Bertrand Competition

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Figure 1: AI image generated by Dall-E 3

Contents

1 Introduction **3**

2 Notation **3**

3 Solving the Classical Model **3**

4 Augmented Demand Function **4**

5 Valid Demand Functions **7**

6 Solving Valid Demand Functions **8**

7 Conclusion **9**

8 Appendix **10**

8.1 Checking the Valid Demand Function Axioms for the Classical Bertrand Model Demand Function 10

1 Introduction

The Bertrand model of competition is a model where two competing companies, producing the same product, have to decide on optimal pricing for their product [1]. While the Cournot model focuses on the quantity that two companies produce, the Bertrand model aims to determine the prices that the companies should set in order to maximise their profits.

In order for the model to be applicable, the following assumptions must be satisfied:

- There is complete information regarding the prices and costs of production for both companies.
- Both companies simultaneously set their price.
- Both companies are producing identical products.
- Both companies are able to supply enough product to meet the demand at any given price.

In our project, we will analyse the classical model of Bertrand Competition, discussing why this is not a suitable model for representing a market. We will then introduce demand functions that are more representative of a real market. Investigating the equilibria generated by these demand functions and graphing how the price set by each company affects their profit will give us a better idea of how two firms can both make a profit within the same market.

2 Notation

We will denote our first company by A and our second company by B.

We define the profit of each company using the following formulas:

$$Profit_A = (P_A - C_A) * D_A(P_A, P_B)$$
$$Profit_B = (P_B - C_B) * D_B(P_A, P_B)$$

where P_A is the price set by company A, C_A is the cost of production for company A, and $D_A(P_A, P_B)$ is the demand for the product sold by company A given that company A sells the product at price P_A and company B sells the product at price P_B. These definitions are analogous for P_B, C_B and $D_B(P_A, P_B)$. For the sake of readability, $D_A(P_A, P_B)$ will be shortened to D_A at times.

3 Solving the Classical Model

Firstly, we will find an equilibrium for the classical model, where we set the costs of production for both companies to zero ($C_A = C_B = 0$). The model assumes that consumers are perfectly rational, meaning that they always buy the product that is the cheapest. Factors such as convenience or brand reputation are disregarded. The corresponding demand function for these assumptions is:

$$D_A = \begin{cases} D_{total}, & \text{if} P_A < P_B \\ \frac{D_{total}}{2}, & \text{if} P_A = P_B \\ 0, & \text{if} P_A > P_B \end{cases}$$

where D_{total} is the total demand. We will first eliminate the three cases where $P_A < 0$ or $P_B < 0$.

1. $\mathrm{P}_A < P_B$
2. $\mathrm{P}_A = P_B$
3. $\mathrm{P}_A > P_B$

In case 1, A has a payoff of $P_A * D_{total} < 0$, so company A would benefit by increasing their price.

In case 2, A has a payoff of $P_A * \frac{D_{total}}{2} < 0$, so company A would again benefit by increasing their price.

In case 3 B has a payoff of $P_B * D_{total} < 0$, so company B would benefit by increasing their price.

Therefore, there is no equilibrium when either P_A or P_B are less than 0.

Let us now consider the case where $P_A > 0$ and $P_B > 0$. Without loss of generality, we can take $P_A > P_B > 0$. In this case, $D_A = 0$ so $g_A(P_A, P_B) = 0$. However, $g_A(\frac{1}{2}P_B, P_B) = \frac{1}{2}P_B D_A > 0$. Therefore A would improve its payoff by changing its strategy to $\frac{1}{2}P_B$ and we do not have a Nash equilibrium.

The only remaining potential equilibrium to check is $P_A = P_B = 0$. This is an equilibrium since even though $g_A(P_A, P_B) = g_B(P_A, P_B) = 0$, both A and B cannot increase their payoffs by changing their strategy.

