CHAPTER

11 MULTIRATE DIGITAL SIGNAL PROCESSING, OVERSAMPLING OF ANALOG-TO-DIGITAL CONVERSION, AND UNDERSAMPLING OF BANDPASS SIGNALS

CHAPTER OUTLINE

11.1 MULTIRATE DIGITAL SIGNAL PROCESSING BASICS

In many areas of digital signal processing (DSP) applications—such as communications, speech, and audio processing—raising or lowering of a sampling rate is required. The principle that deals with changing the sampling rate belongs essentially to *multirate signal processing* (Ifeachor and Jervis, 2002; Proakis and Manolakis, 2007; Porat, 1997; Sorensen and Chen, 1997). As an introduction, we focus on the sampling rate conversion; that is, sampling rate reduction or increase.

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11.1.1 SAMPLING RATE REDUCTION BY AN INTEGER FACTOR

The process of reducing the sampling rate by an integer factor is referred to as *downsampling* of a data sequence. We also refer downsampling as "decimation" (not taking one of ten). The term "decimation" used for the downsampling process has been accepted and used in many textbooks and fields. To downsample a data sequence $x(n)$ by an integer factor of M, we use the following notation:

$$y(m) = x(mM), \tag{11.1}$$

where $y(m)$ is the downsampled sequence, obtained by taking a sample from the data sequence $x(n)$ for every M samples (discarding $M-1$ samples for every M samples). As an example, if the original sequence with a sampling period $T=0.1$ s (sampling rate $=10$ samples per second) is given by

$$x(n): 8\ 7\ 4\ 8\ 9\ 6\ 4\ 2\ -2\ -5\ -7\ -7\ -6\ -4\ldots$$

and we downsample the data sequence by a factor of 3, we obtain the downsampled sequence as

$$y(m): 8\ 8\ 4\ -5\ -6\ldots,$$

with the resultant sampling period $T=3\times 0.1=0.3$ s (the sampling rate now is 3.33 samples per second). Although the example is straightforward, there is a requirement to avoid aliasing noise. We illustrate this next.

From the Nyquist sampling theorem, it is known that aliasing can occur in the downsampled signal due to reduced sampling rate. After downsampling by a factor of M, the new sampling period becomes MT, and therefore the new sampling frequency is

$$f_{sM} = \frac{1}{MT} = \frac{f_s}{M}, \tag{11.2}$$

where f_s is the original sampling rate.

Hence, the folding frequency after downsampling becomes

$$f_{sM}/2 = \frac{f_s}{2M}. \tag{11.3}$$

This tells us that after downsampling by a factor of M, the new folding frequency will be decreased by M times. If the signal to be downsampled has frequency components larger than the new folding frequency, $f > f_s/(2M)$, aliasing noise will be introduced into the downsampled data.

To overcome this problem, it is required that the original signal $x(n)$ be processed by a lowpass filter $H(z)$ before downsampling, which should have a stop frequency edge at $f_s/(2M)$ (Hz). The corresponding normalized stop frequency edge is then converted to be

$$\Omega_{stop} = 2\pi \frac{f_s}{2M} T = \frac{\pi}{M} \text{ radians}. \tag{11.4}$$

In this way, before downsampling, we can guarantee the maximum frequency of the filtered signal satisfies

$$f_{\max} < \frac{f_s}{2M}, \tag{11.5}$$

such that no aliasing noise is introduced after downsampling. A general block diagram of decimation is given in Fig. 11.1, where the filtered output in terms of the z-transform can be written as

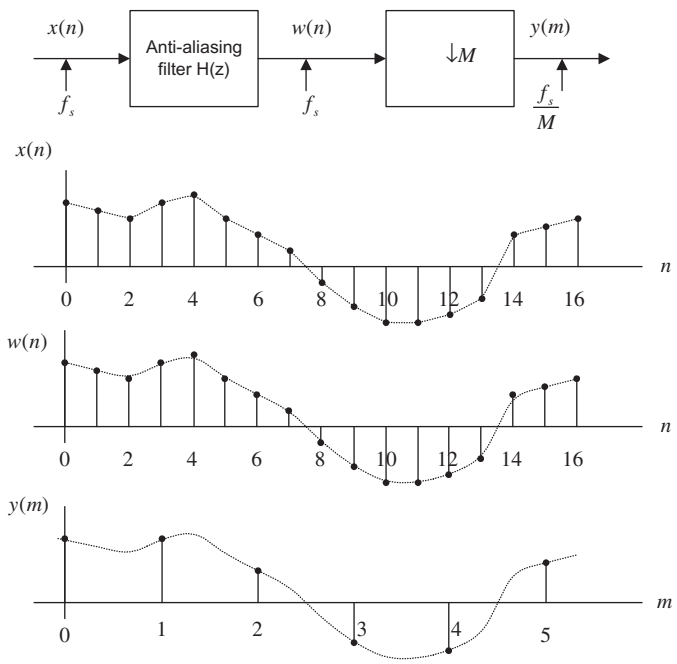


FIG. 11.1

Block diagram of the downsampling process with $M=3$.

$$W(z)=H(z)X(z), \tag{11.6}$$

where $X(z)$ is the z-transform of the sequence to be decimated, $x(n)$, and $H(z)$ is the lowpass filter transfer function. After anti-aliasing filtering, the downsampled signal $y(m)$ takes its value from the filter output as:

$$y(m)=w(mM). \tag{11.7}$$

The process of reducing the sampling rate by a factor of 3 is shown in Fig. 11.1. The corresponding spectral plots for $x(n)$, $w(n)$, and $y(m)$ in general are shown in Fig. 11.2.

To verify this principle, let us consider a signal $x(n)$ generated by the following:

$$x(n)=5\sin\left(\frac{2\pi\times 1000n}{8000}\right)+\cos\left(\frac{2\pi\times 2500}{8000}\right)=5\sin\left(\frac{n\pi}{4}\right)+\cos\left(\frac{5n\pi}{8}\right), \tag{11.8}$$

with a sampling rate of $f_s=8000$ Hz, the spectrum of $x(n)$ is plotted in the first graph in Fig. 11.3A, where we observe that the signal has components at frequencies of 1000 and 2500 Hz. Now we downsample $x(n)$ by a factor of 2, that is, $M=2$. According to Eq. (11.3), we know that the new folding frequency is $4000/2=2000$ Hz. Hence, without using the anti-aliasing lowpass filter, the spectrum

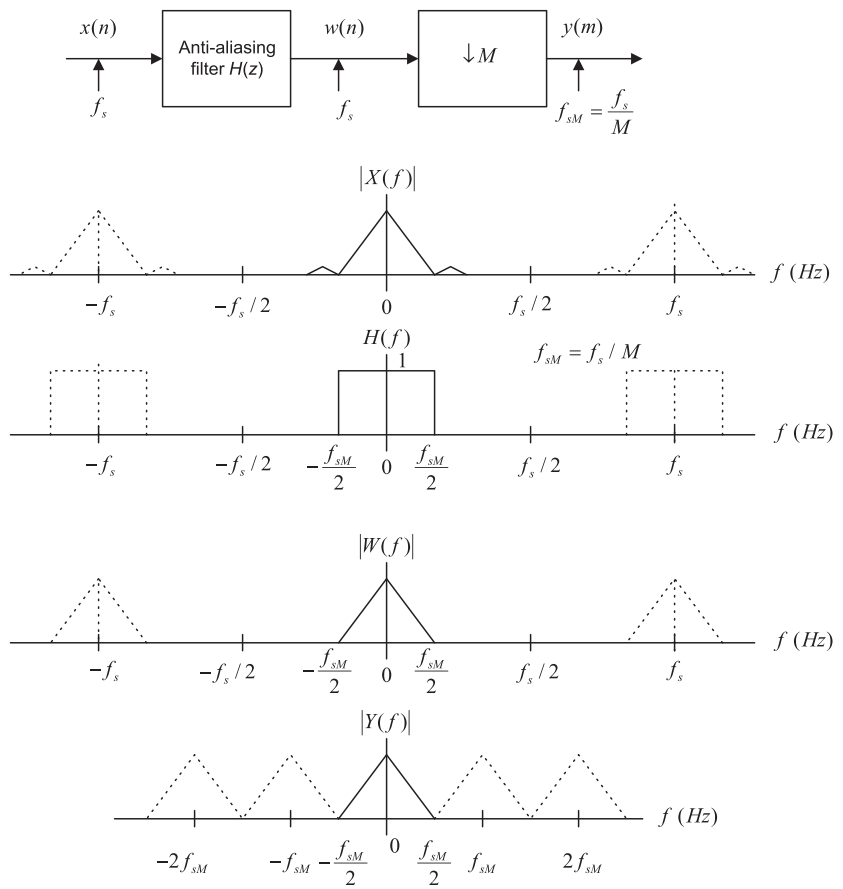


FIG. 11.2

Spectrum after downsampling.

would contain the aliasing frequency of 4–2.5 kHz = 1.5 kHz introduced by 2.5 kHz, plotted in the second graph in Fig. 11.3A.

Now we apply a finite impulse response (FIR) lowpass filter designed with a filter length of $N = 27$ and a cutoff frequency of 1.5 kHz to remove the 2.5 kHz signal before downsampling to avoid aliasing. How to obtain such specifications will be discussed in the later example. The normalized cutoff frequency used for design is given by

$$\Omega_c = 2\pi \times 1500 \times (1/8000) = 0.375\pi.$$

Thus, the aliasing noise is avoided. The spectral plots are given in Fig. 11.3B, where the first plot shows the spectrum of $w(n)$ after anti-aliasing filtering, while the second plot describes the spectrum of $y(m)$ after downsampling. Clearly, we prevent aliasing noise in the downsampled data by sacrificing the original 2.5-kHz signal. Program 11.1 gives the detail of MATLAB implementation.

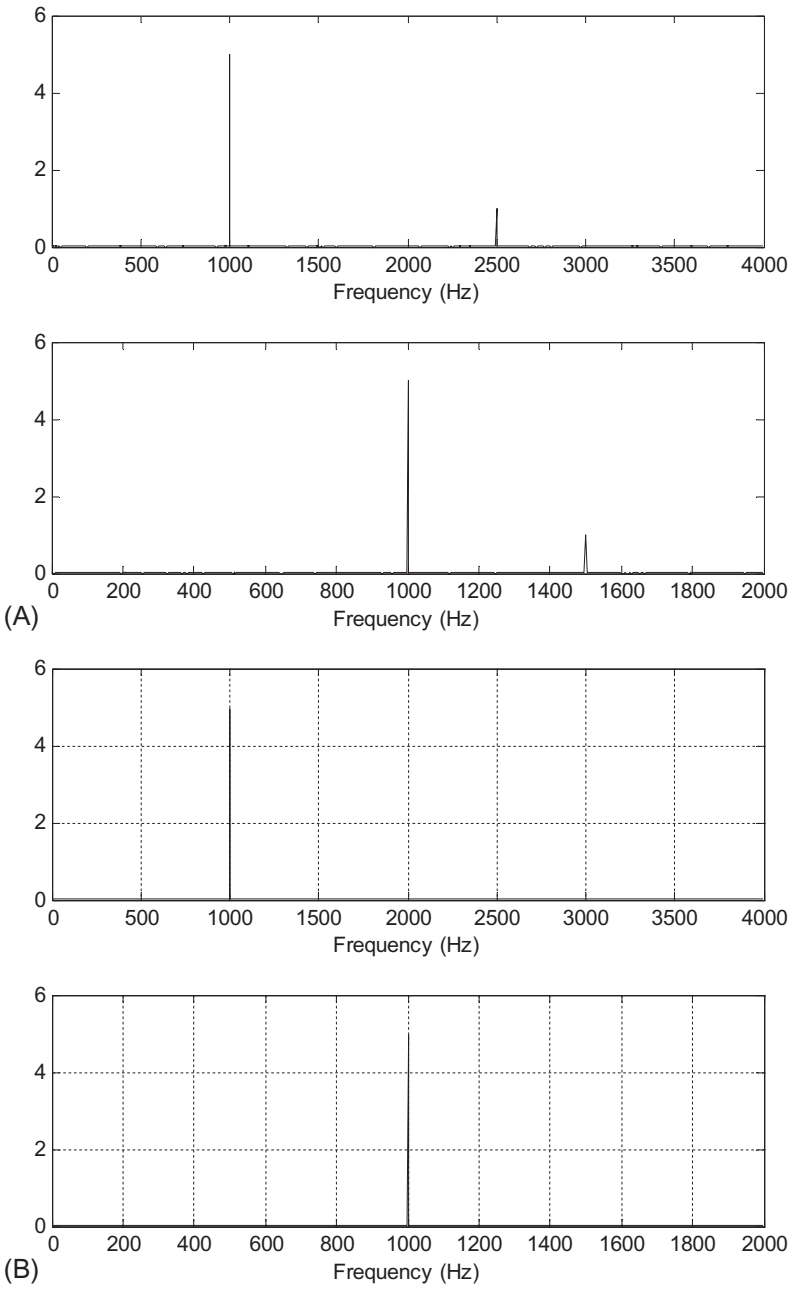


FIG. 11.3

(A) Spectrum before downsampling and spectrum after downsampling without using the anti-aliasing filter.
(B) Spectrum before downsampling and spectrum after downsampling using the anti-aliasing filter.

Program 11.1. MATLAB program for signal decimation.

```
close all; clear all;
% Downsampling filter (see Chapter 7 for FIR filter design)
B=[0.00074961181416  0.00247663033476  0.00146938649416  -0.00440446121505 ...
 -0.00910635730662   0.00000000000000  0.02035676831506  0.02233710562885...
 -0.01712963672810  -0.06376620649567 -0.03590670035210  0.10660980550088...
 0.29014909103794   0.37500000000000  0.29014909103794  0.10660980550088...
 -0.03590670035210  -0.06376620649567  -0.01712963672810 0.02233710562885...
 0.02035676831506  0.00000000000000  -0.00910635730662  -0.00440446121505...
 0.00146938649416  0.00247663033476  0.00074961181416];
% Generate 2048 samples
fs=8000;                          % Sampling rate
N=2048;                           % Number of samples
M=2;                              % Downsample factor
n=0:1:N-1;
x=5*sin(n*pi/4)+cos(5*n*pi/8);
% Compute single-side amplitude spectrum
% AC component will be doubled, and DC component will be kept as the same value
X=2*abs(fft(x,N))/N;X(1)=X(1)/2;
% Map the frequency index up to the folding frequency in Hz
f=[0:1:N/2-1]*fs/N;
%Downsampling
y=x(1:M:N);
NM=length(y);              % Length of the down sampled data
% Compute the single-sided amplitude spectrum for the downsampled signal
Y=2*abs(fft(y,NM))/length(y);Y(1)=Y(1)/2;
% Map the frequency index to the frequency in Hz
fsM=[0:1:NM/2-1]*(fs/M)/NM;
subplot(2,1,1);plot(f,X(1:1:N/2));grid; xlabel('Frequency (Hz)');
subplot(2,1,2);plot(fsM,Y(1:1:NM/2));grid; xlabel('Frequency (Hz)');
figure
w=filter(B,1,x);          % Anti-aliasing filtering
% Compute the single-sided amplitude spectrum for the filtered signal
W=2*abs(fft(w,N))/N;W(1)=W(1)/2;
% Downsampling
y=w(1:M:N);
NM=length(y);
% Compute the single-sided amplitude spectrum for the downsampled signal
Y=2*abs(fft(y,NM))/NM;Y(1)=Y(1)/2;
% Plot spectra
subplot(2,1,1);plot(f,W(1:1:N/2));grid; xlabel('Frequency (Hz)');
subplot(2,1,2);plot(fsM,Y(1:1:NM/2));grid; xlabel('Frequency (Hz)');
```

Now we focus on how to design an anti-aliasing FIR filter, or decimation filter. We discuss this topic via the following example.

EXAMPLE 11.1

Given a DSP downsampling system with the following specifications:
Sampling rate = 6000 Hz
Input audio frequency range = 0–800 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 50 dB
Downsample factor $M = 3$,
Determine the FIR filter length, cutoff frequency, and window type if the window method is used.

Solution:

Specifications are reorganized as:
Anti-aliasing filter operating at the sampling rate = 6000 Hz
Passband frequency range = 0–800 Hz
Stopband frequency range = 1–3 kHz
Passband ripple = 0.02 dB
Stopband attenuation = 50 dB
Filter type = FIR.
The block diagram and specifications are depicted in Fig. 11.4.
The Hamming window is selected, since it provides 0.019 dB ripple and 53 dB stopband attenuation. The normalized transition band is given by

$$\Delta f = \frac{f_{stop} - f_{pass}}{f_s} = \frac{1000 - 800}{6000} = 0.033.$$

The length of the filter and the cutoff frequency can be determined by

$$N = \frac{3.3}{\Delta f} = \frac{3.3}{0.033} = 100.$$

We choose an odd number; that is, $N = 101$, and

$$f_c = \frac{f_{pass} + f_{stop}}{2} = \frac{800 + 1000}{2} = 900 \text{ Hz}.$$

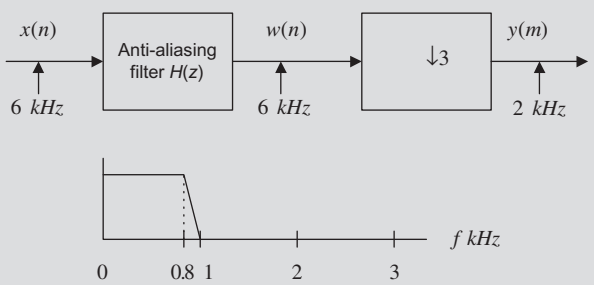


FIG. 11.4

Filter specifications for Example 11.1.

11.1.2 SAMPLING RATE INCREASE BY AN INTEGER FACTOR

Increasing a sampling rate is a process of upsampling by an integer factor of L. This process is described as follows:

$$y(m)=\begin{cases} x\left(\dfrac{m}{L}\right) & m=nL \\ 0 & \text{otherwise} \end{cases}, \tag{11.9}$$

where $n=0, 1, 2, \ldots$, $x(n)$ is the sequence to be upsampled by a factor of L, and $y(m)$ is the upsampled sequence. As an example, suppose that the data sequence is given as follows:

$$x(n): 8\ 8\ 4\ -5\ -6\ldots$$

After upsampling the data sequence $x(n)$ by a factor of 3 (adding L-1 zeros for each sample), we have the upsampled data sequence $w(m)$ as:

$$w(m): 8\ 0\ 0\ 8\ 0\ 0\ 4\ 0\ 0\ -5\ 0\ 0\ -6\ 0\ 0\ldots$$

The next step is to smooth the upsampled data sequence via an interpolation filter. The process is illustrated in Fig. 11.5A.

Similar to the downsampling case, assuming that the data sequence has the current sampling period of T, the Nyquist frequency is given by $f_{\max}=f_s/2$. After upsampling by a factor of L, the new sampling period becomes T/L, thus the new sampling frequency is changed to be

$$f_{sL}=Lf_s. \tag{11.10}$$

This indicates that after upsampling, the spectral replicas originally centered at $\pm f_s$, $\pm 2f_s$, … are included in the frequency range from 0 Hz to the new Nyquist limit $Lf_s/2$ Hz, as shown in Fig. 11.5B. To remove those included spectral replicas, an interpolation filter with a stop frequency edge of $f_s/2$ in Hz must be attached, and the normalized stop frequency edge is given by

$$\Omega_{stop}=2\pi\left(\frac{f_s}{2}\right)\times\left(\frac{T}{L}\right)=\frac{\pi}{L}\,\text{radians}. \tag{11.11}$$

After filtering via the interpolation filter, we will achieve the desired spectrum for $y(n)$, as shown in Fig. 11.5B. Note that since the interpolation is to remove the high-frequency images that are aliased by the upsampling operation, it is essentially an anti-aliasing lowpass filter.

To verify the upsampling principle, we generate the signal $x(n)$ with 1 and 2.5 kHz as follows:

$$x(n)=5\sin\left(\frac{2\pi\times 1000n}{8000}\right)+\cos\left(\frac{2\pi\times 2500n}{8000}\right),$$

with a sampling rate of $f_s=8000$ Hz. The spectrum of $x(n)$ is plotted in Fig. 11.6. Now we upsample $x(n)$ by a factor of 3, that is, $L=3$. We know that the sampling rate is increased to be $3\times 8000=24{,}000$ Hz. Hence, without using the interpolation filter, the spectrum would contain

the image frequencies originally centered at the multiple frequencies of 8 kHz. The top plot in Fig. 11.6 shows the spectrum for the sequence after upsampling and before applying the interpolation filter.

Now we apply an FIR lowpass filter designed with a length of 53, a cutoff frequency of 3250 Hz, and a new sampling rate of 24,000 Hz as the interpolation filter, whose normalized frequency should be

$$\Omega_c = 2\pi \times 3,250 \times \left(\frac{1}{24,000}\right) = 0.2708\pi.$$

The bottom plot in Fig. 11.6 shows the spectrum for $y(m)$ after applying the interpolation filter, where only the original signals with frequencies of 1 and 2.5 kHz are presented. Program 11.2 shows the implementation detail in MTALAB.

