CHAPTER

DIGITAL SIGNALS AND SYSTEMS 3

CHAPTER OUTLINE

3.1 DIGITAL SIGNALS

In our daily lives, analog signals appear in forms such as speech, audio, seismic, biomedical, and communications signals. To process an analog signal using a digital signal processor, the analog signal must be converted into a digital signal, that is, the analog-to-digital conversion (DAC) must take place, as discussed in Chapter 2. Then the digital signal is processed via digital signal processing (DSP) algorithm(s).

A typical digital signal $x(n)$ is shown in Fig. 3.1, where both the time and the amplitude of the digital signal are discrete. Note that the amplitudes of the digital signal samples are given and sketched only at their corresponding time indices, where $x(n)$ represents the amplitude of the nth sample and n is the time index or sample number. From Fig. 3.1, we learn that

$x(0)$: zeroth sample amplitude at the sample number $n=0$,
$x(1)$: first sample amplitude at the sample number $n=1$,
$x(2)$: second sample amplitude at the sample number $n=2$,
$x(3)$: third sample amplitude at the sample number $n=3$, and so on.

Digital Signal Processing. https://doi.org/10.1016/B978-0-12-815071-9.00003-8

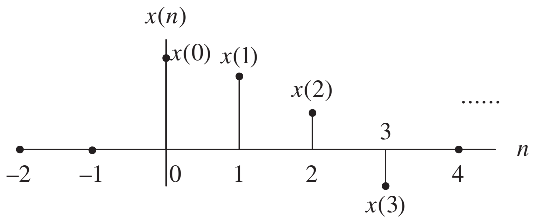

FIG. 3.1

Digital signal notation.

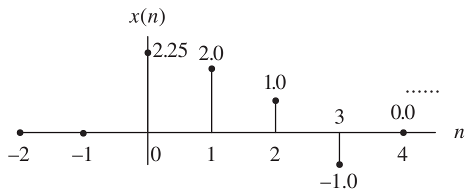

FIG. 3.2

Plot of the digital signal samples.

Furthermore, Fig. 3.2 illustrates the digital samples whose amplitudes are discrete encoded values represented in the digital signal processor. Precision of the data is based on the number of bits used in the DSP system. The encoded data format can be either an integer if a fixed-point digital signal processor is used or a floating-point number if a floating-point digital signal processor is used. As shown in Fig. 3.2 for the floating-point digital signal processor, we can identify the first five sample amplitudes at their time indices as follows:

$$x(0) = 2.25$$
$$x(1) = 2.0$$
$$x(2) = 1.0$$
$$x(3) = -1.0$$
$$x(4) = 0.0$$
$$\ldots$$

Again, note that each sample amplitude is plotted using a vertical bar with a solid dot. This notation is well accepted in the DSP literatures.

3.1.1 COMMON DIGITAL SEQUENCES

Let us study some special digital sequences that are widely used. We define and plot each of them as follows:

Unit-impulse sequence (digital unit-impulse function):

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}. \tag{3.1}$$

The plot of the unit-impulse function is given in Fig. 3.3. The unit-impulse function has the unit amplitude at only $n = 0$ and zero amplitudes at other time indices.

Unit-step sequence (digital unit-step function):

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}. \tag{3.2}$$

The plot is given in Fig. 3.4. The unit-step function has the unit amplitude at $n = 0$ and for all the positive time indices, and amplitudes of zero for all the negative time indices.

The shifted unit-impulse and unit-step sequences are displayed in Fig. 3.5.

As shown in Fig. 3.5, the shifted unit-impulse function $\delta(n-2)$ is obtained by shifting the unit-impulse function $\delta(n)$ to the right by two samples, and the shifted unit-step function $u(n-2)$ is achieved by shifting the unit-step function $u(n)$ to the right by two samples; similarly, $\delta(n+2)$ and $u(n+2)$ are acquired by shifting $\delta(n)$ and $u(n)$ via two samples to the left, respectively.

A sequence $x(n)$ is called a causal sequence if $x(n) = 0$, for $n < 0$. Otherwise, $x(n)$ is called noncausal sequence, that is, $x(n)$ has nonzero value (s) for $n < 0$.

Sinusoidal and exponential sequences are depicted in Figs. 3.6 and 3.7, respectively.

For a sinusoidal sequence $x(n) = A\cos(0.125\pi n)u(n)$ and $A = 10$, we can calculate the digital values for the first eight samples and list their values in Table 3.1. Note that $u(n)$ is used to ensure the sinusoidal sequence $x(n)$ is a causal sequence, and amplitudes of $x(n)$ are discrete-time values (encoded values in the floating format).

For the exponential sequence $x(n) = A(0.75)^n u(n)$, the calculated digital values for the first eight samples with $A = 10$ are listed in Table 3.2.

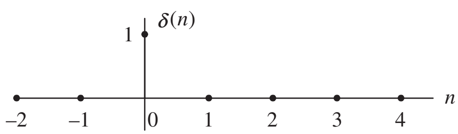

FIG. 3.3

Unit-impulse sequence.

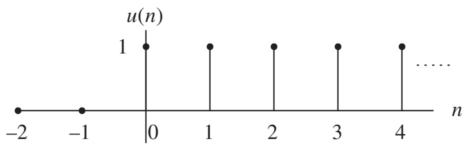

FIG. 3.4

Unit-step sequence.

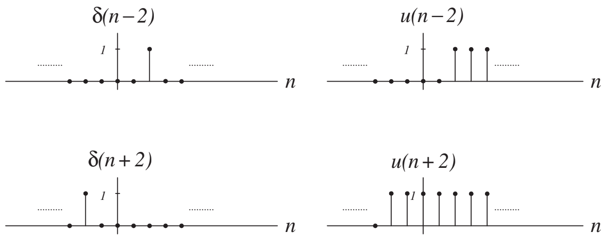

FIG. 3.5

Shifted unit-impulse and unit-step sequences.

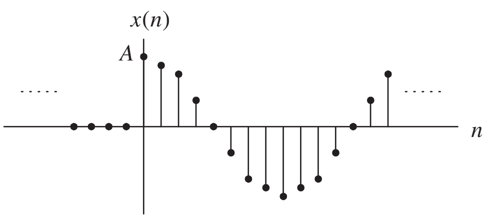

FIG. 3.6

Plot of samples of the sinusoidal function.

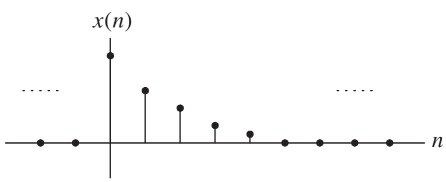

FIG. 3.7

Plot of samples of the exponential function.

Table 3.1 Sample Values Calculated from the Sinusoidal Function

n	$x(n) = 10\cos(0.125\pi n)u(n)$
0	10.0000
1	9.2388
2	7.0711
3	3.8628
4	0.0000
5	−3.8628
6	−7.0711
7	−9.2388

Table 3.2 Sample Values Calculated from the Exponential Function

n	$10(0.75)^n u(n)$
0	10.0000
1	7.5000
2	5.6250
3	4.2188
4	3.1641
5	2.3730
6	1.7798
7	1.3348

EXAMPLE 3.1

Given the following,

$$x(n) = \delta(n+1) + 0.5\delta(n-1) + 2\delta(n-2),$$

Sketch this sequence.

Solution:

According to the shift operation, $\delta(n+1)$ is obtained by shifting $\delta(n)$ to the left by one sample, while $\delta(n-1)$ and $\delta(n-2)$ are yielded by shifting $\delta(n)$ to right by one sample and two samples, respectively. Using the amplitude of each impulse function, we yield the sketch as shown in Fig. 3.8.

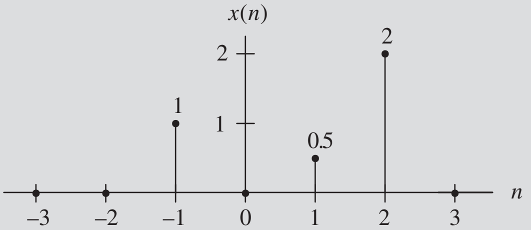

FIG. 3.8

Plot of digital sequence in Example 3.1.

