2-Row Dots and Boxes

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1 Abstract

This paper analyses the game of Dots and Boxes. It investigates irregular grid shapes as opposed to the usual square ones and aims to determine which player wins in each setup. The paper begins by analysing the 2 x 6 version of the game. This is then generalised to the 2 x n dots case. The findings from this research may have applications in the design of other games and algorithms, including the further expansion of this game to an n x n case.

2 Introduction

Dots and Boxes is a game where players take turns drawing a line between two dots on a grid of dots. The line must either be vertical or horizontal and cannot connect more than two dots. When a player completes the fourth side of a square, they win that box and get another turn. The game ends when all possible lines have been drawn and whoever has won the most boxes is the winner of the game.

In this project, we discuss a simple algorithm to always win a 2 x 6 game, and prove it works. We also prove an algorithm for the extension of this game to 2 x n, with different results for when n is even and n is odd.

3 2 x 6 Dots and Boxes

In this section, we present a winning strategy for player 2 in a 2 x 6 game of dots and boxes. We will name it the Mirroring Strategy since it involves player 2 "mirroring" the previous move that player 1 has played. The initial state of the game consists of a vertical line of symmetry which bisects the central box (see Figure 1). Whenever player 1 makes a move, the following move by player 2 is to take the mirror image move with respect to this line of symmetry, with the exception of two edges also shown in Figure 1. The mirror image of these edges are themselves. This represents a problem because if player 1 plays one of these edges, player 2 cannot play on the same edge since it has already been played. Instead, whenever player 1 plays one of these exception edges, player 2 should play the other.

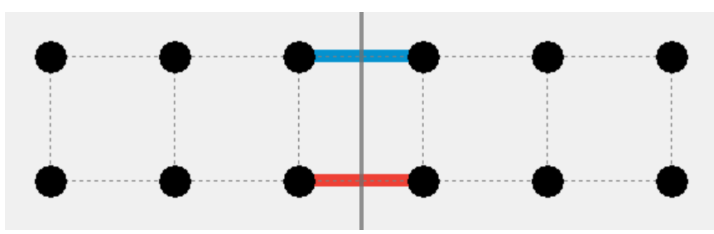

Figure 1: The vertical line of symmetry of the board and the two exception edges which do not have distinct mirror images [1].

Now, we will show that this is a winning strategy for player 2. It is helpful to consider the central box and the non-central boxes (that is, the four remaining boxes) separately. First, we will cover the central box. Whenever player 1 plays an edge of this central box, player 2's move as defined by the Mirroring Strategy will also be an edge of this central box. So, player 1 will play the first and third edges of the box and player 2 will play the second and forth edges of the box. That is, player 2 will complete the central box and earn 1 point towards their final score.

As for the non-central boxes, we now show that both players will complete two of these. When considering the non-central boxes, it is helpful to exclude the two exception edges because they do not contribute to the completion of the non-central boxes. If player 1 plays one of these exception edges in a certain game state, player 2 plays the other exception edge, and then it is player 1's turn again with the four non-central boxes in an identical state.

When we remove these exception edges, the game state will be horizontally symmetric after each move by player 2. Therefore, whenever player 1 completes one of the four boxes not in the centre, player 2 also completes its mirror image box in the next move. So, of the non-central boxes on the sides, two will be blue and two will be red.

Combining our insights on the central box and the non-central box, we conclude that the Mirroring Strategy always forces the game to end with player 1 with a score of 2 and player 2 with a score of 3. Thus, player 2 can always guarantee a victory with this strategy.

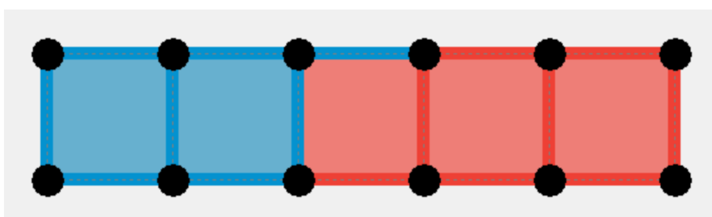

Figure 2: A basic example of a game generated by the Mirroring Strategy [1].

4 2 x n Dots and Boxes

When generalising to 2 x n dots, first considering n odd, we will track backwards from a winning position. We claim that in the position where no boxes have been completed, no boxes can be completed, however, player A must make a move that surrounds a box with 3 lines. Then when is Player A losing in this position and Player B winning?