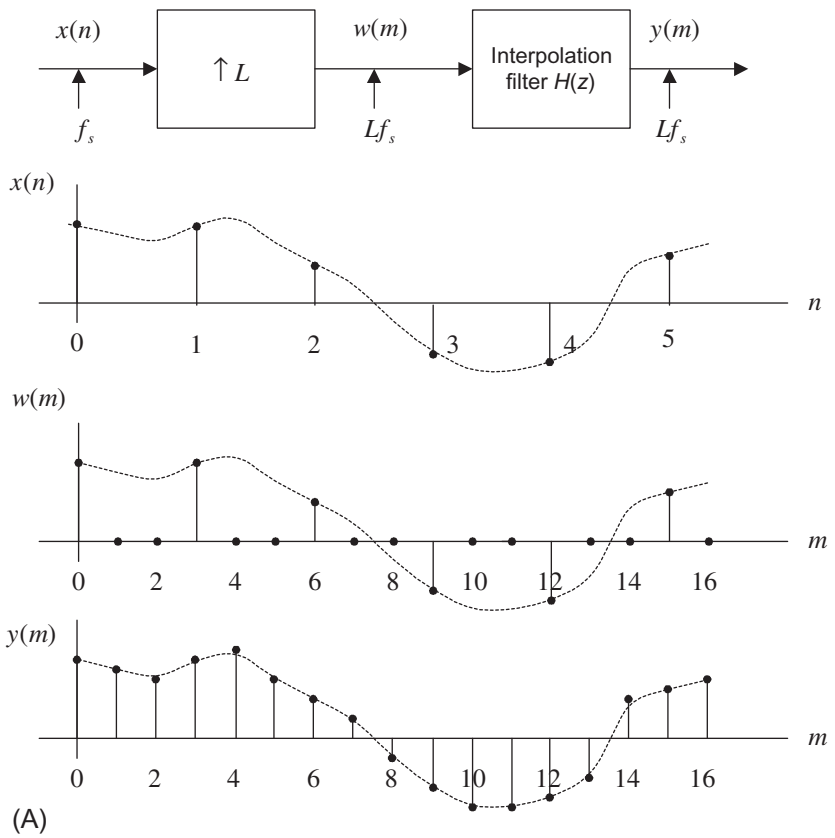


FIG. 11.5

(A) Block diagram for the upsampling process with $L=3$.

Continued

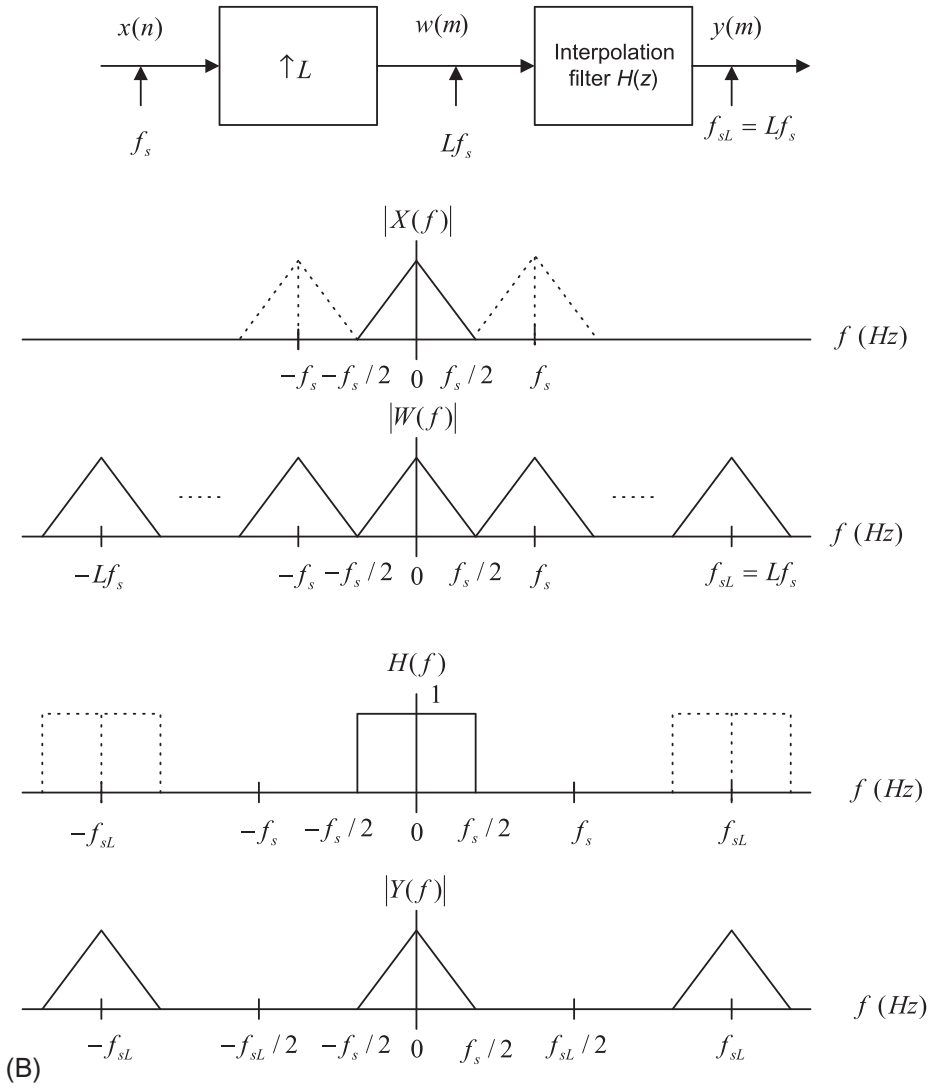


FIG. 11.5, CONT'D

(B) Spectra before and after upsampling.

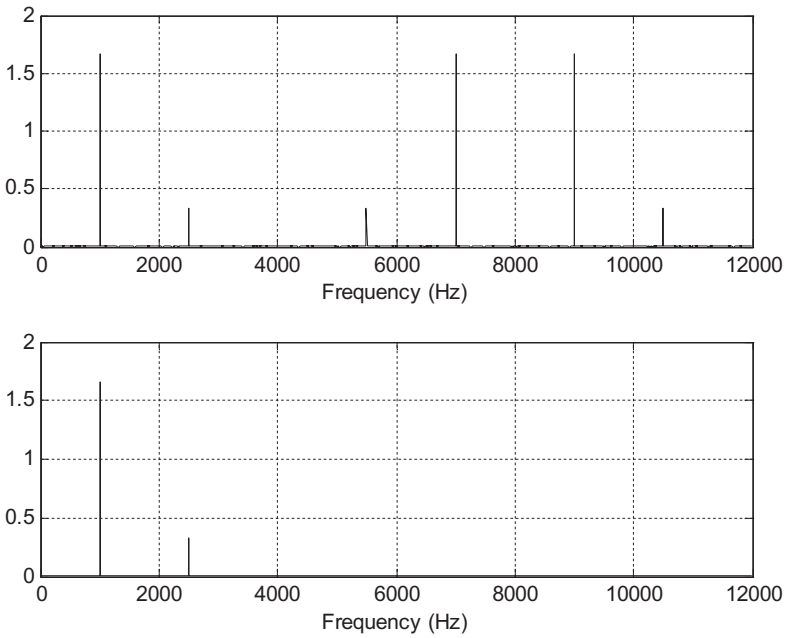


FIG. 11.6

(Top) The spectrum after upsampling and before applying the interpolation filter; (bottom) spectrum after applying the interpolation filter.

Program 11.2. MATLAB program for signal interpolation.

```
close all; clear all
%Upsampling filter (see Chapter 7 for FIR filter design)
B=[-0.00012783931504  0.00069976044649  0.00123831516738  0.00100277549136...
 -0.00025059018468  -0.00203448515158 -0.00300830295487  -0.00174101657599...
  0.00188598835011   0.00578414933758  0.00649330625041  0.00177982369523...
 -0.00670672686935  -0.01319379342716 -0.01116855281442  0.00123034314117...
  0.01775600060894   0.02614700427364  0.01594155162392 -0.01235169936557...
 -0.04334322148505  -0.05244745563466 -0.01951094855292  0.05718573279009...
  0.15568416401644   0.23851539047347  0.27083333333333  0.23851539047347...
  0.15568416401644   0.05718573279009 -0.01951094855292 -0.05244745563466...
 -0.04334322148505  -0.01235169936557  0.01594155162392  0.02614700427364...
  0.01775600060894   0.00123034314117 -0.01116855281442 -0.01319379342716...
 -0.00670672686935   0.00177982369523  0.00649330625041  0.00578414933758...
  0.00188598835011  -0.00174101657599 -0.00300830295487 -0.00203448515158...
 -0.00025059018468   0.00100277549136  0.00123831516738  0.00069976044649...
 -0.00012783931504];
% Generate the 2048 samples with fs=8000Hz
fs=8000;                        % Sampling rate
N=2048;                         % Number of samples
L=3;                            % Upsampling factor
n=0:1:N-1;
x=5*sin(n*pi/4)+cos(5*n*pi/8);
```

```
% Upsampling by a factor of L
w=zeros(1,L*N);
for n=0:1:N-1
w(L*n+1)=x(n+1);
end
NL=length(w);                    % Length of the upsampled data
W=2*abs(fft(w,NL))/NL;W(1)=W(1)/2; %Compute one-sided amplitude spectrum
f=[0:1:NL/2-1]*fs*L/NL; % Map the frequency index to the frequency (Hz)
%Interpolation
y=filter(B,1,w); % Apply interpolation filter
Y=2*abs(fft(y,NL))/NL;Y(1)=Y(1)/2; %Compute the one-sided amplitude spectrum
fsL=[0:1:NL/2-1]*fs*L/NL; % Map the frequency index to the frequency (Hz)
subplot(2,1,1);plot(f,W(1:1:NL/2));grid; xlabel('Frequency (Hz)');
subplot(2,1,2);plot(fsL,Y(1:1:NL/2));grid; xlabel('Frequency (Hz)');
```

Now let us study how to design the interpolation filter via Example 11.2.

EXAMPLE 11.2

Given a DSP upsampling system with the following specifications:
Sampling rate = 6000 Hz
Input audio frequency range = 0–800 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 50 dB
Upsample factor $L = 3$,

Determine the FIR filter length, cutoff frequency, and window type if the window design method is used.

Solution:

The specifications are reorganized as follows:
Interpolation filter operating at the sampling rate = 18,000 Hz
Passband frequency range = 0–800 Hz
Stopband frequency range = 3–9 kHz
Passband ripple = 0.02 dB
Stopband attenuation = 50 dB
Filter type: FIR filter
The block diagram and filter frequency specifications are given in Fig. 11.7.
We choose the Hamming window for this application. The normalized transition band is

$$\Delta f = \frac{f_{stop} - f_{pass}}{f_{sL}} = \frac{3000 - 800}{18{,}000} = 0.1222.$$

The length of the filter and the cutoff frequency can be determined by

$$N = \frac{3.3}{\Delta f} = \frac{3.3}{0.1222} = 27,$$

and the cutoff frequency is given by

$$f_c = \frac{f_{pass} + f_{stop}}{2} = \frac{3000 + 800}{2} = 1900\,\text{Hz}.$$

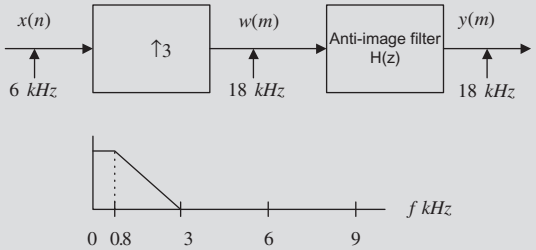


FIG. 11.7

Filter frequency specifications for Example 11.2.

11.1.3 CHANGING SAMPLING RATE BY A NON-INTEGER FACTOR *L/M*

With an understanding of the downsampling and upsampling processes, we now study the sampling rate conversion by a non-integer L/M. This can be viewed as two sampling conversion processes. In step 1, we perform the upsampling process by a factor of integer L following application of an interpolation filter $H_1(z)$; in step 2, we continue filtering the output from the interpolation filter via an anti-aliasing filter $H_2(z)$, and finally execute downsampling. The entire process is illustrated in Fig. 11.8.

Since the interpolation and anti-aliasing filters are in a cascaded form and operate at the same rate, we can select one of them. We choose the one with the lower stop frequency edge and choose the most demanding requirements for passband gain and stopband attenuation for the filter design. A lot of computational savings can be achieved by using one lowpass filter. We illustrate the procedure via the following simulation. Let us generate the signal $x(n)$ by:

$$x(n) = 5\sin\left(\frac{2\pi \times 1000n}{8000}\right) + \cos\left(\frac{2\pi \times 2500n}{8000}\right),$$

with a sampling rate of $f_s = 8000$ Hz and frequencies of 1 and 2.5 kHz.

Now we resample $x(n)$ to 3000 Hz by a non-integer factor of 0.375, that is,

$$\left(\frac{L}{M}\right) = 0.375 = \frac{3}{8}.$$

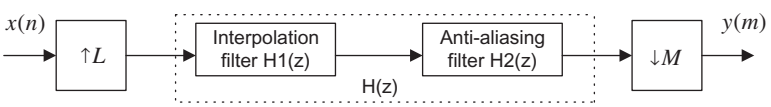


FIG. 11.8

Block diagram for the sampling rate conversion.

Upsampling is at a factor of $L=3$ and the upsampled sequence is filtered by an FIR lowpass filter designed with the filter length $N=53$ and a cutoff frequency of 3250 Hz at the sampling rate of $3 \times 8000 = 24{,}000$ Hz. The spectrum for the upsampled sequence and the spectrum after application of the interpolation filter are plotted in Fig. 11.9A.

The sequence from step 1 can be filtered via another FIR lowpass filter designed with the filter length $N=159$ and a cutoff frequency of 1250 Hz, following by downsampling by a factor of $M=8$. The spectrum after the anti-aliasing filter and the spectrum for the final output $y(m)$ are plotted in Fig. 11.9B. Note that the anti-aliasing filter removes the frequency component of 2.5 kHz to avoid aliasing. This is because after downsampling, the Nyquist limit is 1.5 kHz. As we discussed previously, we can select one filter for implementation. We choose the FIR lowpass filter with $N=159$ and a cutoff frequency of 1250 Hz because its bandwidth is smaller than that of the interpolation filter. The MATLAB implementation is listed in Program 11.3.

Program 11.3. MATLAB program for changing sampling rate with a non-integer factor.

```
close all; clear all;clc;
% Down sampling filter
Bdown=firwd(159,1,2*pi*1250/24000,0,4);
% Generate 2048 samples with fs=8000Hz
fs=8000;                                    % Original sampling rate
N=2048;                                     % The number of samples
L=3;                                        % Upsampling factor
M=8;                                        % Downsampling factor
n=0:1:N-1;                                  % Generate the time index
x=5*sin(n*pi/4)+cos(5*n*pi/8);              % Generate the test signal
% Up sampling by a factor of L
w1=zeros(1,L*N);
for n=0:1:N-1
 w1(L*n+1)=x(n+1);
end
NL=length(w1);                              % Length of up sampled data
W1=2*abs(fft(w1,NL))/NL;W1(1)=W1(1)/2;  % Compute the one-sided
%amplitude spectrum
f=[0:1:NL/2-1]*fs*L/NL; % Map frequency index to its frequency in Hz
subplot(3,1,1);plot(f,W1(1:1:NL/2));grid
xlabel('Frequency (Hz)');
w2=filter(Bdown,1,w1); % Perform the combined anti-aliasing filter
W2=2*abs(fft(w2,NL))/NL;W2(1)=W2(1)/2; % Compute the one-sided
%amplitude spectrum
y2=w2(1:M:NL);
NM=length(y2);                              % Length of the downsampled data
Y2=2*abs(fft(y2,NM))/NM;Y2(1)=Y2(1)/2;% Compute the one-sided
%amplitude spectrum
% Map frequency index to its frequency in Hz before downsampling
fbar=[0:1:NL/2-1]*24,000/NL;
% Map frequency index to its frequency in Hz
fsM=[0:1:NM/2-1]*(fs*L/M)/NM;
subplot(3,1,2);plot(f,W2(1:1:NL/2));grid; xlabel('Frequency (Hz)');
subplot(3,1,3);plot(fsM,Y2(1:1:NM/2));grid; xlabel('Frequency (Hz)');
```

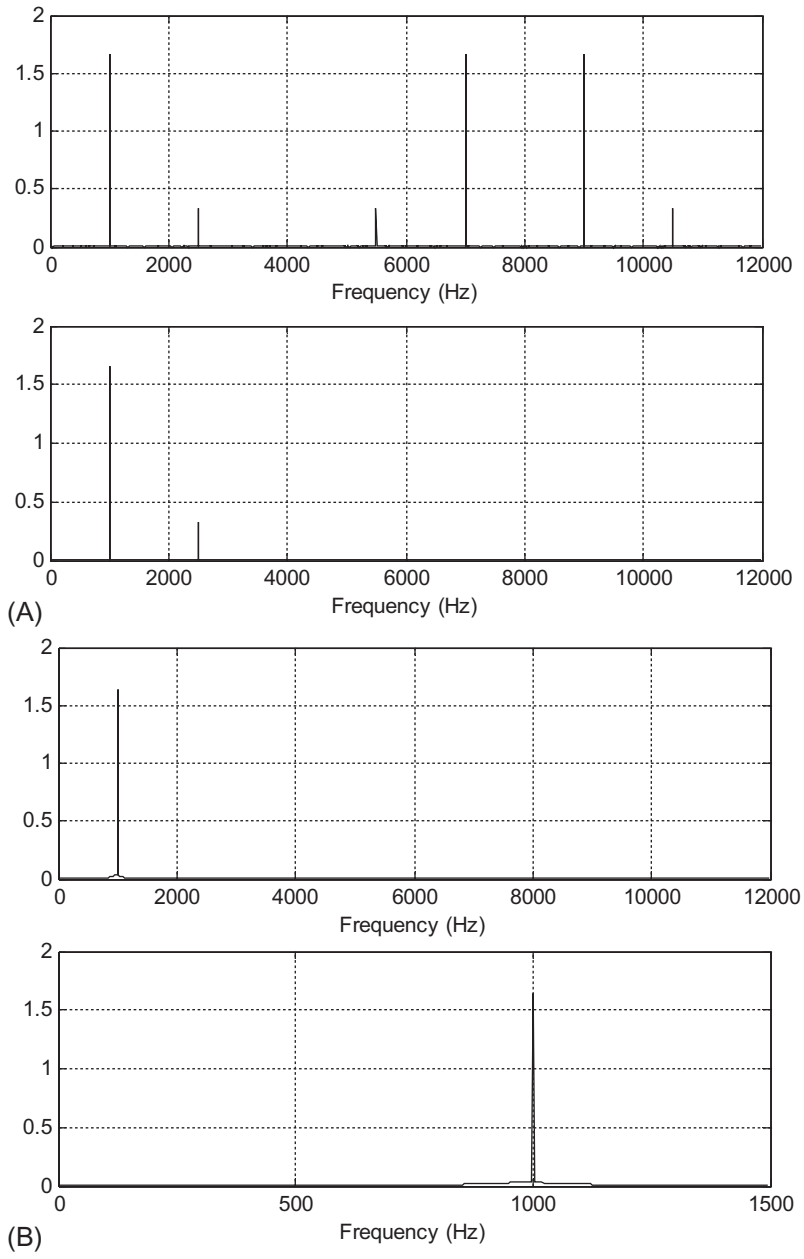


FIG. 11.9

(A) (Top) Spectrum after upsampling and (bottom) spectrum after interpolation filtering. (B) (Top) Spectrum after anti-aliasing filtering and (bottom) spectrum after downsampling.

Therefore, three steps are required to accomplish the process:

1. Upsampling by a factor of $L=3$
2. Filtering the upsampled sequence by an FIR lowpass filter designed with the filter length $N=159$ and a cutoff frequency of 1250 Hz at the sampling rate of $3 \times 8000 = 24{,}000$ Hz
3. Downsampling by a factor of $M=8$.

EXAMPLE 11.3

Given a sampling conversion DSP system (Fig. 11.10A) with the following specifications:
Audio input $x(n)$ is sampled at the rate of 6000 Hz;
Audio output $y(m)$ is operated at the rate of 9000 Hz.
Determine the filter length and cutoff frequency for the combined anti-aliasing filter $H(z)$, and window types, respectively, if the window design method is used.

Solution:

(a) The filter frequency specifications and corresponding block diagram are developed in Fig. 11.10B.