3.1.2 GENERATION OF DIGITAL SIGNALS

Given the sampling rate of a DSP system to sample the analytical function of an analog signal, the corresponding digital function or digital sequence (assuming its sampled amplitudes are encoded to have finite precision) can be found. The digital sequence is often used to

1. Calculate the encoded sample amplitude for a given sample number n;
2. Generate the sampled sequence for simulation.

The procedure to develop a digital sequence from its analog signal function is as follows.

Assuming that an analog signal $x(t)$ is uniformly sampled at the time interval of $\Delta t = T$, where T is the sampling period, the corresponding digital function (sequence) $x(n)$ gives the *instant encoded values* of the analog signal $x(t)$ at all the time instants $t = n\Delta t = nT$ and can be achieved by substituting time $t = nT$ into the analog signal $x(t)$, that is,

$$x(n) = x(t)|_{t=nT} = x(nT). \tag{3.3}$$

Also note that for sampling the unit-step function $u(t)$, we have

$$u(t)|_{t=nT} = u(nT) = u(n). \tag{3.4}$$

The following example will demonstrate the use of Eqs. (3.3) and (3.4).

EXAMPLE 3.2

Assuming a DSP system with a sampling time interval of 125 μs,

(a) Convert each of following analog signal $x(t)$ to the digital signal $x(n)$.
  1. $x(t) = 10e^{-5000t}u(t)$
  2. $x(t) = 10\sin(2000\pi t)u(t)$

(b) Determine and plot the sample values from each obtained digital function.

Solution:

(a) Since $T = 0.000125$ s in Eq. (3.3), substituting $t = nT = n \times 0.000125 = 0.000125n$ into the analog signal $x(t)$ expressed in (1) leads to the digital sequence
  1. $x(n) = x(nT) = 10e^{-5000 \times 0.000125n}u(nT) = 10e^{-0.625n}u(n)$.

     Similarly, the digital sequence for (2) is achieved as follows:
  2. $x(n) = x(nT) = 10\sin(2000\pi \times 0.000125n)u(nT) = 10\sin(0.25\pi n)u(n)$

(b) 1. The first five sample values are calculated and plotted in Fig. 3.9.

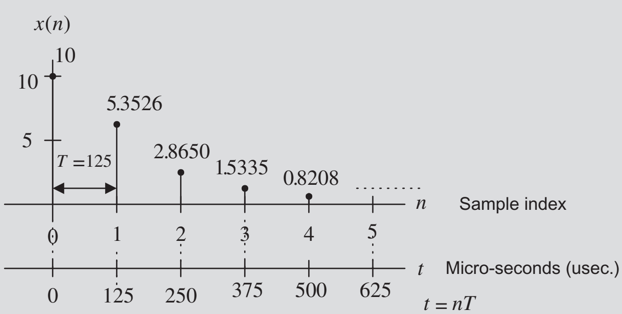

FIG. 3.9

Plot of the digital sequence for (1) in Example 3.2.

$$x(0) = 10e^{-0.625\times 0}u(0) = 10.0$$
$$x(1) = 10e^{-0.625\times 1}u(1) = 5.3526$$
$$x(2) = 10e^{-0.625\times 2}u(2) = 2.8650$$
$$x(3) = 10e^{-0.625\times 3}u(3) = 1.5335$$
$$x(4) = 10e^{-0.625\times 4}u(4) = 0.8208$$

2. The first eight amplitudes are computed and sketched in Fig. 3.10.

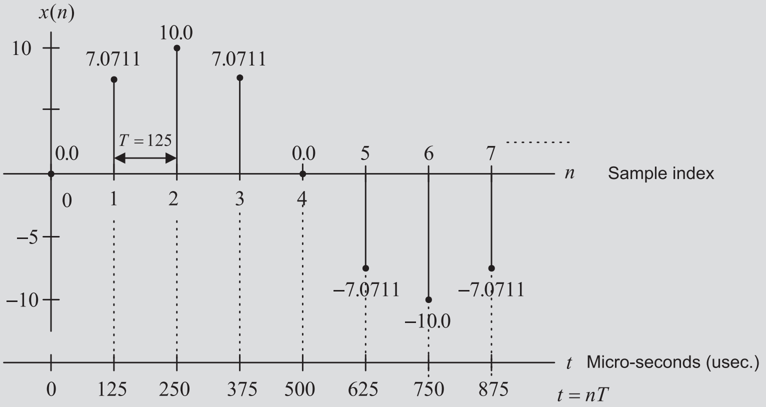

FIG. 3.10

Plot of the digital sequence for (2) in Example 3.2.

$$x(0) = 10\sin(0.25\pi \times 0)u(0) = 0$$
$$x(1) = 10\sin(0.25\pi \times 1)u(1) = 7.0711$$
$$x(2) = 10\sin(0.25\pi \times 2)u(2) = 10.0$$
$$x(3) = 10\sin(0.25\pi \times 3)u(3) = 7.0711$$
$$x(4) = 10\sin(0.25\pi \times 4)u(4) = 0.0$$
$$x(5) = 10\sin(0.25\pi \times 5)u(5) = -7.0711$$
$$x(6) = 10\sin(0.25\pi \times 6)u(6) = -10.0$$
$$x(7) = 10\sin(0.25\pi \times 7)u(7) = -7.0711$$

3.2 LINEAR TIME-INVARIANT, CAUSAL SYSTEMS

In this section, we study linear time-invariant causal systems and focus on properties such as linearity, time invariant, and causality.

3.2.1 LINEARITY

A linear system is illustrated in Fig. 3.11, where $y_1(n)$ is the system output using an input $x_1(n)$, and $y_2(n)$ the system output with an input $x_2(n)$.

Fig. 3.11 illustrates that the system output due to the weighted sum inputs $\alpha x_1(n)+\beta x_2(n)$ is equal to the same weighted sum of the individual outputs obtained from their corresponding inputs, that is,

$$y(n)=\alpha y_1(n)+\beta y_2(n), \tag{3.5}$$

where α and β are constants.

For example, considering a digital amplifier: $y(n)=10x(n)$, where the input is multiplied by 10 to generate the output. Then, the inputs $x_1(n)$ and $x_2(n)$ generate the outputs

$$y_1(n)=10x_1(n) \text{ and } y_2(n)=10x_2(n), \text{respectively}.$$

If, as described in Fig. 3.11, we apply to the system using the combined input $x(n)$, where the first input multiplied by a constant α while the second input multiplied by a constant β, that is,

$$x(n)=\alpha x_1(n)+\beta x_2(n),$$

then the system output due to the combined input is obtained as

$$y(n)=10x(n)=10(\alpha x_1(n)+\beta x_2(n))=10\alpha x_1(n)+10\beta x_2(n). \tag{3.6}$$

If we verify the weighted sum of the individual outputs, we see that

$$\alpha y_1(n)+\beta y_2(n)=\alpha[10x_1(n)]+\beta[10x_2(n)]. \tag{3.7}$$

Comparing Eqs. (3.6) and (3.7) verifies

$$y(n)=\alpha y_1(n)+\beta y_2(n). \tag{3.8}$$

Since this relationship holds for all inputs, system $y(n)=10x(n)$ is a linear system. The linearity means that the system obeys the superposition, as shown in Eq. (3.8). Let us verify a system whose output is a square of its input,

$$y(n)=x^2(n).$$

Applying to the system with the inputs $x_1(n)$ and $x_2(n)$ leads to

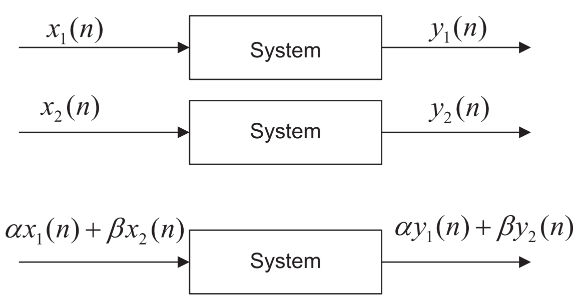

FIG. 3.11

Digital linear system.

$$y_1(n) = x_1^2(n) \text{ and } y_2(n) = x_2^2(n).$$

We can determine the system output using a combined input, which is the weighed sum of the individual inputs with constants α and β, respectively. Working on algebra, we see that

$$\begin{aligned} y(n) &= x^2(n) = (\alpha x_1(n) + \beta x_2(n))^2 \\ &= \alpha^2 x_1^2(n) + 2\alpha\beta x_1(n) x_2(n) + \beta^2 x_2^2(n). \end{aligned} \tag{3.9}$$

Again, we express the weighted sum of the two individual outputs with the same constants α and β as

$$\alpha y_1(n) + \beta y_2(n) = \alpha x_1^2(n) + \beta x_2^2(n). \tag{3.10}$$

It is obvious that

$$y(n) \neq \alpha y_1(n) + \beta y_2(n). \tag{3.11}$$

Hence, the system is a nonlinear system, since the linear property, superposition, does not hold, as shown in Eq. (3.11).

