1. For each of the following, indicate whether or not the inference is allowed. Yes means allowed. a, b are constants, and they do not appear in Γ and Δ.

(a)

1. $Fa \quad \vdash Fa$ .... A
2. $Fa \quad \vdash Gb \vee Fa$ .... 1,∨I
3. $Fa \quad \vdash \exists x Gx \vee Fa$ .... 2,∃I

.... Yes/No

(b)

1. $\Gamma \quad \vdash \exists x(Gx \wedge Fx)$ .... premise
2. $Ga \wedge Fa \quad \vdash \exists x Gx$ .... premise
3. $\Gamma \quad \vdash \exists x Gx$ .... 1,2,∃E

.... Yes/No

(c)

1. $\Gamma \quad \vdash \exists x(Gx \wedge Fx)$ .... premise
2. $\Gamma, Ga \wedge Fa \quad \vdash \exists x Gx \wedge Fa$ .... premise
3. $\Gamma \quad \vdash \exists x Gx \wedge Fa$ .... 1,2,∃E

.... Yes/No

(d)

1. $Fa \quad \vdash Fa$ .... A
2. $Fa \quad \vdash Gb \vee Fa$ .... 1,∨I
3. $Fa \quad \vdash \exists x(Gx \vee Fa)$ .... 2,∃I

.... Yes/No

(e)

1. $Fa \quad \vdash Fa$ .... A
2. $Fa \quad \vdash Gb \vee Fa$ .... 2,∨I
3. $Fa \quad \vdash \exists x Gx \vee Fa$ .... 2,∃I

.... Yes/No

(f)

1. $\forall x(Fx \supset Ga) \quad \vdash \forall x(Fx \supset Ga)$ .... A
2. $\forall x(Fx \supset Ga) \quad \vdash Fa \supset Ga$ .... 1,∀E
3. $\forall x(Fx \supset Ga) \quad \vdash \forall x(Fx \supset Gx)$ .... 2,∀I

.... Yes/No

(g)

1. $\Gamma \vdash \forall x Fx$ .... premise
2. $\Delta \vdash \forall y Gy$ .... premise
3. $\Gamma, \Delta \vdash \forall x \forall y (Fx \wedge Gy)$ .... 1,2,∧I

.... Yes/No

(h)

1. $\Gamma \vdash \forall x Fx \wedge Ga$ .... premise
2. $\Gamma \vdash \forall x \forall y (Fx \wedge Gy)$ .... 1,∀I

.... Yes/No

(i)

1. $\Gamma \vdash \forall x (Fx \wedge Ga)$ .... premise
2. $\Gamma \vdash \forall y \forall x (Fx \wedge Gy)$ .... 1,∀I

.... Yes/No

(j)

1. $\Gamma, \exists x \neg Fx \vdash \exists x Mx$ .... premise
2. $\Delta, \exists x \neg Fx \vdash \neg \exists x Mx$ .... premise
3. $\Gamma, \Delta \vdash \forall x Fx$ .... 1,2,¬I

.... Yes/No

2. Let's prove the theorems known as Quantifier Exchange. Here are the first four.

   For these four, do not appeal to Quantifier Exchange anywhere. Notice we stop one step short of the actual theorem. The theorems in questions are conditionals of the form $s_1 \supset s_2$. Instead of proving that, we stop at $s_1 \vdash s_2$ because the we can get to the conditional trivially by $\supset$I. But the actual theorem is still the conditional!

   (a) $\exists x Fx \vdash \neg\forall x \neg Fx$. The proof of this is in the readings. See if you can reproduce it.

   (b) $\neg\forall x \neg Fx \vdash \exists x Fx$. Add the missing annotations (this is a little more efficient than the one in an earlier exercise).

1.	$\neg\forall x \neg Fx$	$\vdash \neg\forall x \neg Fx$	A
2.	$\neg\exists x Fx$	$\vdash \neg\exists x Fx$	A
3.	Fa	$\vdash Fa$	A
4.	__	$\vdash$ __	__
5.		$\vdash Fa \supset \exists x Fx$	4,$\supset$I
6.		$\vdash$ __	__
7.		$\vdash$ __	5,6,$\supset$E
8.	$\neg\exists x Fx$	$\vdash \neg Fa$	2,7,$\supset$E
9.	$\neg\exists x Fx$	$\vdash \forall x \neg Fx$	__
10.	__	$\vdash$ __	1
11.	__	$\vdash$ __	9, 10, $\neg$I
12.	$\neg\forall x \neg Fx$	$\vdash \exists x Fx$	__

(c) $\forall xFx \vdash \neg\exists x\neg Fx$. Prove this sequent. Hint: assume the sentence in the datum, and assume the denial of the succedent of this sequent.