Specifications for the interpolation filter $H_1(z)$:
Passband frequency range = 0–2500 Hz.
Passband ripples for $H_1(z) = 0.04$ dB
Stopband frequency range = 3000–9000 Hz
Stopband attenuation = 42 dB
Specifications for the anti-aliasing filter $H_2(z)$:
Passband frequency range = 0–2500 Hz.
Passband ripples for $H_2(z) = 0.02$ dB
Stopband frequency range = 4500 Hz–9000 Hz
Stopband attenuation = 46 dB
Combined specifications $H(z)$:
Passband frequency range = 0–2500 Hz.
Passband ripples for $H(z) = 0.02$ dB
Stopband frequency range = 3000 Hz–9000 Hz
Stopband attenuation = 46 dB.
We use an FIR filter with a Hamming window. Since

$$\Delta f = \frac{f_{stop} - f_{pass}}{f_{sL}} = \frac{3{,}000 - 2{,}500}{18{,}000} = 0.0278,$$

the length of the filter and the cutoff frequency can be determined by

$$N = \frac{3.3}{\Delta f} = \frac{3.3}{0.0278} = 118.8.$$

we choose $N=119$, and

$$f_c = \frac{f_{pass} + f_{stop}}{2} = \frac{3000 + 2500}{2} = 2750 \text{ Hz}.$$

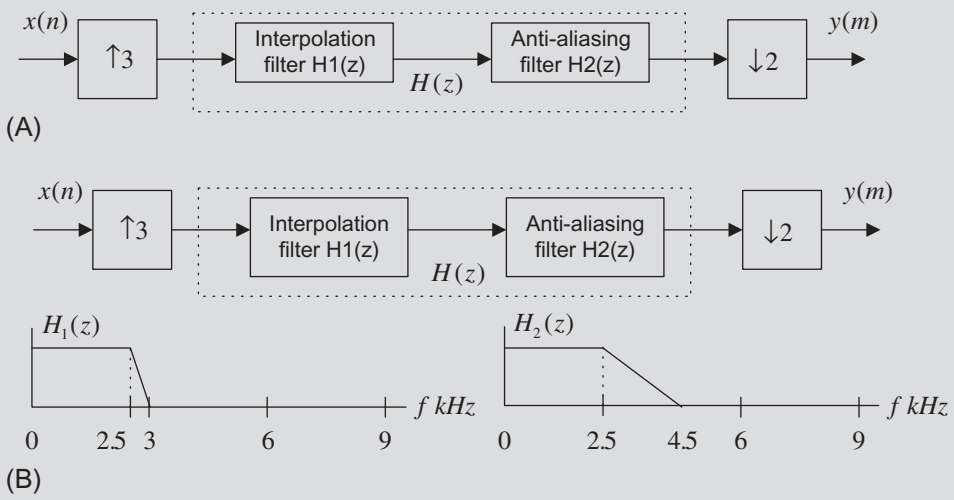


FIG. 11.10

(A) Sampling conversion in Example 11.3. (B) Filter frequency specifications for Example 11.3.

11.1.4 APPLICATION: CD AUDIO PLAYER

In this application example, we discuss principles of the upsampling process and interpolation-filter processes used in the CD audio system to help the reconstruction filter design.

Each raw digital sample recorded on the CD audio system contains 16 bits and is sampled at the rate of 44.1 kHz. Fig. 11.11 describes a portion of one channel of the CD player in terms of a simplified block diagram.

Let us consider the situation without upsampling and application of a digital interpolation filter. We know that the audio signal has a bandwidth of 22.05 kHz, that is, the Nyquist frequency, and digital-to-analog conversion (DAC) produces the sample-and-hold signals that contain the desired audio band and images thereof. To achieve the audio band signal, we need to apply a *reconstruction filter* (also called a smooth filter or anti-image filter) to remove all image frequencies beyond the Nyquist frequency of 22.05 kHz. Due to the requirement of the sharp transition band, a higher-order analog filter design becomes a requirement.

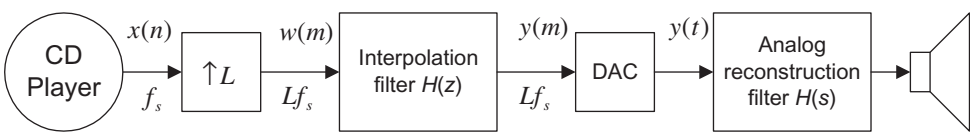


FIG. 11.11

Sample rate conversion in the CD audio player system.

The design of the higher-order analog filter is complex and expensive to implement. In order to relieve such design constraints, we can add the upsampling process before DAC, followed by application of the digital interpolation filter (assume $L=4$). Certainly, the interpolation filter design must satisfy the specifications studied in the previous section on increasing the sampling rate by an integer factor. Again, after digital interpolation, the audio band is kept the same, while the sampling frequency is increased by fourfold ($L=4$), that is, $44.1 \times 4 = 176.4$ kHz.

Since the audio band of 22.05 kHz is now relatively low compared with the new folding frequency (176.4/2 = 88.2 kHz), the use of a simple first-order or second-order analog anti-image filter may be sufficient. Let us look the following simulation.

A test audio signal with a frequency of 16 kHz and a sampling rate of 44.1 kHz are generated using the formula

$$x(n) = \sin\left(\frac{2\pi \times 16{,}000n}{44{,}100}\right).$$

If we use an upsampling factor of 4, then the bandwidth would increase to 88.2 kHz. Based on the audio frequency of 16 kHz, the original Nyquist frequency of 22.05 kHz, and the new sampling rate of 176.4 kHz, we can determine the filter length as

$$\Delta f = \frac{22.05 - 16}{176.4} = 0.0343.$$

Using the Hamming window for FIR filter design leads to

$$N = \frac{3.3}{\Delta f} = 96.2.$$

We choose $N=97$. The cutoff frequency therefore is

$$f_c = \frac{16 + 22.05}{2} = 19.025 \text{ kHz}.$$

The spectrum of the interpolated audio test signal is shown in Fig. 11.12, where the top plot illustrates that after the upsampling, the audio test signal has the frequency of 16 kHz, along with the image frequencies coming from 44.1–16 = 28.1 kHz, 44.1 + 16 = 60.1 kHz, 88.2–16 = 72.2 kHz, and so on. The bottom graph describes the spectrum after the interpolation filter. From lowpass FIR filtering, an interpolated audio signal with a frequency of 16 kHz is observed.

Let us examine the corresponding process in time domain, as shown in Fig. 11.13. The upper left plot shows the original samples. The upper right plot describes the upsampled signals. The lower left plot shows the signals after the upsampling process and digital interpolation filter. Finally, the lower right plot shows the sample-and-hold signals after DAC. Clearly, we can easily design a reconstruction filter to smooth the sample-and-hold signals and obtain the original audio test signal. The advantage of reducing hardware is illustrated. The MATLAB implementation can be seen in Program 11.4.

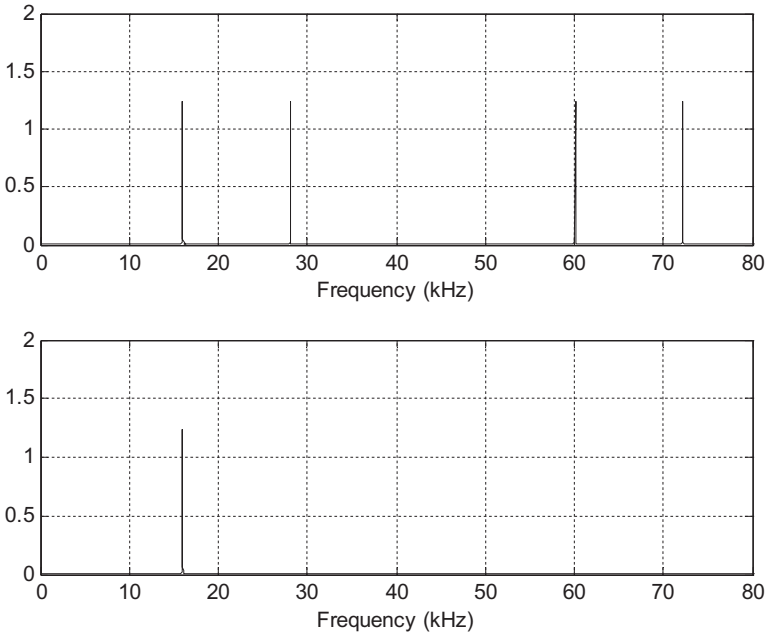


FIG. 11.12

(Top) The spectrum after upsampling and (bottom) the spectrum after applying the interpolation filter.

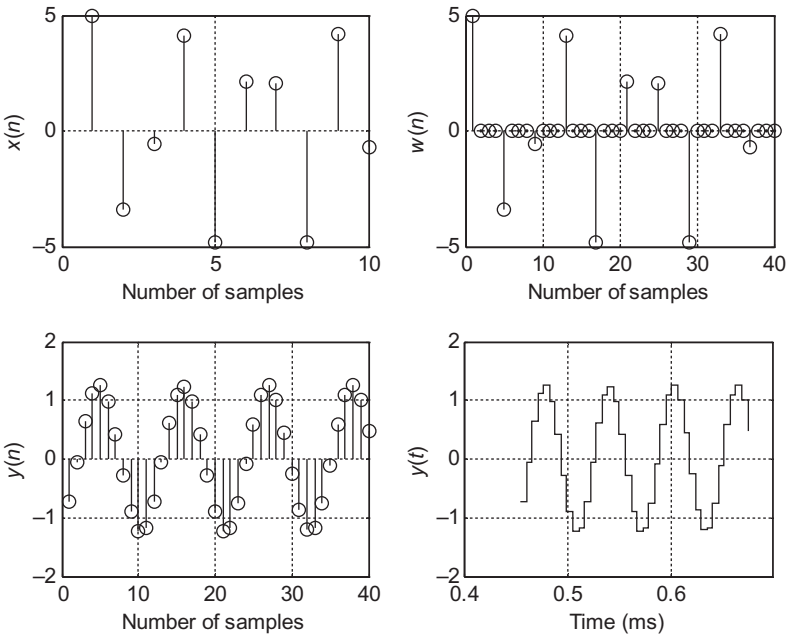


FIG. 11.13

Plots of signals at each stage according to the block diagram in Fig. 11.11.

Program 11.4. MATLAB program for CD player example.

```
close all; clear all; clc
% Generate the 2048 samples with fs=44,100Hz
fs=44,100;                              % Original sampling rate
T=1/fs;                                 % Sampling period
N=2048;                                 % Number of samples
L=4;
fsL=fs*L;                               % Upsampling rate
%Upsampling filter (see Chapter 7 for FIR filter design)
Bup=firwd(97,1,2*19025*pi/fsL,0,4);
n=0:1:N-1;                              % Generate the time indices
x=5*sin(2*pi*16000*n*T);                % Generate the test signal
% Upsampling by a factor of L
w=zeros(1,L*N);
for n=0:1:N-1
w(L*n+1)=x(n+1);
end
NL=length(w);                           % Number of the upsampled data
W=2*abs(fft(w,NL))/NL;W(1)=W(1)/2; % Compute the one-sided
%amplitude spectrum
f=[0:1:NL/2-1]*fs*L/NL; % Map the frequency index to its frequency in Hz
f=f/1000; % Convert to kHz
%Interpolation
y=filter(Bup,1,w);     % Perform the interpolation filter
Y=2*abs(fft(y,NL))/NL;Y(1)=Y(1)/2; % Compute the one-sided
%amplitude spectrum
subplot(2,1,1);plot(f,W(1:1:NL/2));grid;
xlabel('Frequency (kHz)'); axis([0 f(length(f)) 0 2]);
subplot(2,1,2);plot(f,Y(1:1:NL/2));grid;
xlabel('Frequency (kHz)');axis([0 f(length(f)) 0 2]);
figure
subplot(2,2,1);stem(x(21:30));grid
xlabel('Number of Samples');ylabel('x(n)');
subplot(2,2,2);stem(w(81:120));grid
xlabel('Number of Samples'); ylabel('w(n)');
subplot(2,2,3);stem(y(81:120));grid
xlabel('Number of Samples'); ylabel('y(n)')
subplot(2,2,4);stairs([80:1:119]*1000*T,y(81:120));grid
xlabel('Time (ms)'); ylabel('y(t)')
```

11.1.5 MULTISTAGE DECIMATION

The multistage approach for the downsampling rate conversion can be used to dramatically reduce the anti-aliasing filter length. Fig. 11.14 describes a two-stage decimator.

As shown in Fig. 11.14, a total decimation factor is $M = M_1 \times M_2$. Here, even though we develop a procedure for a two-stage case, a similar principle can be applied to general multistage cases.

Using the two-stage decimation in Fig. 11.15, the final Nyquist limit is $\frac{f_s}{2M}$ after final downsampling. So our useful information bandwidth should stop at the frequency edge of $\frac{f_s}{2M}$. Next, we need to

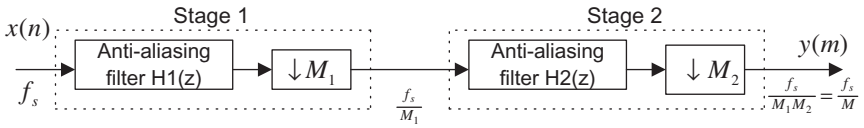


FIG. 11.14

Multistage decimation.

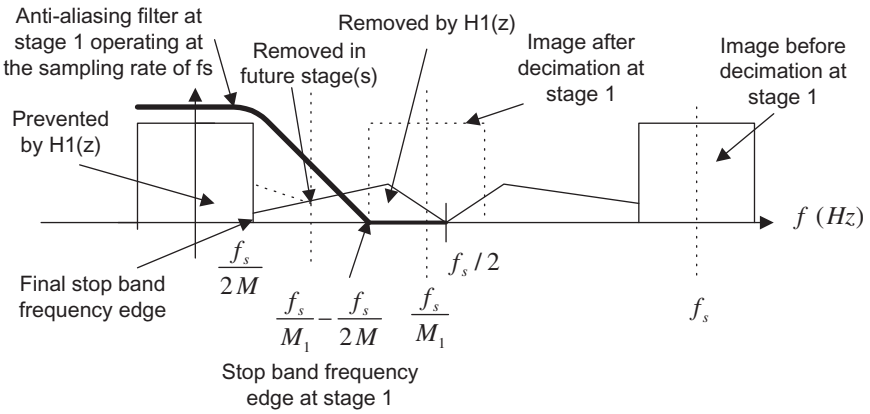


FIG. 11.15

Stopband frequency edge for the anti-aliasing filter at stage 1 for the two-stage decimation.

determine the stop frequency edge for the anti-aliasing lowpass filter at stage 1 before the first decimation process begins. This stop frequency edge is actually the lower frequency edge of the first image replica centered at the sampling frequency of $\frac{f_s}{M_1}$ after the stage 1 decimation. This lower frequency edge of the first image replica is then determined by

$$\frac{f_s}{M_1} - \frac{f_s}{2M}.$$

After downsampling, we expect that some frequency components from $\frac{f_s}{2M_1}$ to $\frac{f_s}{M_1} - \frac{f_s}{2M}$ to be folded over to the frequency band between $\frac{f_s}{2M}$ and $\frac{f_s}{2M_1}$. However, these aliased frequency components do not affect the final useful band between 0 Hz to $\frac{f_s}{2M}$ and will be removed by the anti-aliasing filter(s) in the future stage(s). As illustrated in Fig. 11.15, any frequency components beyond the edge $\frac{f_s}{M_1} - \frac{f_s}{2M}$ can fold over into the final useful information band to create aliasing distortion. Therefore, we can use this frequency as the lower stop frequency edge of the anti-aliasing filter to prevent the aliasing distortion at the final stage. The upper stopband edge (Nyquist limit) for the anti-image filter at stage 1 is clearly $\frac{f_s}{2}$, since the filter operates at f_s samples per second. So the stopband frequency range at stage 1 is therefore from $\frac{f_s}{M_1} - \frac{f_s}{2M}$ to $\frac{f_s}{2}$. The aliasing distortion, introduced into the frequency band from $\frac{f_s}{2M}$ to $\frac{f_s}{2M_1}$, will be filtered out after future decimation stage(s).

Similarly, for stage 2, the lower frequency edge of the first image developed after stage 2 downsampling is

$$\frac{f_s}{M_1 M_2} - \frac{f_s}{2M} = \frac{f_s}{2M}.$$

As is evident in our two-stage scheme, the stopband frequency range for the second anti-aliasing filter at stage 2 should be from $\frac{f_s}{2M}$ to $\frac{f_s}{2M_1}$.

We summarize specifications for the two-stage decimation as follows:

Filter requirement for stage 1:

- Passband frequency range = 0 to f_p
- Stopband frequency range = $\frac{f_s}{M_1} - \frac{f_s}{2M}$ to $\frac{f_s}{2}$
- Passband ripple = $\delta_p/2$, where δ_p is the combined absolute ripple on passband
- Stopband attenuation = δ_s

Filter requirement for stage 2:

- Passband frequency range = 0 to f_p
- Stopband frequency range = $\frac{f_s}{M_1 \times M_2} - \frac{f_s}{2M}$ to $\frac{f_s}{2M_1}$
- Passband ripple = $\delta_p/2$, where δ_p is the combined absolute ripple on passband
- Stopband attenuation = δ_s.

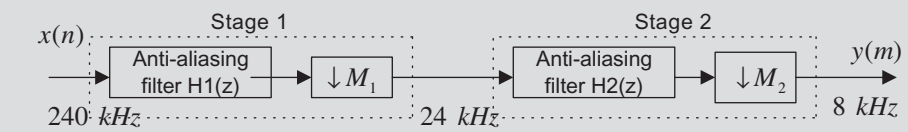
Example 11.4 illustrates the two-stage decimator design.

EXAMPLE 11.4

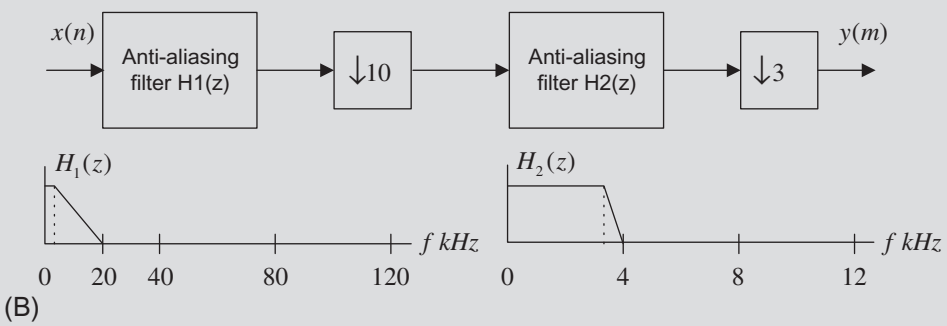
Determine the anti-aliasing FIR filter lengths and cutoff frequencies for the two-stage decimator with the following specifications and block diagram (Fig. 11.16A):

Original sampling rate: $f_s = 240$ kHz

Audio frequency range: 0–3400 Hz



(A)



(B)

FIG. 11.16

(A) Multistage decimation in Example 11.4. (B) Filter frequency specifications for Example 11.4.

Passband ripple: $\delta_p = 0.05$ (absolute)
Stopband attenuation: $\delta_s = 0.005$ (absolute)
FIR filter design using the window method
New sampling rage: $f_{sM} = 8$ kHz

Solution:

$$M = \frac{240\text{kHz}}{8\text{kHz}} = 30 = 10 \times 3$$

We choose $M_1 = 10$ and $M_2 = 3$; there could be other choices. Fig. 11.16B shows the block diagram and filter frequency specifications.
Filter specification for $H_1(z)$:
Passband frequency range: 0–3400 Hz.
Passband ripples: 0.05/2 = 0.025 ($\delta_p\ dB = 20\log_{10}(1+\delta_p)$ =0.212 dB)
Stopband frequency range: 20,000–120,000 Hz
Stopband attenuation: 0.005, $\delta_s\ dB = -20 \times \log_{10}(\delta_s) = 46$ dB
Filter type: FIR, Hamming window.
Note that the lower stopband edge can be determined as

$$f_{stop} = \frac{f_s}{M_1} - \frac{f_s}{2 \times M} = \frac{240{,}000}{10} - \frac{240{,}000}{2 \times 30} = 20{,}000Hz$$

$$\Delta f = \frac{f_{stop} - f_{pass}}{f_s} = \frac{20{,}000 - 3400}{240{,}000} = 0.06917.$$

The length of the filter and the cutoff frequency can be determined by

$$N = \frac{3.3}{\Delta f} = 47.7.$$

We choose $N = 49$, and

$$f_c = \frac{f_{pass} + f_{stop}}{2} = \frac{20{,}000 + 3400}{2} = 11{,}700\,\text{Hz}.$$

Filter specification for $H_2(z)$:
Passband frequency range: 0–3400 Hz.
Passband ripples: 0.05/2 = 0.025 (0.212 dB)
Stopband frequency range: 4000–12,000 Hz
Stopband attenuation: 0.005, $\delta_s\ dB = 46$ dB
Filter type: FIR, Hamming window
Note that

$$\Delta f = \frac{f_{stop} - f_{pass}}{f_{sM1}} = \frac{4000 - 3400}{24{,}000} = 0.025.$$

The length of the filter and the cutoff frequency can be determined by

Continued

EXAMPLE 11.4—CONT'D

$$N = \frac{3.3}{\Delta f} = 132.$$

We choose $N = 133$, and

$$f_c = \frac{f_{pass} + f_{stop}}{2} = \frac{4000 + 3400}{2} = 3700\,\text{Hz}.$$

Reader can verify for the case by using one-stage with a decimation factor of $M = 30$. Using the Hamming window for the FIR filter, the resulting number of taps is 1321, and the cutoff frequency is 3700 Hz. Thus, such a filter requires a huge number of computations and causes a large delay during implementation compared with the two-stage case.