3.2.2 TIME INVARIANCE

The time-invariant system is illustrated in Fig. 3.12, where $y_1(n)$ is the system output for the input $x_1(n)$. Let $x_2(n) = x_1(n - n_0)$ be shifted version of $x_1(n)$ by n_0 samples. The output $y_2(n)$ obtained with the shifted input $x_2(n) = x_1(n - n_0)$ is equivalent to the output $y_2(n)$ acquired by shifting $y_1(n)$ by n_0 samples, $y_2(n) = y_1(n - n_0)$.

This can simply be viewed as the following:

If the system is time invariant and $y_1(n)$ is the system output due to the input $x_1(n)$, then the shifted system input $x_1(n - n_0)$ will produce a shifted system output $y_1(n - n_0)$ by the same amount of time n_0.

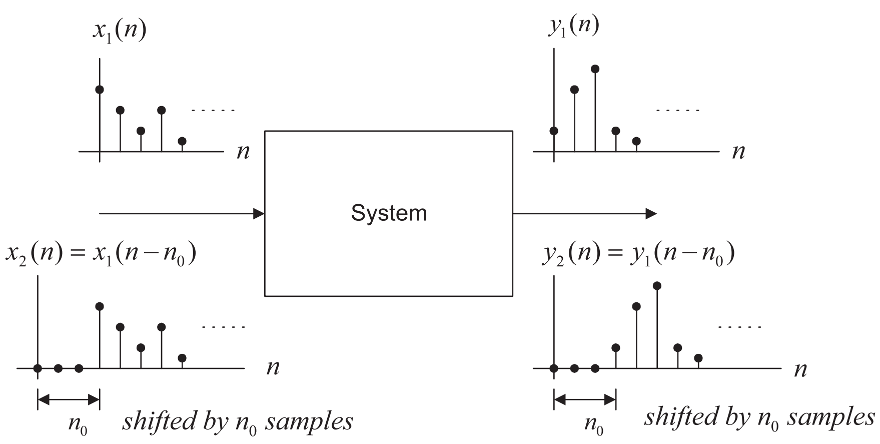

FIG. 3.12

Illustration of the linear time-invariant digital system.

EXAMPLE 3.3

Given the linear systems

(a) $y(n) = 2x(n-5)$

(b) $y(n) = 2x(3n)$,

determine whether each of the following systems is time invariant.

Solution:

(a) Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(n-5)$. Again, let $x_2(n) = x_1(n-n_0)$ be the shifted input and $y_2(n)$ be the output due to the shifted input. We determine the system output using the shifted input as

$$y_2(n) = 2x_2(n-5) = 2x_1(n-n_0-5).$$

Meanwhile, shifting by n_0 samples leads to

$$y_1(n-n_0) = 2x_1(n-5-n_0).$$

We can verify that $y_2(n) = y_1(n-n_0)$. Thus the shifted input of n_0 samples causes the system output to be shifted by the same n_0 samples, thus the system is time invariant.

(b) Let the input and output be $x_1(n)$ and $y_1(n)$, respectively; then the system output is $y_1(n) = 2x_1(3n)$. Again, let the input and output be $x_2(n)$ and $y_2(n)$, where $x_2(n) = x_1(n-n_0)$, a shifted version, and the corresponding output is $y_2(n) = 2x_2(3n)$. Since $x_2(n) = x_1(n-n_0)$, replacing n by $3n$ leads to $x_2(3n) = x_1(3n-n_0)$. We then have

$$y_2(n) = 2x_2(3n) = 2x_1(3n-n_0).$$

On the other hand, if we shift $y_1(n)$ by n_0 samples which replaces n in $y_1(n) = 2x_1(3n)$ by $n-n_0$, we yield

$$y_1(n-n_0) = 2x_1(3(n-n_0)) = 2x_1(3n-3n_0).$$

Clearly, we know that $y_2(n) \neq y_1(n-n_0)$. Since the system output $y_2(n)$ using the shifted input shifted by n_0 samples is not equal to the system output $y_1(n)$ shifted by the same n_0 samples, hence the system is not time invariant, that is, time variant.

3.2.3 CAUSALITY

A causal system is the one in which the output $y(n)$ at time n depends only on the current input $x(n)$ at time n, and its past input sample values such as $x(n-1)$, $x(n-2)$,…. Otherwise, if a system output depends on the future input values such as $x(n+1)$, $x(n+2)$,…, the system is noncausal. The noncausal system cannot be realized in real time.

EXAMPLE 3.4

Given the following linear systems

(a) $y(n) = 0.5x(n) + 2.5x(n-2)$, for $n \geq 0$,

(b) $y(n) = 0.25x(n-1) + 0.5x(n+1) - 0.4y(n-1)$, for $n \geq 0$,

determine whether each is causal.

Solution:

(a) Since for $n \geq 0$, the output $y(n)$ depends on the current input $x(n)$ and its past value $x(n-2)$, the system is causal.

(b) Since for $n \geq 0$, the output $y(n)$ depends on the current input $x(n)$ and its future value $x(n+1)$, the system is a noncausal.

3.3 DIFFERENCE EQUATIONS AND IMPULSE RESPONSES

Now we study the difference equation and its impulse response.

3.3.1 FORMAT OF DIFFERENCE EQUATION

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$$y(n) + a_1 y(n-1) + \cdots + a_N y(n-N) = b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M), \tag{3.12}$$

where $a_1, \ldots, a_N$, and $b_0, b_1, \ldots, b_M$ are the coefficients of the difference equation. M and N are the memory lengths for input $x(n)$ and output $y(n)$, respectively. Eq. (3.12) can further be written as

$$y(n) = -a_1 y(n-1) - \cdots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \cdots + b_M x(n-M), \tag{3.13}$$

or

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j). \tag{3.14}$$

Note that $y(n)$ is the current output which depends on the past output samples $y(n-1), \ldots, y(n-N)$, the current input sample $x(n)$, and the past input samples, $x(n-1), \ldots, x(n-N)$. We will examine the specific difference equations in the following examples.

EXAMPLE 3.5

Given the following difference equation:

$$y(n) = 0.25 y(n-1) + x(n),$$

identify the nonzero system coefficients.

Solution:

Comparison with Eq. (3.13) leads to

$$b_0 = 1$$

$$-a_1 = 0.25$$

that is, $a_1 = -0.25$.

EXAMPLE 3.6

Given a linear system described by the difference equation

$$y(n) = x(n) + 0.5x(n-1),$$

determine the nonzero system coefficients.

Solution:

By comparing Eq. (3.13), we have

$$b_0 = 1 \text{ and } b_1 = 0.5$$

3.3.2 SYSTEM REPRESENTATION USING ITS IMPULSE RESPONSE

A linear time-invariant system can be completely described by its unit-impulse response, which is defined as the system response due to the impulse input $\delta(n)$ with zero-initial conditions, depicted in Fig. 3.13.

With the obtained unit-impulse response $h(n)$, we can represent the linear time-invariant system in Fig. 3.14.

