf{B}, \mathbf{A}, \mathbf{Yi}, \mathbf{Xi})$$

$$\mathbf{y} = \mathbf{filter}(\mathbf{B}, \mathbf{A}, \mathbf{x}, \mathbf{Zi})$$

where B and A are vectors for the coefficients b_j and a_j whose formats are

$$A = [1 \ \ a_1 \ \ a_2 \ \ \cdots \ \ a_N] \text{ and } B = [b_0 \ \ b_1 \ \ b_2 \ \ \cdots \ \ b_M],$$

and x and y are the input data vector and output data vector, respectively.

Note that the filter function **filtic()** is a MATLAB function used to obtain initial states required by the MATLAB filter function **filter()** (requirement by a direct-form II realization) from initial conditions in the difference equation. Hence, Z_i contains initial states required for operating MATLAB function **filter(),** that is,

$$Z_i = [w(-1) \ \ w(-2) \ \ \cdots \ \ ],$$

which can be recovered by another MATLAB function **filtic()**. X_i and Y_i are initial conditions with the length of the greater of M or N, given by.

$$X_i = [x(-1) \ \ x(-2) \ \ \cdots \ \ ] \text{ and } Y_i = [y(-1) \ \ y(-2) \ \ \cdots \ \ ].$$

Especially for zero initial conditions, the syntax is reduced to.

$$\mathbf{y} = \mathbf{filter}(\mathbf{B}, \mathbf{A}, \mathbf{x}).$$

Let us verify the filter operation results in Example 6.1 using the MATLAB functions. The MATLAB codes and results for Example 6.1(a) with the nonzero initial conditions are listed as

```
» B=[0 1]; A=[1 0 -0.5];
» x=[1 0.5 0.25 0.125];
» Xi=[-1 0];Yi=[0 1];
» Zi=filtic(B,A,Yi,Xi);
» y=filter(B,A,x,Zi)
y =
  -0.5000  1.0000  0.2500  0.7500
»
```

For the case of zero initial conditions in Example 6.1(b), the MATLAB codes and results are as follows:

```
» B=[0 1]; A=[1 0 -0.5];
» x=[1 0.5 0.25 0.125];
```

```
» y=filter(B,A,x)
y =
    0  1.0000  0.5000  0.7500
»
```

As we expected, the filter outputs match the ones in Example 6.1.

6.2 DIFFERENCE EQUATION AND TRANSFER FUNCTION

In this section, given a difference equation, we will derive a z-transfer function which is the ratio of the z-transform of the system output over the z-transform of the system input. To proceed, Eq. (6.1) is rewritten as

$$\begin{aligned} y(n) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ - a_1 y(n-1) - \cdots - a_N y(n-N). \end{aligned}$$

With an assumption that all initial conditions of this system are zero, and with $X(z)$ and $Y(z)$ denoting the z-transforms of $x(n)$ and $y(n)$, respectively, taking the z-transform of Eq. (6.1) yields

$$\begin{aligned} Y(z) = b_0 X(z) + b_1 X(z) z^{-1} + \cdots + b_M X(z) z^{-M} \\ - a_1 Y(z) z^{-1} - \cdots - a_N Y(z) z^{-N} \end{aligned}. \tag{6.3}$$

Rearranging Eq. (6.3), we yield

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = \frac{B(z)}{A(z)}, \tag{6.4}$$

where $H(z)$ is defined as the transfer function with its numerator and denominator polynomials defined below:

$$B(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} \tag{6.5}$$

$$A(z) = 1 + a_1 z^{-1} + \cdots + a_N z^{-N}. \tag{6.6}$$

Clearly the z-transfer function is defined as

$$\text{Ratio} = \frac{z - \text{transform of the output}}{z - \text{transform of the input}}.$$

In DSP applications, given the difference equation, we can develop the z-transfer function and represent the digital filter in the z-domain as shown in Fig. 6.3. Then the stability and frequency response can be examined based on the developed transfer function.

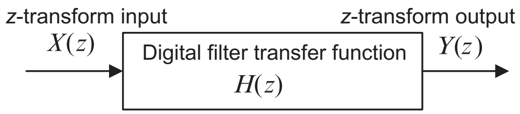


FIG. 6.3

Digital filter transfer function.

EXAMPLE 6.3

A DSP system is described by the following difference equation:

$$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

Solution:

Taking the z-transform on both sides of the previous difference equation, we achieve

$$Y(z) = X(z) - X(z)z^{-2} - 1.3Y(z)z^{-1} - 0.36Y(z)z^{-2}.$$

Moving the last two terms to the left-hand side of the difference equation and factoring $Y(z)$ on the left-hand side and $X(z)$ on the right-hand side, we obtain

$$Y(z)\left(1 + 1.3z^{-1} + 0.36z^{-2}\right) = \left(1 - z^{-2}\right)X(z).$$

Therefore, the transfer function, which is the ratio of $Y(z)$ over $X(z)$, can be found as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}.$$

From the derived transfer function $H(z)$, we could obtain the denominator polynomial and numerator polynomial as

$$A(z) = 1 + 1.3z^{-1} + 0.36z^{-2}, \text{and}$$

$$B(z) = 1 - z^{-2}.$$

The difference equation and its transfer function, as well as the stability issue of the linear time-invariant system, will be discussed in the following sections.

EXAMPLE 6.4

A digital system is described by the following difference equation:

$$y(n) = x(n) - 0.5x(n-1) + 0.36x(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

Solution:

Taking the z-transform on both sides of the previous difference equation, we achieve

$$Y(z) = X(z) - 0.5X(z)z^{-1} + 0.36X(z)z^{-2}.$$

Therefore, the transfer function, that is the ratio of $Y(z)$ to $X(z)$, can be found as

$$H(z) = \frac{Y(z)}{X(z)} = 1 - 0.5z^{-1} + 0.36z^{-2}.$$

From the derived transfer function $H(z)$, it follows that

$$A(z) = 1$$

$$B(z) = 1 - 0.5z^{-1} + 0.36z^{-2}.$$

In DSP applications, the given transfer function of a digital system can be converted into a difference equation for DSP implementation. The following example illustrates the procedure.

EXAMPLE 6.5

Convert each of the following transfer functions into its difference equations.

(a) $H(z) = \dfrac{z^2 - 1}{z^2 + 1.3z + 0.36}$

(b) $H(z) = \dfrac{z^2 - 0.5z + 0.36}{z^2}$

Solution:

(a) Dividing the numerator and denominator by z^2 to obtain the transfer function whose numerator and denominator polynomials have the negative power of z, it follows that

$$H(z) = \frac{(z^2 - 1)/z^2}{(z^2 + 1.3z + 0.36)/z^2} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}.$$

We write the transfer function using the ratio of $Y(z)$ to $X(z)$:

$$\frac{Y(z)}{X(z)} = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}.$$

Then we have

$$Y(z)\left(1 + 1.3z^{-1} + 0.36z^{-2}\right) = X(z)\left(1 - z^{-2}\right).$$

By distributing $Y(z)$ and $X(z)$, we yield

$$Y(z) + 1.3z^{-1}Y(z) + 0.36z^{-2}Y(z) = X(z) - z^{-2}X(z).$$

Applying the inverse z-transform and using the shift property in Eq. (5.3) of Chapter 5, we get

$$y(n) + 1.3y(n-1) + 0.36y(n-2) = x(n) - x(n-2).$$

Writing the output $y(n)$ in terms of inputs and past outputs leads to

$$y(n) = x(n) - x(n-2) - 1.3y(n-1) - 0.36y(n-2).$$

(b) Similarly, dividing the numerator and denominator by z^2, we obtain

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(z^2 - 0.5z + 0.36)/z^2}{z^2/z^2} = 1 - 0.5z^{-1} + 0.36z^{-2}.$$

Thus, $Y(z) = X(z)(1 - 0.5z^{-1} + 0.36z^{-2})$.
By distributing $X(z)$, we yield

$$Y(z) = X(z) - 0.5z^{-1}X(z) + 0.36z^{-2}X(z).$$

Applying the inverse z-transform using the shift property in Eq. (5.3), we obtain

$$y(n) = x(n) - 0.5x(n-1) + 0.36x(n-2).$$

The transfer function $H(z)$ can be factored into the *pole-zero form:*

$$H(z) = \frac{b_0(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}, \tag{6.7}$$

where the zeros z_i can be found by solving roots of the numerator polynomial, while the poles p_i can be solved for the roots of the denominator polynomial.

EXAMPLE 6.6

Given the following transfer functions,

$$H(z) = \frac{1 - z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}},$$

Convert it into the pole-zero form.

Solution:

We first multiply the numerator and denominator polynomials by z^2 to achieve its advanced form in which both the numerator and denominator polynomials have positive powers of z, that is,

$$H(z) = \frac{(1 - z^{-2})z^2}{(1 + 1.3z^{-1} + 0.36z^{-2})z^2} = \frac{z^2 - 1}{z^2 + 1.3z + 0.36}.$$

Letting $z^2 - 1 = 0$, we get $z = 1$ and $z = -1$. Setting $z^2 + 1.3z + 0.36 = 0$ leads $z = -0.4$ and $z = -0.9$. We then can write numerator and denominator polynomials in the factored form to obtain the pole-zero form:

$$H(z) = \frac{(z - 1)(z + 1)}{(z + 0.4)(z + 0.9)}.$$

6.2.1 IMPULSE RESPONSE, STEP RESPONSE, AND SYSTEM RESPONSE

The impulse response $h(n)$ of the DSP system $H(z)$ can be obtained by solving its difference equation using a unit impulse input $\delta(n)$. With the help of the z-transform and noticing that $X(z) = Z\{\delta(n)\} = 1$, we yield

$$h(n) = Z^{-1}\{H(z)X(z)\} = Z^{-1}\{H(z)\}. \tag{6.8}$$

Similarly, for a step input, we can determine step response assuming the zero initial conditions. Letting

$$X(z) = Z[u(n)] = \frac{z}{z - 1},$$

the step response can be found as

$$y(n) = Z^{-1}\left\{H(z)\frac{z}{z - 1}\right\}. \tag{6.9}$$

Furthermore, the z-transform of the general system response is given by

$$Y(z) = H(z)X(z). \tag{6.10}$$

If we know the transfer function $H(z)$ and z-transform of the input $X(z)$, we are able to determine the system response $y(n)$ by finding the inverse z-transform of the output $Y(z)$:

$$y(n) = Z^{-1}\{Y(z)\}. \tag{6.11}$$

EXAMPLE 6.7

Given a transfer function depicting a DSP system

$$H(z) = \frac{z+1}{z-0.5},$$

Determine

1. The impulse response$h(n)$,
2. Step response$y(n)$, and.
3. The system response $y(n)$ if the input is given as $x(n) = (0.25)^n u(n)$.

Solution:

(a) The transfer function can be rewritten as

$$\frac{H(z)}{z} = \frac{z+1}{z(z-0.5)} = \frac{A}{z} + \frac{B}{z-0.5},$$

where $A = \left.\frac{z+1}{(z-0.5)}\right|_{z=0} = -2$, and $B = \left.\frac{z+1}{z}\right|_{z=0.5} = 3.$

Thus we have

$$\frac{H(z)}{z} = \frac{-2}{z} + \frac{3}{z-0.5} \text{ and}$$

$$H(z) = \left(-\frac{2}{z} + \frac{3}{z-0.5}\right)z = -2 + \frac{3z}{z-0.5}.$$

By taking the inverse z-transform as shown in Eq. (6.8), we yield the impulse response

$$h(n) = -2\delta(n) + 3(0.5)^n u(n).$$

(b) For the step input $x(n) = u(n)$ and its z-transform $X(z) = \frac{z}{z-1}$, we can determine the z-transform of the step response as

$$Y(z) = H(z)X(z) = \frac{z+1}{z-0.5}\,\frac{z}{z-1}.$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{z+1}{(z-0.5)(z-1)} = \frac{A}{z-0.5} + \frac{B}{z-1},$$

where

$$A = \left.\frac{z+1}{z-1}\right|_{z=0.5} = -3, \text{and}\, B = \left.\frac{z+1}{z-0.5}\right|_{z=1} = 4.$$

The z-transform step response is therefore

$$Y(z) = \frac{-3z}{z-0.5} + \frac{4z}{z-1}.$$

Applying the inverse z-transform table yields the step response as

$$y(n) = -3(0.5)^n u(n) + 4u(n).$$

(c) To determine the system output response, we first find the z-transform of the input $x(n)$,

$$X(z) = Z\{(0.25)^n u(n)\} = \frac{z}{z-0.25},$$

then $Y(z)$ can be yielded via Eq. (6.10), that is,

$$Y(z) = H(z)X(z) = \frac{z+1}{z-0.5} \cdot \frac{z}{z-0.25} = \frac{z(z+1)}{(z-0.5)(z-0.25)}.$$

Using the partial fraction expansion, we have

$$\frac{Y(z)}{z} = \frac{(z+1)}{(z-0.5)(z-0.25)} = \left(\frac{A}{z-0.5} + \frac{B}{z-0.25}\right)$$

$$Y(z) = \left(\frac{6z}{z-0.5} + \frac{-5z}{z-0.25}\right).$$

Using Eq. (6.11) and Table 5.1 in Chapter 5, we finally yield

$$y(n) = Z^{-1}\{Y(z)\} = 6(0.5)^n u(n) - 5(0.25)^n u(n).$$

The impulse response for (a), step response for (b), and system response for (c) are each plotted in Fig. 6.4.

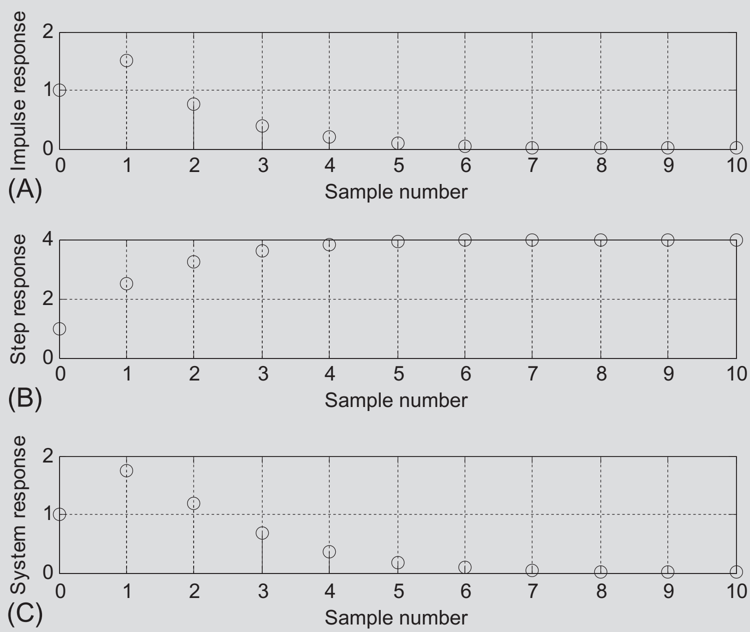


FIG. 6.4

Impulse, step, and system response in Example 6.7.

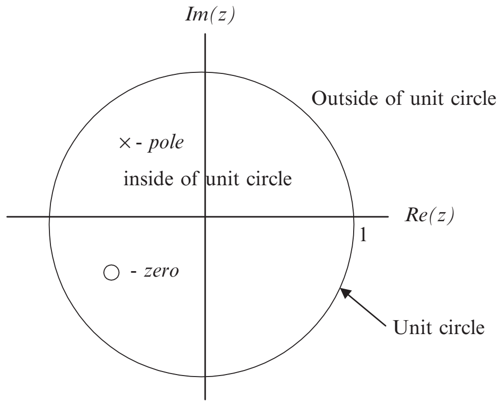


FIG. 6.5

z-Plane and pole-zero plot.

6.3 THE Z-PLANE POLE-ZERO PLOT AND STABILITY

A very useful tool to analyze digital systems is the z-plane pole-zero plot. This graphical technique allows us to investigate characteristics of the digital system shown in Fig. 6.1, including the system stability. In general, a digital transfer function can be written in the pole-zero form as shown in Eq. (6.7), and we can plot the poles and zeros on the z-plane. The z-plane is depicted in Fig. 6.5 and has the following features:

1. The horizontal axis is the real part of the variable z, and the vertical axis represents the imaginary part of the variable z.
2. The z-plane is divided into two parts by a unit circle.
3. Each pole is marked on z-plane using the cross symbol x, while each zero is plotted using the small circle symbol o.

