Systems of Modal Logic

All systems of modal logic are extensions of propositional logic. They accept a common inference rule but differ by their characteristic axioms (more precisely, axiom schemata). To formulate the inference rule and axioms, we introduce the following primitive operator:

$\Box\phi$: "it is necessarily true that ϕ".

It is convenient to also introduce the following operator:

$\Diamond\phi \iff \neg\Box\neg\phi$: "it is possibly true that ϕ" (ϕ is not necessarily false iff. it is possibly true).

Now to the modal systems. The common inference rule is:

Necessitation If ϕ is a truth of logic, infer $\Box\phi$.

$$\frac{\phi}{\Box\phi}$$

The Lewis Modal Systems

System K Characteristic Axiom: $\Box(\phi \supset \psi) \supset (\Box\phi \supset \Box\psi)$

If it is necessarily true that if ϕ then ψ, then: if it is necessarily true that ϕ, then it is also necessarily true that ψ.

System D System K plus

Characteristic Axiom: $\Box\phi \supset \Diamond\phi$

If ϕ is necessarily true, then it is possibly true.

System T System K plus

Characteristic Axiom: $\Box\phi \supset \phi$

If it is necessarily true that ϕ, then it is actually true that ϕ.

System B System T plus

Characteristic Axiom: $\Diamond\Box\phi \supset \phi$

If ϕ is possibly necessarily true, then ϕ is actually true.

System S4 System T plus

Characteristic Axiom: $\Box\phi \supset \Box\Box\phi$

Whatever is necessarily true is necessarily necessary.

System S5 System T plus

Characteristic Axiom: $\Diamond\Box\phi \supset \Box\phi$

Whatever is possibly necessary is necessary.

Exercises

1. Show that the following are provable in all modal systems:
   - $\Box(p \wedge q) \supset (\Box p \supset \Box q)$
   - $(\Box p \vee \Box q) \supset \Box(p \vee q)$
   - $\Box(p \supset q) \supset (\Diamond p \supset \Diamond q)$
2. For each of the formulae in the previous question, explain in plain English (no formal symbols!) why they are plausible.
3. Prove that the characteristic axioms of B and S4 are theorems of S5.
4. An alternative reading of $\Box\phi$ is: it will always be the case that ϕ.
   a) On this reading, what would $\Diamond\phi$ mean?
   b) How plausible or implausible are the various modal systems under this interpretation? Explain for each system.
5. Here is yet another reading of the modal symbols. Let $\Diamond p$ mean that the truth of p is consistent with your knowledge (roughly, for all you know, p could be true).
   a) On this reading, what would $\Box\phi$ mean?
   b) How plausible or implausible are the various modal systems under this interpretation? Explain for each system.