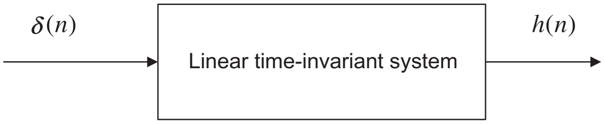

FIG. 3.13

Unit-impulse response of the linear time-invariant system.

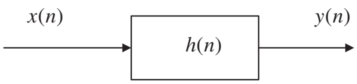

FIG. 3.14

Representation of linear time-invariant system using the impulse response.

EXAMPLE 3.7

Given the linear time-invariant system

$y(n) = 0.5x(n) + 0.25x(n-1)$ with an initial condition $x(-1) = 0$,

(a) Determine the unit-impulse response $h(n)$.

(b) Draw the system block diagram.

(c) Write the output using the obtained impulse response.

Solution:

(a) According to Fig. 3.13, let $x(n) = \delta(n)$, then

$$h(n) = y(n) = 0.5x(n) + 0.25x(n-1) = 0.5\delta(n) + 0.25\delta(n-1).$$

Thus, for this particular linear system, we have

$$h(n) = \begin{cases} 0.5 & n = 0 \\ 0.25 & n = 1 \\ 0 & \text{elsewhere} \end{cases}.$$

(b) The block diagram of the linear time-invariant system is shown in Fig. 3.15.

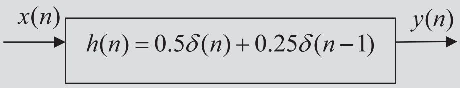

FIG. 3.15

The system block diagram in Example 3.7.

(c) The system output can be rewritten as

$$y(n) = h(0)x(n) + h(1)x(n-1).$$

From the result in Example 3.7, it is noted that if the difference equation without the past output terms, $y(n-1), \ldots, y(n-N)$, that is, the corresponding coefficients $a_1, \ldots, a_N$, are zeros, and the impulse response $h(n)$ has a finite number of terms. We call this a *finite impulse response* (FIR) system. In general, Eq. (3.12) contains the past output terms and resulting impulse response $h(n)$ has an infinite number of terms. We can express the output sequence of a linear time-invariant system from its impulse response and inputs as

$$y(n) = \cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots = \sum_{k=-\infty}^{\infty} h(k)x(n-k). \tag{3.15}$$

Eq. (3.15) is called the *digital convolution sum*, which is explored in a later section.

We can verify Eq. (3.15) by substituting the impulse sequence $x(n) = \delta(n)$ to get the impulse response

$$h(n) = \cdots + h(-1)\delta(n+1) + h(0)\delta(n) + h(1)\delta(n-1) + h(2)\delta(n-2) + \cdots = \sum_{k=-\infty}^{\infty} h(k)\delta(n-k),$$

where *h(k)* are the amplitudes of the impulse response at the corresponding time indices. Now let us look at another example.

EXAMPLE 3.8

Given the difference equation

$$y(n) = 0.25y(n-1) + x(n), \text{ for } n \ge 0 \text{ and } y(-1) = 0,$$

(a) Determine the unit-impulse response $h(n)$.
(b) Draw the system block diagram.
(c) Write the output using the obtained impulse response.
(d) For a step input $x(n) = u(n)$, verify and compare the output responses for the first three output samples using the difference equation and digital convolution sum (Eq. 3.15).

Solution:

(a) Let $x(n) = \delta(n)$, then

$$h(n) = 0.25h(n-1) + \delta(n).$$

To solve for $h(n)$, we evaluate

$$h(0) = 0.25h(-1) + \delta(0) = 0.25 \times 0 + 1 = 1$$

$$h(1) = 0.25h(0) + \delta(1) = 0.25 \times 1 + 0 = 0.25$$

$$h(2) = 0.25h(1) + \delta(2) = 0.25 \times 0.5 + 0 = 0.0625$$

With the calculated results, we can predict the impulse response as

$$h(n) = (0.25)^n u(n) = \delta(n) + 0.25\delta(n-1) + 0.0625\delta(n-2) + \cdots.$$

(b) The system block diagram is given in Fig. 3.16.

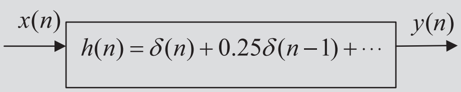

FIG. 3.16

The system block diagram in Example 3.8.

(c) The output sequence is a sum of infinite terms expressed as

$$\begin{aligned} y(n) &= h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots \\ &= x(n) + 0.25x(n-1) + 0.0625x(n-2) + \cdots \end{aligned}$$

(d) From the difference equation and using the zero-initial condition, we have

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \ge 0 \text{ and } y(-1) = 0$$

$$n = 0,\ y(0) = 0.25y(-1) + x(0) = u(0) = 1$$

$$n = 1,\ y(1) = 0.25y(0) + x(1) = 0.25 \times u(0) + u(1) = 1.25$$

$$n = 2,\ y(2) = 0.25y(1) + x(2) = 0.25 \times 1.25 + u(2) = 1.3125$$

Applying the convolution sum in Eq. (3.15) yields

$$y(n) = x(n) + 0.25x(n-1) + 0.0625x(n-2) + \cdots$$

$$\begin{aligned} n = 0,\ y(0) &= x(0) + 0.25x(-1) + 0.0625x(-2) + \cdots \\ &= u(0) + 0.25 \times u(-1) + 0.125 \times u(-2) + \cdots = 1 \end{aligned}$$

$$\begin{aligned} n = 1,\ y(1) &= x(1) + 0.25x(0) + 0.0625x(-1) + \cdots \\ &= u(1) + 0.25 \times u(0) + 0.125 \times u(-1) + \cdots = 1.25 \end{aligned}$$

$$\begin{aligned} n = 2,\ y(2) &= x(2) + 0.25x(1) + 0.0625x(0) + \cdots \\ &= u(2) + 0.25 \times u(1) + 0.0625 \times u(0) + \cdots = 1.3125 \\ &\cdots \end{aligned}$$

Comparing the results, we verify that a linear time-invariant system can be represented by the convolution sum using its impulse response and input sequence. Note that we verify only the causal system for simplicity, and the principle works for both causal and noncausal systems.

Note that this impulse response $h(n)$ contains an infinite number of terms in its duration due to the past output term $y(n-1)$. Such a system as described in the preceding example is called an *infinite impulse response* (IIR) system, which is studied in the later chapters.

3.4 DIGITAL CONVOLUTION

Digital convolution plays an important role in digital filtering. As we verifies in the last section, a linear time-invariant system can be represented by a digital convolution sum. Given a linear time-invariant system, we can determine its unit-impulse response $h(n)$, which relates the system input and output. To find the output sequence $y(n)$ for any input sequence $x(n)$, we write the digital convolution as shown in Eq. (3.15) as:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \cdots + h(-1)x(n+1) + h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + \cdots . \end{aligned} \tag{3.16}$$

The sequences $h(k)$ and $x(k)$ in Eq. (3.16) are interchangeable. In Eq. (3.16), let $m = n - k$, we have an alternative form as

$$\begin{aligned} y(n) &= \sum_{m=\infty}^{-\infty} h(n-m)x(m) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \cdots + x(-1)h(n+1) + x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) + \cdots . \end{aligned} \tag{3.17}$$

Using a conventional notation, we express the digital convolution as

$$y(n) = h(n)*x(n). \tag{3.18}$$

Note that for a causal system, which implies its impulse response

$$h(n) = 0 \text{ for } n < 0.$$

The lower limit of the convolution sum begins at 0 instead of -∞, that is

$$y(n)=\sum_{k=0}^{\infty}h(k)x(n-k). \tag{3.19}$$

The alternative for Eq. (3.19) can be expressed as

$$y(n)=\sum_{k=-\infty}^{n}x(k)h(n-k). \tag{3.20}$$

We will focus on evaluating the convolution sum based on Eq. (3.17). Let us examine first a few outputs from Eq. (3.17):

$$y(0)=\sum_{k=-\infty}^{\infty}x(k)h(-k)=\cdots+x(-1)h(1)+x(0)h(0)+x(1)h(-1)+x(2)h(-2)+\cdots$$

$$y(1)=\sum_{k=-\infty}^{\infty}x(k)h(1-k)=\cdots+x(-1)h(2)+x(0)h(1)+x(1)h(0)+x(2)h(-1)+\cdots$$

$$y(2)=\sum_{k=-\infty}^{\infty}x(k)h(2-k)=\cdots+x(-1)h(3)+x(0)h(2)+x(1)h(1)+x(2)h(0)+\cdots$$

We see that the convolution sum requires the sequence $h(n)$ to be reversed and shifted. The graphical, formula, and table methods are discussed for evaluating the digital convolution via the several examples. To begin with evaluating the convolution sum graphically, we need to apply the reversed sequence and shifted sequence. The reversed sequence is defined as follows: If $h(n)$ is the given sequence, $h(-n)$ is the reversed sequence. The reversed sequence is a mirror image of the original sequence, assuming the vertical axis as the mirror. Let us study the reversed sequence and shifted sequence via the following example.