Let us investigate the z-plane pole-zero plot of a digital filter system via the following example.

EXAMPLE 6.8

Given the digital transfer function

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}},$$

Plot poles and zeros.

Solution:

Converting the transfer function to its advanced form by multiplying both numerator and denominator by z^2, it follows that

$$H(z) = \frac{(z^{-1} - 0.5z^{-2})z^2}{(1 + 1.2z^{-1} + 0.45z^{-2})z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45}.$$

By setting $z^2 + 1.2z + 0.45 = 0$ and $z - 0.5 = 0$, we obtain two poles

$$p_1 = -0.6 + j0.3$$

$$p_2 = p_1^* = -0.6 - j0.3$$

and a zero $z_1 = 0.5$, which are plotted on the z-plane shown in Fig. 6.6. According to the form of Eq. (6.7), we also yield the pole-zero form as

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}} = \frac{(z - 0.5)}{(z + 0.6 - j0.3)(z + 0.6 + j0.3)}.$$

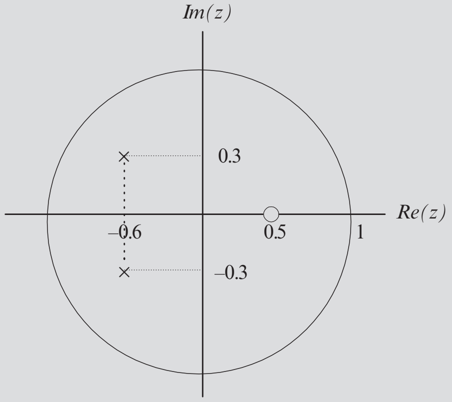


FIG. 6.6

The z-plane pole-zero plot of Example 6.8.

Having zeros and poles plotted on the z-plane, we are able to study the system stability. We first establish the relationship between the s-plane in Laplace domain and the z-plane in z-transform domain, as illustrated in Fig. 6.7.

As shown in Fig. 6.7, the sampled signal, which is not quantized, with a sampling period of T is written as

$$x_s(t) = \sum_{n=0}^{\infty} x(nT)\delta(t - nT) = x(0)\delta(t) + x(T)\delta(t - T) + x(2T)\delta(t - 2T) + \cdots. \tag{6.12}$$

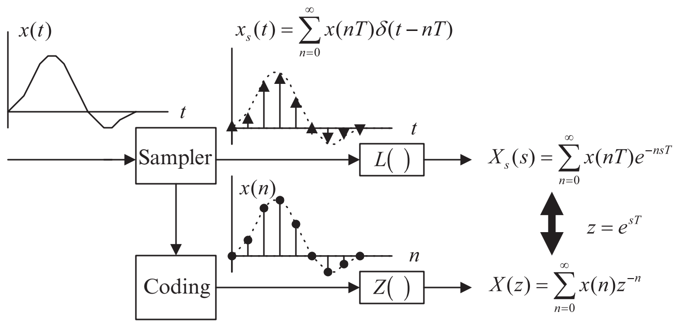


FIG. 6.7

Relationship between Laplace transform and z-transform.

Taking the Laplace transform and using the Laplace shift property (Appendix B, Table B.5) as

$$L(\delta(t-nT))=e^{-nTs} \tag{6.13}$$

leads to

$$X_s(s)=\sum_{n=0}^{\infty} x(nT)e^{-nTs}=x(0)e^{-0\times Ts}+x(T)e^{-Ts}+x(2T)e^{-2Ts}+\cdots. \tag{6.14}$$

Comparing Eq. (6.14) with the definition of a one-sided z-transform of the data sequence $x(n)$ from analog-to-digital conversion (ADC):

$$X(z)=Z(x(n))=\sum_{n=0}^{\infty} x(n)z^{-n}=x(0)z^{-0}+x(1)z^{-1}+x(2)z^{-2}+\cdots. \tag{6.15}$$

Clearly, we see the relationship of the sampled system in Laplace domain and its digital system in z-transform domain by the following mapping:

$$z=e^{sT}. \tag{6.16}$$

Substituting $s=-\alpha\pm j\omega$ into Eq. (6.16), it follows that $z=e^{-\alpha T\pm j\omega T}$. In the polar form, we have

$$z=e^{-\alpha T}\angle \pm \omega T. \tag{6.17}$$

Eqs. (6.16) and (6.17) give the following important conclusions.

If $\alpha>0$, this means $|z|=e^{-\alpha T}<1$. Then the left-hand side half plane (LHHP) of the s-plane is mapped to the inside of the unit circle of the z-plane. When $\alpha=0$, this causes $|z|=e^{-\alpha T}=1$. Thus the $j\omega$ axis of the s-plane is mapped on the unit circle of the z-plane, as shown in Fig. 6.8. Obviously, the right-hand half plane (RHHP) of the s-plane is mapped to the outside of the unit cycle in the z-plane. A stable system

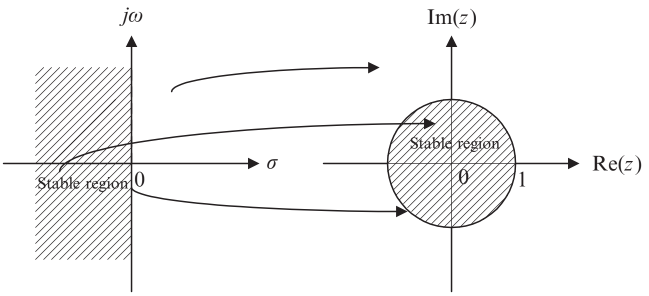


FIG. 6.8

Mapping between *s*-plane and *z*-plane.

means that for a given bounded input, the system output must be bounded. Similar to the analog system, the digital system requires that its all poles plotted on the z-plane must be inside the unit circle. We summarize the rules for determining the stability of a DSP system as follows:

(a) If the outmost pole(s) of the z-transfer function $H(z)$ describing the DSP system is(are) inside the unit circle on the z-plane pole-zero plot, then the system is stable.
(b) If the outmost pole(s) of the z-transfer function $H(z)$ is(are) outside the unit circle on the z-plane pole-zero plot, the system is unstable.
(c) If the outmost pole(s) is(are) first-order pole(s) of the z-transfer function $H(z)$ and on the unit circle on the z-plane pole-zero plot, then the system is marginally stable.
(d) If the outmost pole(s) is(are) multiple-order pole(s) of the z-transfer function $H(z)$ and on the unit circle on the z-plane pole-zero plot, then the system is unstable.
(e) The zeros do not affect the system stability.

Note that the following facts apply to a stable system [bounded-in/bounded-out (BIBO) stability discussed in Chapter 3]:

1. If the input to the system is bounded, then the output of the system will also be bounded, or the impulse response of the system will go to zero in a finite number of steps.
2. An unstable system is one in which the output of the system will grow without bound due to any bounded input, initial condition, or noise, or its impulse response will grow without bound.
3. The impulse response of a marginally stable system stays at a constant level or oscillates between the two finite values.

Examples illustrating these rules are shown in Fig. 6.9.

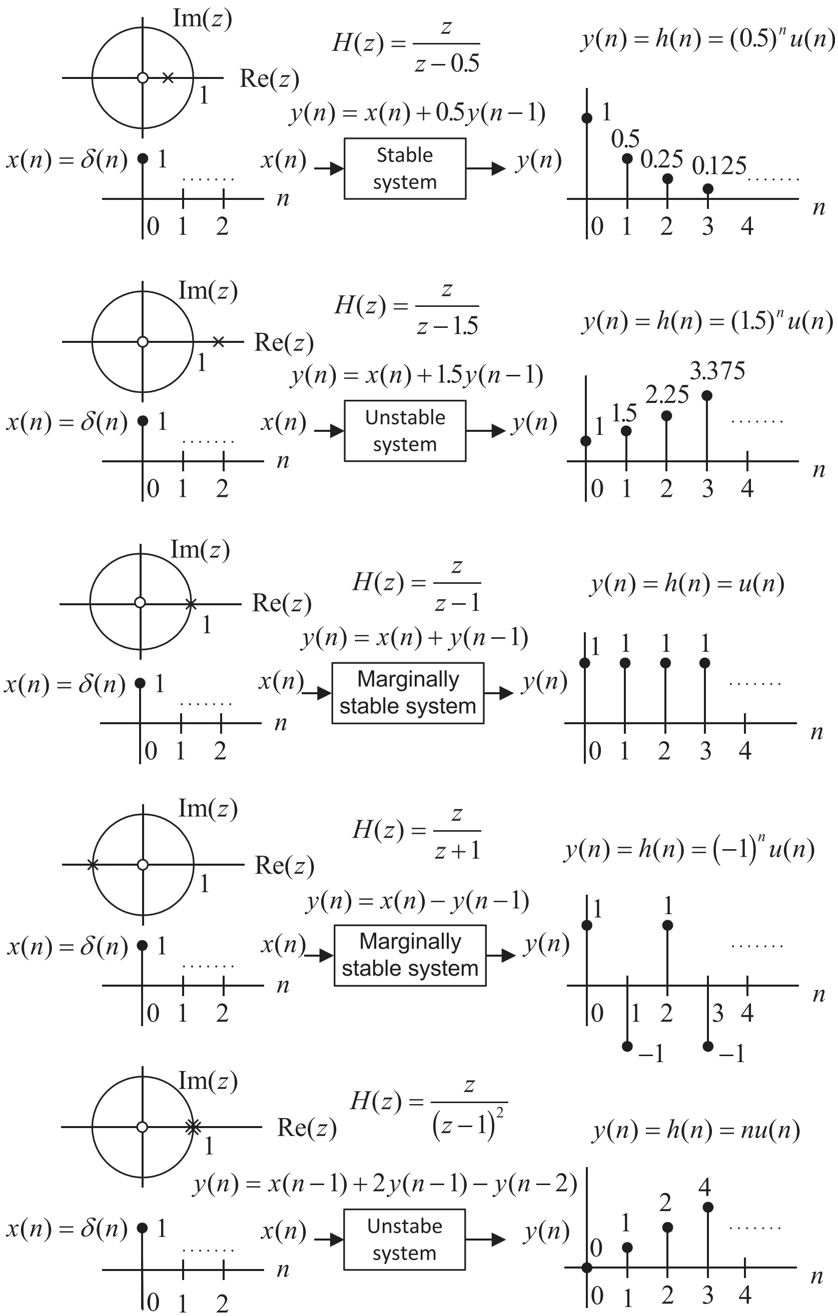


FIG. 6.9

Stability illustrations.

EXAMPLE 6.9

The following transfer functions describe digital systems.

$$H(z) = \frac{z+0.5}{(z-0.5)(z^2+z+0.5)}$$

$$H(z) = \frac{z^2+0.25}{(z-0.5)(z^2+3z+2.5)}$$

$$H(z) = \frac{z+0.5}{(z-0.5)(z^2+1.4141z+1)}$$

$$H(z) = \frac{z^2+z+0.5}{(z-1)^2(z+1)(z-0.6)}$$

For each, sketch the z-plane pole-zero plot and determine the stability status for the digital system.

Solution:

(a) A zero is found to be $z = -0.5$.
Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -0.5 \pm j0.5$,

$$|z| = \sqrt{(-0.5)^2 + (\pm 0.5)^2} = 0.707 < 1.$$

The plot of poles and a zero is shown in Fig. 6.10. Since the outmost poles are inside the unit circle, the system is stable.

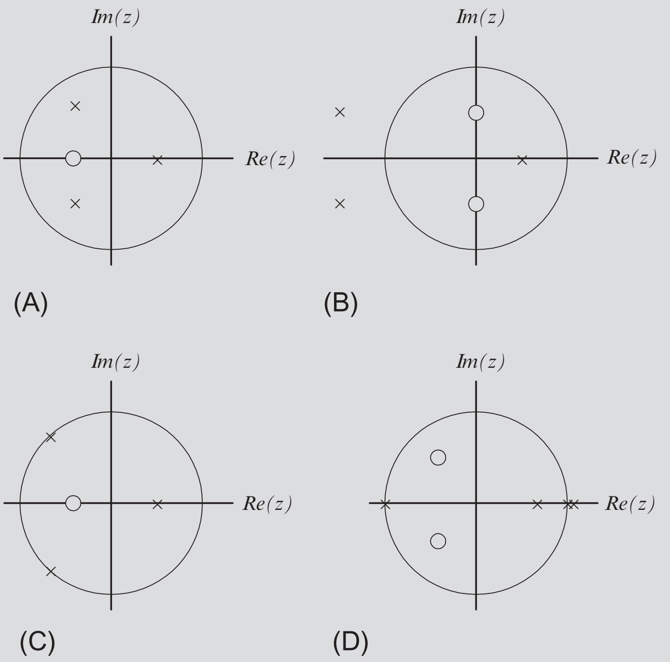


FIG. 6.10

Pole-zero plots for Example 6.9.

Continued

EXAMPLE 6.9—CONT'D

(b) Zeros are $z = \pm j0.5$.
Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -1.5 \pm j0.5$,

$$|z| = \sqrt{(1.5)^2 + (\pm 0.5)^2} = 1.5811 > 1.$$

The plot of poles and zeros is shown in Fig. 6.10. Since we have two poles at $z = -1.5 \pm j0.5$, which are outside the unit circle, the system is unstable.

(c) A zero is found to be $z = -0.5$.
Poles: $z = 0.5$, $|z| = 0.5 < 1$; $z = -0.707 \pm j0.707$,

$$|z| = \sqrt{(0.707)^2 + (\pm 0.707)^2} = 1$$

The zero and poles are plotted in Fig. 6.10. Since the outmost poles are first order at $z = -0.707 \pm j0.707$ and are on the unit circle, the system is marginally stable.

(d) Zeros: $z = -0.5 \pm j0.5$.
Poles: $z = 1$, $|z| = 1$; $z = 1$, $|z| = 1$; $z = -1$, $|z| = 1$; $z = 0.6$, $|z| = 0.6 < 1$.
The zeros and poles are plotted in Fig. 6.10. Since the outmost pole is multiple-order (second order) pole at $z = 1$ and is on the unit circle, the system is unstable.

6.4 DIGITAL FILTER FREQUENCY RESPONSE

From the Laplace transfer function, we can achieve the analog filter steady-state frequency response $H(j\omega)$ by substituting $s = j\omega$ into the transfer function $H(s)$, that is,

$$H(s)|_{s=j\omega} = H(j\omega).$$

Then we can study the magnitude frequency response $|H(j\omega)|$ and phase response $\angle H(j\omega)$. Similarly, in a DSP system, using the mapping Eq. (6.16), we substitute $z = e^{sT}|_{s=j\omega} = e^{j\omega T}$ into the z-transfer function $H(z)$ to acquire the digital frequency response, which is converted into the magnitude frequency response $|H(e^{j\omega T})|$ and phase response $\angle|H(e^{j\omega T})|$, that is,

$$H(z)|_{z=e^{j\omega T}} = H\left(e^{j\omega T}\right) = \left|H\left(e^{j\omega T}\right)\right| \angle H\left(e^{j\omega T}\right). \tag{6.18}$$

Let us introduce a normalized digital frequency in radians in digital domain

$$\Omega = \omega T. \tag{6.19}$$

Then the digital frequency response in Eq. (6.18) would become

$$H\left(e^{j\Omega}\right) = H(z)\big|_{z=e^{j\Omega}} = \left|H\left(e^{j\Omega}\right)\right| \angle H\left(e^{j\Omega}\right). \tag{6.20}$$

The formal derivation for Eqs. (6.20) can be found in Appendix D.

Now we verify the frequency response via the following simple digital filter. Consider a digital filter with a sinusoidal input of the amplitude K (Fig. 6.11):

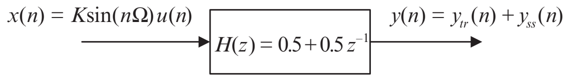


FIG. 6.11

System transient and steady-state frequency responses.