As a result, the demand function for the classical model gives that in any scenario, $g_A(P_A, P_B)$ and $g_B(P_A, P_B)$ are both equal to 0, resulting in neither company making a profit. This is known as the Bertrand Paradox [2]. This result does not correspond well with reality. In real economies, multiple companies in a sector can coexist and generate profit. One reason for the failings of the Bertrand model is that it relies on poor assumptions when defining the demand function. For example, real consumers will not always favour the company with the cheaper product. In the next section, we will create a demand function that gives more realistic equilibria.

4 Augmented Demand Function

We will now consider a pair of modified demand functions, with the goal of reaching more realistic solutions. As before, we will require that if $P_A < P_B$ then $D_A > D_B$. But unlike the previous case, D_A will be diminished instead of set to 0 when P_A grows. We also have that the total demand decreases if both P_A and P_B increase. The functions are shown below:

$$D_A = \frac{P_B}{(P_A + P_B)^n} \tag{1}$$

$$D_B = \frac{P_A}{(P_A + P_B)^n} \tag{2}$$

The corresponding profit functions are:

$$Profit_A = \frac{P_B(P_A - C_A)}{(P_A + P_B)^n}$$

$$Profit_B = \frac{P_A(P_B - C_B)}{(P_A + P_B)^n}$$

To find the Nash Equilibrium, we will find the points at which both companies are at a best response to each other. We first take the derivative of our profit functions with respect to the price set by a company.

$$\frac{\partial Profit_A}{\partial P_A} = \frac{P_B}{(P_A + P_B)^n} - \frac{3P_A P_B}{(P_A + P_B)^{n+1}} + \frac{3C_A P_B}{(P_A + P_B)^{n+1}}$$

$$\frac{\partial Profit_B}{\partial P_B} = \frac{P_A}{(P_B + P_A)^n} - \frac{3P_B P_A}{(P_B + P_A)^{n+1}} + \frac{3C_B P_A}{(P_B + P_A)^{n+1}}$$

Setting both equations to 0, and solving for P_A and P_B we get:

$$P_A = \frac{C_B + (n-1)C_A}{n-2}$$

$$P_B = \frac{C_A + (n-1)C_B}{n-2}$$

An interesting point that comes from this specific demand function is that $P_A - C_A = P_B - C_B$, this is shown by rearranging the equations above.

$$P_A - C_A = \frac{C_A + C_B}{n-2} = P_B - C_B$$

Now let's generalise this for any demand function. Taking the demand function D_A, if we have a global maximum $P_A^* \in \mathbb{R}^+$, for any $C_A, P_B \in \mathbb{R}^+$, then we can generalise the equilibria set (P_A, P_B).

$$Profit_A = (P_A - C_A) * D_A(P_A, P_B)$$

$$\frac{\partial}{\partial P_A} Profit_A = (P_A - C_A) * \frac{\partial}{\partial P_A} D_A(P_A, P_B) + D_A(P_A, P_B)$$

Setting the RHS to 0 and rearranging for P_A we get:

$$P_A = C_A - \frac{D_A(P_A, P_B)}{\frac{\partial}{\partial P_A} D_A(P_A, P_B)}$$

By symmetry, we also have:

$$P_B = C_B - \frac{D_B(P_A, P_B)}{\frac{\partial}{\partial P_B} D_B(P_A, P_B)}$$

We can now use both formulae to create an iterative algorithm that will find equilibria for any Bertrand Competition model with differentiable demand functions, and global maxima in the region $P_A \geq C_A \geq 0, P_B \geq C_B \geq 0$. We setup the iterative steps below:

$$P_A^{k+1} = C_A - \frac{D_A(P_A^k, P_B^k)}{\frac{\partial}{\partial P_A} D_A(P_A^k, P_B^k)}$$

$$P_B^{k+1} = C_A - \frac{D_B(P_A^{k+1}, P_B^k)}{\frac{\partial}{\partial P_B} D_B(P_A^{k+1}, P_B^k)}$$