EXAMPLE 3.9

Given a sequence,

$$h(k)=\begin{cases}3, & k=0,1\\ 1, & k=2,3\\ 0 & \text{elsewhere}\end{cases}$$

where k is the time index or sample number,

(a) Sketch the sequence $h(k)$ and reversed sequence $h(-k)$.

(b) Sketch the shifted sequences $h(-k+3)$ and $h(-k-2)$.

Solution:

(a) Since $h(k)$ is defined, we plot it in Fig. 3.17.

Next, we need to find the reversed sequence $h(-k)$. We examine the following for

$$k>0, h(-k)=0$$

$$k=0, h(-0)=h(0)=3$$

$$k=-1, h(-k)=h(-(-1))=h(1)=3$$

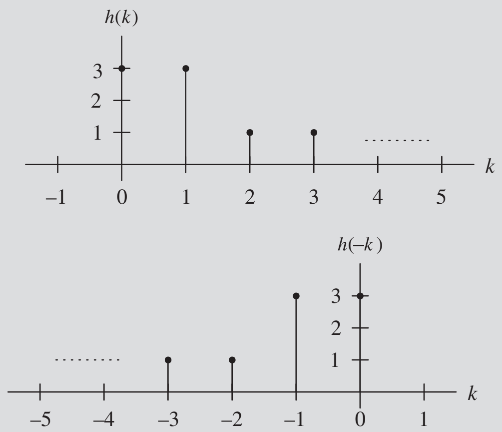

FIG. 3.17

Plots of the digital sequence and its reversed sequence in Example 3.9.

$$k=-2, h(-k)=h(-(-2))=h(2)=1$$

$$k=-3, h(-k)=h(-(-3))=h(3)=1$$

One can verify that $k \le -4$, $h(-k)=0$. Then the reversed sequence $h(-k)$ is shown as the second plot in Fig. 3.17.

As shown in the sketches, $h(-k)$ is just a mirror image of the original sequence $h(k)$.

(b) Based on the definition of the original sequence, we know that

$h(0)=h(1)=3$, $h(2)=h(3)=1$, and the others are zeros. The time indices correspond to the following:

$$-k+3=0, k=3$$

$$-k+3=1, k=2$$

$$-k+3=2, k=1$$

$$-k+3=3, k=0$$

Thus we can sketch $h(-k+3)$ as shown in Fig. 3.18.

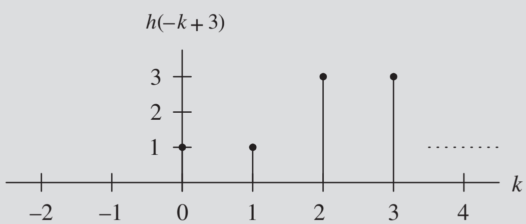

FIG. 3.18

Plot of the sequence $h(-k+3)$ in Example 3.9.

Continued

EXAMPLE 3.9—CONT'D

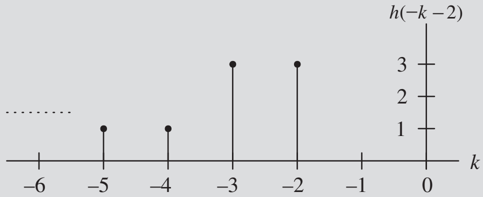

FIG. 3.19

Plot of the sequence $h(-k-2)$ in Example 3.9.

Similarly, $h(-k-2)$ is yielded in Fig. 3.19.

We can get $h(-k+3)$ by shifting $h(-k)$ to the right by three samples, and we can obtain $h(-k-2)$ by shifting $h(-k)$ to the left by two samples.

In summary, given $h(-k)$, we can obtain $h(n-k)$ by shifting $h(-k)$ n samples to the right or the left, depending on weather n is positive or negative.

Once we understand the shifted sequence and reversed sequence, we can perform digital convolution of two sequences $h(k)$ and $x(k)$, defined in Eq. (3.17) graphically. From that equation, we see that each convolution value $y(n)$ is the sum of the products of two sequences $x(k)$ and $h(n-k)$, the latter of which is the shifted version of the reversed sequence $h(-k)$ by $|n|$ samples. Hence, we can summarize the graphical convolution procedure in Table 3.3.

Table 3.3 Digital Convolution Using the Graphical Method

Step 1. Obtain the reversed sequence $h(-k)$.

Step 2. Shift $h(-k)$ by $|n|$ samples to get $h(n-k)$. If $n \geq 0$, $h(-k)$ will be shifted to right by n samples; but if $n<0$, $h(-k)$ will be shifted to the left by $|n|$ samples.

Step 3. Perform the convolution sum that is the sum of products of two sequences $x(k)$ and $h(n-k)$ to get $y(n)$.

Step 4. Repeat steps (1)–(3) for the next convolution value $y(n)$.

We illustrate digital convolution sum via the following example.

EXAMPLE 3.10

Using the following sequences defined in Fig. 3.20, evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

(a) By the graphical method.
(b) By applying the formula directly.

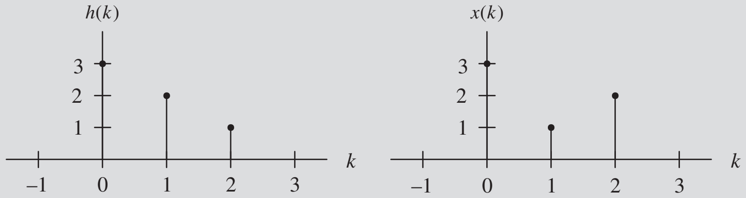

FIG. 3.20

Plots of digital input sequence and impulse sequence in Example 3.10.

Solution:

(a) To obtain $y(0)$, we need the reversed sequence $h(-k)$; and to obtain $y(1)$, we need the reversed sequence $h(1-k)$, and so on. Using the technique we have discussed, sequences $h(-k)$, $h(-k+1)$, $h(-k+2)$, $h(-k+3)$, and $h(-k+4)$ are achieved and plotted in Fig. 3.21, respectively.

Again, using the information in Figs. 3.20 and 3.21, we can compute the convolution sum as:

$$\text{Sum of product of } x(k) \text{ and } h(-k): y(0) = 3 \times 3 = 9$$

$$\text{Sum of product of } x(k) \text{ and } h(1-k): y(1) = 1 \times 3 + 3 \times 2 = 9$$

$$\text{Sum of product of } x(k) \text{ and } h(2-k): y(2) = 2 \times 3 + 1 \times 2 + 3 \times 1 = 11$$

$$\text{Sum of product of } x(k) \text{ and } h(3-k): y(3) = 2 \times 2 + 1 \times 1 = 5$$

$$\text{Sum of product of } x(k) \text{ and } h(4-k): y(4) = 2 \times 1 = 2$$

Sum of product of $x(k)$ and $h(5-k): y(n)=0$ for $n>4$, since sequences $x(k)$ and $h(n-k)$ do not overlap.

Finally, we sketch the output sequence $y(n)$ in Fig. 3.22.