We can determine the system output $y(n)$, which consists of the transient response $y_{tr}(n)$ and the steady-state response $y_{ss}(n)$. We find the z-transform output as

$$Y(z) = \left(\frac{0.5z + 0.5}{z}\right)\frac{Kz\sin\Omega}{z^2 - 2z\cos\Omega + 1} \tag{6.21}$$

To perform the inverse z-transform to find the system output, we further rewrite Eq. (6.21) as

$$\frac{Y(z)}{z} = \left(\frac{0.5z+0.5}{z}\right)\frac{K\sin\Omega}{(z-e^{j\Omega})(z-e^{-j\Omega})} = \frac{A}{z} + \frac{B}{z-e^{j\Omega}} + \frac{B^*}{z-e^{-j\Omega}},$$

where A, B, and the complex conjugate B^* are the constants for the partial fractions. Applying the partial fraction expansion leads to

$$A = 0.5K\sin\Omega$$

$$B = \left.\frac{0.5z+0.5}{z}\right|_{z=e^{j\Omega}}\frac{K}{2j} = \left|H\left(e^{j\Omega}\right)\right|e^{j\angle H\left(e^{j\Omega}\right)}\frac{K}{2j}.$$

Note that the first part of constant B is a complex function, which is obtained by substituting $z = e^{j\Omega}$ into the filter z-transfer function. We can also express the complex function in terms of the polar form:

$$\left.\frac{0.5z+0.5}{z}\right|_{z=e^{j\Omega}} = 0.5 + 0.5z^{-1}\big|_{z=e^{j\Omega}} = H(z)|_{z=e^{j\Omega}} = H\left(e^{j\Omega}\right) = \left|H\left(e^{j\Omega}\right)\right|e^{j\angle H\left(e^{j\Omega}\right)},$$

where $H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega}$, and we call this complex function the steady-state frequency response. Based on the complex conjugate property, we get another residue as

$$B^* = \left|H\left(e^{j\Omega}\right)\right|e^{-j\angle H\left(e^{j\Omega}\right)}\frac{K}{-j2}.$$

The z-transform system output is then given by

$$Y(z) = A + \frac{Bz}{z-e^{j\Omega}} + \frac{B^*z}{z-e^{-j\Omega}}.$$

Taking the inverse z-transform, we achieve the following system transient and steady-state responses:

$$y(n) = \underbrace{0.5K\sin\Omega\delta(n)}_{y_{tr}(n)} + \underbrace{\left|H\left(e^{j\Omega}\right)\right|e^{j\angle H\left(e^{j\Omega}\right)}\frac{K}{j2}e^{jn\Omega}u(n) + \left|H\left(e^{j\Omega}\right)\right|e^{-j\angle H\left(e^{j\Omega}\right)}\frac{K}{-j2}e^{-jn\Omega}u(n)}_{y_{ss}(n)}.$$

Simplifying the response yields the form

$$y(n) = 0.5K\sin\Omega\delta(n) + \left|H\left(e^{j\Omega}\right)\right|K\frac{e^{jn\Omega+j\angle H\left(e^{j\Omega}\right)}u(n) - e^{-jn\Omega-j\angle H\left(e^{j\Omega}\right)}u(n)}{j2}.$$

We can further combine the last term using Euler's formula to express the system response as

$$y(n) = \underbrace{0.5K\sin\Omega\delta(n)}_{y_{tr}(n)\text{ will decay to zero after the first sample}} + \underbrace{\left|H\left(e^{j\Omega}\right)\right|K\sin\left(n\Omega + \angle H\left(e^{j\Omega}\right)\right)u(n)}_{y_{ss}(n)}$$

Finally, the steady-state response is identified as

$$y_{ss}(n) = K\left|H\left(e^{j\Omega}\right)\right|\sin\left(n\Omega + \angle H\left(e^{j\Omega}\right)\right)u(n).$$

For this particular filter, the transient response exists for only the first sample in the system response. By substituting $n=0$ into $y(n)$ and after simplifying algebra, we achieve the response for the first output sample.

$$y(0) = y_{tr}(0) + y_{ss}(0) = 0.5K\sin(\Omega) - 0.5K\sin(\Omega) = 0.$$

Note that the first output sample of the transient response cancels the first output sample of the steady-state response, so the combined first output sample has a value of zero for this particular filter. The system response reaches the steady-state response after the first output sample. At this point, we can conclude:

$$\text{Steady state magnitude frequency response} = \frac{\text{Peak amplitude of steady state response at }\Omega}{\text{Peak amplitude of sinusoidal input at }\Omega}$$

$$= \frac{\left|H(e^{j\Omega})\right|K}{K} = \left|H\left(e^{j\Omega}\right)\right|$$

$$\text{Steady state phase frequency response} = \text{Phase difference} = \angle H\left(e^{j\Omega}\right).$$

Fig. 6.12 shows the system response with sinusoidal inputs at $\Omega=0.25\pi$, $\Omega=0.5\pi$, and $\Omega=0.75\pi$, respectively.

Next, we examine the properties of the filter frequency response $H(e^{j\Omega})$. From Euler's identity and trigonometric identity, we know that

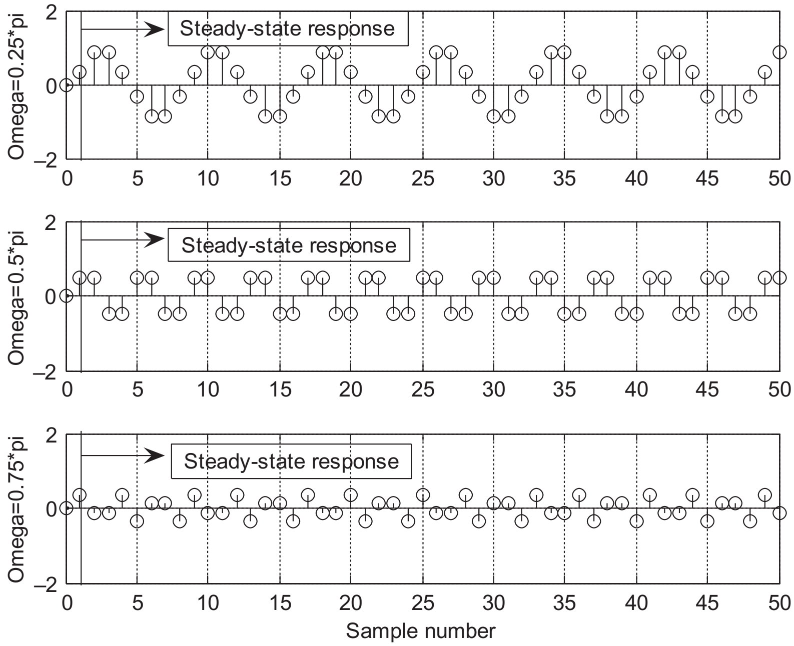


FIG. 6.12

Digital filter responses to different input sinusoids.

$$e^{j(\Omega+k2\pi)} = \cos(\Omega+k2\pi) + j\sin(\Omega+k2\pi)$$
$$= \cos\Omega + j\sin\Omega = e^{j\Omega},$$

where k is an integer taking the values $k=0,\pm1,\pm2,\cdots$. Then the frequency response has the following properties (assuming all input sequences are real):

1. Periodicity.
   (a) Frequency response: $H(e^{j\Omega})=H(e^{j(\Omega+k2\pi)})$.
   (b) Magnitude frequency response: $|H(e^{j\Omega})|=|H(e^{j(\Omega+k2\pi)})|$.
   (c) Phase response: $\angle H(e^{j\Omega})=\angle H(e^{j\Omega+k2\pi})$.

The second property is given without proof (see proof in Appendix D).

2. Symmetry.
   (a) Magnitude frequency response: $|H(e^{-j\Omega})|=|H(e^{j\Omega})|$.
   (b) Phase response: $\angle H(e^{-j\Omega})=-\angle H(e^{j\Omega})$.

Since the maximum frequency in a DSP system is the folding frequency, $f_s/2$, where $f_s=1/T$, and T designates the sampling period, the corresponding maximum normalized frequency of the system frequency can be calculated as.

$$\Omega=\omega T=2\pi\frac{f_s}{2}\times T=\pi \text{ radians}. \tag{6.22}$$

The frequency response $H(e^{j\Omega})$ for $|\Omega|>\pi$ consists of the image replicas of $H(e^{j\Omega})$ for $|\Omega|\le\pi$ and will be removed via the reconstruction filter later. Hence, we need to evaluate $H(e^{j\Omega})$ only for the positive normalized frequency range from $\Omega=0$ to $\Omega=\pi$ radians. The frequency, in Hz, can be determined by

$$f=\frac{\Omega}{2\pi}f_s. \tag{6.23}$$

The magnitude frequency response, often expressed in decibels, is defined as

$$\left|H\left(e^{j\Omega}\right)\right|_{dB}=20\times\log_{10}\left(\left|H\left(e^{j\Omega}\right)\right|\right). \tag{6.24}$$

The DSP system stability, magnitude response, and phase response are investigated via the following examples.

EXAMPLE 6.10

Given the following digital system with a sampling rate of 8000 Hz,

$$y(n)=0.5x(n)+0.5x(n-1),$$

Determine the frequency response.

Solution:

Taking the z-transform on both sides of the difference equation leads to

$$Y(z)=0.5X(z)+0.5z^{-1}X(z).$$

Then the transfer function describing the system is easily found as

$$H(z)=\frac{Y(z)}{X(z)}=0.5+0.5z^{-1}.$$

Continued

EXAMPLE 6.10—CONT'D

Substituting $z = e^{j\Omega}$, we have the frequency response as

$$H(e^{j\Omega}) = 0.5 + 0.5e^{-j\Omega}$$
$$= 0.5 + 0.5\cos(\Omega) - j0.5\sin(\Omega).$$

Therefore, the magnitude frequency response and phase response are given by

$$|H(e^{j\Omega})| = \sqrt{(0.5 + 0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}$$

and

$$\angle H(e^{j\Omega}) = \tan^{-1}\left(\frac{-0.5\sin(\Omega)}{0.5 + 0.5\cos(\Omega)}\right).$$

Several points for the magnitude response and phase response are calculated and illustrated in Table 6.1.

According to the data, we plot the frequency response and phase response of the DSP system as shown in Fig. 6.13.

Table 6.1 Frequency Response Calculations for Example 6.10

Ω (radians)	$f = \frac{\Omega}{2\pi} f_s$ (Hz)	$\lvert H(e^{j\Omega})\rvert$	$\lvert H(e^{j\Omega})\rvert_{dB}$	$\angle H(e^{j\Omega})$
0	0	1.000	0 dB	0^0
0.25π	1000	0.924	$--0.687$ dB	-22.5^0
0.50π	2000	0.707	$--3.012$ dB	-45.00^0
0.75π	3000	0.383	$--8.336$ dB	-67.50^0
1.00π	4000	0.000	$-\infty$	-90^0

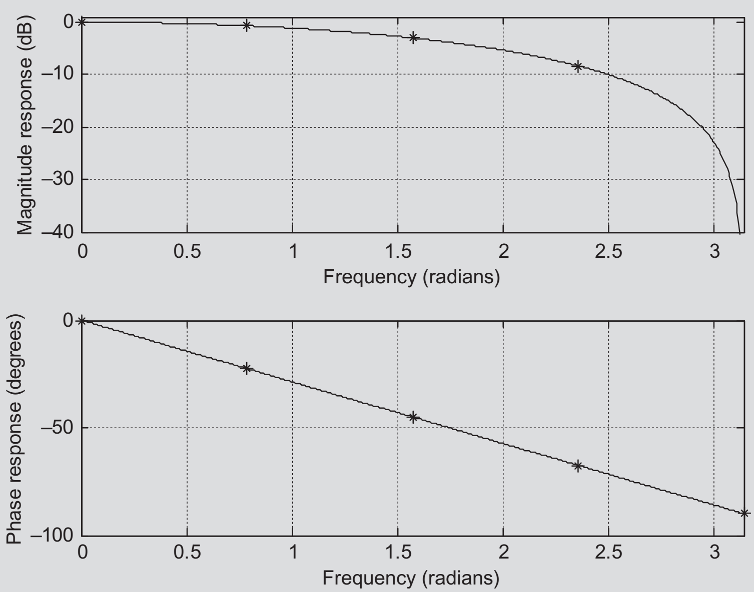


FIG. 6.13

Frequency responses of the digital filter in Example 6.10.

It is observed that when the frequency increases, the magnitude response decreases. The DSP system acts like a digital lowpass filter, and its phase response is linear.

We can also verify the periodicity for $|H(e^{j\Omega})|$ and $\angle H(e^{j\Omega})$ when $\Omega = 0.25\pi + 2\pi$:

$$\left|H\left(e^{j(0.25\pi+2\pi)}\right)\right| = \sqrt{(0.5+0.5\cos(0.25\pi+2\pi))^2 + (0.5\sin(0.25\pi+2\pi))^2}$$
$$= 0.924 = \left|H\left(e^{j0.25\pi}\right)\right|$$

$$\angle H\left(e^{j(0.25\pi+2\pi)}\right) = \tan^{-1}\left(\frac{-0.5\sin(0.25\pi+2\pi)}{0.5+0.5\cos(0.25\pi+2\pi)}\right) = -22.5^0 = \angle H\left(e^{j0.25\pi}\right).$$

For $\Omega = -0.25\pi$, we can verify the symmetry property as

$$\left|H\left(e^{-j0.25\pi}\right)\right| = \sqrt{(0.5+0.5\cos(-0.25\pi))^2 + (0.5\sin(-0.25\pi))^2}$$
$$= 0.924 = \left|H\left(e^{j0.25\pi}\right)\right|$$

$$\angle H\left(e^{-j0.25\pi}\right) = \tan^{-1}\left(\frac{-0.5\sin(-0.25\pi)}{0.5+0.5\cos(-0.25\pi)}\right) = 22.5^0 = -\angle H\left(e^{j0.25\pi}\right).$$

The properties can be observed in Fig. 6.14, where the frequency range is chosen from $\Omega = -2\pi$ to $\Omega = 4\pi$ radians. As shown in the figure, the magnitude and phase responses are periodic with a period of 2π. For a period between $\Omega = -\pi$ and $\Omega = \pi$, the magnitude responses for the portion $\Omega = -\pi$ to $\Omega = 0$ and the portion $\Omega = 0$ to $\Omega = \pi$ are same, while the phase responses are opposite. Since the magnitude and phase responses calculated for the range from $\Omega = 0$ to $\Omega = \pi$ are sufficient to the frequency response information, this range is only required for generating the frequency response plots.

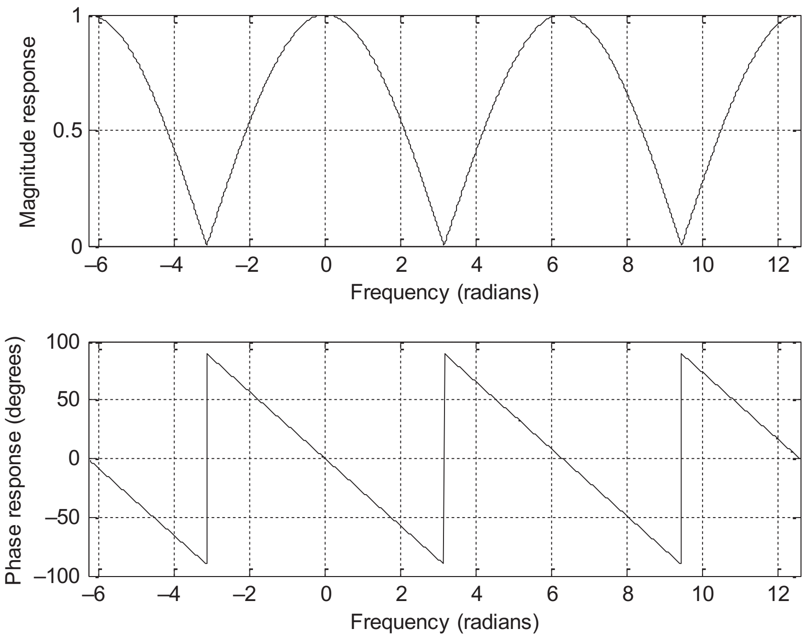


FIG. 6.14

Periodicity of the magnitude response and phase response in Example 6.10.