The multistage scheme is very helpful for the sampling rate conversion between audio systems. For example, to convert the CD audio at the sampling rate of 44.1 kHz to the MP3 or digital audio type system (professional audio rate), in which the sampling rate of 48 kHz is used, the conversion factor $L/M = 48/44.1 = 160/147$ is required. Using the single-stage scheme may cause impractical FIR filter sizes for interpolation and downsampling. However, since $L/M = 160/147 = (4/3)(8/7)(5/7)$, we may design an efficient 3-stage system, in which stages 1, 2, and 3 use the conversion factors as $L/M = 8/7$, $L/M = 5/7$, and $L/M = 4/3$, respectively.

11.2 POLYPHASE FILTER STRUCTURE AND IMPLEMENTATION

Due to the nature of the decimation and interpolation processes, polyphase filter structures can be developed to efficiently implement the decimation and interpolation filters (using fewer number of multiplications and additions). As we will explain, these filters are all-pass filters with different phase shifts (Proakis and Manolakis, 2007), thus we call them *polyphase filters*.

Here, we skip their derivations and illustrate implementations of decimation and interpolation using simple examples. Consider the interpolation process shown in Fig. 11.17, where $L = 2$.

We assume that the FIR interpolation filter has four taps, shown as

$$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3}$$

and the filter output is

$$y(m) = h(0)w(m) + h(1)w(m-1) + h(2)w(m-2) + h(3)w(m-3).$$

For the purpose of comparison, the direct interpolation process shown in Fig. 11.17 is summarized in Table 11.1, where $w(m)$ is the upsampled signal and $y(m)$ the interpolated output. Processing each input

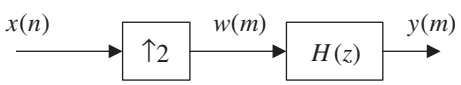


FIG. 11.17

Upsampling by a factor of 2 and a four-tap interpolation filter.

Table 11.1 Results of the Direct Interpolation Process in Fig. 11.17 (8 Multiplications and 6 Additions for Processing Each Input Sample $x(n)$)

n	$x(n)$	m	$w(m)$	$y(m)$
$n=0$	$x(0)$	$m=0$	$w(0)=x(0)$	$y(0)=h(0)x(0)$
		$m=1$	$w(1)=0$	$y(1)=h(1)x(0)$
$n=1$	$x(1)$	$m=2$	$w(2)=x(1)$	$y(2)=h(0)x(1)+h(2)x(0)$
		$m=3$	$w(3)=0$	$y(3)=h(1)x(1)+h(3)x(0)$
$n=2$	$x(2)$	$m=4$	$w(4)=x(2)$	$y(4)=h(0)x(2)+h(2)x(1)$
		$m=5$	$w(5)=0$	$y(5)=h(1)x(2)+h(3)x(1)$
…	…	…	…	…

sample $x(n)$ requires applying the difference equation twice to obtain $y(0)$ and $y(1)$. Hence, for this example, we need eight multiplications and six additions.

The output results in Table 11.1 can be easily obtained using the polyphase filters shown in Fig. 11.18.

In general, there are L polyphase filters. Having the designed interpolation filter $H(z)$ of N taps, we can determine each bank of filter coefficients as follows:

$$\rho_k(n)=h(k+nL)\text{ for }k=0,1,\ldots,L-1\text{ and }n=0,1,\ldots,\frac{N}{L}-1. \tag{11.12}$$

For our example, $L=2$ and $N=4$, we have $L-1=1$, and $N/L-1=1$, respectively. Hence, there are two filter banks, $\rho_0(z)$ and $\rho_1(z)$, each having a length of 2, as illustrated in Fig. 11.18. When $k=0$ and $n=1$, the upper limit of time index required for $h(k+nL)$ is $k+nL=0+1\times 2=2$. When $k=1$ and $n=1$, the upper limit of the time index for $h(k+nL)$ is 3. Hence, the first filter $\rho_0(z)$ has the coefficients $h(0)$ and $h(2)$. Similarly, the second filter $\rho_1(z)$ has coefficients $h(1)$ and $h(3)$. In fact, the filter coefficients of $\rho_0(z)$ are a decimated version of $h(n)$ starting at $k=0$, while the filter coefficients of $\rho_1(z)$ are a decimated version of $h(n)$ starting at $k=1$, and so on.

As shown in Fig. 11.18, we can reduce the computational complexity from eight multiplications and six additions down to four multiplications and three additions for processing each input sample $x(n)$. Generally, the computation can be reduced by a factor of L as compared with the direct process.

The commutative model for the polyphase interpolation filter is given in Fig. 11.19.

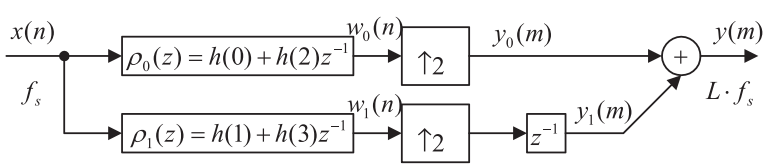


FIG. 11.18

Polyphase filter implementation for the interpolation in Fig. 11.17 (4 multiplications and 3 additions for processing each input sample $x(n)$).

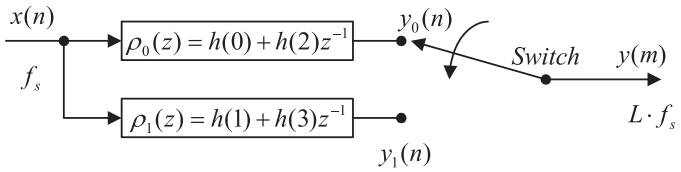


FIG. 11.19

Commutative model for the polyphase interpolation filter.

EXAMPLE 11.5

Verify $y(1)$ in Table 11.1 using the polyphase filter implementation in Figs. 11.18 and 11.19, respectively.

Solution:

Applying the polyphase interpolation filter as shown in Fig. 11.18 leads to

$$w_0(n) = h(0)x(n) + h(2)x(n-1)$$
$$w_1(n) = h(1)x(n) + h(3)x(n-1);$$

when $n = 0$,

$$w_0(0) = h(0)x(0)$$
$$w_1(0) = h(1)x(0).$$

After interpolation, we have

$$y_0(m) : w_0(0)\ 0\ \cdots$$

and

$$y_1(m) : 0\ \ w_1(0)\ \ 0\ \ \cdots$$

Note: there is a unit delay for the second filter bank. Hence

$$m = 0, y_0(0) = h(0)x(0), y_1(0) = 0$$
$$m = 1, y_0(1) = 0, y_1(1) = h(1)x(0).$$

Combining two channels, we finally get

$$m = 0, y(0) = y_0(0) + y_1(0) = h(0)x(0),$$
$$m = 1, y(1) = y_0(1) + y_1(1) = h(1)x(0).$$

Therefore, $y(1)$ matches that in the direct interpolation process given in Table 11.1.

Applying the polyphase interpolation filter using the commutative model in Fig. 11.19, we have

$$y_0(n) = h(0)x(n) + h(2)x(n-1)$$
$$y_1(n) = h(1)x(n) + h(3)x(n-1);$$

when $n=0$,

$$m=0, y(0)=y_0(0)=h(0)x(0)+h(2)x(-1)=h(0)x(0)$$

$$m=1, y(1)=y_1(0)=h(1)x(0)+h(3)x(-1)=h(1)x(0).$$

Clearly, $y(1)=h(1)x(0)$ matches $y(1)$ the result in Table 11.1.

Next, we explain the properties of polyphase filters (i.e., they have all-pass gain and possible different phases). Each polyphase filter $\rho_k(n)$ operating at the original sampling rate f_s (assuming 8 kHz) is a downsampled version of the interpolation filter $h(n)$ operating at the upsampling rate Lf_s (32 kHz assuming an interpolation factor of $L=4$). Considering that the designed interpolation FIR filter coefficients $h(n)$ are the impulse response sequence with a flat frequency spectrum up to a bandwidth of $f_s/2$ (assume a bandwidth of 4 kHz with a perfect flat frequency magnitude response, theoretically) at a sampling rate of Lf_s (32 kHz), we then downsample $h(n)$ to obtain polyphase filters by a factor of $L=4$ and operate them at a sampling rate of f_s (8 kHz).

The Nyquist frequency after downsampling should be $(Lf_s/2)/L=f_s/2$ (4 kHz); at the same time, each downsampled sequence $\rho_k(n)$ operating at f_s (8 kHz) has a flat spectrum up to $f_s/2$ (4 kHz) due to the $f_s/2$ (4 kHz) bandlimited sequence of $h(n)$ at the sampling rate of Lf_s (32 kHz). Hence, all of the polyphase filters are all-pass filters. Since each polyphase $\rho_k(n)$ filter has different coefficients, each may have a different phase. Therefore, these polyphase filters are the all-pass filters having possible different phases, theoretically.

Next, consider the following decimation process in Fig. 11.20.

Assuming a three-tap decimation filter, we have

$$H(z)=h(0)+h(1)z^{-1}+h(2)z^{-2}$$

$$w(n)=h(0)x(n)+h(1)x(n-1)+h(2)x(n-2).$$

The direct decimation process is shown in Table 11.2 for the purpose of comparison. Obtaining each output $y(m)$ requires processing filter difference equations twice, resulting in six multiplications and four additions for this particular example.

The efficient way to implement a polyphase filter is given in Fig. 11.21.

Similarly, there are M polyphase filters. With the designed decimation filter $H(z)$ of N taps, we can obtain filter bank coefficients by

$$\rho_k(n)=h(k+nM) \text{ for } k=0,1,\cdots,M-1 \text{ and } n=0,1,\cdots,\frac{N}{M}-1. \tag{11.13}$$

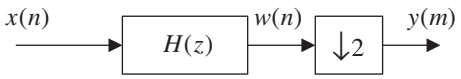


FIG. 11.20

Decimation by a factor of 2 and a three-tap anti-aliasing filter.

Table 11.2 Results of Direct Decimation Process in Fig. 11.20 (6 Multiplications and 4 Additions for Obtaining Each Output $y(m)$).

n	$x(n)$	$w(n)$	m	$y(m)$
$n=0$	$x(0)$	$w(0)=h(0)x(0)$	$m=0$	$y(0)=h(0)x(0)$
$n=1$	$x(1)$	$w(1)=h(0)x(1)+h(1)x(0)$ discard		
$n=2$	$x(2)$	$w(2)=h(0)x(2)+h(1)x(1)+h(2)x(0)$	$m=1$	$y(1)=h(0)x(2)+h(1)x(1)+h(2)x(0)$
$n=3$	$x(3)$	$w(3)=h(0)x(3)+h(1)x(2)+h(2)x(1)$ discard		
$n=4$	$x(5)$	$w(4)=h(0)x(4)+h(1)x(3)+h(2)x(2)$	$m=2$	$y(2)=h(0)x(4)+h(1)x(3)+h(2)x(2)$
$n=5$	$x(6)$	$w(5)=h(0)x(5)+h(1)x(4)+h(2)x(3)$ discard		
…	…	…	…	…

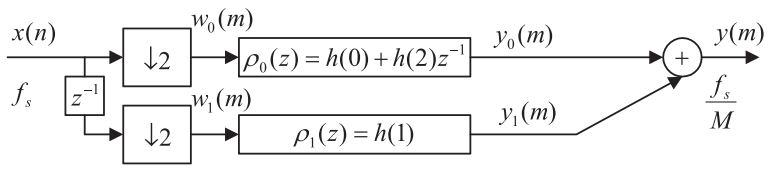


FIG. 11.21

Polyphase filter implementation for the decimation in Fig. 11.20 (3 multiplications and 1 addition for obtaining each output $y(m)$).

For our example, we see that $M-1=1$ and $N/M-1=1$(roundedup). Thus, we have two filter banks. Since $k=0$ and $n=1$, $k+nM=0+1\times 2=2$. The time index upper limit required for $h(k+nM)$ is 2 for the first filter bank $\rho_0(z)$. Hence $\rho_0(z)$ has filter coefficients $h(0)$ and $h(2)$. However, when $k=1$ and $n=1$, $k+nM=1+1\times 2=3$, the time index upper limit required for $h(k+nM)$ is 3 for the second filter bank, and the corresponding filter coefficients are required to be $h(1)$ and $h(3)$. Since our direct interpolation filter $h(n)$ does not contain the coefficient $h(3)$, we set $h(3)=0$ to get the second filter bank with one tap only, as shown in Fig. 11.21. Also as shown in that figure, achieving each $y(m)$ needs three multiplications and one addition. In general, the number of multiplications is reduced by a factor of M.

The commutative model for the polyphase decimator is shown in Fig. 11.22.

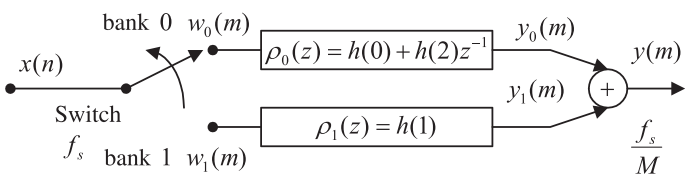


FIG. 11.22

Commutative model for the polyphase decimation filter.

EXAMPLE 11.6

Verify $y(1)$ in Table 11.2 using the polyphase decimation filter implementation in Fig. 11.21.

Solution:

Using Fig. 11.21, we write the difference equations as

$$y_0(m) = h(0)w_0(m) + h(2)w_0(m-1)$$

$$y_1(m) = h(1)w_1(m).$$

Assume $n=0$, $n=1$, $n=2$, and $n=3$, we have the inputs as $x(0)$, $x(1)$, $x(2)$, and $x(3)$, and.
$w_0(m)$: $x(0)$ $x(2)$ $\cdots$
Delaying $x(n)$ by one sample and decimate it by a factor or 2 leads to.
$w_1(m)$: 0 $x(1)$ $x(3)$ $\cdots$.
Hence, applying the filter banks yields the following:
$m=0$, we have inputs for each filter as

$$w_0(0) = x(0) \text{ and } w_1(0) = 0$$

then

$$y_0(0) = h(0)w_0(0) + h(2)w_0(-1) = h(0)x(0)$$

$$y_1(0) = h(1)w_1(0) = h(1) \times 0 = 0.$$

Combining two channels, we obtain

$$y(1) = y_0(1) + y_1(1) = h(0)x(0) + 0 = h(0)x(0)$$

$m=1$, we get inputs for each filter as

$$w_0(1) = x(2) \text{ and } w_1(1) = x(1),$$

then

$$y_0(1) = h(0)w_0(1) + h(2)w_0(0) = h(0)x(2) + h(2)x(0)$$

$$y_1(1) = h(1)w_1(1) = h(1)x(1).$$

Combining two channels leads to

$$y(1) = y_0(1) + y_1(1) = h(0)x(2) + h(2)x(0) + h(1)x(1).$$

We note that $y(1)$ is the same as that shown in Table 11.2. Similar analysis can be done for the commutative model shown in Fig. 11.22.

Program 11.5 demonstrates the polyphase implementation of decimation. The program is modified based on Program 11.1.

Program 11.5. Decimation using polyphase implementation.

```
close all; clear all;
% Downsampling filter (see Chapter 7 for FIR filter design)
B =[0.00074961181416  0.00247663033476  0.00146938649416  -0.00440446121505 ...
-0.00910635730662  0.00000000000000  0.02035676831506  0.02233710562885...
-0.01712963672810  -0.06376620649567  -0.03590670035210  0.10660980550088...
0.29014909103794  0.37500000000000  0.29014909103794  0.10660980550088...
-0.03590670035210  -0.06376620649567  -0.01712963672810  0.02233710562885...
0.02035676831506  0.00000000000000  -0.00910635730662  -0.00440446121505...
0.00146938649416 0.00247663033476 0.00074961181416];
% Generate 2048 samples
fs=8000;                                                    % Sampling rate
N=2048;                                                     % Number of samples
M=2;                                                        % Down sample factor
n=0:1:N-1;
x=5*sin(n*pi/4)+cos(5*n*pi/8);
% Compute the single-sided amplitude spectrum
% AC component will be doubled, and DC component will be kept the
% same value
X=2*abs(fft(x,N))/N;X(1)=X(1)/2;
% Map the frequency index up to the folding frequency in Hz
f=[0:1:N/2-1]*fs/N;
% Decimation
w0=x(1:M:N); p0=B(1:2:length(B)); % Downsampling
w1=filter([0 1],1,x); % Delay one sample
w1=w1(1:M:N); p1=B(2:M:length(B)) % Downsampling
y=filter(p0,1,w0)+filter(p1,1,w1);
NM=length(y);                                          % Length of the downsampled data
% Compute the single-sided amplitude spectrum for the downsampled
% signal
Y=2*abs(fft(y,NM))/NM;Y(1)=Y(1)/2;
% Map the frequency index to the frequency in Hz
fsM=[0:1:NM/2-1]*(fs/M)/NM;
% Plot spectra
subplot(2,1,1);plot(f,X(1:1:N/2));grid; xlabel('Frequency (Hz)');
subplot(2,1,2);plot(fsM,Y(1:1:NM/2));grid; xlabel('Frequency (Hz)');
```

Program 11.6 demonstrates polyphase implementation of interpolation using the information in Program 11.2.

Program 11.6. Interpolation using polyphase implementation.

```
close all; clear all
%Upsampling filter (see Chapter 7 for FIR filter design)
B =[-0.00012783931504  0.00069976044649  0.00123831516738  0.00100277549136...
  -0.00025059018468  -0.00203448515158 -0.00300830295487 -0.00174101657599...
  0.00188598835011    0.00578414933758  0.00649330625041  0.00177982369523...
  -0.00670672686935  -0.01319379342716 -0.01116855281442  0.00123034314117...
  0.01775600060894    0.02614700427364  0.01594155162392 -0.01235169936557...
  -0.04334322148505  -0.05244745563466 -0.01951094855292  0.05718573279009...
```

```
0.15568416401644   0.23851539047347   0.27083333333333   0.23851539047347...
0.15568416401644   0.05718573279009  -0.01951094855292  -0.05244745563466...
-0.04334322148505 -0.01235169936557   0.01594155162392   0.02614700427364...
0.01775600060894   0.00123034314117  -0.01116855281442  -0.01319379342716...
-0.00670672686935  0.00177982369523   0.00649330625041   0.00578414933758...
0.00188598835011  -0.00174101657599  -0.00300830295487  -0.00203448515158...
-0.00025059018468  0.00100277549136   0.00123831516738   0.00069976044649...
-0.00012783931504];
% Generate 2048 samples with fs=8000Hz
fs=8000;                                                    % Sampling rate
N=2048;                                                     % Number of samples
L = 3;                                                      % Upsampling factor
n=0:1:N-1;
x=5*sin(n*pi/4)+cos(5*n*pi/8);
p0=B(1:L:length(B)); p1=B(2:L:length(B)); p2=B(3:L:length(B));
% Interpolation
w0=filter(p0,1,x);
w1=filter(p1,1,x);
w2=filter(p2,1,x);
y0=zeros(1,L*N);y0(1:L:length(y0))=w0;
y1=zeros(1,L*N);y1(1:L:length(y1))=w1;
y1=filter([0 1],1,y1);
y2=zeros(1,L*N);y2(1:L:length(y2))=w2;
y2=filter([001],1,y2);
y=y0+y1+y2; % Interpolated signal
NL = length(y);                                          % Length of the upsampled data
X=2*abs(fft(x,N))/N;X(1)=X(1)/2; %Compute the one-sided amplitude
% spectrum
f=[0:1:N/2-1]*fs/N; % Map the frequency index to the frequency (Hz)
Y=2*abs(fft(y,NL))/NL;Y(1)=Y(1)/2; %Compute the one-sided amplitude %spectrum
fsL=[0:1:NL/2-1]*fs*L/NL; % Map the frequency index to the frequency %(Hz)
subplot(2,1,1);plot(f,X(1:1:N/2));grid; xlabel('Frequency (Hz)');
subplot(2,1,2);plot(fsL,Y(1:1:NL/2));grid; xlabel('Frequency (Hz)');
```

Note that wavelet transform and subband coding are also in the area of multirate signal processing. We discuss these subjects in Chapter 12.

11.3 OVERSAMPLING OF ANALOG-TO-DIGITAL CONVERSION

Oversampling of the analog signal has become more popular in DSP industry to improve resolution of analog-to-digital conversion (ADC). Oversampling uses a sampling rate, which is much higher than the Nyquist rate. We can define an oversampling ratio as

$$\frac{f_s}{2f_{\max}} >> 1. \tag{11.14}$$

The benefits from an oversampling ADC include:

1. helping to design a simple analog anti-aliasing filter before ADC, and
2. reducing the ADC noise floor with possible noise shaping so that a low resolution ADC can be used.

11.3.1 OVERSAMPLING AND ADC RESOLUTION

To begin with developing the relation between oversampling and ADC resolution, we first summarize the regular ADC and some useful definitions discussed in Chapter 2:

$$\text{Quantization noise power} = \sigma_q^2 = \frac{\Delta^2}{12} \tag{11.15}$$

$$\text{Quantization step} = \Delta = \frac{A}{2^n} \tag{11.16}$$

$A =$ full range of the analog signal to be digitized
$n =$ number of bits per sample (ADC resolution).