(b) Applying Eq. (3.20) with zero-initial conditions leads to

$$y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2)$$

$$n=0,\ y(0) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) = 3 \times 3 + 1 \times 0 + 2 \times 0 = 9$$

$$n=1,\ y(1) = x(0)h(1) + x(1)h(0) + x(2)h(-1) = 3 \times 2 + 1 \times 3 + 2 \times 0 = 9$$

$$n=2,\ y(2) = x(0)h(2) + x(1)h(1) + x(2)h(0) = 3 \times 1 + 1 \times 2 + 2 \times 3 = 11$$

$$n=3,\ y(3) = x(0)h(3) + x(1)h(2) + x(2)h(1) = 3 \times 0 + 1 \times 1 + 2 \times 2 = 5$$

$$n=4,\ y(4) = x(0)h(4) + x(1)h(3) + x(2)h(2) = 3 \times 0 + 1 \times 0 + 2 \times 1 = 2$$

$$n \geq 5,\ y(n) = x(0)h(n) + x(1)h(n-1) + x(2)h(n-2) = 3 \times 0 + 1 \times 0 + 2 \times 0 = 0.$$

In simple cases such as Example 3.10, it is not necessary to use the graphical or formula methods. We can compute the convolution by treating the input sequence and impulse response as number sequences and sliding the reversed impulse response past the input sequence, cross-multiplying, and summing the nonzero overlap terms at each step. The procedure and calculated results are listed in Table 3.4.

Continued

EXAMPLE 3.10—CONT'D

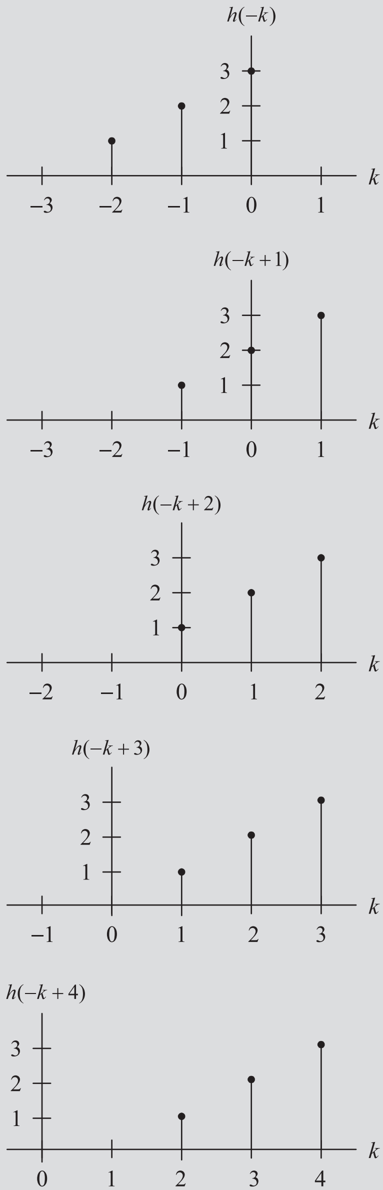

FIG. 3.21

Illustration of convolution of two sequences $x(k)$ and $h(k)$ in Example 3.10.

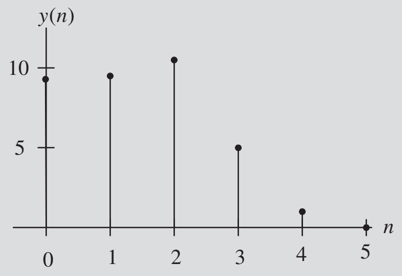

FIG. 3.22

Plot of the convolution sum in Example 3.10.

Table 3.4 Convolution Sum Using the Table Method

k:	−2	−1	0	1	2	3	4	5	
$x(k)$:			**3**	**1**	**2**				
$h(-k)$:	1	2	3						$y(0)=3\times3=9$
$h(1-k)$		1	2	3					$y(1)=3\times2+1\times3=9$
$h(2-k)$			1	2	3				$y(2)=3\times1+1\times2+2\times3=11$
$h(3-k)$				1	2	3			$y(3)=1\times1+2\times2=5$
$h(4-k)$					1	2	3		$y(4)=2\times1=2$
$h(5-k)$						1	2	3	$y(5)=0$ (no overlap)

We can see that the calculated results using all the methods are consistent. The steps using the table method are concluded in Table 3.5.

Table 3.5 Digital Convolution Steps via the Table

Step 1. List the index k covering a sufficient range.

Step 2. List the input $x(k)$.

Step 3. Obtain the reversed sequence $h(-k)$, and align the rightmost element of $h(n-k)$ to the leftmost element of $x(k)$.

Step 4. Cross-multiply and sum the nonzero overlap terms to produce $y(n)$.

Step 5. Slide $h(n-k)$ to the right by one position.

Step 6. Repeat Step 4; stop if all the output values are zero or if required.

EXAMPLE 3.11

Given the following two rectangular sequences,

$$x(n) = \begin{cases} 1 & n = 0,1,2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & \text{otherwise} \end{cases},$$

Convolve them using the table method.

Solution:

Using Table 3.5 as a guide, we list the operations and calculations in Table 3.6.

Table 3.6 Convolution Sum in Example 3.11

k:	**−2**	**−1**	**0**	**1**	**2**	**3**	**4**	**5**	…	
$x(k)$:			**1**	**1**	**1**				…	
$h(-k)$:	1	1	0							$y(0)=0$ (no overlap)
$h(1-k)$		1	1	0						$y(1)=1\times 1=1$
$h(2-k)$			1	1	0					$y(2)=1\times 1+1\times 1=2$
$h(3-k)$				1	1	0				$y(3)=1\times 1+1\times 1=2$
$h(4-k)$					1	1	0			$y(4)=1\times 1=1$
$h(n-k)$						1	1	0		$y(n)=0$, $n\geq 5$ (no overlap) Stop

Note that the output should show the trapezoidal shape.

Let us examine convolving a finite long sequence with an infinite long sequence.

EXAMPLE 3.12

A system representation using the unit-impulse response for the linear system

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0$$

is determined in Example 3.8 as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k),$$

where $h(n) = (0.25)^n u(n)$. For a step input $x(n) = u(n)$, determine the output response for the first three output samples using the table method.

Solution:

Using Table 3.5 as a guide, we list the operations and calculations in Table 3.7.

Table 3.7 Convolution Sum in Example 3.13

k:	−2	−1	0	1	2	3	…	
$x(k)$:			1	1	1	1	…	
$h(-k)$:	0.0625	0.25	1					$y(0)=1\times 1=1$
$h(1-k)$		0.0625	0.25	1				$y(1)=1\times 0.25+1\times 1=1.25$
$h(2-k)$			0.0625	0.25	1			$y(2)=1\times 0.0625+1\times 0.25$ $+1\times 1=1.3125$
								Stop as required

As expected, the output values are the same as those obtained in Example 3.8.

3.5 BOUNDED-INPUT AND BOUNDED-OUTPUT STABILITY

We are interested in designing and implementing stable linear time-invariant systems. A stable system is one for which every bounded input produces a bounded output (BIBO). There are many other stability definitions. To find the stability criterion for the linear time-invariant system, consider the linear time-invariant representation with the bounded input as $|x(n)|<M$, where M is a positive finite number. Taking absolute value of Eq. (3.15) leads to the following inequality:

$$|y(n)|=\left|\sum_{k=-\infty}^{\infty} x(k)h(n-k)\right|<\sum_{k=-\infty}^{\infty}|x(k)||h(n-k)|. \tag{3.21}$$

Using the bounded input, we obtain

$$|y(n)|<M(\cdots+|h(-1)|+|h(0)|+|h(1)|+|h(2)|+\cdots). \tag{3.22}$$

If the absolute sum in Eq. (3.22) is a finite number, the product of the absolute sum and the maximum input value is therefore a finite number. Hence, we obtain a bounded output with a bounded input. This concludes that a linear time-invariant system is stable if only if the sum of its absolute impulse response coefficients is a finite positive number, that is,

$$S=\sum_{k=-\infty}^{\infty}|h(k)|=\cdots+|h(-1)|+|h(0)|+|h(1)|+\cdots<\infty. \tag{3.23}$$

Fig. 3.23 illustrates a linear time-invariant stable system, where the impulse response decreases to zero in finite amount of time so that the summation of its absolute impulse response coefficients is guaranteed to be finite.