(d) $\neg\exists x\neg Fx \vdash \forall xFx$. Prove this sequent. This was in an earlier exercise. See if you can reproduce it.

3. Let's prove the remaining theorems known as Quantifier Exchange. You may appeal to the theorems proven in the previous question. Hint: for the latter three, the first one below gives you something like a template.

   (a) $\exists x \neg Fx \vdash \neg\forall x Fx$. Fill in the missing items.

1.	$\exists x \neg Fx$	$\vdash \exists x \neg Fx$	............................__
2.	$\forall x Fx$	$\vdash \forall x Fx$	..............................__
3.		$\vdash \forall x Fx \supset \neg\exists x \neg Fx$	...................__
4.	$\forall x Fx$	$\vdash \neg\exists x \neg Fx$	.........................__
5.	$\exists x \neg Fx, \forall x Fx$	$\vdash \exists x \neg Fx$	............................__
6.	$\exists x \neg Fx$	$\vdash \neg\forall x Fx$	..........................__

   (b) $\neg\forall x Fx \vdash \exists x \neg Fx$. Prove this sequent.

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   (c) $\forall x \neg Fx \vdash \neg\exists x Fx$. Prove this.

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(d) $\neg\exists x Fx \vdash \forall x \neg Fx$. Prove this.

4. Let's see some cases of interactions between the conditional and quantifiers. (Hint for constructing proofs in this section: The proofs have a lot in common).

   (a) $\forall xFx \supset \forall xGx \vdash \exists x\neg Fx \vee \forall xGx$. Add the missing annotations:

1.	$\forall xFx \supset \forall xGx$	$\vdash$	$\forall xFx \supset \forall xGx$	.... __
2.		$\vdash$	$(\forall xFx \supset \forall xGx) \supset (\neg\forall xFx \vee \forall xGx)$	. __
3.	$\forall xFx \supset \forall xGx$	$\vdash$	$\neg\forall xFx \vee \forall xGx$	.... __
4.	$\neg\forall xFx$	$\vdash$	$\neg\forall xFx$	.... __
5.		$\vdash$	$\neg\forall xFx \supset \exists x\neg Fx$	.... __
6.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx$	.... __
7.	$\neg\forall xFx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	.... __
8.	$\forall xGx$	$\vdash$	$\forall xGx$	.... __
9.	$\forall xGx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	.... __
10.	$\forall xFx \supset \forall xGx$	$\vdash$	$\exists x\neg Fx \vee \forall xGx$	.... __

   (b) $\forall xFx \supset \exists xGx \vdash \exists x\neg Fx \vee \exists xGx$. Prove this.

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(c) $\exists x Fx \supset \forall x Gx \vdash \forall x \neg Fx \lor \forall x Gx$. Prove this.

(d) $\exists x Fx \supset \exists x Gx \vdash \forall x \neg Fx \lor \exists x Gx$. Prove this.

5. Brackets matter.

(a) Prove $\exists x(Fx \supset \forall y Gy) \vdash \exists x \neg Fx \vee \forall x Gx$. (Just so you see the importance of brackets, compare this to 3c.)

6. Construct models that fit the given specifications:

   (a) $\exists x Fx$ true, $\forall x Gx$ true.

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   (b) $\exists x Fx$ true, $\forall x Gx$ false.

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   (c) $\exists x Fx$ false, $\forall x Gx$ true.

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   (d) $\exists x Fx \supset \forall x Gx$ false.

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7. Explain why it is not possible to construct a model that fits the given specifications?

   (a) $\forall x(Fx \supset Gx)$ true, $\forall x(\neg Fx \vee Gx)$ false.

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   (b) $\forall x(Fx \supset Gx)$ true, $\exists xFx$ true, $\exists xGx$ false.

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   (c) $\exists xFx$ false, $\exists xGx$ false, $\exists x(Fx \vee Gx)$ true.

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