Again, note that the phase plot shows a sawtooth shape instead of a linear straight line for this particular filter. This is due to the phase wrapping at $\Omega = 2\pi$ radians since $e^{j(\Omega+k2\pi)} = e^{j\Omega}$ is used in the calculation. However, the phase plot shows that the phase is linear in the useful information range from $\Omega = 0$ to $\Omega = \pi$ radians.

EXAMPLE 6.11

Given a digital system with a sampling rate of 8000 Hz

$$y(n) = x(n) - 0.5y(n-1),$$

Determine the frequency response

Solution:

Taking the z-transform on both sides of the difference equation leads to

$$Y(z) = X(z) - 0.5z^{-1}Y(z).$$

Then the transfer function describing the system is easily found as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+0.5z^{-1}} = \frac{z}{z+0.5}.$$

Substituting $z = e^{j\Omega}$, we have the frequency response as

$$H(e^{j\Omega}) = \frac{1}{1+0.5e^{-j\Omega}}$$
$$= \frac{1}{1+0.5\cos(\Omega) - j0.5\sin(\Omega)}.$$

Therefore, the magnitude frequency response and phase response are given by

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\sqrt{(1+0.5\cos(\Omega))^2 + (0.5\sin(\Omega))^2}}$$

and

$$\angle H\left(e^{j\Omega}\right) = -\tan^{-1}\left(\frac{-0.5\sin(\Omega)}{1+0.5\cos(\Omega)}\right).$$

Several points for the magnitude response and phase response are calculated and illustrated in Table 6.2.

Table 6.2 Frequency Response Calculations in Example 6.11

Ω (radians)	$f = \frac{\Omega}{2\pi}f_s$ **(Hz)**	$\lvert H(e^{j\Omega})\rvert$	$\lvert H(e^{j\Omega})\rvert_{dB}$	$\angle H(e^{j\Omega})$
0	0	0.670	−3.479 dB	0^0
0.25π	1000	0.715	−2.914 dB	14.64^0
0.50π	2000	0.894	−0.973 dB	26.57^0
0.75π	3000	1.357	2.652 dB	28.68^0
1.00π	4000	2.000	6.021 dB	0^0

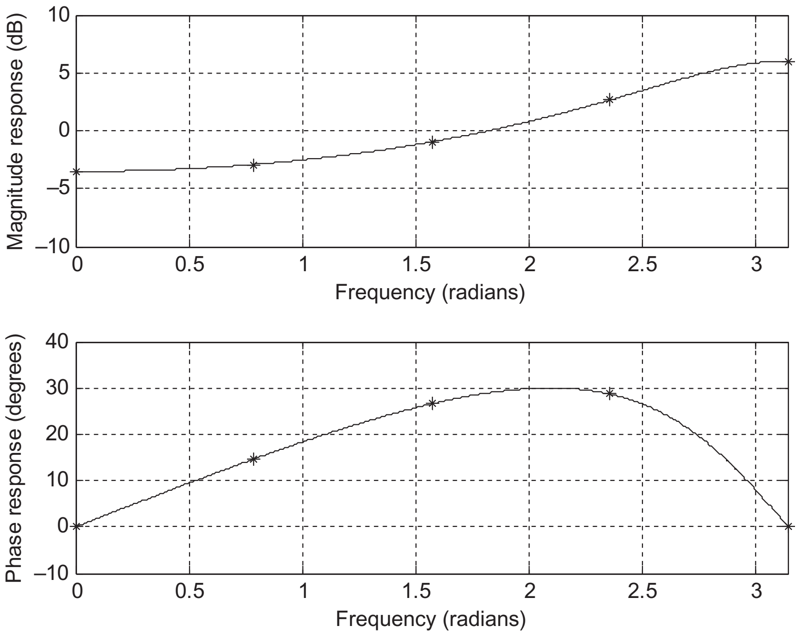


FIG. 6.15

Frequency responses of the digital filter in Example 6.11.

According to the achieved data, the magnitude response and phase response of the DSP system are roughly plotted in Fig. 6.15.

From Table 6.2 and Fig. 6.15, we can see that when the frequency increases, the magnitude response increases. The DSP system actually performs digital highpass filtering.

If all the coefficients a_i for $i = 0, 1, \ldots, M$ in Eq. (6.1) are zeros, Eq. (6.2) is reduced to

$$\begin{aligned} y(n) &= \sum_{i=0}^{M} b_i x(n-i) \\ &= b_0 x(n) + b_1 x(n-1) + \cdots + b_K x(n-M). \end{aligned} \tag{6.25}$$

Note that b_i is the ith impulse response coefficient. Also, since M is a finite positive integer, b_i in this particular case is a finite set, $H(z) = B(z)$; note that the denominator $A(z) = 1$. Such systems are called *finite impulse response* (FIR) systems. If not all a_i in Eq. (6.1) are zeros, the impulse response $h(i)$ would consist of an infinite number of coefficients. Such systems are called *infinite impulse response* (IIR) systems. The z-transform of the IIR $h(i)$, in general, is given by $H(z) = \frac{B(z)}{A(z)}$, where $A(z) \neq 1$.

6.5 BASIC TYPES OF FILTERING

The basic filter types can be classified into four categories: *lowpass*, *highpass*, *bandpass*, and *bandstop*. Each of them finds a specific application in DSP. One of the objectives in applications may involve the design of digital filters. In general, the filter is designed based on the specifications primarily for the passband, stopband, and transition band of the filter frequency response. The filter passband is

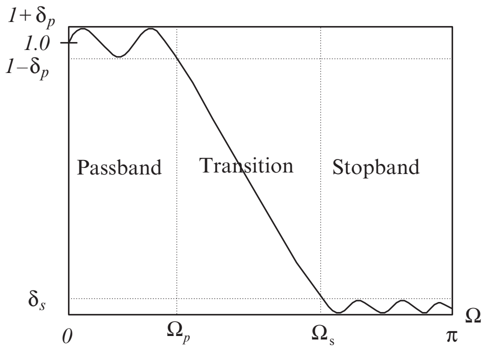


FIG. 6.16

Magnitude response of the normalized lowpass filter.

the frequency range with the amplitude gain of the filter response being approximately unity. The filter stopband is defined as the frequency range over which the filter magnitude response is attenuated to eliminate the input signal whose frequency components are within that range. The transition band denotes the frequency range between the passband and the stopband.

The design specifications of the lowpass filter are illustrated in Fig. 6.16, where the low-frequency components are passed through the filter while the high-frequency components are attenuated. As shown in Fig. 6.16, Ω_p and Ω_s are the passband cutoff frequency and the stopband cutoff frequency, respectively; δ_p is the design parameter to specify the ripple (fluctuation) of the frequency response in the passband, while δ_s specifies the ripple of the frequency response in the stopband.

The highpass filter remains the high-frequency components and rejects low-frequency components. The magnitude frequency response for the highpass filter is demonstrated in Fig. 6.17.

The bandpass filter attenuates both low- and high-frequency components while keeping the middle-frequency components, as shown in Fig. 6.18.

As illustrated in Fig. 6.18, Ω_{pL} and Ω_{sL} are the lower passband cutoff frequency and lower stopband cutoff frequency, respectively. Ω_{pH} and Ω_{sH} are the upper passband cutoff frequency and upper stopband cutoff frequency, respectively. δ_p is the design parameter to specify the ripple of the frequency response in the passband, while δ_s specifies the ripple of the frequency response in the stopband.

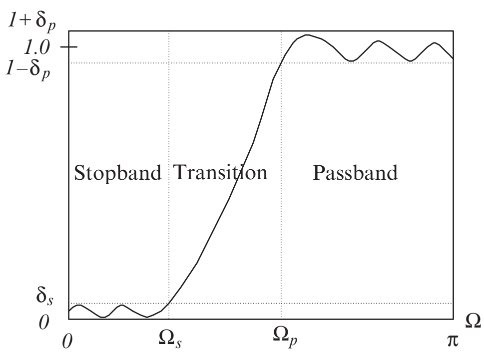


FIG. 6.17

Magnitude response of the normalized highpass filter.

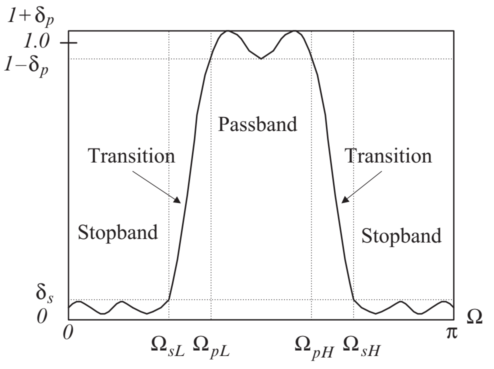


FIG. 6.18

Magnitude response of the normalized bandpass filter.

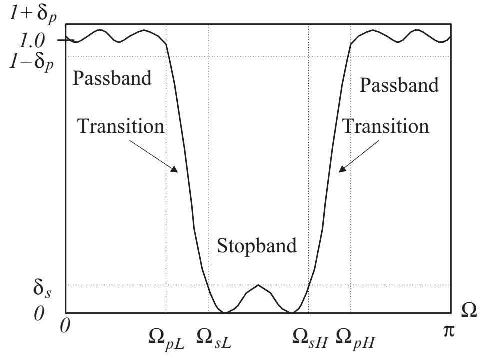


FIG. 6.19

Magnitude of the normalized bandstop filter.

Finally, the bandstop (bandreject or notch) filter shown in Fig. 6.19 rejects the middle-frequency components and accepts both the low- and the high-frequency components.

As a matter of fact, all kinds of digital filters are implemented using FIR or IIR systems. Furthermore, the FIR and IIR system can each be realized by various filter configurations, such as direct-forms, cascade forms, and parallel forms. Such topics will be included in the next section.

Given a transfer function, the MATLAB function **freqz()** can be used to determine the frequency response. The syntax is given by.

$$[\mathbf{h}, \mathbf{w}] = \mathbf{freqz}(\mathbf{B}, \mathbf{A}, \mathbf{N})$$

where all the parameters are defined as

h = an output vector containing frequency response.

w = an output vector containing normalized frequency values distributed in the range from 0 to π radians.

B = an input vector for numerator coefficients.

A = an input vector for denominator coefficients.

N = the number of normalized frequency points used for calculating the frequency response.

Let's consider Example 6.12.

EXAMPLE 6.12

Given each of the following digital transfer functions,

(a) $H(z) = \dfrac{z}{z - 0.5}$

(b) $H(z) = 1 - 0.5z^{-1}$

(c) $H(z) = \dfrac{0.5z^2 - 0.32}{z^2 - 0.5z + 0.25}$

(d) $H(z) = \dfrac{1 - 0.9z^{-1} + 0.81z^{-2}}{1 - 0.6z^{-1} + 0.36z^{-2}}$,

1. Plot the poles and zeros on the z-plane.
2. Use MATLAB function **freqz()** to plot the magnitude frequency response and phase response for each transfer function.
3. Identify the corresponding filter type such as lowpass, highpass, bandpass, or bandstop.

Solution:

1. The pole-zero plot for each transfer function is demonstrated in Fig. 6.20. The transfer functions of (a) and (c) need to be converted into the standard form (delay form) required by the MATLAB function **freqz(),** in which both the numerator and denominator polynomials have negative powers of z. Hence, we obtain

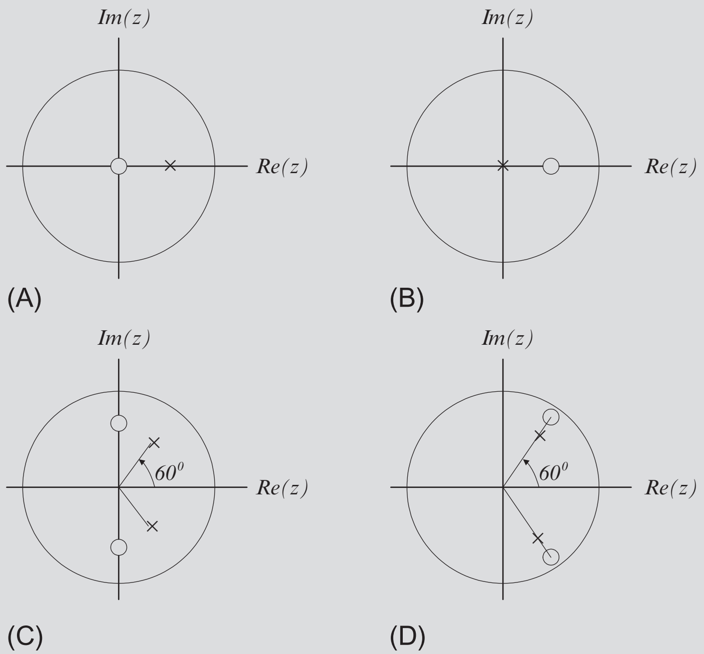


FIG. 6.20

Pole-zero plots of Example 6.12.

$$H(z) = \frac{z}{z-0.5} = \frac{1}{1-0.5z^{-1}}$$

$$H(z) = \frac{0.5z^2-0.32}{z^2-0.5z+0.25} = \frac{0.5-0.32z^{-2}}{1-0.5z^{-1}+0.25z^{-2}},$$

while the transfer functions of (b) and (d) are already in their standard forms (delay forms).

2. The MATLAB program for plotting the magnitude frequency response and the phase response for each case is listed in Program 6.2.
3. From the plots in Figs. 6.21(a–d) of magnitude frequency responses for all cases, we can conclude that case (a) is a lowpass filter, (b) is a highpass filter, (c) is a bandpass filter, and (d) is a bandstop (bandreject) filter.

Program 6.2. MATLAB program for Example 6.12.

```
% Example 6.12
% Plot the frequency response and phase response
% Case a
 figure (1)
 [h w]=freqz([1],[1-0.5],1024);  % Calculate frequency response
 phi=180*unwrap(angle(h))/pi;
 subplot(2,1,1), plot(w,abs(h)),grid;
 xlabel('Frequency (radians)'), ylabel('Magnitude')
 subplot(2,1,2), plot(w,phi),grid;
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
% Case b
 figure (2)
 [h w]=freqz([1-0.5],[1],1024);  %Calculate frequency response
 phi=180*unwrap(angle(h))/pi;
 subplot(2,1,1), plot(w,abs(h)),grid;
xlabel('Frequency (radians)'), ylabel('Magnitude')
 subplot(2,1,2), plot(w,phi),grid;
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
% Case c
 figure (3)
 [h w]=freqz([0.5 0-0.32],[1-0.5 0.25],1024); % Calculate frequency response
 phi=180*unwrap(angle(h))/pi;
 subplot(2,1,1), plot(w,abs(h)),grid;
xlabel('Frequency (radians)'), ylabel('Magnitude')
 subplot(2,1,2), plot(w,phi),grid;
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
% Case d
 figure (4)
 [h w]=freqz([1-0.9 0.81], [1-0.6 0.36],1024); %Calculate frequency response
 phi=180*unwrap(angle(h))/pi;
 subplot(2,1,1), plot(w,abs(h)),grid;
xlabel('Frequency (radians)'), ylabel('Magnitude')
 subplot(2,1,2), plot(w,phi),grid;
xlabel('Frequency (radians)'), ylabel('Phase (degrees)')
%
```

