APPENDIX

SINUSOIDAL STEADY-STATE RESPONSE OF DIGITAL FILTERS

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D.1 SINUSOIDAL STEADY-STATE RESPONSE

Analysis of the sinusoidal steady-state response of the digital filters will lead to the development of the magnitude and phase responses of digital filters. Let us look at the following digital filter with a digital transfer function $H(z)$ and a complex sinusoidal input

$$x(n) = Ve^{j(\Omega n + \varphi_x)}, \tag{D.1}$$

where $\Omega = \omega T$ is the normalized digital frequency, while T is the sampling period and $y(n)$ denotes the digital output, as shown in Fig. D.1.

The z-transform output from the digital filter is then given by

$$Y(z) = H(z)X(z). \tag{D.2}$$

Since $X(z) = \frac{Ve^{j\varphi_x}z}{z - e^{j\Omega}}$, we have

$$Y(z) = \frac{Ve^{j\varphi_x}z}{z - e^{j\Omega}}H(z). \tag{D.3}$$

Based on the partial fraction expansion, $Y(z)/z$ can be expanded as the following form:

$$\frac{Y(z)}{z} = \frac{Ve^{j\varphi_x}}{z - e^{j\Omega}}H(z) = \frac{R}{z - e^{j\Omega}} + \text{sum of the rest of partial fractions}. \tag{D.4}$$

Multiplying the factor $(z - e^{j\Omega})$ on both sides of Eq. (D.4) yields

$$Ve^{j\phi_x}H(z) = R + \left(z - e^{j\Omega}\right)(\text{sum of the rest of partial fractions}). \tag{D.5}$$

Substituting $z = e^{j\Omega}$, we get the residue as

$$R = Ve^{j\phi_x}H\left(e^{j\Omega}\right).$$

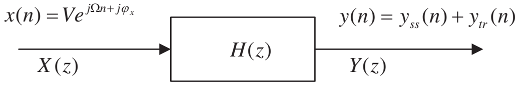

FIG. D.1

Steady-state response of the digital filter.

Then substituting $R = Ve^{j\phi_x}H(e^{j\Omega})$ back into Eq. (D.4) results in

$$\frac{Y(z)}{z} = \frac{Ve^{j\varphi_x}H(e^{j\Omega})}{z - e^{j\Omega}} + \text{sum of the rest of partial fractions,} \tag{D.6}$$

and multiplying z on both sides of Eq. (D.6) leads to

$$Y(z) = \frac{Ve^{j\varphi_x}H(e^{j\Omega})z}{z - e^{j\Omega}} + z \times \text{sum of the rest of partial fractions.} \tag{D.7}$$

Taking the inverse z-transform leads to two parts of the solution:

$$y(n) = Ve^{j\varphi_x}H\left(e^{j\Omega}\right)e^{j\Omega n} + Z^{-1}(\text{z} \times \text{sum of the rest of partial fractions}). \tag{D.8}$$

From Eq. (D.8), we have the steady-state response

$$y_{ss}(n) = Ve^{j\varphi_x}H\left(e^{j\Omega}\right)e^{j\Omega n} \tag{D.9}$$

and the transient response

$$y_{tr}(n) = Z^{-1}(\text{z} \times \text{sum of the rest of partial fractions}). \tag{D.10}$$

Note that since the digital filter is a stable system, and the locations of its poles must be inside the unit circle on the z-plane, the transient response will be settled to zero eventually. To develop filter magnitude and phase responses, we write the digital steady-state response as

$$y_{ss}(n) = V\left|H\left(e^{j\Omega}\right)\right|e^{j\Omega + j\varphi_x + \angle H\left(e^{j\Omega}\right)}. \tag{D.11}$$

Comparing Eq. (D.11) and Eq. (D.1), it follows that

$$\begin{aligned} \text{Magnitude response} &= \frac{\text{Amplitude of the steady state output}}{\text{Amplitude of the sinusoidal input}} \\ &= \frac{V\left|H(e^{j\Omega})\right|}{V} = \left|H\left(e^{j\Omega}\right)\right| \end{aligned} \tag{D.12}$$

$$\text{Phase response} = \frac{e^{j\varphi_x + j\angle H\left(e^{j\Omega}\right)}}{e^{j\varphi_x}} = e^{j\angle H\left(e^{j\Omega}\right)} = \angle H\left(e^{j\Omega}\right). \tag{D.13}$$

Thus we conclude that

$$\text{Frequency response} = H\left(e^{j\Omega}\right) = H(z)|_{z=e^{j\Omega}}. \tag{D.14}$$

