Exercises for 3.5–3.11

These exercises are intended to get you used to using sequents.

1. A derivation is a series of sequents. Because of that, derivations contain information that the standardized presentation of arguments do not because that style only tracks the succedent side explicitly. The inference rules of our proof system tell you, among other things, how to keep track of things on the datum side.

   Consider the following derivation which is missing the datum on line 3:

1.	Γ	$\vdash A \supset B$	premise
2.	Δ	$\vdash A$	premise
3.	__	$\vdash B$	1,2,⊃E

   What goes in the datum of line 3? You can see in the annotation that line 3 got there by applying Conditional Elimination to lines 1 and 2. Line 1's datum is Γ and line 2's is Δ. Conditional Elimination tells us that in that case the datum of line 3 is Γ, Δ so that's what goes in there.

   Let's do a couple more of these to get used to paying attention to the datum.

   Fill in the missing datums in the following two derivations:

   (a)

1.	Γ	$\vdash (P \vee Q) \supset R$	premise
2.	Δ	$\vdash P$	premise
3.	__	$\vdash P \vee Q$	2,∨I
4.	__	$\vdash R$	1,3,⊃E

   (b)

1.	Γ	$\vdash (P \vee Q) \supset R$	premise
2.	__	$\vdash P$	A
3.	__	$\vdash P \vee Q$	2,∨I
4.	__	$\vdash R$	1,3,⊃E

   (c) You can think of the argument (a) as formalizing something like 'The college catalog says that if Masha has taken logic or has taken calculus, then she has satisfied the formal reasoning requirement. Her transcript says that she have taken logic. It follows that Masha has satisfied the formal reasoning requirement.' What would argument (b) be formalizing?

2. Our inference rules keep track of *what* supports *what* —the datum is the former what, the succedent the latter. Let's practice using our inference rules to keep track of the succedent side.

   (a) We know that $P \wedge Q$ and $Q \wedge P$ are logically equivalent. That means that if Γ supports $P \wedge Q$ it also supports $Q \wedge P$. Any decent proof system should tells us that, and ours does. Add the missing succedents in the following derivation:

   1. Γ $\vdash P \wedge Q$ .... premise
   2. Γ $\vdash$ __ .... 1,$\wedge$E
   3. Γ $\vdash Q$ .... 1,$\wedge$E
   4. Γ, Γ $\vdash$ __ .... 2,3,$\wedge$I
   5. Γ $\vdash Q \wedge P$ 4

   Hint: according to the annotations, line 5 is a rewrite of line 4. That tells you what the succedent of line 4 is.

   (b) We also know that $P \vee Q$ and $Q \vee P$ are logically equivalent. That means that if Γ supports $P \vee Q$ it also supports $Q \vee P$. Our proof system shows that, too. Add the missing succedents in the following derivation:

   1. Γ $\vdash P \vee Q$ .... premise
   2. P $\vdash$ __ .... A
   3. P $\vdash$ __ .... 2,$\vee$I
   4. Q $\vdash Q$ .... A
   5. Q $\vdash$ __ .... 4,$\vee$I
   6. Γ $\vdash Q \vee P$ .... 1,3,5,$\vee$E

   Notice that there are several lines with exactly the same succedent as the concluding line. An argument in the standardized form would make it much more difficult to discern when one has actually reached the desired conclusion because the standardized form only gives us the succedent side.

3. Comprehending a derivation requires comprehending how the various sequents work together to enable us to infer to the conclusion. Annotations are there to guide our comprehension. In presenting your derivation, it is crucial to make sure that your annotation are correct. Let's practice annotating.

   $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ are logically equivalent. So if we have evidence for the former sentence, we have evidence for the latter. We can show that. Fill in the missing annotations in the following derivation:

1.	Γ	$\vdash P \wedge (Q \vee R)$	premise
2.	Γ	$\vdash P$	1,__
3.	Γ	$\vdash Q \vee R$	__
4.	Q	$\vdash Q$	__
5.	Γ, Q	$\vdash P \wedge Q$	__
6.	Γ, Q	$\vdash (P \wedge Q) \vee (P \wedge R)$	5,∨I
7.	R	$\vdash R$	__
8.	Γ, R	$\vdash P \wedge R$	__
9.	Γ, R	$\vdash (P \wedge Q) \vee (P \wedge R)$	__
10.	Γ, Γ, Γ	$\vdash (P \wedge Q) \vee (P \wedge R)$	__
11.	Γ	$\vdash (P \wedge Q) \vee (P \wedge R)$	__

4. Fill in the missing datums, succedents, annotations in the following derivation from $\Gamma \vdash (P \vee Q) \vee R)$ to $\Gamma \vdash P \vee (Q \vee R)$.

1.	Γ	$\vdash (P \vee Q) \vee R$	premise
2.	__	$\vdash P \vee Q$	A
3.	P	$\vdash P$	A
4.	__	$\vdash P \vee (Q \vee R)$	3,∨I
5.	Q	$\vdash$ __	A
6.	__	$\vdash Q \vee R$	5,∨I
7.	Q	$\vdash$ __	6,∨I
8.	$P \vee Q$	$\vdash P \vee (Q \vee R)$	2,4,7,∨E
9.	R	$\vdash R$	A

10.	R	$\vdash Q \vee R$	............................9,∨I
11.	R	$\vdash P \vee (Q \vee R)$	........................__
12.	Γ	$\vdash P \vee (Q \vee R)$	.................1,8,11,∨E

5. I mentioned that ∨E need not require three sequents. Here is an example of that:

1.	Γ	$\vdash P \vee P$	........................premise
2.	Δ, P	$\vdash Q$	............................premise
3.	Γ, Δ	$\vdash Q$	............................1,2,2,∨E

Notice that in the third line of the annotation, line 2 is referred to twice. It is used once as $\Lambda_2, s_1 \vdash s_3$ and again as $\Lambda_3, s_2 \vdash s_3$. This is possible because two of the sequents that must be matched for using ∨E have the same form. You can do the same using ∧I to derive from $\Gamma \vdash P$ to $\Gamma \vdash P \wedge P$. Add the missing annotations:

1.	Γ	$\vdash P$	...................................__
2.	Γ	$\vdash P \wedge P$	............................__

6. Construct the following derivations:

   (a) From $\Gamma \vdash Q \wedge P$ to $\Gamma \vdash Q \vee P$.
   (b) From $\Gamma \vdash \neg Q \wedge (Q \wedge P)$ to $\Gamma \vdash Q$.
   (c) From $\Gamma \vdash P$ to $\Gamma \vdash (P \wedge P) \vee Q$