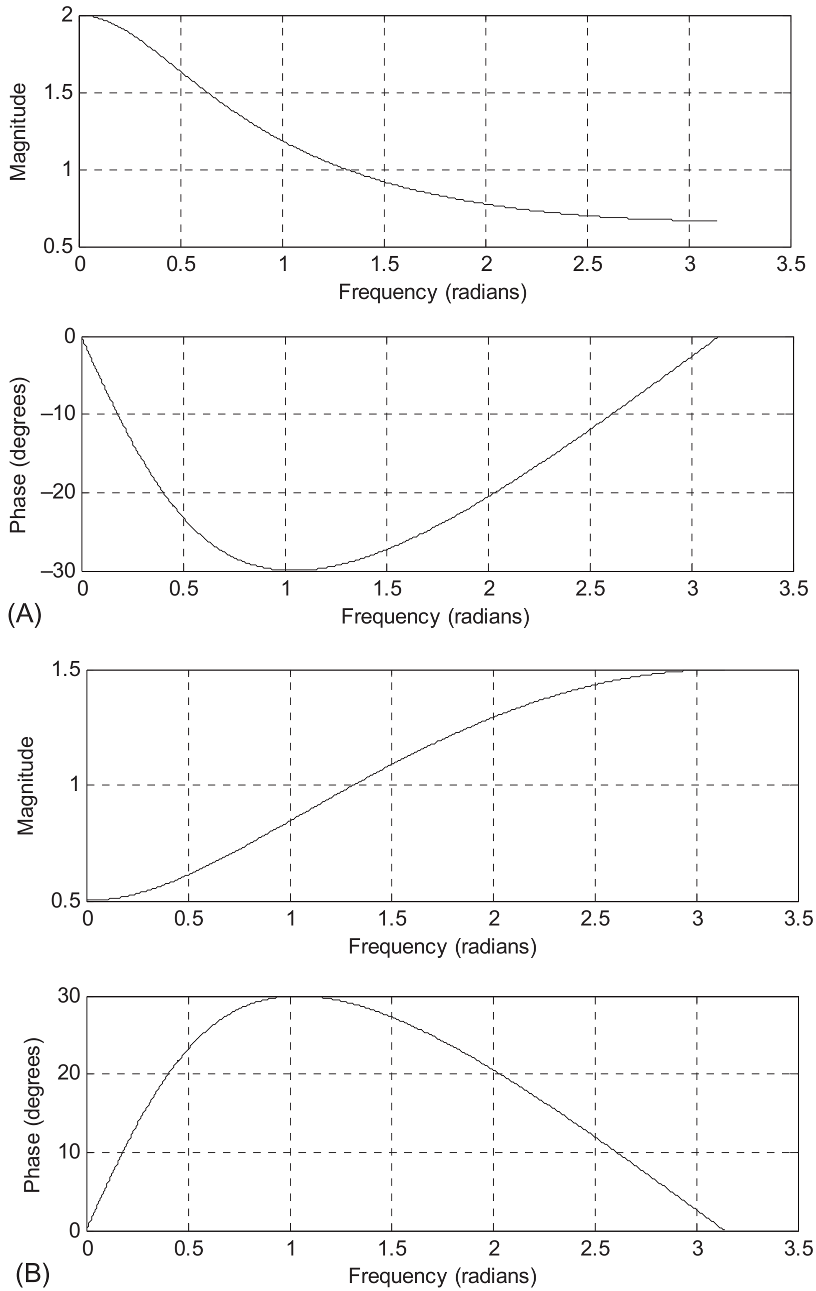


FIG. 6.21

Plots of frequency responses for Example 6.12: (A) for (a), (B) for (b),

(Continued)

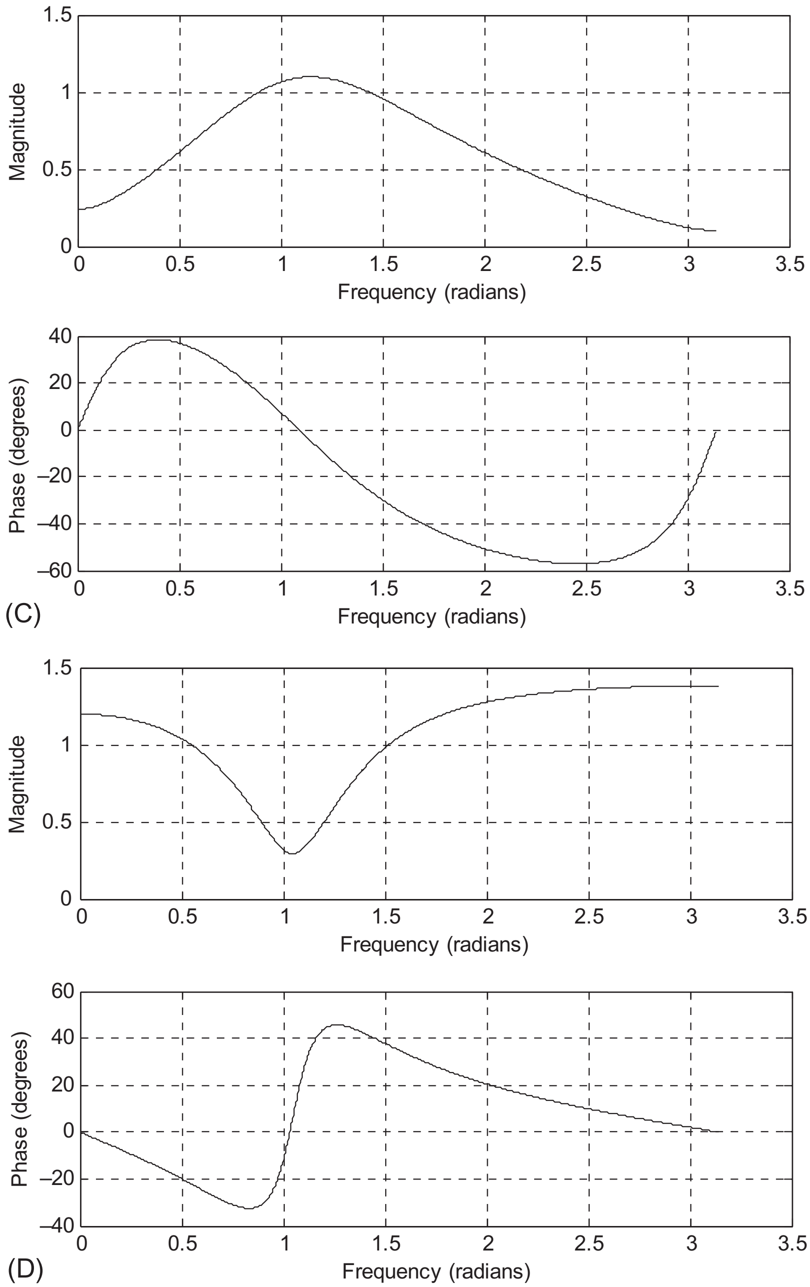


FIG. 6.21, CONT'D

(C) for (c), and (D) for (d).

6.6 REALIZATION OF DIGITAL FILTERS

In this section, basic realization methods for digital filters are discussed. Digital filters described by the transfer function $H(z)$ may be generally realized into the following forms:

- direct-form I
- direct-form II
- cascade
- parallel

The reader can explore various lattice realizations in the textbook by Stearns and Hush [1990]. The lattice filter realization has a good numerical property and easy for the control of filter stability. Lattice filters are widely used in speech processing and a favored structure for implementing adaptive filters.

6.6.1 DIRECT-FORM I REALIZATION

As we know, a digital filter transfer function, $H(z)$, is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}. \tag{6.26}$$

Let $x(n)$ and $y(n)$ be the digital filter input and output, respectively. We can express the relationship in z-transform domain as

$$Y(z) = H(z)X(z), \tag{6.27}$$

where $X(z)$ and $Y(z)$ are the z-transforms of $x(n)$ and $y(n)$, respectively. If we substitute Eq. (6.26) into $H(z)$ in Eq. (6.27), we have

$$Y(z) = \left(\frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}\right) X(z). \tag{6.28}$$

Taking the inverse of the z-transform in Eq. (6.28), we yield the relationship between input $x(n)$ and output $y(n)$ in time domain, as follows:

$$\begin{aligned} y(n) &= b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M) \\ &\quad - a_1 y(n-1) - a_2 y(n-2) - \cdots - a_N y(n-N). \end{aligned} \tag{6.29}$$

This difference equation thus can be implemented by a direct-form I realization shown in Fig. 6.22(a). Fig. 6.22(b) illustrates the realization of the second-order IIR filter ($M = N = 2$). Note that the notation used in Figs. 6.22(a) and (b) are defined in Fig. 22(c) and will be applied for discussion of other realizations.

Also, note that any of the a_j and b_i can be zero, thus all the paths are not required to exist for the realization.

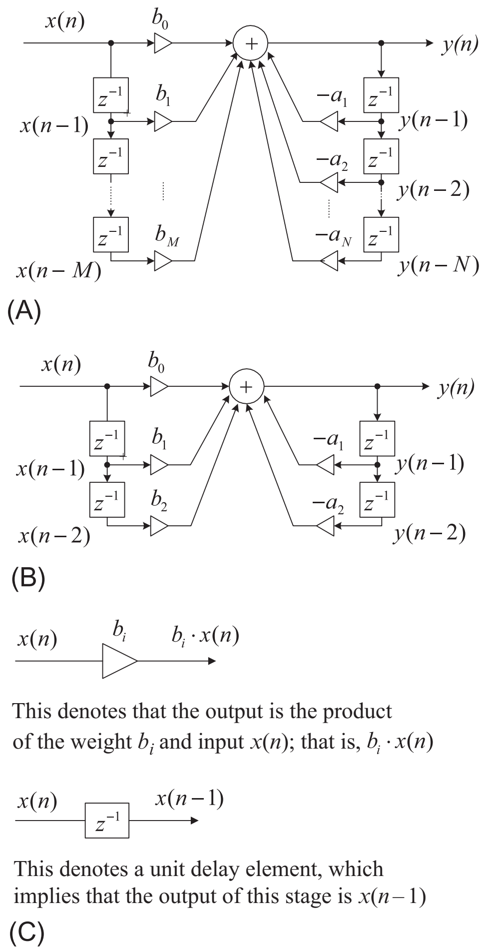


FIG. 6.22

(A) Direct-form I realization, (B) Direct-form I realization with $M=2$, (C) Notation.

6.6.2 DIRECT-FORM II REALIZATION

Considering Eqs. (6.26) and (6.27) with $N=M$, we can express

$$
\begin{aligned}
Y(z) &= H(z)X(z) = \frac{B(z)}{A(z)}X(z) = B(z)\left(\frac{X(z)}{A(z)}\right) \\
&= \left(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}\right)\underbrace{\left(\frac{X(z)}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}}\right)}_{W(z)}.
\end{aligned}
\tag{6.30}
$$

Also, define a new z-transform function as

$$W(z) = \frac{X(z)}{1 + a_1 z^{-1} + \cdots + a_M z^{-M}}, \tag{6.31}$$

we have

$$Y(z) = \left(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}\right) W(z). \tag{6.32}$$

The corresponding difference equations for Eqs. (6.31) and (6.32), respectively, become

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - \cdots - a_M w(n-M) \tag{6.33}$$

and

$$y(n) = b_0 w(n) + b_1 w(n-1) + \ldots + b_M w(n-M). \tag{6.34}$$

Realization of Eqs. (6.33) and (6.34) becomes another direct-form II realization, which is demonstrated in Fig. 6.23A. Again, the corresponding realization of the second-order IIR filter is described in Fig. 6.23B. Note that in Fig. 6.23A, the variables $w(n)$, $w(n-1)$, $w(n-2)$, …, $w(n-M)$ are different from the filter inputs $x(n-1)$, $x(n-2)$, …,$x(n-M)$.

On comparing structures between direct-form I realization and direct-form II realization, it can be seen that both realizations require the same number of multiplications while the direct-form II realization requires two accumulators. One of the benefits from the direct-form II structure is the use of the

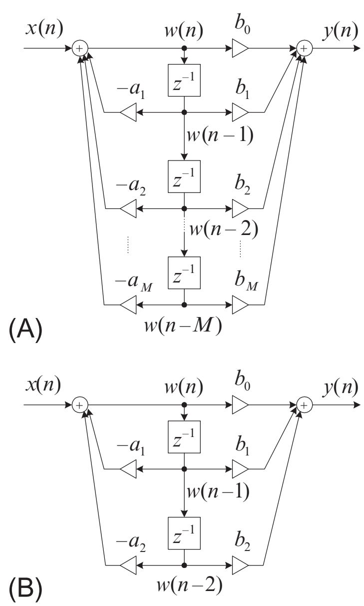


FIG. 6.23

(A) Direct-form II realization, (B) direct-form II realization with $M=2$.

reduced number of delay elements (saving memory). The other benefit relies on the fixed-point realization, where the filtering process involves only integer operations. Scaling of filter coefficients in the fixed-point implementation is required to avoid the overflow in the accumulator. For the direct-form II structure, the numerator and denominator filter coefficients are scaled separately so that the overflow problem for each accumulator can easily be controlled. More details will be illustrated in Chapter 14.

6.6.3 CASCADE (SERIES) REALIZATION

An alternate way to filter realization is to cascade the factorized $H(z)$ in the following form:

$$H(z) = H_1(z) \cdot H_2(z) \cdots H_k(z), \tag{6.35}$$

where $H_k(z)$ is chosen to be the first- or second-order transfer function (section), which is defined by

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1}} \tag{6.36}$$

or

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1} + b_{k2}z^{-2}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}, \tag{6.37}$$

respectively. The block diagram of the cascade, or series, realization is depicted in Fig. 6.24.

6.6.4 PARALLEL REALIZATION

Now we convert $H(z)$ into the following form

$$H(z) = H_1(z) + H_2(z) + \cdots + H_k(z), \tag{6.38}$$

where $H_k(z)$ is defined as the first- or second-order transfer function (section) given by

$$H_k(z) = \frac{b_{k0}}{1 + a_{k1}z^{-1}} \tag{6.39}$$

or

$$H_k(z) = \frac{b_{k0} + b_{k1}z^{-1}}{1 + a_{k1}z^{-1} + a_{k2}z^{-2}}, \tag{6.40}$$

respectively. The resulting parallel realization is illustrated in the block diagram in Fig. 6.25.

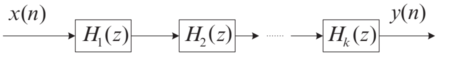


FIG. 6.24

Cascade realization.

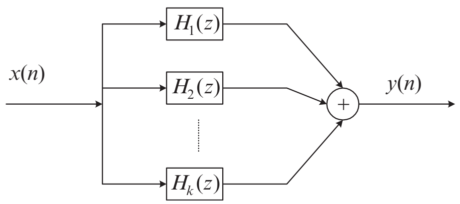


FIG. 6.25

Parallel realization.

EXAMPLE 6.13

Given a second-order transfer function

$$H(z) = \frac{0.5(1 - z^{-2})}{1 + 1.3z^{-1} + 0.36z^{-2}},$$

perform the filter realizations and write difference equations using the following realizations:

1. Direct-form I and direct-form II
2. Cascade form via first-order sections
3. Parallel form via first-order sections

Solution:

1. To perform the filter realizations using the direct-form I and direct-form II, we rewrite the given second-order transfer function as

$$H(z) = \frac{0.5 - 0.5z^{-2}}{1 + 1.3z^{-1} + 0.36z^{-2}}$$

and identify that.

$a_1 = 1.3$, $a_2 = 0.36$, $b_0 = 0.5$, $b_1 = 0$, and $b_2 = -0.5$.

Based on realizations in Fig. 6.22, we sketch the direct-form I realization as shown in Fig. 6.26.

The difference equation for the direct-form I realization is given by

$$y(n) = 0.5x(n) - 0.5x(n-2) - 1.3y(n-1) - 0.36y(n-2).$$

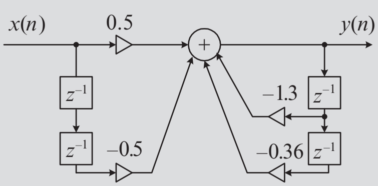


FIG. 6.26

Direct-form I realization for Example 6.13.

Using the direct-form II realization shown in Fig. 6.23, we have the realization depicted in Fig. 6.27.

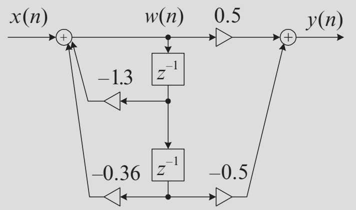


FIG. 6.27

Direct-form II realization for Example 6.13.