We implemented this in Python using the following code:

```
import sympy as smp

x, y = smp.symbols('A B', real=True)
f = y/((x+y)**3)

df = smp.diff(f, x)

A = 7
B = 7
C_A = 1
C_B = 2
for i in range(100):
    A = C_A - f.subs([(x, A), (y, B)]).evalf()/df.subs([(x, A), (y, B)]).evalf()
    B = C_B - f.subs([(x, B), (y, A)]).evalf()/df.subs([(x, B), (y, A)]).evalf()
print(A, B)
```

Figure 2: Code used to find equilibria.

f is our demand function here. We took the example above and set $n = 3$. With $C_A = 1$ and $C_B = 2$, using equations (10) and (11), we would expect our code to output $P_A = 4$ and $P_B = 5$, which it does.

```
P_A = 4.00000000000000, P_B = 5.00000000000000
```

Equations 1 and 2 were our first attempts at creating augmented demand functions. After further testing, we noticed some weaknesses in our model. With our demand functions, we have that if $P_B \to \infty$ then $D_A \to 0$, which is counter-intuitive and contrary to real world market dynamics. The appearance of this flaw in our model led us to develop a more rigorous restriction on which demand functions are suitable for Bertrand Competition. In the next section, we will define the necessary conditions for demand functions which more accurately model real competition.

5 Valid Demand Functions

The Bertrand Paradox which is outlined in section 3 motivates the development of better demand functions, with the goal of improving upon the current model. In this section, we introduce the concept of valid demand functions. This family of demand functions will have a greater correspondence with real-world economies, avoiding the pitfalls of the classical model demand function.

Definition 5.1. A function D_A on $P_A \in (0, \infty)$ and $P_B \in (0, \infty)$ is a **valid demand function** for A if and only if:

- (V1) D_A is continuous on P_A, P_B.
- (V2) $P_A \to \infty, \quad D_A \to 0$.
- (V3) $P_A \to 0, \quad D_A \to \infty$.
- (V4) D_A is decreasing on P_A.
- (V5) D_A is non-decreasing on P_B.

V1 is a result of small fluctuations in price having negligible effects on demand in the real world. V2 and V3 are due to the fact that if the price of a product is very high, no consumers will want to purchase it, and hence $D_A \to 0$. Similarly, if a product is free, every consumer will want to purchase it, and hence $D_A \to \infty$. V4 guarantees that when the price of a product is increased, the demand for it will decrease. V5 codifies that if B increases the price of its product, no consumers will stop buying from A and start buying from B. Therefore, the demand for A's product does not decrease when B increases the price of their product.

Note that the demand function in the classical model violates V1, V2, V3, and V4 (V5 holds). This provides an explanation for why the classical Bertrand model does not produce realistic results. See subsection 8.1 in the appendix for evidence that the classical model's demand function fails the criteria required to be a valid demand function.

The augmented demand functions introduced in the previous section are an improvement over the classical model's demand function. They satisfy V1, V2, V3 and V4, but violate V5. Therefore, they are not valid demand functions. The next section will study a pair of valid demand functions in order to further improve our model.

6 Solving Valid Demand Functions

$$D_A = \frac{P_B^2}{P_A^2(P_A + P_B)^2}$$
$$D_B = \frac{P_A^2}{P_B^2(P_A + P_B)^2}$$

These functions satisfy all our requirements for being a valid demand function. Finding an equilibrium for these demand functions by hand is extremely difficult due to the complex algebra involved. However, we can use our Python algorithm to solve it numerically. Running this function through our algorithm, with $C_A = 1$ and $C_B = 2$, we find an equilibrium at:

```
1.58451111471209 2.87367966116040
```

We verify this result by checking both values are local maxima of their respective profit functions. To do so, we implement the following code to plot both profit functions as well as find an approximate maximum for each.

```
x_list = np.arange(1, 8, 0.01)
z_1 = (x_list - 1) * (B**2)/(x_list**2 * (x_list + B)**2)
plt.plot(x_list, z_1, label="Company A")
z_2 = (x_list - 2) * (A**2)/(x_list**2 * (x_list + A)**2)
plt.plot(x_list, z_2, label="Company B")
plt.legend(loc="upper right")
plt.xlabel("Price set.")
plt.ylabel("Profit made.")
print(x_list[np.argmax(z_1)])
print(x_list[np.argmax(z_2)])
plt.savefig('Both_profit_functions.png', format='png')
```

Figure 3: Profit against price code for the above demand function.