From this position, player A initially chooses which chain (¡- need to define chain) either player B can complete, then player B chooses which chain A can complete, and so on. Or player after player A chooses this chain, player B can allow player A to complete another chain. We will call the lengths of these chains $c_i, i = 1, 2, 3, ..., m$ ordered in ascending order (i.e. $c_{i+1} > c_i$) and the points player A gets is $\sum_{j \in S_A} c_j$, where S_A is defined as all i such that chain i is awarded to A. We will use the same definition for player B. If there are an odd number of chains (m odd) then $|S_A| = |S_B| + 1$, with this as player

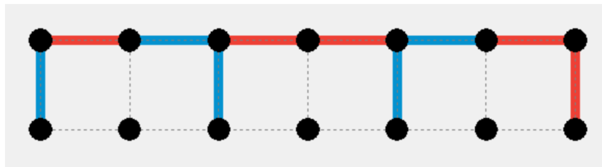

Figure 3: Example of last possible position before 3 lines around a box [1].

A will always choose the smallest chain for B to complete and vice versa so points awarded to B are $\sum_{j \in S_B} c_j = \sum_{i=1}^{\frac{m+1}{2}} c_(2i-1)$, and points awarded to A are $\sum j \in S_B = \sum_{i=1}^{\frac{m}{2}} c_{2i}$. We see that $Points_A = c_2 + c_4 + ... + c_{m-1} < c_1 + c_3 + ... + c_m = Points_B$. If m is even, this argument cannot hold, for example $c_1 = 1, c_2 = 3, c_3 = 3, c_4 = 3$. Here $Points_B = c_1 + c_3 = 4$ and $Points_A = c_2, c_4 = 6$ so Player A wins.

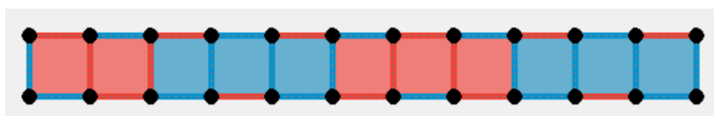

Figure 4: Completed game where B completes the first chain and still loses [1].

With this Player B always wants there to be an odd number of chains. How can they enforce this? They can by ensuring there is an even number of vertical lines not on the edges. This splits the game into $l+1$ chains, where l is the number of vertical lines not on the edges. In practice, a strategy to ensure this is the same as the strategy mentioned in Section 3, where you copy the opponent's moves. When the opponent places a vertical line in the interior so do you, l is even making m odd, note this can only be done for an even n, so player 2 is not necessarily winning when n is odd. However, is winning when n is even.

When n is odd, each player still wants to employ the strategy above, so player A wants l odd and player B wants l even, this means on each turn, with optimal play, Player A will draw an interior vertical line, and so will player B, until we get to the position.

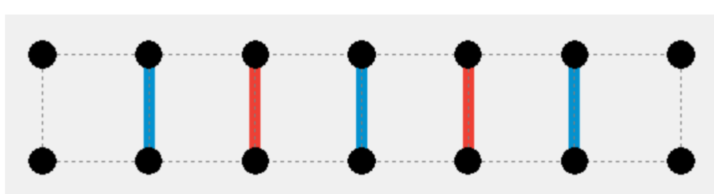

Figure 5: Necessary position for an even n game played perfectly [1].

We have an odd number of interior vertical lines, and given all are filled there are n-1 chains with $c_i = 1$. Player A and B will take turns filling in the boxes until $Points_A = Points_B = \frac{n-1}{2}$. So for n odd, we always get a draw. A completed game is shown below.

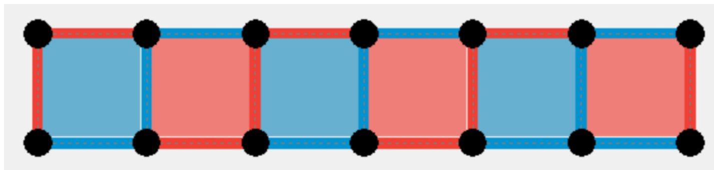

Figure 6: Completed game with even n (a draw) [1].

5 Conclusion

Here we have explored a unique way to show a 2 x 6 dots and boxes game is always losing at the start. We used the knowledge gained through exploring this game and realised we could extend it to solve any game with a width of 2 dots. This knowledge is invaluable and can be used to extend further into possibly any size dots and boxes game. We can make use of the fact that 2 losing games is losing and summing 2 winning games is winning, to solve m, 2 x n dots and boxes game. This would help solve a 2m x n game for generalised m and n.

References

[1] Anwar A. Dots-and-Boxes;. Available from: https://github.com/aqeelanwar/Dots-and-Boxes. (Accessed: 14.12.2023).