The difference equations for the direct-form II realization are expressed as

$$w(n) = x(n) - 1.3w(n-1) - 0.36w(n-2)$$

$$y(n) = 0.5w(n) - 0.5w(n-2)$$

2. To achieve the cascade (series) form realization, we factor $H(z)$ into two first-order sections to yield

$$H(z) = \frac{0.5(1 - z^{-2})}{1 + 1.3z^{-1} + 0.36z^{-2}} = \frac{0.5 - 0.5z^{-1}}{1 + 0.4z^{-1}} \frac{1 + z^{-1}}{1 + 0.9z^{-1}},$$

where $H_1(z)$ and $H_2(z)$ are chosen to be

$$H_1(z) = \frac{0.5 - 0.5z^{-1}}{1 + 0.4z^{-1}}$$

$$H_2(z) = \frac{1 + z^{-1}}{1 + 0.9z^{-1}}.$$

Note that the obtained $H_1(z)$ and $H_2(z)$ are not the unique selections for realization. For example, there is another way of choosing $H_1(z) = \frac{0.5 - 0.5z^{-1}}{1 + 0.9z^{-1}}$ and $H_2(z) = \frac{1 + z^{-1}}{1 + 0.4z^{-1}}$ to yield the same $H(z)$. Using $H_1(z)$ and $H_2(z)$ we have obtained, and with the direct-form II realization, we achieve the cascade form depicted in Fig. 6.28.

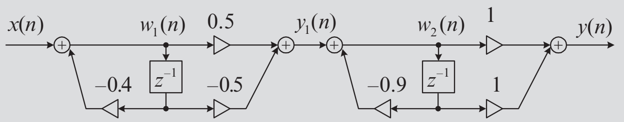


FIG. 6.28

Cascade realization for Example 6.13.

The difference equations for the direct-form II realization have two cascaded sections, expressed as.

Section 1:

$$w_1(n) = x(n) - 0.4w(n-1)$$

$$y_1(n) = 0.5w_1(n) - 0.5w_1(n-1)$$

Continued

EXAMPLE 6.13—CONT'D

Section 2:

$$w_2(n) = y_1(n) - 0.9w_2(n-1)$$

$$y(n) = w_2(n) + w_2(n-1).$$

3. In order to yield the parallel form of realization, we need to make use of the partial fraction expansion, and will first let

$$\frac{H(z)}{z} = \frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)} = \frac{A}{z} + \frac{B}{z+0.4} + \frac{C}{z+0.9},$$

where

$$A = z\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=0} = \frac{0.5(z^2-1)}{(z+0.4)(z+0.9)}\bigg|_{z=0} = -1.39$$

$$B = (z+0.4)\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=-0.4} = \frac{0.5(z^2-1)}{z(z+0.9)}\bigg|_{z=-0.4} = 2.1$$

$$C = (z+0.9)\left(\frac{0.5(z^2-1)}{z(z+0.4)(z+0.9)}\right)\bigg|_{z=-0.9} = \frac{0.5(z^2-1)}{z(z+0.4)}\bigg|_{z=-0.9} = -0.21.$$

Therefore

$$H(z) = -1.39 + \frac{2.1z}{z+0.4} + \frac{-0.21z}{z+0.9} = -1.39 + \frac{2.1}{1+0.4z^{-1}} + \frac{-0.21}{1+0.9z^{-1}}.$$

Again, using the direct-form II for each section, we obtain the parallel realization shown in Fig. 6.29.

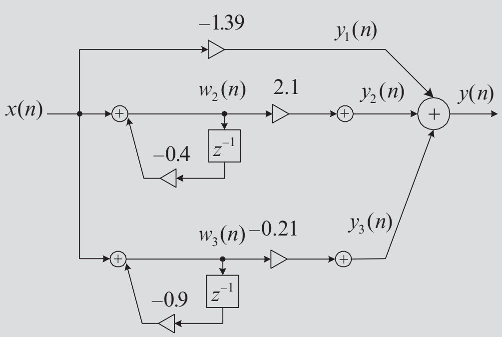


FIG. 6.29

Parallel realization for Example 6.13.

The difference equations for the direct-form II realization have three parallel sections, expressed as

$$y_1(n) = -1.39x(n)$$
$$w_2(n) = x(n) - 0.4w_2(n-1)$$
$$y_2(n) = 2.1w_2(n)$$
$$w_3(n) = x(n) - 0.9w_3(n-1)$$
$$y_3(n) = -0.21w_3(n)$$
$$y(n) = y_1(n) + y_2(n) + y_3(n).$$

In practice, the second-order filter module using the direct-form I or direct-form II is used. The high-order filter can be factored in the cascade form with the first- or second-order sections. In case the first-order filter is required, we can still modify the second-order filter module by setting the corresponding filter coefficients to be zero.

6.7 APPLICATION: SIGNAL ENHANCEMENT AND FILTERING

This section investigates the applications of signal enhancement using a preemphasis filter and speech filtering using a bandpass filter. Enhancement also includes the biomedical signals such as an electrocardiogram (ECG) signal.

6.7.1 PREEMPHASIS OF SPEECH

A speech signal may have frequency components that falloff at high frequencies. In some applications such as speech coding, to avoid overlooking the high frequencies, the high-frequency components are compensated using preemphasis filtering. A simple digital filter used for such compensation is given as

$$y(n) = x(n) - \alpha x(n-1), \tag{6.41}$$

where α is the positive parameter to control the degree of preemphasis filtering and usually is chosen to be <1. The filter described in Eq. (6.41) is essentially a highpass filter. Applying z-transform on both sides of Eq. (6.41) and solving for the transfer function, we have

$$H(z) = 1 - \alpha z^{-1}. \tag{6.42}$$

The magnitude and phase responses adopting the preemphasis parameter $\alpha = 0.9$ and the sampling rate $f_s = 8,000$ Hz are plotted in Fig. 6.30A using MATLAB.

Fig. 6.30B compares the original speech waveform and the preemphasized speech using the filter in Eq. (6.42). Again, we apply the fast Fourier transform (FFT) to estimate the spectrum of the original speech and the spectrum of the preemphasized speech. The plots are displayed in Fig. 6.31.

From Fig. 6.31, we can conclude that the filter does its job to boost the high-frequency components and attenuate the low-frequency components. We can also try this filter with different values of α to examine the degree of preemphasis filtering of the digitally recorded speech. The MATLAB list is given in Program 6.3.

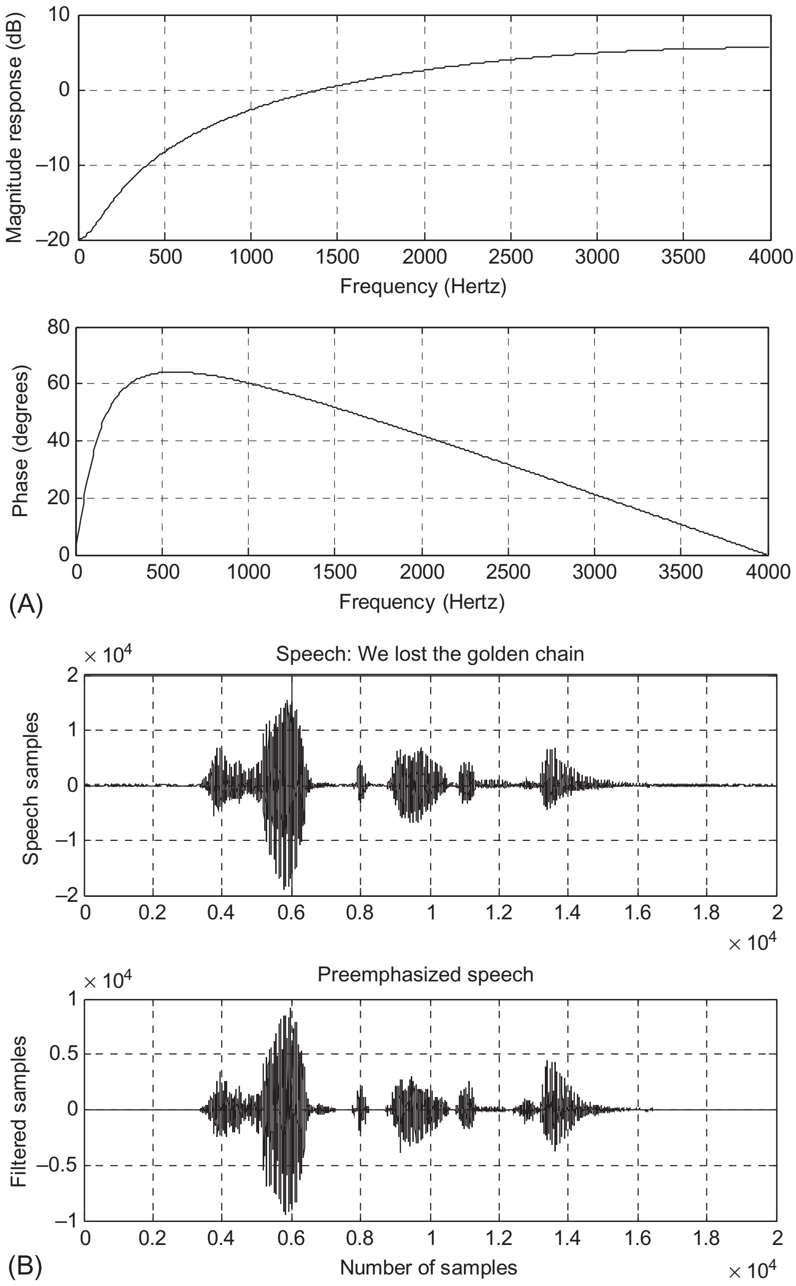


FIG. 6.30

(A) Frequency responses of the preemphasis filter. (B) Original speech and preemphasized speech waveforms.

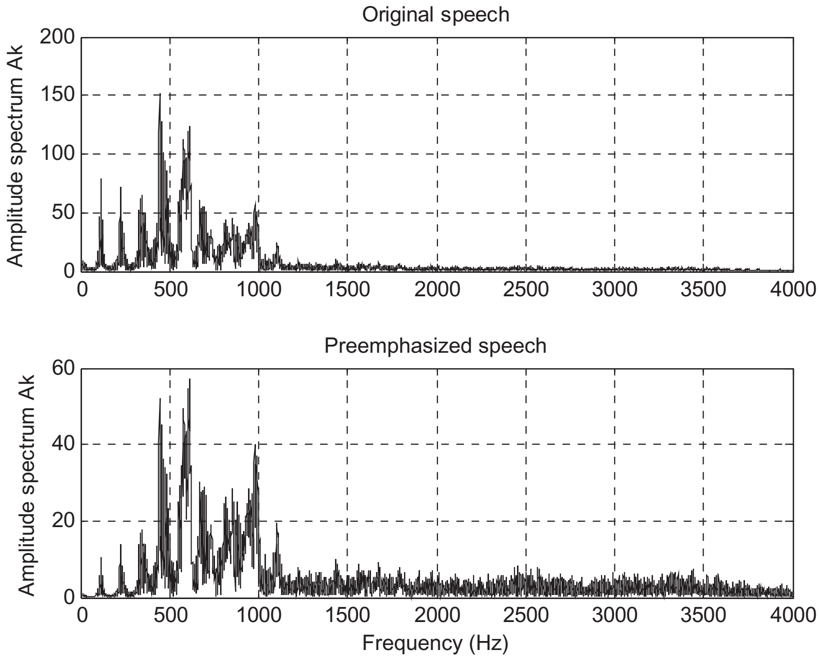


FIG. 6.31

Amplitude spectral plots for the original speech and preemphasized speech.

Program 6.3. MALAB program for preemphasis of speech

```
% Matlab program for Figs. 6.30 and 6.31
close all;clear all
fs=8000;                        % Sampling rate
alpha =0.9;                     % Degree of pre-emphasis
figure(1);
freqz([1 -alpha],1512,fs);        % Calculate and display frequency response
load speech.dat
figure(2);
y=filter([1 -alpha],1,speech);    % Filtering speech
subplot(2,1,1),plot(speech);grid;
ylabel('Speech samples')
title('Speech: We lost the golden chain.')
subplot(2,1,2),plot(y);grid
ylabel('Filtered samples')
xlabel('Number of samples');
title('Preemphasized speech.')
figure(3);
```

```
N=length(speech);                        % Length of speech
Axk=abs(fft(speech.*hamming(N)'))/N;  % Two-sided spectrum of speech
Ayk=abs(fft(y.*hamming(N)'))/N;       % Two-sided spectrum of preemphasized speech
f=[0:N/2]*fs/N;
Axk(2:N)=2*Axk(2:N);              % Get one-side spectrum of speech
Ayk(2:N)=2*Ayk(2:N);              % Get one-side spectrum of filtered speech
subplot(2,1,1),plot(f,Axk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
title('Original speech');
subplot(2,1,2),plot(f,Ayk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
xlabel('Frequency (Hz)');
title('Preemphasized speech');
%
```

6.7.2 BANDPASS FILTERING OF SPEECH

Bandpass filtering plays an important role in DSP applications. It can be used to pass the signals according to the specified frequency passband and reject the frequency other than the passband specification. Then the filtered signal can further be used for the signal feature extraction. Filtering can also be applied to perform applications such as noise reduction, frequency boosting, digital audio equalizing, and digital crossover, among others.

Let us consider the following digital fourth-order bandpass Butterworth filter with a lower cutoff frequency of 1000 Hz, an upper cutoff frequency of 1400 Hz (i.e., the bandwidth is 400 Hz), and a sampling rate of 8000 Hz:

$$H(z) = \frac{0.0201 - 0.0402z^{-2} + 0.0201z^{-4}}{1 - 2.1192z^{-1} + 2.6952z^{-2} - 1.6924z^{-3} + 0.6414z^{-4}}. \tag{6.43}$$

Converting the z-transfer function into the DSP difference equation yields

$$\begin{aligned} y(n) = {} & 0.0201x(n) - 0.0402x(n-2) + 0.0201x(n-4) \\ & + 2.1192y(n-1) - 2.6952y(n-2) + 1.6924y(n-3) - 0.6414y(n-4). \end{aligned} \tag{6.44}$$

The filter frequency responses are computed and plotted in Fig. 6.32A with MATLAB. Fig. 6.32B shows the original speech and filtered speech, while Fig. 6.32C displays the spectral plots for the original speech and filtered speech.

As shown in Fig. 6.32C, the designed bandpass filter significantly reduces low-frequency components, which are <1000 Hz, and the high-frequency components above 1400 Hz, while letting the signals with the frequencies ranging from 1000 to 1400 Hz pass through the filter. Similarly, we can design and implement other types of filters, such as lowpass, highpass, bandpass, and bandreject to filter the signals and examine the performances of their designs. MATLAB implementation detail is given in Program 6.4.

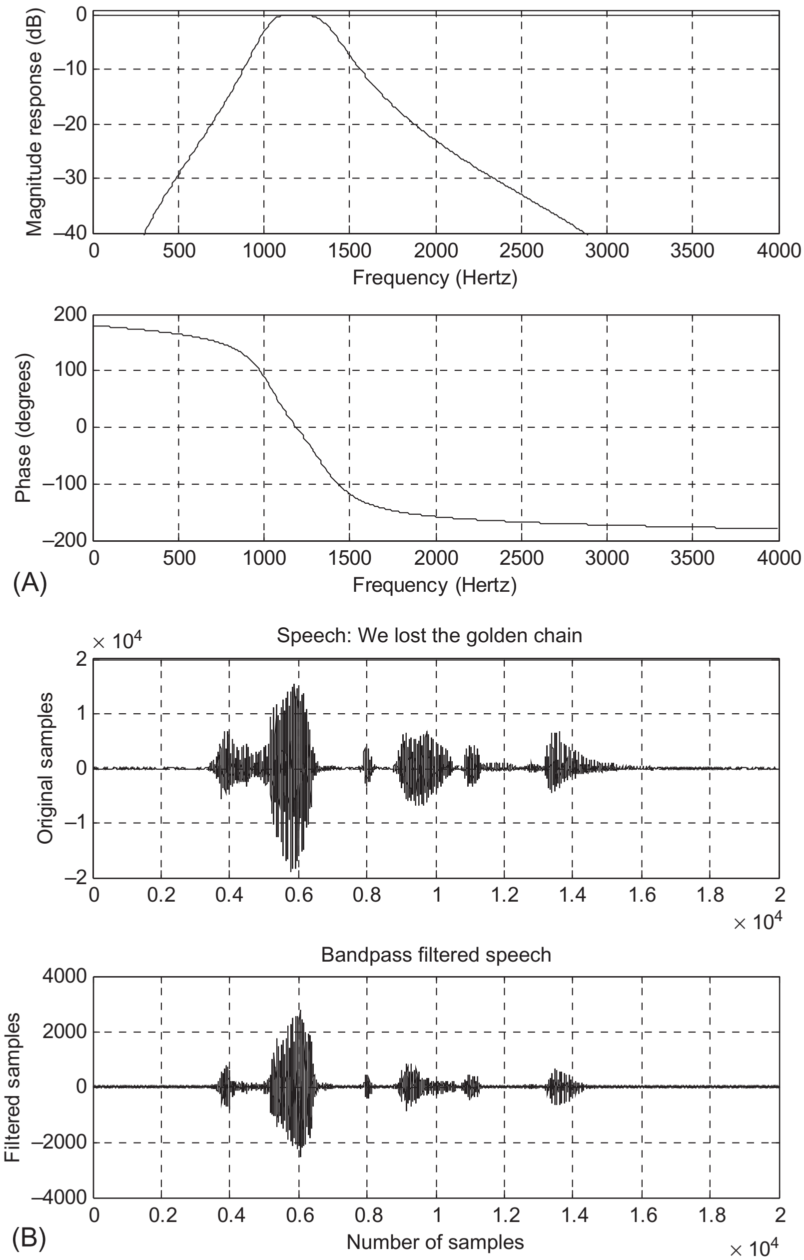


FIG. 6.32

(A) Frequency responses of the designed bandpass filter. (B) Plots of the original speech and filtered speech.