The two resulting values we obtained from this code are (1.5800000000000005, 2.870000000000002), which are within 0.01 of our computed equilibria, and the following graph shows our maxima are exactly where we predicted.

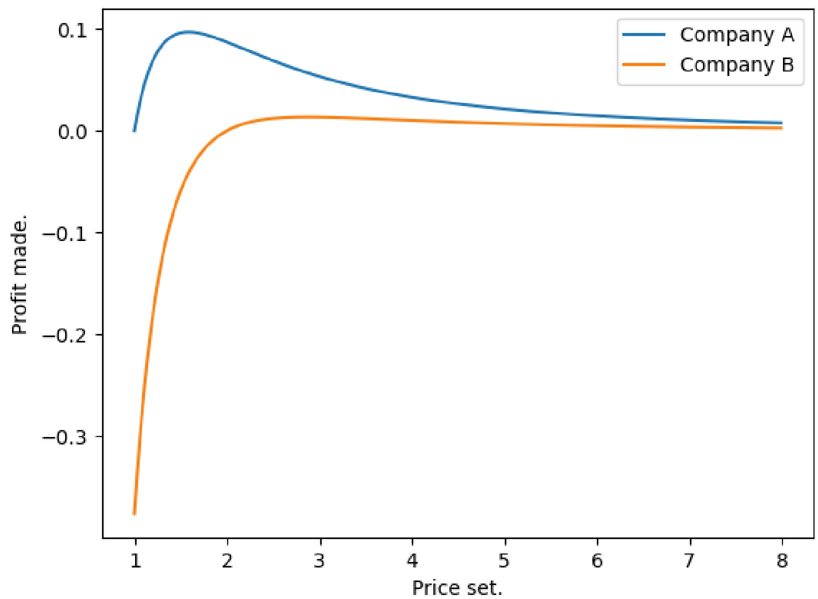

Figure 4: Profit against price graph for the above demand function.

7 Conclusion

We have explored an extension of the Bertrand Competition, where we consider improved demand functions with the aim of producing a better model of competition. We first analysed the classical model, identifying its flaws, and then proposed the notion of a valid demand function, which helps us to produce more realistic results.

In addition, we developed a computational method to find Nash equilibria for arbitrary demand functions. These demand functions can be set in order to encompass many different situations across various product markets. For example, a company with a reputation for producing products of lower quality would have a larger reduction in demand at higher prices.

Our concept of valid demand functions is a guide to setting up a useful and realistic competition model, and our Python script can be employed to easily solve the resulting games to find their Nash equilibria. As a result, our methodology will be useful to a wide variety of economic settings involving competition, and will aid the further study of Bertrand Competition.

8 Appendix

8.1 Checking the Valid Demand Function Axioms for the Classical Bertrand Model Demand Function

Trivially, the classical Bertrand model demand function violates V1.
For V2, we provide a counter-example:

Set $P_B = P_A + 1$. Therefore, $D_A = 1$.
Now take $P_A \to \infty$.
Then $P_A \to \infty$, $D_A = 1$, which violates V2.

V3 is also violated because D_A is always 0, $\frac{1}{2}$ or 1.
V4 is violated because D_A is the same for $P_A = P_B + 1$ and for $P_A = P_B + 2$, for arbitrary P_B. Therefore, D_A does not always decrease when P_A is increased, meaning D_A is not decreasing on P_A.

However, the demand function does satisfy V5. When P_B increases, D_A either stays the same or decreases.

References

[1] Salish MS. Bertrand Competition;. Available from: https://inomics.com/terms/bertrand-competition-1504578. (Accessed: 16.11.2023).

[2] Walker M. Bertrand (Nash) equilibrium;. Available from: https://www.concurrences.com/en/dictionary/bertrand-nash-equilibrium. (Accessed: 16.11.2023).