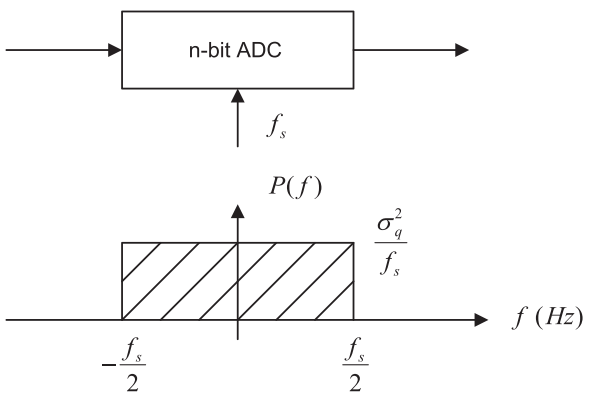
Substituting Eq. (11.16) into Eq. (11.15), we have:

$$\text{Quantization noise power} = \sigma_q^2 = \frac{A^2}{12} \times 2^{-2n} \tag{11.17}$$

The power spectral density of the quantization noise with an assumption of uniform probability distribution is shown in Fig. 11.23. Note that this assumption is true for quantizing a uniformly distributed signal in a full range with a sufficiently long duration. It is not generally true in practice, see research papers authored by Lipshiz et al. (1992) and Maher (1992). However, using the assumption will guide us for some useful results for oversampling systems.

The quantization noise power is the area obtained from integrating the power spectral density function in the range from $-f_s/2$ to $f_s/2$. Now let us examine the oversampling ADC, where the sampling rate is much bigger than that of the regular ADC; that is $f_s >> 2f_{\max}$. The scheme is shown in Fig. 11.24.

As we can see, oversampling can reduce the level of noise power spectral density. After the decimation process with the decimation filter, only a portion of quantization noise power in the range from $-f_{\max}$ and $f_{\max}$ is kept in the DSP system. We call this an *in-band frequency range*.



$$\sigma_q^2 = \int_{-\infty}^{\infty} P(f)df = area = \frac{\sigma_q^2}{f_s} \times (f_s)$$

FIG. 11.23

Regular ADC system.

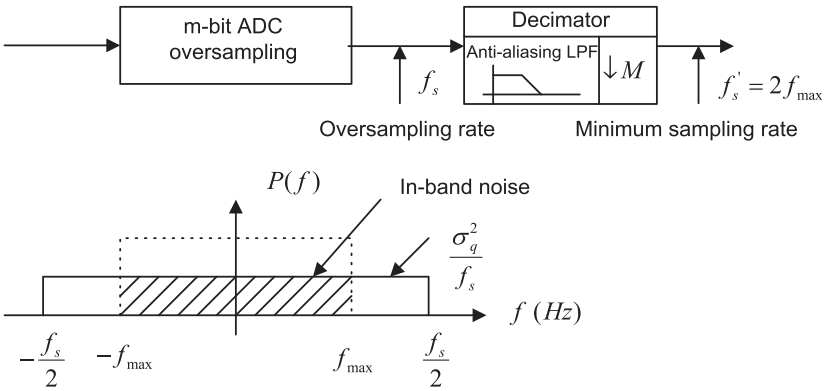


FIG. 11.24

Oversampling ADC system.

In Fig. 11.24, the shaded area, which is the quantization noise power, is given by

$$\text{Quantization Noise Power} = \int_{-\infty}^{\infty} P(f)df = \frac{2f_{\max}}{f_s} \times \sigma_q^2 = \frac{2f_{\max}}{f_s} \times \frac{A^2}{12} \times 2^{-2m}. \tag{11.18}$$

Assuming that the regular ADC shown in Fig. 11.23 and the oversampling ADC shown in Fig. 11.24 are equivalent, we set their quantization noise powers to be the same to obtain

$$\frac{A^2}{12} \times 2^{-2n} = \frac{2f_{\max}}{f_s} \times \frac{A^2}{12} \times 2^{-2m}. \tag{11.19}$$

Eq. (11.19) leads to two useful equations for applications:

$$n = m + 0.5 \times \log_2\left(\frac{f_s}{2f_{\max}}\right) \text{ and} \tag{11.20}$$

$$f_s = 2f_{\max} \times 2^{2(n-m)}, \tag{11.21}$$

where

f_s = sampling rate in the oversampling DSP system,

$f_{\max}$ = maximum frequency of the analog signal,

m = number of bits per sample in the oversampling DSP system,

n = number of bits per sample in the regular DSP system using the minimum sampling rate.

From Eq. (11.20) and given the number of bits (m) used in the oversampling scheme, we can determine the number of bits per sample equivalent to the regular ADC. On the other hand, given the number of bits in the oversampling ADC, we can determine the required oversampling rate so that the oversampling ADC is equivalent to the regular ADC with the larger number of bits per sample (n). Let us look at the following examples.

EXAMPLE 11.7

Given an oversampling audio DSP system with maximum audio input frequency of 20 kHz and ADC resolution of 14 bits, determine the oversampling rate to improve the ADC resolution to 16-bit resolution.

Solution:

Based on the specifications, we have

$$f_{\max} = 20\,\text{kHz}, m = 14\,\text{bits and}\, n = 16\,\text{bits}.$$

Using Eq. (11.21) leads to

$$f_s = 2f_{\max} \times 2^{2(n-m)} = 2 \times 20 \times 2^{2(16-14)} = 640\,\text{kHz}.$$

Since $fs/(2f_{\max}) = 2^4$, we see that each doubling of the minimum sampling rate ($2f_{\max} = 40$ kHz) will increase the resolution by a half bit.

EXAMPLE 11.8

Given an oversampling audio DSP system with the following specifications:
Maximum audio input frequency = 4 kHz
ADC resolution = 8 bits.
Sampling rate = 80 MHz,
Determine the equivalent ADC resolution.

Solution:

Since $f_{\max} = 4$ kHz, $f_s = 80$ kHz, and $m = 8$ bits, applying Eq. (11.20) yields

$$n = m + 0.5 \times \log_2\left(\frac{f_s}{2f_{\max}}\right) = 8 + 0.5 \times \log_2\left(\frac{80{,}000\text{kHz}}{2 \times 4\text{kHz}}\right) \approx 15\,\text{bits}.$$

The MATLAB program shown in Program 11.7 validates the oversampling technique. We consider the following signal

$$x(t) = 1.5\sin(2\pi \times 150t) + 0.9\sin(2\pi \times 175t + \pi/6) + 0.6\sin(2\pi \times 200t + \pi/4) \tag{11.22}$$

with a regular sampling rate of 1 kHz. The oversampling rate of 4 kHz is used and each sample is quantized using a 3-bit code. The anti-aliasing lowpass filter is designed with a cutoff frequency of $\Omega = 2\pi f_{\max} T = 2\pi \times 500/4000 = 0.25\pi$ rad. Fig. 11.25 shows the frequency responses of the designed filter while Fig. 11.26 compares the signals in the time and frequency domains, respectively, where $x(t)$ denotes the continuous version, $x_q(n)$ is the quantized version using a regular sampling rate of 1 kHz, and $y_q(n)$ is the enhanced version using the oversampling system with $L = 4$. The detailed amplitude comparisons are given in Fig. 11.27. The measured SNRs are 14.3 dB using the regular sampling system and 21.0 dB using the oversampling system. Since $L = 4$, the achieved signal is expected to have a 4-bit quality ($0.5 \times \log_2 4 = 1$ bit improvement). From simulation, we achieve approximately 6-dB SNR improvement. The improvement will stop when L increases due to the fact that when the

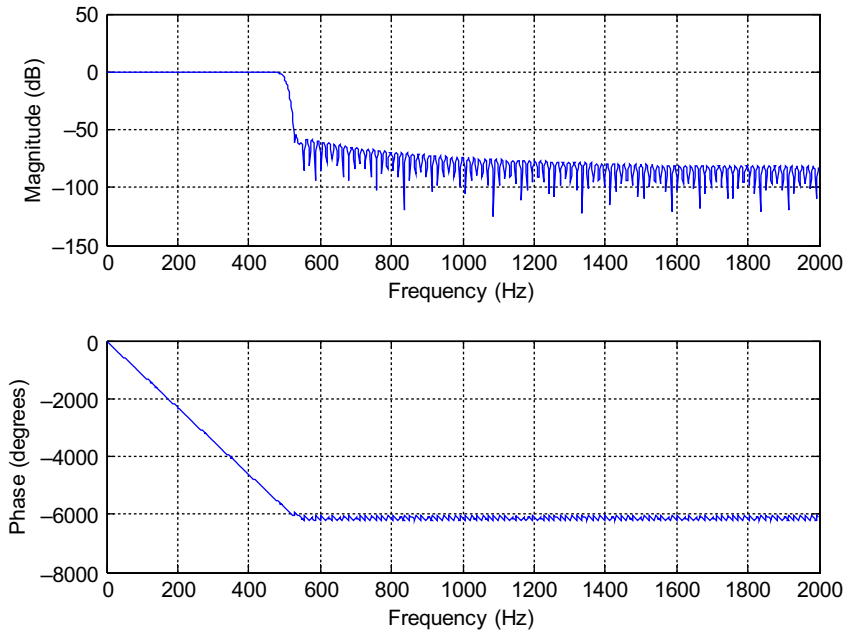


FIG. 11.25

Frequency responses of the designed filter.

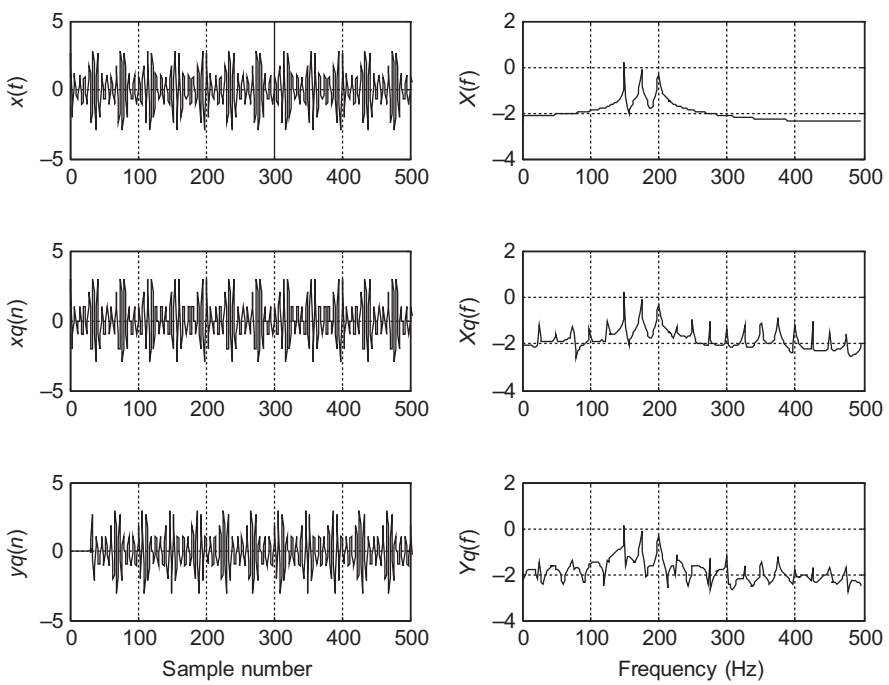


FIG. 11.26

Signal comparisons in both time and frequency domains.

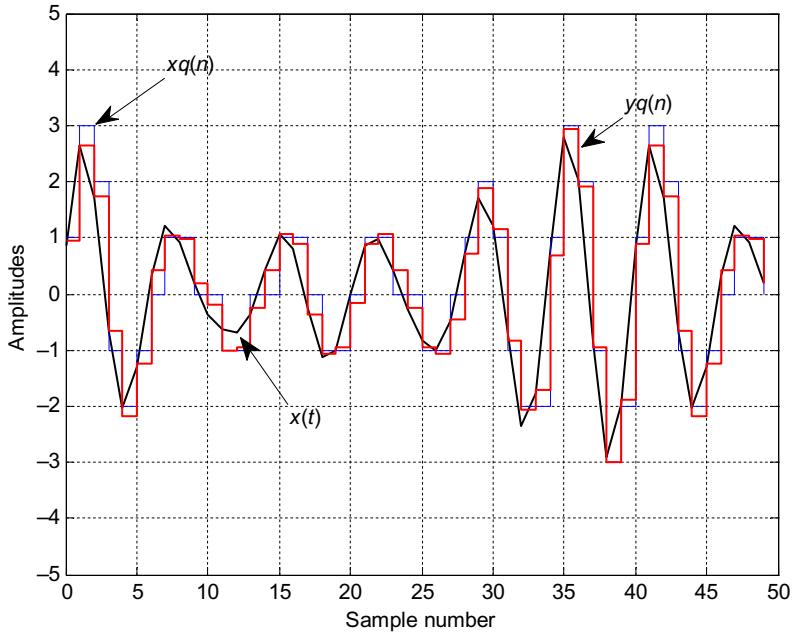


FIG. 11.27

Comparisons of continuous, regular sampled, and oversampled signal amplitudes.

sampling increases the quantization error may have correlation with the sinusoidal signal. The degradation performance can be cured using the dithering technique (Tan and Wang, 2011), which is beyond our scope.

Program 11.7. Oversampling implementation.

```
clear all; close all,clc
ntotal=512;
n=0:ntotal; % Number of samples
L=4; % Oversampling factor
nL=0:ntotal*L; % Number of samples for oversampling
numb=3; % Number of bits
A=2^(numb-1)-1; %Peak value
f1=150;C1=0.5*A;f2=175;C2=A*0.3;f3=200;C3=A*0.2; %Frequencies and amplitudes
fmax=500;fs=1000;T=1/fs; % Maximum frequency, sampling rate, sampling period
fsL=L*fs;TL=1/fsL;%Oversampling rate, and oversampling period
% Sampling at fs=1000Hz
x=C1*sin(2*pi*f1*n*T)+C2*sin(2*pi*f2*T*n+pi/6)+C3*sin(2*pi*f3*T*n+pi/4);
xq=round(x); %Quantized signal at the minimum sampling rate
NN=length(n);
f=[0:ntotal-1]*fs/NN;
M=32*L;nd=M/L; %Number of delay in samples due to anti-aliasing filtering
B=firwd(2*M+1,1,2*pi*fmax/fsL,0,4); % Anti-aliasing filter design (5% transition
```

```
bandwidth)
Figure(1);
freqz(B,1,1000,fsL)
% Oversampling
xx=C1*sin(2*pi*f1*nL*TL)+C2*sin(2*pi*f2*nL*TL+pi/6)+C3*sin(2*pi*f3*nL*TL+
pi/4);
xxq=round(xx); % Quantized signal
% Down sampling
y=filter(B,1,xxq);%Anti-aliasing filtering
yd.=y(1:L:length(y));% Down sample
Figure (2)
subplot(3,2,1);plot(n,x,'k');grid;axis([0500-5 5]);ylabel('x(t)')
Ak=2*abs(fft(x))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,2);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid;ylabel('X(f)'); axis
([0500-4 2])
subplot(3,2,3);plot(n,xq,'k');grid;axis([0500-5 5]);ylabel('xq(n)');
Ak=2*abs(fft(xq))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,4);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid;ylabel('Xq(f)'); axis
([0500-4 2])
subplot(3,2,5);plot(n,yd.,'k');grid;axis([0500-5 5]);ylabel('yq(n)');
xlabel('Sample number');
Ak=2*abs(fft(yd))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,6);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid;ylabel('Yq(f)'); axis
([0500-4 2])
xlabel('Frequency (Hz)');
Figure (3)
plot(n(1:50),x(1:50),'k','LineWidth',2); hold % Plot of first 50 samples
stairs(n(1:50),xq(1:50),'b');
stairs(n(1:50),yd.(1+nd:50+nd),'r','LineWidth',2);grid
axis([0 50-5 5]);xlabel('Sample number');ylabel('Amplitudes')
snr(x,xq);
snr(x(1:ntotal-nd),yd.(1+nd:ntotal));
```

11.3.2 SIGMA-DELTA MODULATION ADC

To further improve ADC resolution, *sigma-delta modulation* (SDM) ADC is used. The principles of the first-order SDM are described in Fig. 11.28.

First, the analog signal is sampled to obtain the discrete-time signal $x(n)$. This discrete-time signal is subtracted by the analog output from the m-bit DAC, converting the m bit oversampled digital signal $y(n)$. Then the difference is sent to the discrete-time analog integrator, which is implemented by the switched-capacitor technique, for example. The output from the discrete-time analog integrator is converted using an m-bit ADC to produce the oversampled digital signal. Finally, the decimation filter removes outband quantization noise. Further decimation process can change the oversampling rate back to the desired sampling rate for the output digital signal $w(m)$.

To examine the SDM, we need to develop a DSP model for the discrete-time analog filter described in Fig. 11.29.

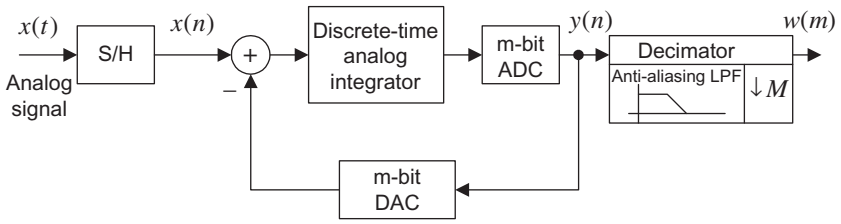


FIG. 11.28

Block diagram of SDM ADC.

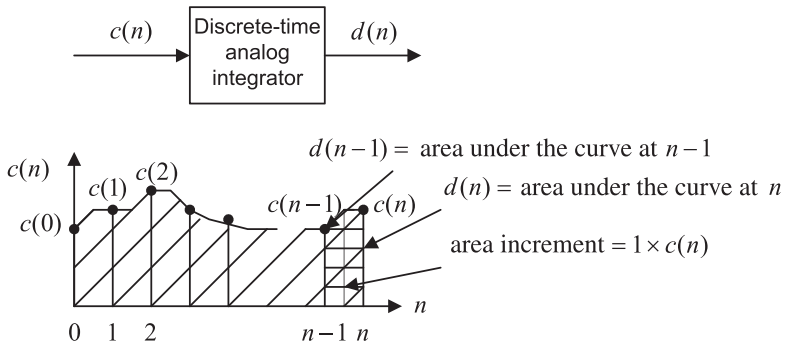


FIG. 11.29

Illustration of discrete-time analog integrator.

As shown in Fig. 11.29, the input signal $c(n)$ designates the amplitude at time instant n, while the output $d(n)$ is the area under the curve at time instant n, which can be expressed as a sum of the area under the curve at time instant $n-1$ and area increment:

$$d(n) = d(n-1) + \text{area incremetal}.$$

Using the extrapolation method, we have

$$d(n) = d(n-1) + 1 \times c(n). \tag{11.23}$$

Applying the z-transform to Eq. (11.23) leads to a transfer function of the discrete-time analog filter as

$$H(z) = \frac{D(z)}{C(z)} = \frac{1}{1-z^{-1}}. \tag{11.24}$$

Again, considering that the m-bit quantization requires one sample delay, we get the DSP model for the first-order SDM depicted in Fig. 11.30, where $y(n)$ is the oversampling data encoded by m bits each, and $e(n)$ represents quantization error.

The SDM DSP model represents a feedback control system. Appling the z-transform leads to

$$Y(z) = \frac{1}{1-z^{-1}}\left(X(z) - z^{-1}Y(z)\right) + E(z). \tag{11.25}$$

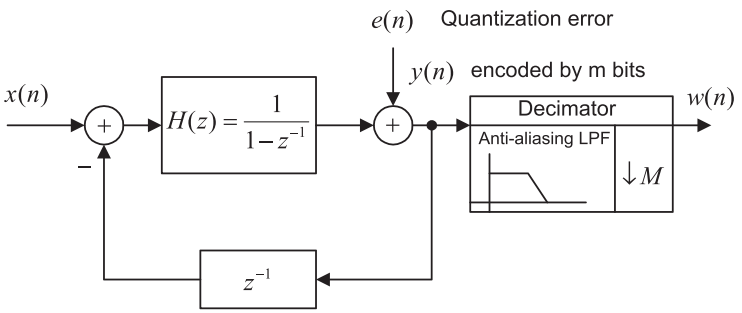


FIG. 11.30

DSP model for the first-order SDM ADC.

After simple algebra, we have

$$Y(z) = \underbrace{X(z)}_{\substack{\text{Original}\\ \text{digitalsignal}\\ \text{transform}}} + \underbrace{\left(1 - z^{-1}\right)}_{\substack{\text{Highpass}\\ \text{filter}}} \times \underbrace{E(z)}_{\substack{\text{Quantization}\\ \text{error}\\ \text{transform}}} \tag{11.26}$$

In Eq. (11.26), the indicated highpass filter pushes quantization noise to the high-frequency range, where later the quantization noise can be removed by the decimation filter. Thus we call this highpass filter $(1 - z^{-1})$ as the *noise shaping filter,* illustrated in Fig. 11.31.