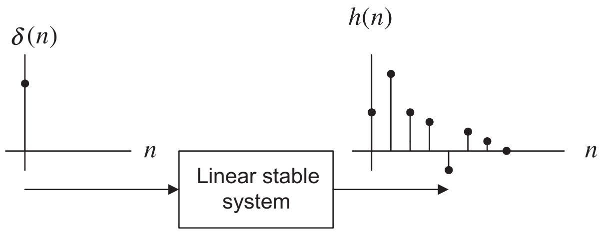

FIG. 3.23

Illustration of stability of the digital linear time-invariant system.

EXAMPLE 3.13

Given the linear time-invariant system in Example 3.8,

$$y(n) = 0.25y(n-1) + x(n) \text{ for } n \geq 0 \text{ and } y(-1) = 0,$$

which is described by the unit-impulse response

$$h(n) = (0.25)^n u(n),$$

determine whether this system is stable or not.

Solution:

Using Eq. (3.23), we have

$$S = \sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} \left|(0.25)^k u(k)\right|.$$

Applying the definition of the unit-step function $u(k)$ for $u(k)$, we have

$$S = \sum_{k=0}^{\infty} (0.25)^k = 1 + 0.25 + 0.25^2 + \cdots$$

Using the formula for a sum of the geometric series (see Appendix H),

$$\sum_{k=0}^{\infty} a^k = \frac{1}{1-a},$$

where $a = 0.25 < 1$, we conclude

$$S = \sum_{k=0}^{\infty} (0.25)^k = 1 + 0.25 + 0.25^2 + \cdots = \frac{1}{1-0.25} = \frac{4}{3} < \infty.$$

Since the summation is a finite number, the linear system is stable.

3.6 SUMMARY

1. Concepts of digital signals are explained. Digital signal samples are sketched, using their encoded amplitude versus sample numbers with vertical bars topped by solid circles located at their

sampling instants, respectively. Impulse sequence, unit-step sequence, and their shifted versions are sketched in this notation.

2. An analog signal function can be sampled to its digital (discrete time) version by substituting time $t = nT$ into the analog function, that is,

$$x(n) = x(t)|_{t=nT} = x(nT).$$

The digital function values can be calculated for the given time index (sample number).

3. The DSP system we wish to design is typically a linear, time invariant, causal system. Linearity means that the superposition principle exists. Time-invariance requires that the shifted input generates the corresponding shifted output with the same amount of time. Causality indicates that the system output depends on only its current input sample and past input sample(s).
4. The difference equation describing a linear, time-invariant system has a format such that the current output depends on the current input, past input(s), and past output (s) in general.
5. The unit-impulse response can be used to fully describe a linear, time-invariant system. Given the impulse response, the system output is the sum of the products of the impulse response coefficients and corresponding input samples, called the digital convolution sum.
6. Digital convolution sum, which represents a DSP system, is evaluated in three ways: the graphical method, evaluation of the formula, and the table method. The table method is found to be most effective.
7. BIBO is a type of stability in which a bounded input will produce a bounded output. The condition for a BIBO linear time-invariant system requires that the sum of the absolute impulse response coefficients be a finite positive number.

3.7 PROBLEMS

3.1 Sketch each of the following special digital sequences:
- **(a)** $5\delta(n)$
- **(b)** $-2\delta(n-5)$
- **(c)** $-5u(n)$
- **(d)** $5u(n-2)$

3.2 Calculate the first eight sample values and sketch each of the following sequences:
- **(a)** $x(n) = 0.5^n u(n)$
- **(b)** $x(n) = 5\sin(0.2\pi n)u(n)$
- **(c)** $x(n) = 5\cos(0.1\pi n + 30^0)u(n)$
- **(d)** $x(n) = 5(0.75)^n \sin(0.1\pi n)u(n)$

3.3 Sketch each of the following special digital sequences:
- **(a)** $8\delta(n)$
- **(b)** $-3.5\delta(n-4)$
- **(c)** $4.5u(n)$
- **(d)** $-6u(n-3)$

3.4 Calculate the first eight sample values and sketch each of the following sequences:
- **(a)** $x(n) = 0.25^n u(n)$

(b) $x(n) = 3\sin(0.4\pi n)u(n)$
(c) $x(n) = 6\cos(0.2\pi n + 30^0)u(n)$
(d) $x(n) = 4(0.5)^n \sin(0.1\pi n)u(n)$

3.5 Sketch the following sequence:
(a) $x(n) = 3\delta(n+2) - 0.5\delta(n) + 5\delta(n-1) - 4\delta(n-5)$
(b) $x(n) = \delta(n+1) - 2\delta(n-1) + 5\delta(n-4)$

3.6 Given the digital signals $x(n)$ in Figs. 3.24 and 3.25, write an expression for each digital signal using the unit-impulse sequence and its shifted sequences.

3.7 Sketch the following sequences:
(a) $x(n) = 2\delta(n+3) - 0.5\delta(n+1) - 5\delta(n-2) - 4\delta(n-5)$
(b) $x(n) = 2\delta(n+2) - 2\delta(n+1) + 5u(n-3)$

3.8 Given the digital signals $x(n)$ as shown in Figs. 3.26 and 3.27, write an expression for each digital signal using the unit-impulse sequence and its shifted sequences.

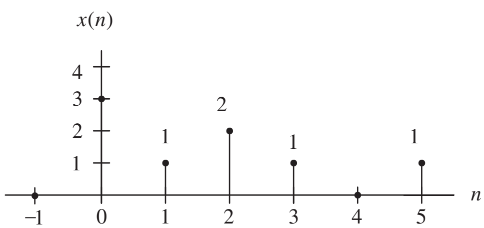

FIG. 3.24

The first digital signal in Problem 3.6.

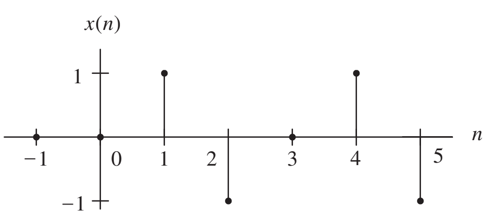

FIG. 3.25

The second digital signal in Problem 3.6.

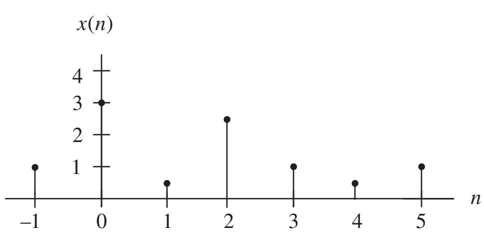

FIG. 3.26

The first digital signal in Problem 3.8.

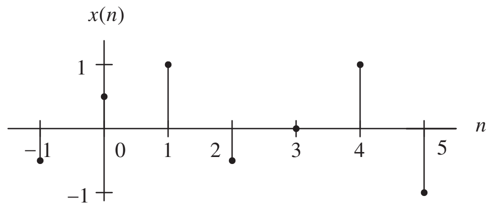

FIG. 3.27

The second digital signal in Problem 3.8.

3.9 Assume that a digital signal processor with a sampling time interval of 0.01 s converts each of the following analog signals $x(t)$ to a digital signal $x(n)$, determine the digital sequences for each of the following analog signals.

(a) $x(t)=e^{-50t}u(t)$

(b) $x(t)=5\sin(20\pi t)u(t)$

(c) $x(t)=10\cos(40\pi t+30^0)u(t)$

(d) $x(t)=10e^{-100t}\sin(15\pi t)u(t)$

3.10 Determine which of the following systems is a linear system.

(a) $y(n)=5x(n)+2x^2(n)$

(b) $y(n)=x(n-1)+4x(n)$

(c) $y(n)=4x^3(n-1)-2x(n)$

3.11 Assume that a digital signal processor with a sampling time interval of 0.005 s converts each of the following analog signals $x(t)$ to a digital signal $x(n)$, determine the digital sequences for each of the following analog signals.