(Continued)

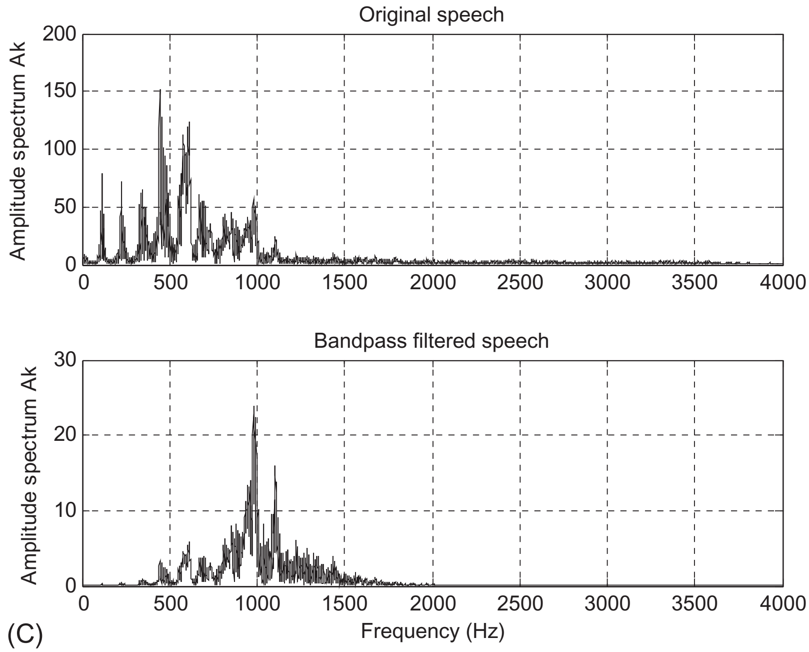


FIG. 6.32, CONT'D

(C) Amplitude spectra of the original speech and bandpass filtered speech.

Program 6.4. MATLAB program for bandpass filtering of speech

```
fs=8000;              % Sampling rate
freqz([0.0201 0.00-0.0402 0 0.0201],[1-2.1192 2.6952-1.6924 0.6414],512,fs);
axis([0 fs/2-40 1]);              % Frequency response of bandpass filter
figure
load speech.dat
y=filter([0.0201 0.00-0.0402 0.0201],[1-2.1192 2.6952-1.6924 0.6414],speech);
subplot(2,1,1),plot(speech); grid;            % Filtering speech
ylabel('Origibal Samples')
title('Speech: We lost the golden chain.')
subplot(2,1,2),plot(y);grid
xlabel('Number of Samples');ylabel('Filtered Samples')
title('Bandpass filtered speech.')
figure
N=length(speech);
Axk=abs(fft(speech.*hamming(N)'))/N;  % One-sided spectrum of speech
Ayk=abs(fft(y.*hamming(N)'))/N;       % One-sided spectrum of filtered speech
f=[0:N/2]*fs/N;
Axk(2:N)=2*Axk(2:N);Ayk(2:N)=2*Ayk(2:N);  % One-sided spectra
subplot(2,1,1),plot(f,Axk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak')
title('Original speech');
subplot(2,1,2),plot(f,Ayk(1:N/2+1));grid
ylabel('Amplitude spectrum Ak');xlabel('Frequency (Hz)');
title('Bandpass filtered speech');
```

6.7.3 ENHANCEMENT OF ECG SIGNAL USING NOTCH FILTERING

A notch filter is a bandreject filter with a very narrow bandwidth. It can be applied to enhance the ECG signal corrupted during the data acquisition stage, where the signal contains a 60-Hz interference induced from the power line. Let us consider the following digital second-order notch filter with a notch frequency of 60 Hz and the digital system has a sampling frequency of 500 Hz. We obtain a notch filter (details can be found in Chapter 8) as follows:

$$H(z) = \frac{1 - 1.4579z^{-1} + z^{-2}}{1 - 1.3850z^{-1} + 0.9025z^{-2}}. \tag{6.45}$$

The DSP difference equation is expressed as

$$y(n) = x(n) - 1.4579x(n-1) + x(n-2) + 1.3850y(n-1) - 0.9025y(n-2). \tag{6.46}$$

The frequency responses are computed and plotted in Fig. 6.33. Comparison of the raw ECG signal corrupted by the 60-Hz interference with the enhanced ECG signal for both time domain and frequency domain are displayed in Figs. 6.34 and 6.35, respectively. As we can see, the notch filter completely removes the 60-Hz interference.

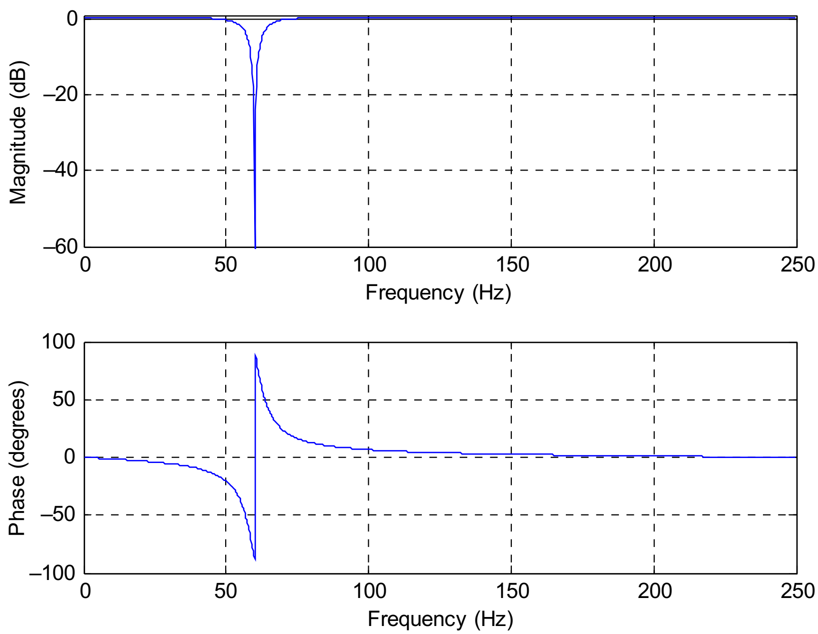


FIG. 6.33

Notch filter frequency responses.

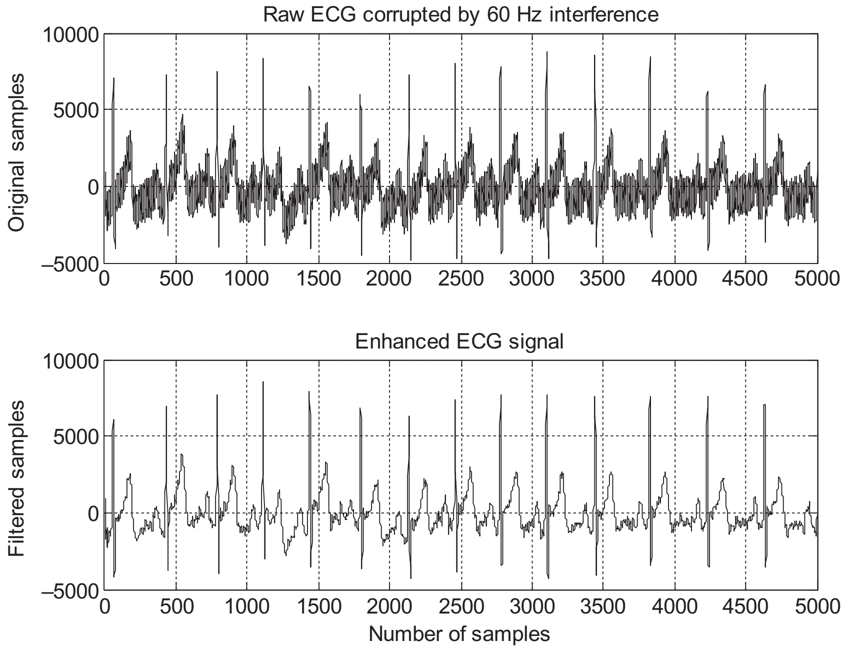


FIG. 6.34

The corrupted ECG signal and the enhanced ECG signal.

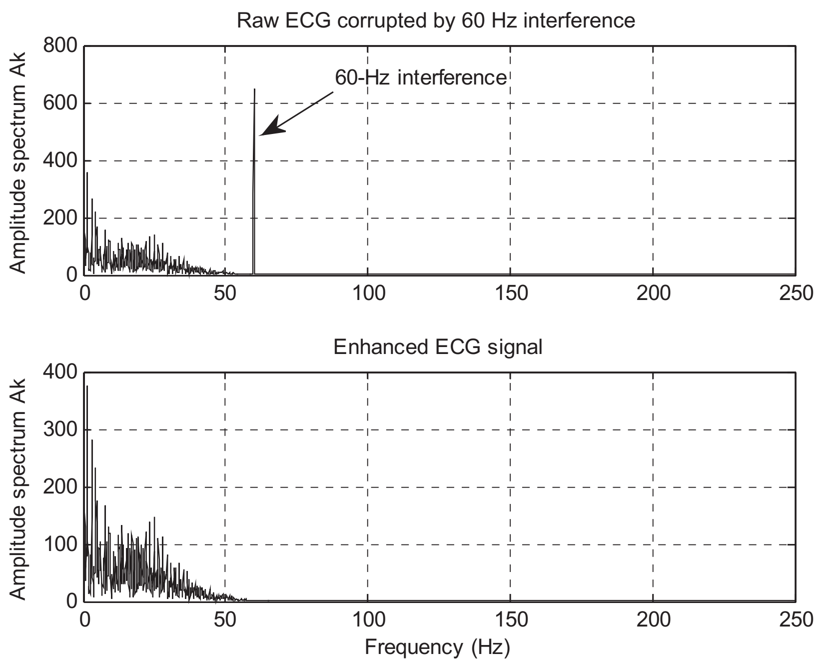


FIG. 6.35

The corrupted ECG signal spectrum and the enhanced ECG signal spectrum.

6.8 SUMMARY

1. The digital filter (DSP system) is represented by a difference equation, which is linear, time invariant.
2. The filter output depends on the filter current input, past input(s), and past output(s) in general. Given the arbitrary inputs and nonzero or zero initial conditions, operating the difference equation can generate the filter output recursively.
3. System responses such as the impulse response and step response can be determined analytically using the z-transform.
4. Transfer function can be obtained by applying z-transform to the difference equation to determine the ratio of the output z-transform over the input z-transform. A digital filter (DSP system) can be represented by its transfer function.
5. System stability can be studied using a very useful tool, a z-plane pole-zero plot.
6. The frequency response of a DSP system was developed and illustrated to investigate magnitude and phase responses. In addition, the FIR and IIR systems were defined.
7. Digital filters and their specifications, such as lowpass, highpass, bandpass, and bandstop, were reviewed.
8. A digital filter can be realized using standard realization methods such as the direct-form I; direct-form II; cascade, or series form; and parallel form.
9. Digital processing of speech using the preemphasis filter and bandpass filter was investigated to study spectral effects of the processed digital speech. The preemphasis filter boosts the high-frequency components, while bandpass filtering keeps the mid-band frequency components and rejects other lower- and upper-band frequency components.

6.9 PROBLEMS

6.1 Given a difference equation

$$y(n) = x(n) - 0.5y(n-1),$$

(a) Calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and initial condition: $y(-1) = 1$.

(b) Calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and zero initial condition: $y(-1) = 0$.

6.2 Given a difference equation

$$y(n) = 0.5x(n-1) + 0.6y(n-1),$$

(a) Calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and initial conditions: $x(-1) = -1$, and $y(-1) = 1$.

(b) Calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input $x(n) = (0.5)^n u(n)$ and zero initial conditions: $x(-1) = 0$, and $y(-1) = 0$.

6.3 Given a difference equation

$$y(n) = x(n-1) - 0.75y(n-1) - 0.125y(n-2),$$

(a) Calculate the system response $y(n)$ for $n=0, 1, \ldots, 4$ with the input $x(n)=(0.5)^n u(n)$ and initial conditions: $x(-1)=-1$, $y(-2)=2$, and $y(-1)=1$.
(b) Calculate the system response $y(n)$ for $n=0, 1, \ldots, 4$ with the input $x(n)=(0.5)^n u(n)$ and zero initial conditions: $x(-1)=0$, $y(-2)=0$, and $y(-1)=0$.

6.4 Given the following difference equation,

$$y(n)=0.5x(n)+0.5x(n-1),$$

(a) Find $H(z)$.
(b) Determine the impulse response $y(n)$ if the input $x(n)=4\delta(n)$.
(c) Determine the step response $y(n)$ if the input $x(n)=10u(n)$.

6.5 Given the following difference equation,

$$y(n)=x(n)-0.5y(n-1),$$

(a) Find $H(z)$.
(b) Determine the impulse response $y(n)$ if the input$x(n)=\delta(n)$.
(c) Determine the step response $y(n)$ if the input$x(n)=u(n)$.

6.6 A digital system is described by the following difference equation:

$$y(n)=x(n)-0.25x(n-2)-1.1y(n-1)-0.28y(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial$B(z)$.

6.7 A digital system is described by the following difference equation:

$$y(n)=0.5x(n)+0.5x(n-1)-0.6y(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.8 A digital system is described by the following difference equation:

$$y(n)=0.25x(n-2)+0.5y(n-1)-0.2y(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.9 A digital system is described by the following difference equation:

$$y(n)=x(n)-0.3x(n-1)+0.28x(n-2).$$

Find the transfer function $H(z)$, the denominator polynomial $A(z)$, and the numerator polynomial $B(z)$.

6.10 Convert each of the following transfer functions into the difference equations:

(a) $H(z)=0.5+0.5z^{-1}$

(b) $H(z)=\dfrac{1}{1-0.3z^{-1}}$

6.11 Convert each of the following transfer functions into the difference equations:

(a) $H(z)=0.1+0.2z^{-1}+0.3z^{-2}$

(b) $H(z)=\dfrac{0.5-0.5z^{-2}}{1-0.3z^{-1}+0.8z^{-2}}$

6.12 Convert each of the following transfer functions into the difference equations:

(a) $H(z) = \dfrac{z^2 - 0.25}{z^2 + 1.1z + 0.18}$

(b) $H(z) = \dfrac{z^2 - 0.1z + 0.3}{z^3}$

6.13 Convert each of the following transfer functions into its pole-zero form:

(a) $H(z) = \dfrac{z^2 + 2z + 1}{z^2 + 5z + 6}$

(b) $H(z) = \dfrac{1 - 0.16z^{-2}}{1 + 0.7z^{-1} + 0.1z^{-2}}$

(c) $H(z) = \dfrac{z^2 + 4z + 5}{z^3 + 2z^2 + 6z}$

6.14 A transfer function depicting a discrete-time system is given by

$$H(z) = \frac{10(z+1)}{(z+0.75)}.$$

(a) Determine the impulse response $h(n)$ and step response.