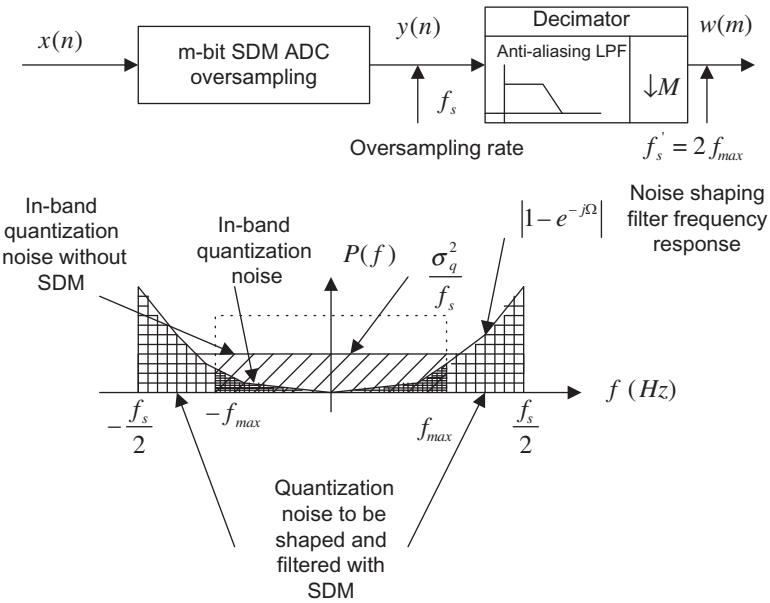


FIG. 11.31

Noise shaping of quantization noise for SDM ADC.

Shaped-in-band noise power after use of decimation filter can be estimated by the solid area under the curve. We have

$$\text{Shaped-in-band noise power} = \int_{-\Omega_{\max}}^{\Omega_{\max}} \frac{\sigma_q^2}{2\pi} \left|1 - e^{-j\Omega}\right|^2 d\Omega \tag{11.27}$$

Using the Maclaurin series expansion and neglecting the higher-order terms due to the small value of $\Omega_{\max}$, we yield

$$1 - e^{-j\Omega} = 1 - \left(1 + \frac{(-j\Omega)}{1!} + \frac{(-j\Omega)^2}{2!} + \cdots\right) \approx j\Omega.$$

Applying this approximation to Eq. (11.27) leads to

$$\text{Shaped-in-band noise power} \approx \int_{-\Omega_{\max}}^{\Omega_{\max}} \frac{\sigma_q^2}{2\pi} |j\Omega|^2 d\Omega = \frac{\sigma_q^2}{3\pi} \Omega_{\max}^3. \tag{11.28}$$

After simple algebra, we have

$$\text{Shaped-in-band noise power} \approx \frac{\pi^2 \sigma_q^2}{3} \left(\frac{2f_{\max}}{f_s}\right)^3 = \frac{\pi^2}{3} \times \frac{A^2 2^{-2m}}{12} \left(\frac{2f_{\max}}{f_s}\right)^3. \tag{11.29}$$

If we let the shaped-in-band noise power equal the quantization noise power from the regular ADC using a minimum sampling rate, we have

$$\frac{\pi^2}{3} \times \frac{A^2 2^{-2m}}{12} \left(\frac{2f_{\max}}{f_s}\right)^3 = \frac{A^2}{12} \times 2^{-2n}. \tag{11.30}$$

We modify Eq. (11.30) into the following useful formats for applications:

$$n = m + 1.5 \times \log_2 \left(\frac{f_s}{2f_{\max}}\right) - 0.86 \tag{11.31}$$

$$\left(\frac{f_s}{2f_{\max}}\right)^3 = \frac{\pi^2}{3} \times 2^{2(n-m)}. \tag{11.32}$$

EXAMPLE 11.9

Given the following DSP system specifications:
Over sampling rate system
First-order SDM with 2-bit ADC
Sampling rate = 4 MHz
Maximum audio input frequency = 4 kHz,
Determine the equivalent ADC resolution.

Solution:

Since $m=2$ bits, and

$$\frac{f_s}{2f_{\max}}=\frac{4000kHz}{2\times 4kHz}=500.$$

we calculate

$$n=m+1.5\times\log_2\left(\frac{f_s}{2f_{\max}}\right)-0.86=2+1.5\times\log_2(500)-0.86\approx 15\,\text{bits}.$$

We can also extend the first-order SDM DSP model to the second-order SDM DSP model by cascading one section of the first-order discrete-time analog filter as depicted in Fig. 11.32.

Similarly to the first-order SDM DSP model, applying the z-transform leads the following relationship:

$$Y(z)=\underbrace{X(z)}_{\substack{\text{Original}\\ \text{digital signal}\\ \text{transform}}}+\underbrace{\left(1-z^{-1}\right)^2}_{\substack{\text{Highpass}\\ \text{noise shaping}\\ \text{filter}}}\times\underbrace{E(z)}_{\substack{\text{Quantization}\\ \text{error}\\ \text{transform}}}. \tag{11.33}$$

Note that the noise shape filter becomes to a second-order highpass filter; hence, the more quantization noise is pushed to the high-frequency range, the better ADC resolution is expected to be. In a similar analysis to the first-order SDM, we get the following useful formulas:

$$n=m+2.5\times\log_2\left(\frac{f_s}{2f_{\max}}\right)-2.14 \tag{11.34}$$

$$\left(\frac{f_s}{2f_{\max}}\right)^5=\frac{\pi^4}{5}\times 2^{2(n-m)}. \tag{11.35}$$

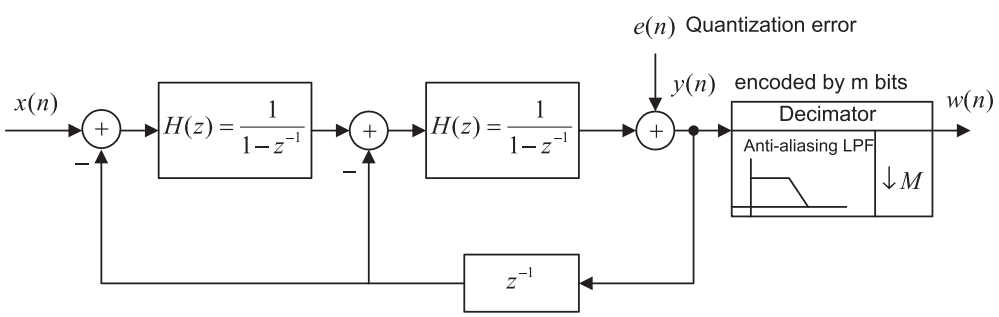


FIG. 11.32

DSP model for the second-order SDM ADC.

In general, the Kth-order SDM DSP model and ADC resolution formulas are given as

$$Y(z) = \underbrace{X(z)}_{\substack{\text{Original} \\ \text{digital signal} \\ \text{transform}}} + \underbrace{\left(1 - z^{-1}\right)^{K}}_{\substack{\text{Highpass} \\ \text{noise shaping} \\ \text{filter}}} \times \underbrace{E(z)}_{\substack{\text{Quantization} \\ \text{error} \\ \text{transform}}} \tag{11.36}$$

$$n = m + 0.5 \times (2K+1) \times \log_2\left(\frac{f_s}{2f_{\max}}\right) - 0.5 \times \log_2\left(\frac{\pi^{2K}}{2K+1}\right) \tag{11.37}$$

$$\left(\frac{f_s}{2f_{\max}}\right)^{2K+1} = \frac{\pi^{2K}}{2K+1} \times 2^{2(n-m)}. \tag{11.38}$$

EXAMPLE 11.10

Given the oversampling rate DSP system with the following specifications:
second-order SDM = 1 bit ADC
sampling rate = 1 MHz
maximum audio input frequency = 4 kHz,
Determine the effective ADC resolution.

Solution:

$$n = 1 + 2.5 \times \log_2\left(\frac{1{,}000kHz}{2 \times 4kHz}\right) - 2.14 \approx 16 \text{ bits}.$$

We implement the first-order SDM system using the same continuous signal in Eq. (11.22). The continuous signal is originally sampled at 1 kHz and each sample is encoded using 3 bits. The SDM system uses an oversampling rate of 8 kHz ($L = 8$) and each sample is quantized using a 3-bit code. The anti-aliasing lowpass filter is designed with a cutoff frequency of $\Omega = 2\pi f_{\max} T = 2\pi \times 500/8,000 = \pi/8$ rad. Fig. 11.33 shows the frequency responses of the designed filter while Fig. 11.34 compares the time and frequency domain signals, where $x(t)$ designates the continuous version, $x_q(n)$ denotes the quantized version using a regular sampling rate ($L = 1$) while $y_q(n)$ is the enhanced version using $L = 8$. The detailed amplitude comparisons are given in Fig. 11.35. The measured SNRs are 14.3 dB in the regular sampling system while 33.83 dB in the oversampling SDM system. We can observe a significant SNR improvement with 19.5 dB. The detailed implementation using MATLAB is given in Program 11.8.

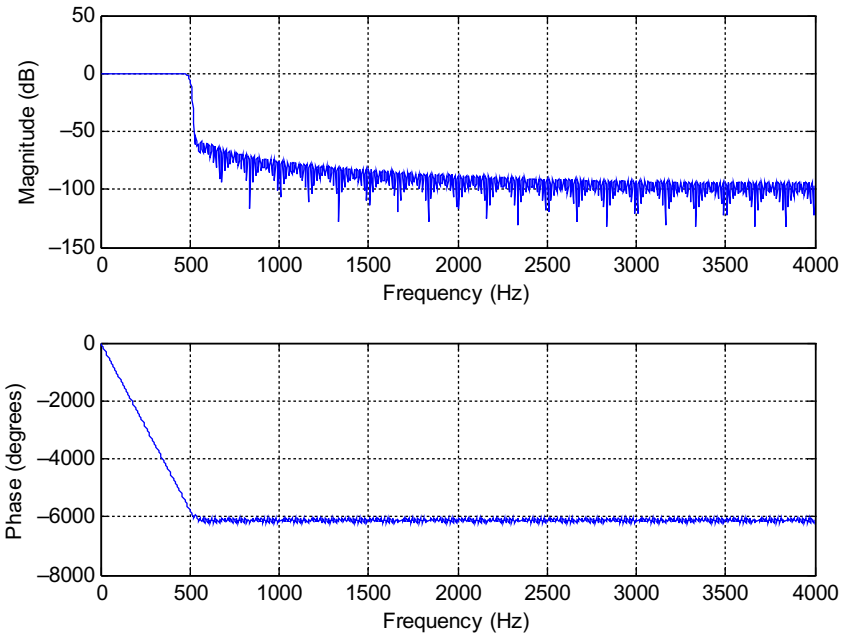


FIG. 11.33

Frequency responses of the designed filter.

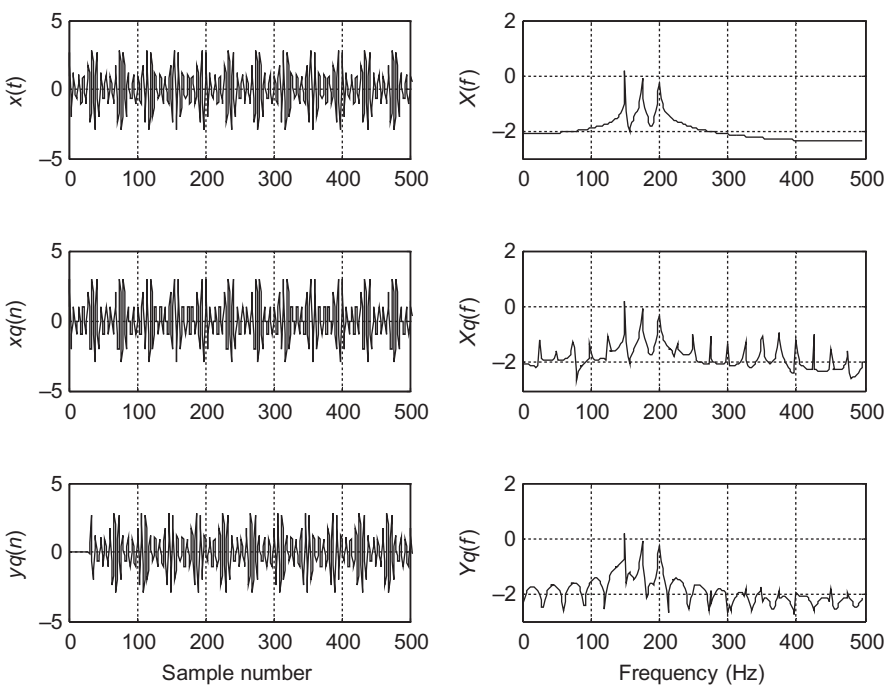


FIG. 11.34

Signal comparisons in both time and frequency domains.

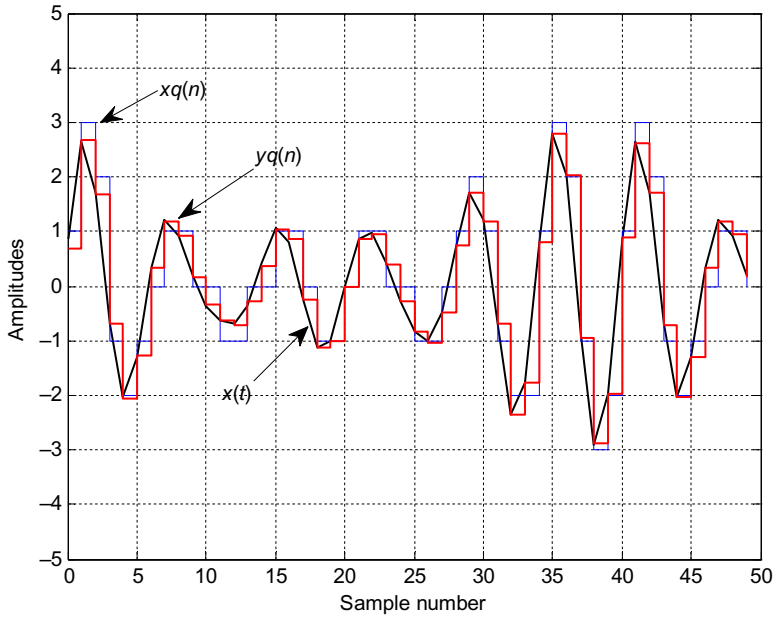


FIG. 11.35

Comparisons of continuous, regular sampled, and oversampled signal amplitudes.

Program 11.8. First-order SDM oversampling implementation.

```
clear all; close all;clc
ntotal=512; % Number of samples
n=0:ntotal;
L=8; %Oversampling factor
nL=0:ntotal*L;numb=3;A=2^(numb-1)-1; %Peak value
f1=150;C1=0.5*A;f2=175;C2=A*0.3;f3=200;C3=A*0.2;%Frequencies and amplitudes
fmax=500;fs=1000; T=1/fs; % Sampling rate and sampling period
fsL=L*fs;TL=1/fsL; % Oversampling rate and oversampling period
% Sampling at fs=1000Hz
x=C1*sin(2*pi*f1*n*T)+C2*sin(2*pi*f2*T*n+pi/6)+C3*sin(2*pi*f3*T*n+pi/4);
xq=round(x); %Quantization
NN=length(n);
M=32*L;nd=M/L; %Delay in terms of samples for anti-aliasing filtering
B=firwd(2*M+1,1,2*pi*fmax/fsL,0,4);% Design of an anti-aliasing filter
Figure(1)
freqz(B,1,1000,fsL);
% Oversampling
xx=C1*sin(2*pi*f1*nL*TL)+C2*sin(2*pi*f2*nL*TL+pi/6)+C3*sin(2*pi*f3*nL*TL+pi/4);
% The first-order SDM processing
  yq=zeros(1,ntotal*L+1+1); %Initializing the buffer
  y=yq;
```

```
for i=1:ntotal*L
  y(i+1)=(xx(i+1)-yq(i))+y(i);
  yq(i+1)=round(y(i+1));
end
xxq=yq(1:ntotal*L+1); %Signal Quantization
% Downsampling
y=filter(B,1,xxq);
yd.=y(1:L:length(y));
f=[0:ntotal-1]*fs/NN;
Figure (2)
subplot(3,2,1);plot(n,x,'k');grid;axis([0500-5 5]);ylabel('x(t)');
Ak=2*abs(fft(x))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,2);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid;
axis([0500-3 2]);ylabel('X(f)');
subplot(3,2,3);plot(n,xq,'k');grid;axis([0500-5 5]);ylabel('xq(n)');
Ak=2*abs(fft(xq))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,4);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid
axis([0500-3 2]);ylabel('Xq(f)');
subplot(3,2,5);plot(n,yd.,'k');grid;axis([0500-5 5]);ylabel('yq(n)');
xlabel('Sample number');
Ak=2*abs(fft(yd))/NN; Ak(1)=Ak(1)/2;
subplot(3,2,6);plot(f(1:NN/2),log10(Ak(1:NN/2)),'k');grid
axis([0500-3 2]);ylabel('Yq(f)');xlabel('Frequency (Hz)');
Figure (3)
plot(n(1:50),x(1:50),'k','LineWidth',2); hold
stairs(n(1:50),xq(1:50),'b');
stairs(n(1:50),yd.(1+nd:50+nd),'r','LineWidth',2);
axis([0 50-5 5]);grid;xlabel('Sample number');ylabel('Amplitudes');
snr(x,xq);
snr(x(1:ntotal-nd),yd.(1+nd:ntotal));
```

Next, we review the application of the oversampling ADC used in the industry. Fig. 11.36 illustrates a function diagram for the MAX1402 low-power, multichannel oversampling sigma-delta ADC used in industry. It applies a sigma-delta modulator with a digital decimation filter to achieve 16-bit accuracy. The device offers three fully differential input channels, which can be independently programmed. It can also be configured as five pseudo-differential input channels. It comprises two chopper buffer amplifiers and a programmable gain amplifier, a DAC unit with predicted input subtracted from the analog input to acquire the differential signal, and a second-order switched-capacitor sigma-delta modulator.

The chip produces a 1-bit data stream, which will be filtered by the integrated digital filter to complete ADC. The digital filter's user-selectable decimation factor offers flexibility as conversion resolution can be reduced in exchange for a higher data rate or vice versa. The integrated digital lowpass filter is the first-order or third-order Sinc infinite impulse response filter. Such a filter offers notches corresponding to its output data rate and its frequency harmonics, so it can effectively reduce the developed image noises in the frequency domain. (The Sinc filter is beyond the scope of our discussion.) The MAX1402 can provide 16-bit accuracy at 480 samples per second and 12-bit accuracy at 4800 samples per second. The chip finds a wide application in sensors and instrumentation. Its detailed features can be found in the MAX1402 data sheet (Maxim Integrated Products, 2018).

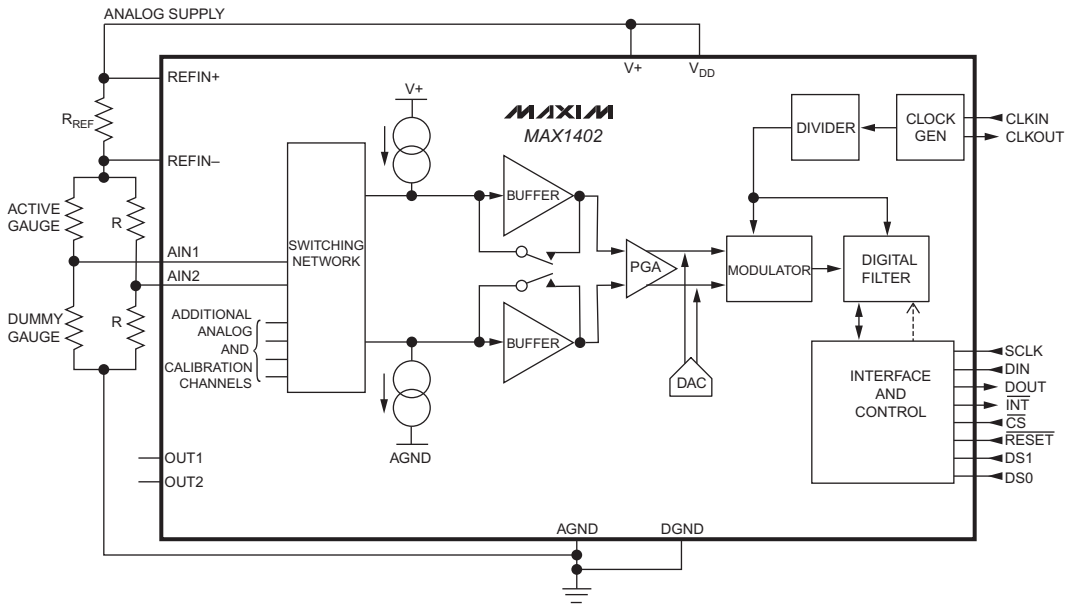


FIG. 11.36

Functional Diagram for the Sigma-delta ADC.

11.4 APPLICATION EXAMPLE: CD PLAYER

Fig. 11.37 illustrates a CD playback system, also described earlier. A laser optically scans the tracks on a CD to produce a digital signal. The digital signal is then demodulated, and parity bits are used to detect bit errors due to manufacturing defects, dust, and so on, and to correct them. The demodulated signal is again oversampled by a factor of 4 and hence the sampling rate is increased to 176.4 kHz for each channel. Each digital sample then passes through a 14-bit DAC, which produces the sample-and-hold voltage signals that pass the anti-image lowpass filter. The output from each analog filter is fed to its corresponding loudspeaker. Oversampling relaxes the design requirements of the analog anti-image lowpass filter, which is used to smooth out the voltage steps.