Since $H(e^{j\Omega}) = |H(e^{j\Omega})| \angle H(e^{j\Omega})$

$$\text{Magnitude response} = \left|H\left(e^{j\Omega}\right)\right| \tag{D.15}$$

$$\text{Phase response} = \angle H\left(e^{j\Omega}\right). \tag{D.16}$$

D.2 PROPERTIES OF THE SINUSOIDAL STEADY-STATE RESPONSE

From Euler's identity and trigonometric identity, we know that

$$\begin{aligned} e^{j(\Omega + k2\pi)} &= \cos(\Omega + k2\pi) + j\sin(\Omega + k2\pi) \\ &= \cos\Omega + j\sin\Omega = e^{j\Omega}, \end{aligned} \tag{D.17}$$

where k is an integer taking values of $k=0,\pm 1,\pm 2,\cdots$. Then:

$$\text{Frequency response}: H\left(e^{j\Omega}\right)=H\left(e^{j(\Omega+k2\pi)}\right) \tag{D.18}$$

$$\text{Magnitude frequency response}: \left|H\left(e^{j\Omega}\right)\right|=\left|H\left(e^{j(\Omega+k2\pi)}\right)\right| \tag{D.19}$$

$$\text{Phase response}: \angle H\left(e^{j\Omega}\right)=\angle H\left(e^{j\Omega+2k\pi}\right). \tag{D.20}$$

Clearly, the frequency response is periodical, with a period of 2π. Next, let us develop the symmetric properties. Since the transfer function is written as

$$H(z)=\frac{Y(z)}{X(z)}=\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}, \tag{D.21}$$

Substituting $z=e^{j\Omega}$ into Eq. (D.21) yields

$$H\left(e^{j\Omega}\right)=\frac{b_0+b_1e^{-j\Omega}+\cdots+b_Me^{-jM\Omega}}{1+a_1e^{-j\Omega}+\cdots+a_Ne^{-jN\Omega}}. \tag{D.22}$$

Using Euler's identity, $e^{-j\Omega}=\cos\Omega-j\sin\Omega$, we have

$$H\left(e^{j\Omega}\right)=\frac{(b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega)-j(b_1\sin\Omega+\cdots+b_M\sin M\Omega)}{(1+a_1\cos\Omega+\cdots+a_N\cos N\Omega)-j(a_1\sin\Omega+\cdots+a_N\sin N\Omega)}. \tag{D.23}$$

Similarly,

$$H\left(e^{-j\Omega}\right)=\frac{(b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega)+j(b_1\sin\Omega+\cdots+b_M\sin M\Omega)}{(1+a_1\cos\Omega+\cdots+a_N\cos N\Omega)+j(a_1\sin\Omega+\cdots+a_N\sin N\Omega)}. \tag{D.24}$$

Then the magnitude response and phase response can be expressed as

$$\left|H\left(e^{j\Omega}\right)\right|=\frac{\sqrt{(b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega)^2+(b_1\sin\Omega+\cdots+b_M\sin M\Omega)^2}}{\sqrt{(1+a_1\cos\Omega+\cdots+a_N\cos N\Omega)^2+(a_1\sin\Omega+\cdots+a_N\sin N\Omega)^2}} \tag{D.25}$$

$$\begin{aligned}\angle H\left(e^{j\Omega}\right)&=\tan^{-1}\left(\frac{-(b_1\sin\Omega+\cdots+b_M\sin M\Omega)}{b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega}\right)\\&-\tan^{-1}\left(\frac{-(a_1\sin\Omega+\cdots+a_N\sin N\Omega)}{1+a_1\cos\Omega+\cdots+a_N\cos N\Omega}\right)\end{aligned} \tag{D.26}$$

Based in Eq. (D.24), we also have the magnitude and phase response for $H(e^{-j\Omega})$ as

$$\left|H\left(e^{-j\Omega}\right)\right|=\frac{\sqrt{(b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega)^2+(b_1\sin\Omega+\cdots+b_M\sin M\Omega)^2}}{\sqrt{(1+a_1\cos\Omega+\cdots+a_N\cos N\Omega)^2+(a_1\sin\Omega+\cdots+a_N\sin N\Omega)^2}} \tag{D.27}$$

$$\begin{aligned}\angle H\left(e^{-j\Omega}\right)&=\tan^{-1}\left(\frac{b_1\sin\Omega+\cdots+b_M\sin M\Omega}{b_0+b_1\cos\Omega+\cdots+b_M\cos M\Omega}\right)\\&-\tan^{-1}\left(\frac{a_1\sin\Omega+\cdots+a_N\sin N\Omega}{1+a_1\cos\Omega+\cdots+a_N\cos N\Omega}\right).\end{aligned} \tag{D.28}$$

Comparing (D.25) with (D.27) and (D.26) with (D.28), respectively, we conclude the symmetric properties as

$$\left|H\left(e^{-j\Omega}\right)\right| = \left|H\left(e^{j\Omega}\right)\right| \tag{D.29}$$

$$\angle H\left(e^{-j\Omega}\right) = -\angle H\left(e^{j\Omega}\right) \tag{D.30}$$