(b) Determine the system response $y(n)$ if the input is$x(n) = (0.25)^n u(n)$.

6.15 Given each of the following transfer functions that describe the digital systems, sketch the z-plane pole-zero plot and determine the stability for each digital system.

(a) $H(z) = \dfrac{z - 0.5}{(z + 0.25)(z^2 + z + 0.8)}$

(b) $H(z) = \dfrac{z^2 + 0.25}{(z - 0.5)(z^2 + 4z + 7)}$

(c) $H(z) = \dfrac{z + 0.95}{(z + 0.2)(z^2 + 1.414z + 1)}$

(d) $H(z) = \dfrac{z^2 + z + 0.25}{(z - 1)(z + 1)^2 (z - 0.36)}$

6.16 Given the following digital system with a sampling rate of 8000 Hz,

$$y(n) = 0.5x(n) + 0.5x(n-2),$$

(a) Determine the frequency response.

(b) Calculate and plot the magnitude and phase frequency responses.

(c) Determine the filter type based on the magnitude frequency response.

6.17 Given the following digital system with a sampling rate of 8000 Hz,

$$y(n) = 0.5x(n-1) + 0.5x(n-2),$$

(a) Determine the frequency response.

(b) Calculate and plot the magnitude and phase frequency responses.

(c) Determine the filter type based on the magnitude frequency response.

6.18 For the following digital system with a sampling rate of 8000 Hz,

$$y(n) = 0.5x(n) + 0.5y(n-1),$$

(a) Determine the frequency response.
(b) Calculate and plot the magnitude and phase frequency responses.
(c) Determine the filter type based on the magnitude frequency response.

6.19 For the following digital system with a sampling rate of 8000 Hz,

$$y(n) = x(n) - 0.5y(n-2),$$

(a) Determine the frequency response.
(b) Calculate and plot the magnitude and phase frequency responses.
(c) Determine the filter type based on the magnitude frequency response.

6.20 Given the following difference equation:

$$y(n) = x(n) - 2 \cdot \cos(\alpha)x(n-1) + x(n-2) + 2\gamma \cdot \cos(\alpha)y(n-1) - \gamma^2 y(n-2),$$

where $\gamma = 0.8$ and $\alpha = 60^0$,
(a) Find the transfer function $H(z)$.
(b) Plot the poles and zeros on the z-plan with the unit circle.
(c) Determine the stability of the system from the pole-zero plot.
(d) Calculate the amplitude (magnitude) response of $H(z)$.
(e) Calculate the phase response of $H(z)$.

6.21 For each of the following difference equations,
(a) $y(n) = 0.5x(n) + 0.5x(n-1)$
(b) $y(n) = 0.5x(n) - 0.5x(n-1)$
(c) $y(n) = 0.5x(n) + 0.5x(n-2)$
(d) $y(n) = 0.5x(n) - 0.5x(n-2)$,
  1. Find $H(z)$.
  2. Calculate the magnitude response.
  3. Specify the filtering type based on the calculated magnitude response.

6.22 An IIR system is expressed as

$$y(n) = 0.5x(n) + 0.2y(n-1), y(-1) = 0.$$

(a) Find $H(z)$.
(b) Find the system response $y(n)$ due to the input $x(n) = (0.5)^n u(n)$.

6.23 Given the following IIR system with zero initial conditions:

$$y(n) = 0.5x(n) - 0.7y(n-1) - 0.1y(n-2),$$

(a) Find $H(z)$.
(b) Find the unit step response.

6.24 Given the first-order IIR system

$$H(z) = \frac{1 + 2z^{-1}}{1 - 0.5z^{-1}},$$

realize $H(z)$ and develop the difference equations using the following forms:

(a) direct-form I

(b) direct-form II

6.25 Given the second-order IIR filter

$$H(z) = \frac{1 - 0.9z^{-1} - 0.1z^{-2}}{1 + 0.3z^{-1} - 0.04z^{-2}},$$

realize $H(z)$ and develop difference equations using the following forms:

(a) direct-form I

(b) direct-form II

(c) cascade (series) form via the first-order sections

(d) parallel form via the first-order sections

6.26 Given the following preemphasis filters:

$$H(z) = 1 - 0.5z^{-1}$$

$$H(z) = 1 - 0.7z^{-1}$$

$$H(z) = 1 - 0.9z^{-1},$$

(a) Write the difference equation for each.

(b) Determine which emphasizes high-frequency components most.

MATLAB Problems

6.27 Given a filter

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.5z^{-1} + 0.25z^{-2}},$$

use MATLAB to plot

(a) its magnitude frequency response;

(b) its phase response.

6.28 Given a difference equation

$$y(n) = x(n-1) - 0.75y(n-1) - 0.125y(n-2),$$

(a) Use the MATLAB functions **filter()** and **filtic()** to calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input of $x(n) = (0.5)^n u(n)$ and initial conditions: $x(-1) = -1$, $y(-2) = 2$, and $y(-1) = 1$.

(b) Use the MATLAB function **filter()** to calculate the system response $y(n)$ for $n = 0, 1, \ldots, 4$ with the input of $x(n) = (0.5)^n u(n)$ and zero initial conditions: $x(-1) = 0$, $y(-2) = 0$, and $y(-1) = 0$.

6.29 Given a filter

$$H(z) = \frac{1 - z^{-1} + z^{-2}}{1 - 0.9z^{-1} + 0.81z^{-2}},$$

(a) Plot the magnitude frequency response and phase response using MATLAB.

(b) Specify the type of filtering.

(c) Find the difference equation.

(d) Perform filtering, that is, calculate $y(n)$ for first 1000 samples for each of the following inputs and plot the filter outputs using MATLAB, assuming that all initial conditions are zeros and the sampling rate is 8000 Hz:

$$x(n) = \cos\left(\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

$$x(n) = \cos\left(\frac{8}{3}\pi \cdot 10^3 \frac{n}{8{,}000}\right)$$

$$x(n) = \cos\left(6\pi \cdot 10^3 \frac{n}{8{,}000}\right);$$

(e) Repeat (d) using the MATLAB function **filter()**.

6.30 Repeat (d) in Problem 6.29 using direct-II form structure.

MATLAB Projects

6.31 Sound effects of preemphasis filtering:

A preemphasis filter is shown in Fig. 6.36 with a selective parameter $0 \le \alpha < 1$, which controls the degree of preemphasis filtering.

Assuming that the system has a sampling rate of 8000 Hz, plot the frequency responses for $\alpha = 0$, $\alpha = 0.4$, $\alpha = 0.8$, $\alpha = 0.95$, and $\alpha = 0.99$, respectively. For each case, apply the preemphasis filter to the given speech ("speech.dat") and discuss the sound effects.

6.32 Echo generation (sound regeneration):

Echo is the repetition of sound due to sound wave reflection from the objects. It can easily be generated using an FIR filter shown in Fig. 6.37:where $|\alpha| < 1$ is an attenuation factor and R is the delay of the echo. However, a single echo generator may not be useful, so a multiple-echo generator using an IIR filter is usually applied, as shown in Fig. 6.38.

As shown in Fig. 6.38, an echo signal is generated by the sum of delayed versions of sound with attenuation and the non-delayed version given by

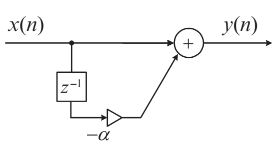


FIG. 6.36

A preemphasis filter.

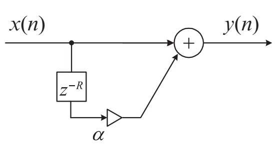


FIG. 6.37

A single echo generator using an FIR filter.

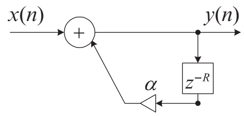


FIG. 6.38

A multiple-echo generator using an IIR filter.

$$y(n) = x(n) + \alpha x(n-R) + \alpha^2 x(n-2R) + \cdots = \sum_{k=0}^{\infty} \alpha^k x(n-kR)$$

where α is the attenuation factor. Applying z-transform, it follows that.
$Y(z) = X(z)\sum_{k=0}^{\infty}\left(\alpha z^{-R}\right)^k = X(z)\frac{1}{1-\alpha z^{-R}}$ for $|\alpha z^{-R}| < 1$
Thus, we yield the transfer function and difference equation below:

$$H(z) = \frac{1}{1-\alpha z^{-R}}$$

and $y(n) = x(n) + \alpha y(n-1)$.

(a) Assuming that the system has a sampling rate of 8000 Hz, plot the IIR filter frequency responses for the following cases: $\alpha = 0.5$ and $R = 1$; $\alpha = 0.6$ and $R = 4$; $\alpha = 0.7$ and $R = 10$, and characterize the frequency responses.

(b) After implementing the multiple-echo generator using the following code:

$$\mathrm{y = filter([1], [1\ zeros(1, R-1) - alpha], x)}$$

Evaluate the sound effects of processing the speech file ("sppech.dat") for the following cases: $\alpha = 0.5$ and $R = 500$ (62.5 ms); $\alpha = 0.7$ and $R = 1,\ 000$ (125 ms); $\alpha = 0.5$, $R = 2,\ 000$ (250 ms), and $\alpha = 0.5$, $R = 4,\ 000$ (500 ms).

Advanced Problems

6.33 Let $x(n) = \{a_N, a_{N-1}, \cdots, a_0, a_1, \cdots, a_{N-1}, a_N\}$ be a finite-duration sequence, which is real and even. Show that if $z = re^{j\theta}$ is a zero of $X(z)$, then $z = (1/r)e^{-j\theta}$ is also a zero.

6.34 Show that the following two systems are equivalent:

(a) $y(n) = x(n) - 0.5x(n-1)$

(b) $y(n) = 0.2y(n-1) + x(n) - 0.7x(n-1) + 0.1x(n-2)$

6.35 A causal linear time-invariant system has the time-domain input and z-transform output as shown below:

$$x(n) = (0.5)^n u(n)$$

and

$$Y(z) = \frac{z^{-1}}{(1-0.5z^{-1})(1-0.25z^{-1})}$$

(a) Determine the system transfer function $H(z)$.

(b) Determine the system output $y(n)$.

6.36 Given a linear-phase FIR system as

$$y(n) = b_0 x(n) + b_1 x(n-1) + b_1(n-2)$$

if its frequency response is normalized to $H(e^{j0}) = 1$ and it rejects completely a frequency component at $\Omega_0 = 2\pi/3$, then

(a) determine the transfer function $H(z)$;

(b) compute and sketch the magnitude and phase responses of the filter.

6.37 Let $z = re^{j\Omega}$ be a zero inside of the unit circle, where $0 < r < 1$. Consider a system whose transfer function as

$$H(z) = 1 - re^{j\theta}z^{-1}$$

(a) Show that the magnitude response is

$$\left|H\left(e^{j\Omega}\right)\right| = \sqrt{1 - 2r\cos(\Omega - \theta) + r^2};$$

(b) Show that the phase response is

$$\angle H\left(e^{j\Omega}\right) = \tan^{-1}\left(\frac{r\sin(\Omega - \theta)}{1 - r\cos(\Omega - \theta)}\right);$$

(c) Plot the magnitude response for $r = 0.8$, $\theta = 60^0$, and $\Omega = 0, \pi/4, \pi/2, 3\pi/4, \pi$, respectively.

6.38 Let $z = re^{j\Omega}$ be a zero inside of the unit circle, where $0 < r < 1$. Consider a system whose transfer function as

$$H(z) = \frac{1}{1 - re^{j\theta}z^{-1}}$$

(a) Show that the magnitude response is

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{1}{\sqrt{1 - 2r\cos(\Omega - \theta) + r^2}};$$

(b) Show that the phase response is

$$\angle H\left(e^{j\Omega}\right) = -\tan^{-1}\left(\frac{r\sin(\Omega - \theta)}{1 - r\cos(\Omega - \theta)}\right);$$

(c) Plot the magnitude response for $r = 0.8$, $\theta = 60^0$, and $\Omega = 0, \pi/4, \pi/2, 3\pi/4, \pi$, respectively.

6.39 Determine the 3-dB bandwidth for the following filters:

(a) $H(z) = \dfrac{1 - a}{1 - az^{-1}}$

(b) $H(z) = \dfrac{1 - a}{2}\dfrac{1 + z^{-1}}{1 - az^{-1}}$

where $0 < a < 1$, which is a better lowpass filer.

6.40 Given a system as

$$y(n) = e^{J\Omega_0}y(n-1) + x(n),$$

Show that for $x(n) = \delta(n)$, the response is

$$y(n) = \cos(\Omega_0 n)u(n) + j\sin(\Omega_0 n)u(n).$$

6.41 Convert the first-order highpass filter with the system function

$$H(z) = \frac{1 - z^{-1}}{1 - az^{-1}}$$

into a lowpass filter, where $0 < a < 1$.

(a) Determine the difference equation.

(b) Sketch the pole-zero pattern.

(c) Sketch magnitude and phase responses.

6.42 Determine the magnitude and phase responses of the multipath channel

$$y(n) = x(n) + x(n - M)$$

Find the frequencies when $H(e^{j\Omega}) = 0$.

6.43 Consider a system whose transfer function as

$$H(z) = \frac{\left(1 - e^{j\theta} z^{-1}\right)}{\left(1 - re^{j\theta} z^{-1}\right)}$$

(a) Show that the magnitude response is

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{\sqrt{2 - 2\cos(\Omega - \theta)}}{\sqrt{1 - 2r\cos(\Omega - \theta) + r^2}};$$

(b) Show that the phase response is

$$\Omega = \tan^{-1}\left(\frac{\sin(\Omega - \theta)}{1 - \cos(\Omega - \theta)}\right) - \tan^{-1}\left(\frac{r\sin(\Omega - \theta)}{1 - r\cos(\Omega - \theta)}\right);$$

(c) Plot the magnitude response for $r = 0.8$, $\theta = 45^\circ$, and $\Omega = 0, \pi/4, \pi/2, 3\pi/4, \pi$, respectively.

6.44 Consider a system whose transfer function as

$$H(z) = \frac{\left(1 - e^{j\theta} z^{-1}\right)\left(1 - e^{-j\theta} z^{-1}\right)}{\left(1 - re^{j\theta} z^{-1}\right)\left(1 - re^{-j\theta} z^{-1}\right)}$$

(a) Show that

$$H(z) = \frac{1 - 2\cos(\theta) z^{-1} + z^{-2}}{1 - 2r\cos(\theta) z^{-1} + r^2 z^{-2}};$$

(b) Show that the magnitude response is

$$\left|H\left(e^{j\Omega}\right)\right| = \frac{\sqrt{2 - 2\cos(\Omega - \theta)}}{\sqrt{1 - 2r\cos(\Omega - \theta) + r^2}} \frac{\sqrt{2 - 2\cos(\Omega + \theta)}}{\sqrt{1 - 2r\cos(\Omega + \theta) + r^2}};$$

(c) Plot the magnitude response for $r = 0.8$, $\theta = 45^\circ$, and $\Omega = 0, \pi/4, \pi/2, 3\pi/4, \pi$, respectively.

6.45 Echoes and reverberations can be generated by delaying and scaling the signal $x(n)$ as

$$y(n) = \sum_{k=0}^{\infty} b_k x(n - kD)$$

where $0 < b_k < 1$ and D is the integer delay.

(a) Show that for $b_k = a^k$

$$H(z) = \frac{1}{1 - az^{-D}};$$

(b) Show that if we generate reverberations using

$$y(n) = \left(\frac{1}{a} - a\right)\sum_{k=0}^{\infty} a^k x(n - kD) - \frac{1}{a}x(n)$$

Then

$$H(z) = \frac{z^{-D} - a}{1 - az^{-D}},$$

and the magnitude frequency response is allpass filter, that is,

$$\left|H\left(e^{j\Omega}\right)\right| = 1.$$