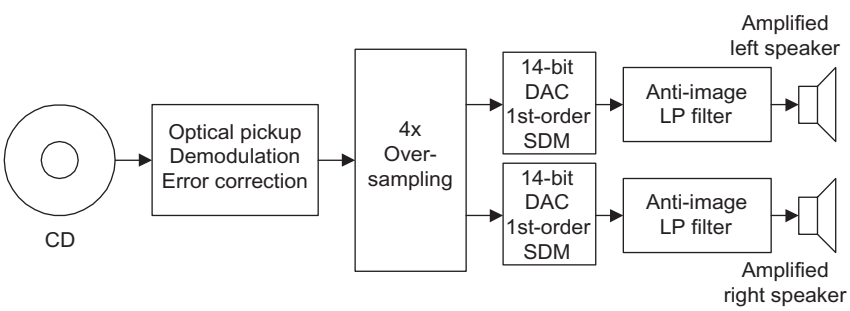


FIG. 11.37

Simplified decoder of a CD recording system.

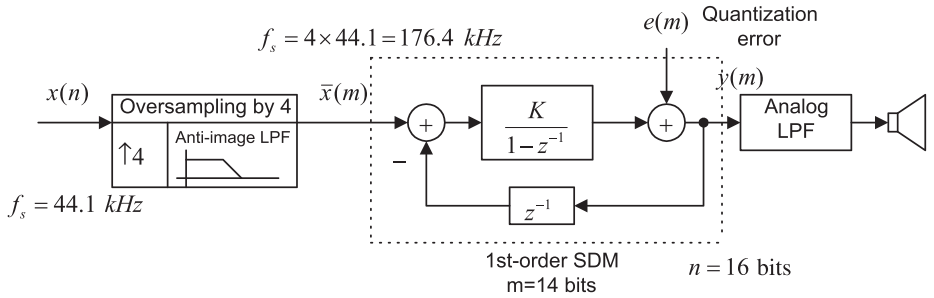


FIG. 11.38

Illustration of oversampling and SDM ADC used in the decoder of a CD recording system.

The earliest system used a third-order Bessel filter with a 3-dB gain attenuation at 30 kHz. Note that the first-order SDM is added to the 14-bit DAC unit to further improve the 14-bit DAC to 16-bit DAC.

Let us examine the single-channel DSP portion as shown in Fig. 11.38.

The spectral plots for the oversampled and interpolated signal $\bar{x}(n)$, the 14-bit SDM output $y(n)$, and the final analog output audio signal are given in Fig. 11.39. As we can see in plot (a) in the figure, the quantization noise is uniformly distributed, and only in-band quantization noise (0–22.05 kHz) is

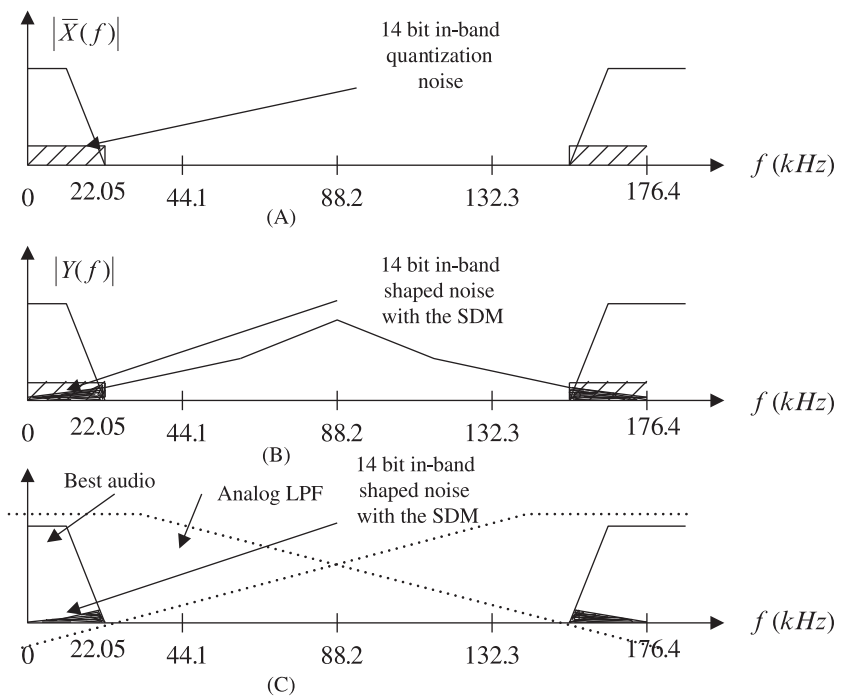


FIG. 11.39

Spectral illustrations for the oversampling and SDM ADC used in the decoder of a CD recording system.

expected. Again, 14 bits for each sample are kept after oversampling. Without using the first-order SDM, we expect the effective ADC resolution due to oversampling to be

$$n = 14 + 0.5 \times \log_2\left(\frac{176.4}{44.1}\right) = 15 \text{ bits},$$

which is fewer than 16 bits. To improve quality further, the first-order SDM is used. The in-band quantization noise is then shaped. The first-order SDM pushes quantization noise to the high-frequency range, as illustrated in plot (b) in Fig. 11.39. The effective ADC resolution now becomes

$$n = 14 + 1.5 \times \log_2\left(\frac{176.4}{44.1}\right) - 0.86 \approx 16 \text{ bits}.$$

Hence, 16-bit ADC audio quality is preserved. On the other hand, from plot (c) in Fig. 11.39, the audio occupies a frequency range up to 22.05 kHz, while the DSP Nyquist limit is 88.2 kHz, so the low-order analog anti-image filter can satisfy the design requirement.

11.5 UNDERSAMPLING OF BANDPASS SIGNALS

As we discussed in Chapter 2, the sampling theorem requires that the sampling rate be twice as large as the highest frequency of the analog signal to be sampled. The sampling theorem ensures the complete reconstruction of the analog signal without aliasing distortion. In some applications, such as the modulated signals in communications systems, the signal exists in only a small portion of the bandwidth. Fig. 11.40 shows an amplitude modulated (AM) signal in both time domain and frequency domain. Assuming that the message signal has a bandwidth of 4 kHz and a carrier frequency of 96 kHz, the upper frequency edge the AM signal is therefore 100 kHz (f_c+B). Then the traditional sampling process

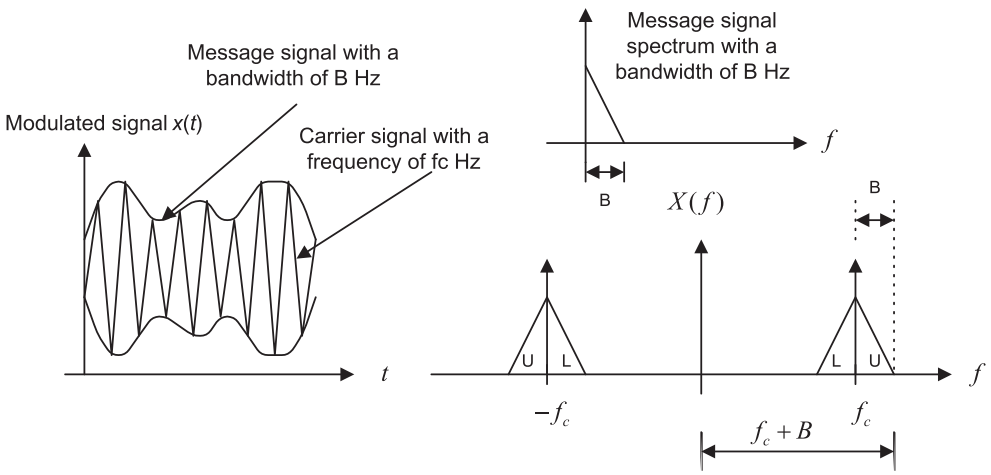


FIG. 11.40

Message signal, modulated signal, and their spectra.

requires that the sampling rate be larger than 200 kHz $2(f_c+B)$, resulting in at a high processing cost. Note that sampling the baseband signal of 4 kHz only requires a sampling rate of 8 kHz ($2B$).

If a certain condition is satisfied at the undersampling stage, we are able to make use of the aliasing signal to recover the message signal, since the aliasing signal contains the folded original message information (which we used to consider as distortion). The reader is referred to the undersampling technique discussed in Ifeachor and Jervis (2002) and Porat (1997). Let the message to be recovered have a bandwidth of B, the theoretical minimum sampling rate be $f_s=2B$, and the carrier frequency of the modulated signal be f_c. We discuss the following cases.

Case 1.

If $f_c=$ even integer $\times B$ and $f_c=2B$, the sampled spectrum with all the replicas will be as shown in Fig. 11.41A.

As an illustrative example in time domain for Case 1, suppose we have a bandpass signal with a carrier frequency of 20 Hz; that is,

$$x(t)=\cos(2\pi\times 20t)m(t), \tag{11.39}$$

where $m(t)$ is the message signal with a bandwidth of 2 Hz. Using a sampling rate of 4 Hz by substituting $t=nT$, where $T=1/f_s$ into Eq. (11.39), we get the sampled signal as

$$x(nT)=\cos(2\pi\times 20t)m(t)|_{t=nT}=\cos(2\pi\times 20n/4)m(nT). \tag{11.40}$$

Since $10n\pi=5n(2\pi)$ is the multiple of 2π,

$$\cos(2\pi\times 20n/4)=\cos(10\pi n)=1, \tag{11.41}$$

we obtain the undersampled signal as

$$x(nT)=\cos(2\pi\times 20n/4)m(nT)=m(nT), \tag{11.42}$$

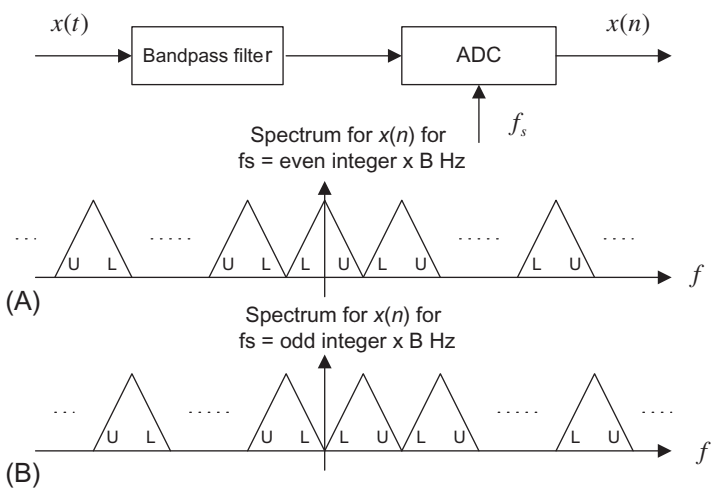


FIG. 11.41

Spectrum of the undersampled signal.

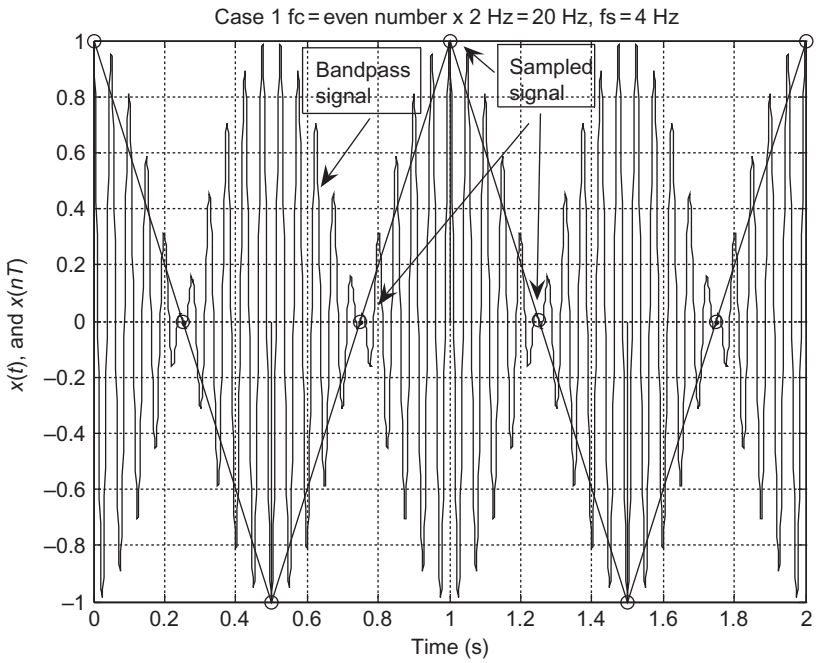


FIG. 11.42

Plots of the bandpass signal and sampled signal for Case 1.

which is a perfect digital message signal. Fig. 11.42 shows the bandpass signal and its sampled signals when the message signal is 1 Hz, given as

$$m(t) = \cos(2\pi t). \tag{11.43}$$

Case 2.

If $f_c =$ odd integer $\times$ B and $f_c = 2B$, the sampled spectrum with all the replicas will be as shown in Fig. 11.41B, where the spectral portions L and U are reversed. Hence, the frequency reversal will occur. Then a further digital modulation in which the signal is multiplied by the digital oscillator with a frequency of B Hz can be used to adjust the spectrum to be the same as that in Case 1.

As another illustrative example for Case 2, let us sample the following the bandpass signal with a carrier frequency of 22 Hz, given by

$$x(t) = \cos(2\pi \times 22t)m(t). \tag{11.44}$$

Applying undersampling using a sampling rate of 4 Hz, it follows that

$$x(nT) = \cos(2\pi \times 22n/4)m(nT) = \cos(11n\pi)m(nT). \tag{11.45}$$

Since $11n\pi$ can be either an odd or an even integer multiple of π, we have

$$\cos(11\pi n) = \begin{cases} -1 & n = \text{odd} \\ 1 & n = \text{even}. \end{cases} \tag{11.46}$$

We see that Eq. (11.46) causes the message samples to change sign alternatively with a carrier frequency of 22 Hz, which is the odd integer multiple of the message bandwidth of 2 Hz. This in fact will reverse the baseband message spectrum. To correct the spectrum reversal, we multiply an oscillator with a frequency of $B = 2$ Hz by the bandpass signal, that is

$$x(t)\cos(2\pi \times 2t) = \cos(2\pi \times 22t)m(t)\cos(2\pi \times 2t). \tag{11.47}$$

Then the undersampled signal is then given by

$$\begin{aligned} x(nT)\cos(2\pi \times 2n/4) &= \cos(2\pi \times 22n/4)m(nT)\cos(2\pi \times 2n/4) \\ &= \cos(11n\pi)m(nT)\cos(n\pi) \end{aligned} \tag{11.48}$$

Since

$$\cos(11\pi n)\cos(\pi n) = 1, \tag{11.49}$$

it follows that

$$x(nT)\cos(2\pi \times 2n/4) = \cos(\pi \times 11n)m(nT)\cos(\pi \times n) = m(nT), \tag{11.50}$$

which is the recovered message signal. Fig. 11.43 shows the sampled bandpass signals with the reversed message spectrum and the corrected message spectrum, respectively, for a message signal having a frequency of 0.5 Hz, that is,

$$m(t) = \cos(2\pi \times 0.5t). \tag{11.51}$$

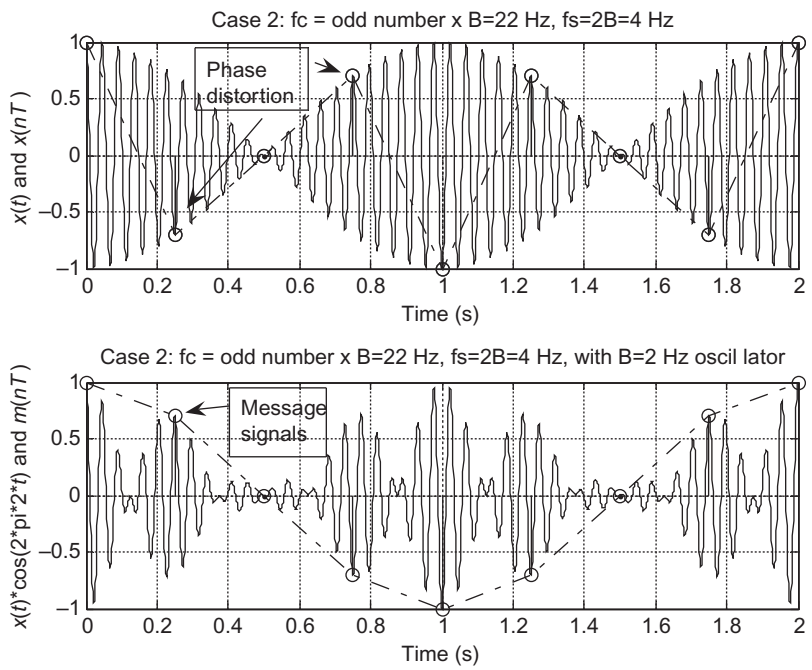


FIG. 11.43

Plots of the bandpass signals and sampled signals for Case 2.

Case 3.

If $f_c =$ non interger $\times B$, we can extend the bandwidth B to $\overline{B}$ such that

$$f_c = \text{integer} \times \overline{B} \text{ and } f_s = 2\overline{B}. \tag{11.52}$$

Then we can apply Case 1 or Case 2. An illustration of Case 3 is included in the following example.

EXAMPLE 11.11

Given a bandpass signal with the spectrum and the carry frequency f_c shown in Fig. 11.44A–C, respectively, and assuming the baseband bandwidth $B = 4$ kHz, select the sampling rate and sketch the sampled spectrum ranging from 0 Hz to the carrier frequency for each of the following carrier frequencies:

1. $f_c = 16$ kHz
2. $f_c = 12$ kHz
3. $f_c = 18$ kHz

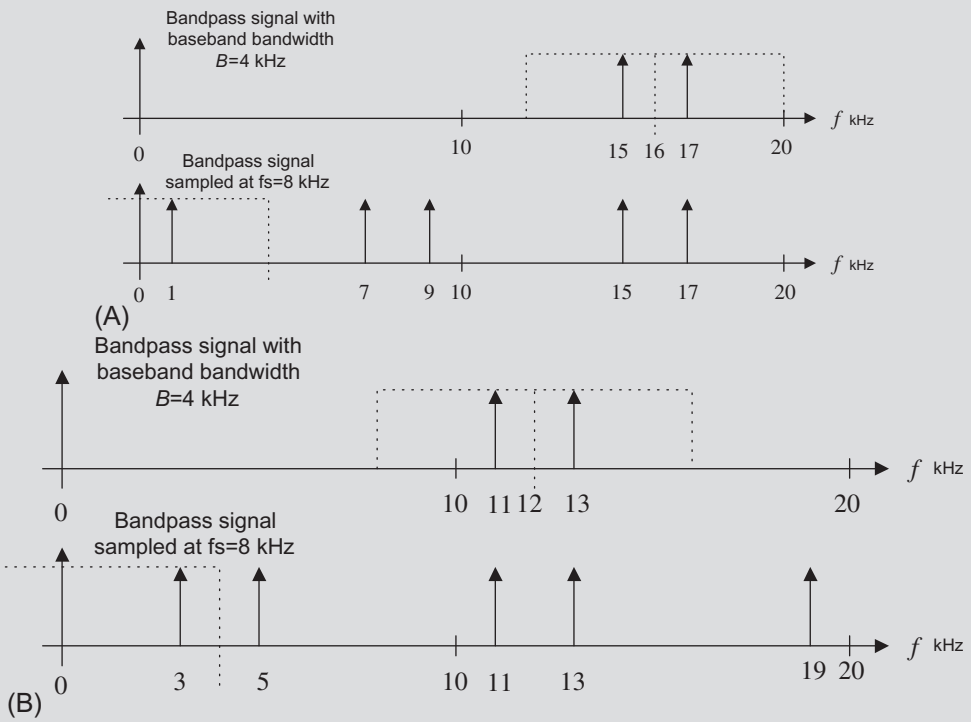


FIG. 11.44

(A) Sampled signal spectrum for $f_c = 16$ kHz. (B) Sampled signal spectrum for $f_c = 12$ kHz.

Continued

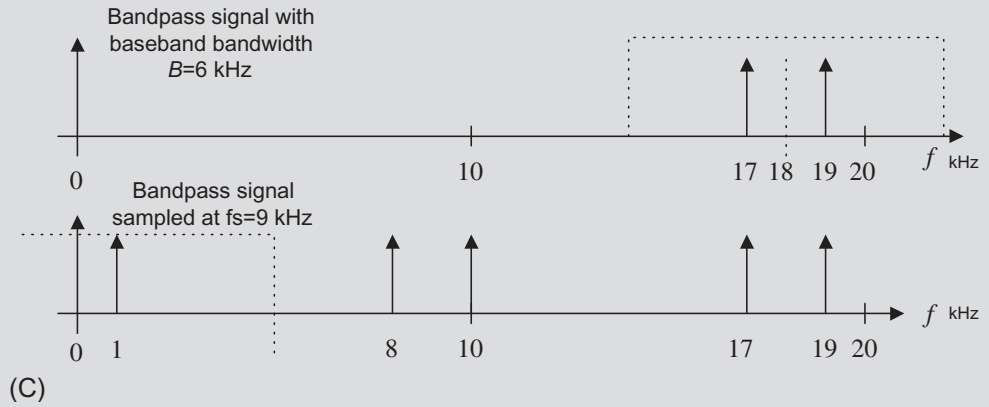


FIG. 11.44, CONT'D

(C) Sampled signal spectrum for $f_c = 18$ kHz.