(a) $x(t)=e^{-100t}u(t)$

(b) $x(t)=4\sin(60\pi t)u(t)$

(c) $x(t)=7.5\cos(20\pi t+60^0)u(t)$

(d) $x(t)=20e^{-200t}\sin(60\pi t)u(t)$

3.12 Determine which of the following systems is a linear system.

(a) $y(n)=4x(n)+8x^3(n)$

(b) $y(n)=x(n-3)+3x(n)$

(c) $y(n)=5x^2(n-1)-3x(n)$

3.13 Given the following linear systems, find which one is time invariant.

(a) $y(n)=-5x(n-10)$

(b) $y(n)=4x(n^2)$

3.14 Determine which of the following linear systems is causal.

(a) $y(n)=0.5x(n)+100x(n-2)-20x(n-10)$

(b) $y(n)=x(n+4)+0.5x(n)-2x(n-2)$

3.15 Determine the causality for each of the following linear systems.

(a) $y(n)=0.5x(n)+20x(n-2)-0.1y(n-1)$

(b) $y(n)=x(n+2)-0.4y(n-1)$

(c) $y(n)=x(n-1)+0.5y(n+2)$

3.16 Find the unit-impulse response for each of the following linear systems.

(a) $y(n)=0.5x(n)-0.5x(n-2)$; for $n\geq 0$, $x(-2)=0$, $x(-1)=0$

(b) $y(n) = 0.75y(n-1) + x(n)$; for $n \geq 0$, $y(-1) = 0$
(c) $y(n) = -0.8y(n-1) + x(n-1)$; for $n \geq 0$, $x(-1) = 0$, $y(-1) = 0$

3.17 Determine the causality for each of the following linear systems.
(a) $y(n) = 5x(n) + 10x(n-4) - 0.1y(n-5)$
(b) $y(n) = 2x(n+2) - 0.2y(n-2)$
(c) $y(n) = 0.1x(n+1) + 0.5y(n+2)$

3.18 Find the unit-impulse response for each of the following linear systems.
(a) $y(n) = 0.2x(n) - 0.3x(n-2)$; for $n \geq 0$, $x(-2) = 0$, $x(-1) = 0$
(b) $y(n) = 0.5y(n-1) + 0.5x(n)$; for $n \geq 0$, $y(-1) = 0$
(c) $y(n) = -0.6y(n-1) - x(n-1)$; for $n \geq 0$, $x(-1) = 0$, $y(-1) = 0$

3.19 For each of the following linear systems, find the unit-impulse response, and draw the block diagram.
(a) $y(n) = 5x(n-10)$
(b) $y(n) = x(n) + 0.5x(n-1)$

3.20 Given the sequence

$$h(k) = \begin{cases} 2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & \text{elsewhere} \end{cases}$$

where k is the time index or sample number,
(a) sketch the sequence $h(k)$ and the reverse sequence $h(-k)$;
(b) sketch the shifted sequences $h(-k+2)$ and $h(-k-3)$.

3.21 Given the sequence

$$h(k) = \begin{cases} -1 & k = 0,1 \\ 2 & k = 2,3 \\ -2 & k = 4 \\ 0 & \text{elsewhere} \end{cases}$$

where k is the time index or sample number,
(a) sketch the sequence $h(k)$and the reverse sequence$h(-k)$;
(b) sketch the shifted sequences and $h(-k-2)$.

3.22 Using the following sequence definitions,

$$h(k) = \begin{cases} 2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad x(k) = \begin{cases} 2, & k = 0 \\ 1, & k = 1,2 \\ 0 & \text{elsewhere} \end{cases}$$

evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

(a) using the graphical method;
(b) using the table method;
(c) applying the convolution formula directly.

3.23 Using the sequence definitions

$$h(k) = \begin{cases} 2, & k = 0,1,2 \\ 1, & k = 3,4 \\ 0 & \text{elsewhere} \end{cases} \quad \text{and} \quad x(k) = \begin{cases} 2, & k = 0 \\ 1, & k = 1,2 \\ 0 & \text{elsewhere} \end{cases}$$

evaluate the digital convolution

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

(a) the graphical method;
(b) the table method;
(c) applying the convolution formula directly.

3.24 Convolve the following two rectangular sequences:

$$x(n) = \begin{cases} 1 & n = 0,1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad h(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1,2 \\ 0 & \text{otherwise} \end{cases}$$

using the table method.

3.25 Determine the stability for the following linear system.

$$y(n) = 0.5x(n) + 100x(n-2) - 20x(n-10)$$

3.26 For each of the following linear systems, find the unit-impulse response, and draw the block diagram.
(a) $y(n) = 2.5x(n-5)$
(b) $y(n) = 2x(n) + 1.2x(n-1)$

3.27 Determine the stability for the following linear system.

$$y(n) = 0.5x(n) + 100x(n-2) - 20x(n-10)$$

3.28 Determine the stability for each of the following linear systems.
(a) $y(n) = \sum_{k=0}^{\infty} 0.75^k x(n-k)$
(b) $y(n) = \sum_{k=0}^{\infty} 2^k x(n-k)$

3.29 Determine the stability for each of the following linear systems.
(a) $y(n) = \sum_{k=0}^{\infty} (-1.5)^k x(n-k)$
(b) $y(n) = \sum_{k=0}^{\infty} (-0.5)^k x(n-k)$

Advanced Problems

3.30 Given each of the following discrete-time systems,

(a) $y(n) = x(-n+3)$

(b) $y(n) = x(n-1) + 0.5y(n-2)$

(c) $y(n) = nx(n-1) + x(n)$

(d) $y(n) = |x(n)|$

determine if the system is (1) linear or nonlinear; (2) time invariant or time varying; (3) causal or noncausal; and (4) stable or unstable.

3.31 Given each of the following discrete-time systems,

(a) $y(n) = \text{sign}[x(n)]$, where $\text{sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$

(b) $y(n) = \text{truncate}[x(n)]$

(c) $y(n) = \text{round}[x(n)]$

determine if the system is (1) linear or nonlinear; (2) time invariant or time varying; (3) causal or noncausal; and (4) stable or unstable.

3.32 Given each of the following discrete-time systems,

(a) $y(n) = x(n)x(n-1)$

(b) $y(n) = x(n) - 0.2x(n-1)y(n-2)$

determine if the system is (1) linear or nonlinear; (2) time invariant or time varying; (3) causal or noncausal; and (4) stable or unstable.

3.33 For $N > M$, show that

(a) $\sum_{k=M}^{N} r^k = \begin{cases} \dfrac{r^M - r^{N+1}}{1-r} & \text{for } r \neq 1 \\ N - M + 1 & \text{for } r = 1 \end{cases}$

(b) For $|r| < 1$, $\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$

3.34 Given a relaxed discrete-time system,

$$y(n) - ay(n-1) = x(n),$$

(a) Show that the impulse response is

$$h(n) = a^n u(n);$$

(b) If the impulse response of a relaxed discrete-time system is found as

$$h(n) = \begin{cases} a^n & n \geq 0, n = \text{even} \\ 0 & \text{otherwise} \end{cases},$$

determine the discrete-time system equation.

3.35 Given the following relaxed discrete-time system with a causal input $x(n)$, where $x(n) = 0$, for $n < 0$:

$$y(n) - \frac{n}{n+1} y(n-1) = \frac{1}{n+1} x(n),$$

show that the impulse response is $h(n) = 1/(n+1)$.

3.36 Given $x(n) = a^n u(n)$, where $|a| < 1$, and $h(n) = u(n)$

determine

(a) $y(n) = h(n) * x(n)$

(b) $y(n) = h(n) * x(-n)$.

3.37 A causal system output $y(n)$ is expressed as

$$y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k),$$

where $h(n)$ and $x(n)$ are the impulse response and system input, respectively. Show that

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k).$$

3.38 A system output $y(n)$ is expressed as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k).$$

If both $h(n)$ and $x(n)$ are causal, show that

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k) = \sum_{k=0}^{n} x(k)h(n-k).$$