Solution:

1. Since $f_c/B = 4$ is an even number, which is Case 1, we select $f_s = 8$ kHz and sketch the sampled spectrum shown in Fig. 11.44A.
2. Since $f_c/B = 3$ is an odd number, we select $f_s = 8$ kHz and sketch the sampled spectrum shown in Fig. 11.44B.
3. Now, $f_c/B = 4.5$ which is a non-integer. We extend the band width $\overline{B} = 4.5$ kHz, so $f_c/\overline{B} = 4$ and $f_s = 2\overline{B} = 9$ kHz. Then the sketched spectrum is shown in Fig. 11.44C.

Simulation Example

An AM with a 1-kHz message signal is given as:

$$x(t) = [1 + 0.8 \times \sin(2\pi \times 1000t)]\cos(2\pi \times f_c t). \tag{11.53}$$

Assuming a message bandwidth of 4 kHz, determine the sampling rate, use MATLAB to sample the AM signal, and sketch the sampled spectrum up to the sampling frequency for each the following carrier frequencies:

1. $f_c = 96$ kHz
2. $f_c = 100$ kHz
3. $f_c = 99$ kHz

1. For this case, $f_c/B = 24$ is an even number. We select $f_s = 8$ kHz. Fig. 11.45A describes the simulation, where the upper left plot is the AM signal, the upper right plot is the spectrum of the AM signal, the lower left plot is the undersampled signal, and the lower right plot is the spectrum of the undersampled signal displayed from 0 to 8 kHz.

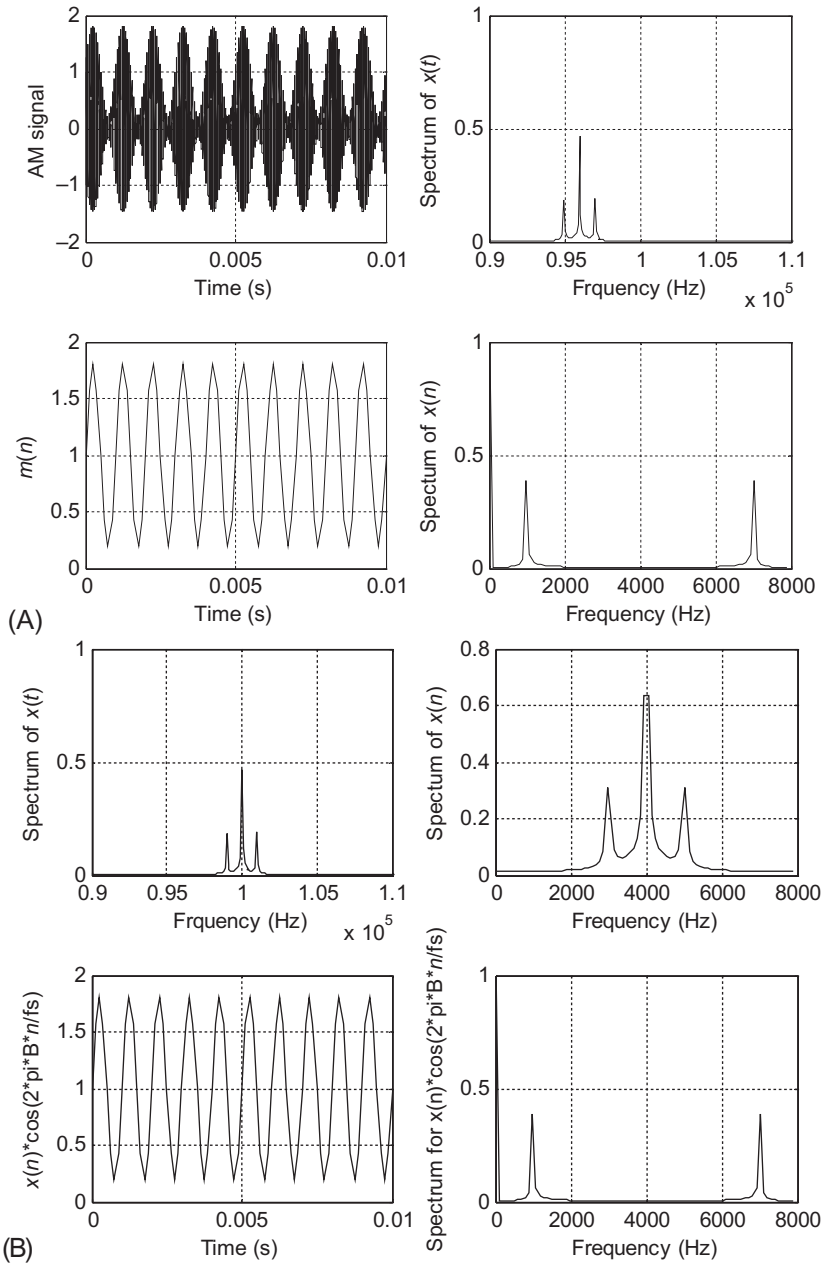


FIG. 11.45

(A) Sampled AM signal and spectrum for $f_c = 96$ kHz. (B) Sampled AM signal and spectrum for $f_c = 100$ kHz.

(Continued)

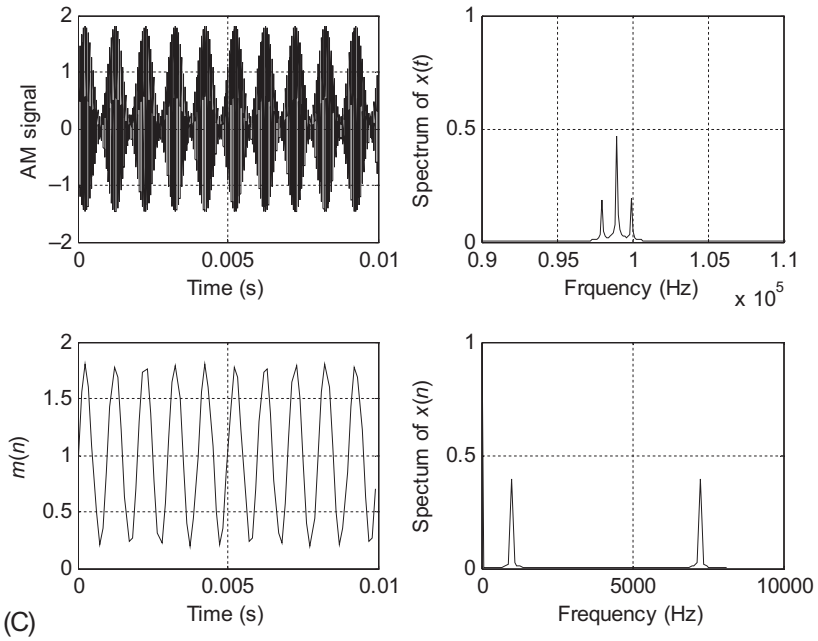


FIG. 11.45, CONT'D

(C) Sampled AM signal and spectrum for $f_c = 99$ kHz.

2. $f_c/B = 25$ is an odd number, we choose $f_s = 8$ kHz, and a further process is needed. We can multiply the undersampled signal by a digital oscillator with a frequency of $B = 4$ kHz to achieve the 1-kHz baseband signal. The plots of the AM signal spectrum, undersampled signal spectrum, and the oscillator mixed signal and its spectrum are shown in Fig. 11.45B.
3. For $f_c = 99$ kHz, $f_c/B = 24.75$. We extend the bandwidth to $\overline{B} = 4.125$ so that $f_c/\overline{B} = 24$. Hence, the undersampling rate is used as $f_s = 8.25$ kHz. Fig. 11.45C shows the plots of the AM signal, the AM signal spectrum, the undersampled signal based on the extended baseband width, and sampled signal spectrum ranging from 0 to 8.25 kHz, respectively.

This example verifies principles of undersampling of bandpass signals.

11.6 SUMMARY

1. Downsampling (decimation) by an integer factor of M means taking one sample from the data sequence $x(n)$ for every M samples and discard the last $M-1$ samples.
2. Upsampling (Interpolation) by an integer factor of L means inserting $L-1$ zeros for every sample in the data sequence $x(n)$.
3. Downsampling requires a decimation (anti-aliasing) filter to avoid frequency aliasing before downsampling.

4. Upsampling requires an interpolation (anti-image) filter to remove the images after interpolation.
5. Changing the sampling rate by a non-integer factor of L/M requires two stages: an interpolation stage and a downsampling stage.
6. Two-stage decimation can dramatically reduce the anti-aliasing filter length.
7. Polyphase implementations of the decimation filter and interpolation filter can reduce complexity of the filter operations, that is, fewer multiplications and additions.
8. Using oversampling can improve the regular ADC resolution. SDM ADC can achieve even higher ADC resolution, using noise-shaping effect for further reduction of quantization noise.
9. The audio CD player uses multirate signal processing and oversampling.
10. Undersampling can be used to sample the bandpass signal and find its application in communications.

11.7 PROBLEMS

11.1 For a single-stage decimator with the following specifications:
Original sampling rate = 1 kHz
Decimation factor $M = 2$
Frequency of interest = 0–100 Hz
Passband ripple = 0.015 dB
Stopband attenuation = 40 dB,
(a) Draw the block diagram for the decimator;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the anti-aliasing FIR filter design.

11.2 For a single-stage interpolator with the following specifications:
Original sampling rate = 1 kHz
Interpolation factor $L = 2$
Frequency of interest = 0–150 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 45 dB,
(a) Draw the block diagram for the interpolator;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the anti-image FIR filter design.

11.3 For a single-stage decimator with the following specifications:
Original sampling rate = 8 kHz.
Decimation factor $M = 4$
Frequency of interest = 0–800 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 46 dB,
(a) Draw the block diagram for the decimator;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the anti-aliasing FIR filter design.

11.4 For a single-stage interpolator with the following specifications:
Original sampling rate = 8 kHz.
Interpolation factor $L = 3$
Frequency of interest = 0–3400 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 46 dB,
(a) Draw the block diagram for the interpolator;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the anti-image FIR filter design.

11.5 For the sampling conversion from 4 to 3 kHz with the following specifications:
Original sampling rate = 4 kHz.
Interpolation factor $L = 3$
Decimation factor $M = 2$
Frequency of interest = 0–400 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 46 dB,
(a) Draw the block diagram for the interpolator;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the combined FIR filter $H(z)$.

11.6 For the design of a two-stage decimator with the following specifications:
Original sampling rate = 32 kHz
Frequency of interest = 0–250 Hz
Passband ripple = 0.05 (absolute)
Stopband attenuation = 0.005 (absolute)
Final sampling rate = 1000 Hz,
(a) Draw the decimation block diagram;
(b) Specify the sampling rate for each stage;
(c) Determine the window type, filter length, and cutoff frequency for the first stage if the window method is used for anti-aliasing FIR filter design ($H_1(z)$);
(d) Determine the window type, filter length, and cutoff frequency for the second stage if the window method is used for the anti-aliasing FIR filter design ($H_2(z)$).

11.7 For the sampling conversion from 6 to 8 kHz with the following specifications:
Original sampling rate = 6 kHz
Interpolation factor $L = 4$
Decimation factor $M = 3$
Frequency of interest = 0–2400 Hz
Passband ripple = 0.02 dB
Stopband attenuation = 46 dB,
(a) Draw the block diagram for the processor;
(b) Determine the window type, filter length, and cutoff frequency if the window method is used for the combined FIR filter $H(z)$.

11.8 For the design of a two-stage decimator with the following specifications:
Original sampling rate = 320 kHz
Frequency of interest = 0–3400 Hz

Passband ripple = 0.05 (absolute)
Stopband attenuation = 0.005 (absolute)
Final sampling rate = 8000 Hz

(a) Draw the decimation block diagram;
(b) Specify the sampling rate for each stage;
(c) determine the window type, filter length, and cutoff frequency for the first stage if the window method is used for anti-aliasing FIR filter design ($H_1(z)$);
(d) determine the window type, filter length, and cutoff frequency for the second stage if the window method is used for anti-aliasing FIR filter design ($H_2(z)$).

11.9 **(a)** Given an interpolator filter as

$$H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2},$$

draw the block diagram for interpolation polyphase filter implementation for the case of $L = 2$.

(b) Given a decimation filter as

$$H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3},$$

draw the block diagram for the decimation polyphase filter implementation for the case of $M = 2$.

11.10 Using the commutative models for the polyphase interpolation and decimation filters,

(a) draw the block diagram for the interpolation polyphase filter implementation for the case of $L = 2$, and $H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2}$;
(b) draw the block diagram for the decimation polyphase filter implementation for the case of $M = 2$, and $H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3}$.

11.11 **(a)** Given an interpolator filter as

$$H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3} + 0.7z^{-4} + 0.6z^{-5},$$

draw the block diagram for the interpolation polyphase filter implementation for the case of $L = 4$.

(b) Given a decimation filter as

$$H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3} + 0.5z^{-3} + 0.4z^{-4},$$

draw the block diagram for decimation polyphase filter implementation for the case of $M = 4$.

11.12 Using the commutative models for the polyphase interpolation and decimation filters,

(a) Draw the block diagram for interpolation polyphase filter implementation for the case of $L = 4$, and $H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3} + 0.7z^{-4} + 0.6z^{-5}$;
(b) Draw the block diagram for decimation polyphase filter implementation for the case of $M = 4$, and $H(z) = 0.25 + 0.4z^{-1} + 0.5z^{-2} + 0.6z^{-3} + 0.5z^{-3} + 0.4z^{-4}$.

11.13 Given a speech system with the following specifications:
Speech input frequency range: 0–4 kHz.
ADC resolution = 16 bits.
Current sampling rate = 8 kHz,

(a) Determine the oversampling rate if a 12-bit ADC chip is used to replace the speech system;
(b) Draw the block diagram.

11.14 Given a speech system with the following specifications:
Speech input frequency range: 0–4 kHz.
ADC resolution = 6 bits.
Oversampling rate = 4 MHz,
(a) Draw the block diagram;
(b) Determine the actual effective ADC resolution (number of bits per sample).

11.15 Given an audio system with the following specifications:
Audio input frequency range: 0–15 kHz.
ADC resolution = 16 bits.
Current sampling rate = 30 kHz,
(a) Determine the oversampling rate if a 12-bit ADC chip is used to replace the audio system;
(b) Draw the block diagram.

11.16 Given an audio system with the following specifications:
Audio input frequency range: 0–15 kHz.
ADC resolution = 6 bits.
Oversampling rate = 45 MHz,
(a) Draw the block diagram;
(b) Determine the actual effective ADC resolution (number of bits per sample).

11.17 Given the following specifications of an oversampling DSP system:
Audio input frequency range: 0–4 kHz
First-order SDM with a sampling rate of 128 kHz
ADC resolution in SDM = 1 bit,
(a) draw the block diagram using the DSP model;
(b) determine the equivalent (effective) ADC resolution.

11.18 Given the following specifications of an oversampling DSP system:
Audio input frequency range: 0–20 kHz
Second-order SDM with a sampling rate of 160 kHz
ADC resolution in SDM = 10 bits,
(a) Draw the block diagram using the DSP model;
(b) Determine the equivalent (effective) ADC resolution.

11.19 Given the following specifications of an oversampling DSP system:
Signal input frequency range: 0–500 Hz
First-order SDM with a sampling rate of 128 kHz
ADC resolution in SDM = 1 bit,
(a) Draw the block diagram using the DSP model;
(b) Determine the equivalent (effective) ADC resolution.

11.20 Given the following specifications of an oversampling DSP system:
Signal input frequency range: 0–500 Hz
Second-order SDM with a sampling rate of 16 kHz
ADC resolution in SDM = 8 bits,
(a) Draw the block diagram using the DSP model;
(b) Determine the equivalent (effective) ADC resolution.

11.21 Given a bandpass signal with its spectrum shown in Fig. 11.46, and assuming the bandwidth $B = 5$ kHz, select the sampling rate, and sketch the sampled spectrum ranging from 0 Hz to the carrier frequency for each of the following carrier frequencies:

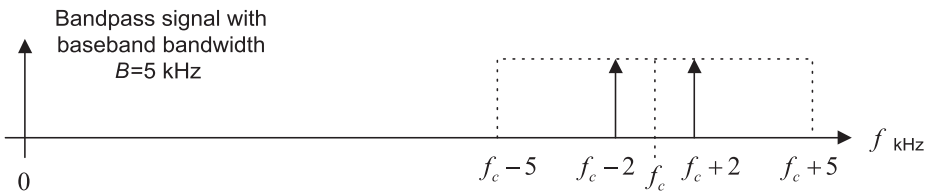


FIG. 11.46

Spectrum of the bandpass signal in Problem 11.21.

(a) $f_c = 30$ kHz
(b) $f_c = 25$ kHz
(c) $f_c = 33$ kHz

11.22 Given a bandpass signal with a spectrum shown in Fig. 11.46, and assuming $f_s = 10$ kHz, select the sampling rate and sketch the sampled spectrum ranging from 0 Hz to the carrier frequency for each of the following carrier frequency f_c
(a) $f_c = 15$ kHz
(b) $f_c = 20$ kHz

11.23 Given a bandpass signal with a spectrum shown in Fig. 11.46, and assuming $B = 5$ kHz, select the sampling rate and sketch the sampled spectrum ranging from 0 Hz to the carrier frequency for each of the following carrier frequency f_c
(a) $f_c = 35$ kHz,
(b) $f_c = 40$ kHz,
(c) $f_c = 22$ kHz.

11.8 MATLAB PROBLEMS

Use MATLAB to solve Problems 11.24–11.30.

11.24 Generate a sinusoid with a 1000 Hz for 0.05 s using a sampling rate of 8 kHz,
(a) Design a decimator to change the sampling rate to 4 kHz with specifications below:
Signal frequency range: 0–1800 Hz.
Hamming window required for FIR filter design
(b) Write a MATLAB program to implement the downsampling scheme, and plot the original signal and the downsampled signal versus the sample number, respectively.

11.25 Generate a sinusoid with a 1000 Hz for 0.05 s using a sampling rate of 8 kHz,
(a) Design an interpolator to change the sampling rate to 16 kHz with following specifications:
Signal frequency range: 0–3600 Hz
Hamming window required for FIR filter design
(b) Write a MATLAB program to implement the upsampling scheme, and plot the original signal and the upsampled signal versus the sample number, respectively.

FIG. 11.47

Decimators in Problem 11.31.

11.26 Generate a sinusoid with a frequency of 500 Hz for 0.1 s using a sampling rate of 8 kHz,

(a) Design an interpolation and decimation processing algorithm to change the sampling rate to 22 kHz
Signal frequency range: 0–3400 Hz.
Hamming window required for FIR filter design

(b) Write a MATLAB program to implement the scheme, and plot the original signal and the sampled signal at the rate of 22 kHz versus the sample number, respectively.

11.27 Repeat Problem 11.24 using the polyphase form for the decimator.

11.28 Repeat Problem 11.25 using the polyphase form for the interpolator.

11.29 **(a)** Use MATLAB to create a 1-s sinusoidal signal using the sampling rate of $f_s = 1000$ Hz

$$x(t) = 1.8\cos(2\pi \times 100t) + 1.0\sin(2\pi \times 150t + \pi/4),$$

where each sample $x(t)$ can be round off using 3-bit signed integer (directly round off the calculated $x(t)$) and evaluate the SQNR.

(b) Use MATLAB to design an oversampling system including the anti-aliasing filter with a selectable integer factor L using the same equation for the input $x(t)$.

(c) Recover the signal using the quantized 3-bit signal and measure the SQNRs for the following integer factors: $L = 2$, $L = 4$, $L = 8$, $L = 16$, and $L = 32$. From the results, explain which one offers better quality for the recovered signals.

11.30 **(a)** Use MATLAB to create a 1-s sinusoidal signal using the sampling rate $f_s = 1000$ Hz

$$x(t) = 1.8\cos(2\pi \times 100t) + 1.0\sin(2\pi \times 150t + \pi/4),$$

where each sample $x(t)$ can be round off using 3-bit signed integer (directly round off the calculated $x(t)$) and evaluate the SQNR.

(b) Use MATLAB to implement the first-order SDM system including anti-aliasing filter with an oversampling factor of 16. Measure the SQNR.

(c) Use MATLAB to implement the second-order SDM system including anti-aliasing filter with an oversampling factor of 16. Measure the SQNR. Compare the SQNR with the one obtained in (b).

11.31 Show that the two decimators are equivalent (Vaidyanathan, 1990, 1993) in Fig. 11.47.

11.32 Show that the following two interpolators are equivalent (Vaidyanathan, 1990, 1993) in Fig. 11.48.

FIG. 11.48

Interpolators in Problem 11.32.

MATLAB PROJECT

Problem 11.33 Audio-Rate Conversion System

Given a 16-bit stereo audio file ("No9seg.wav") with a sampling rate of 44.1 kHz, design a multistage conversion system and implement the designed system to convert the audio file from 44.1 to 48 kHz. Listen and compare the quality of the original audio with the converted audio.