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A geometric approach to nonlocality in the Bohm model of quantum mechanics

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The nonlocality of Bohm's model of quantum mechanics is demonstrated geometrically. This is done by using the properties that in the Bohm model the particle trajectories in configuration space do not cross, yet the predictions of quantum mechanics must be reproduced. This approach provides a simple demonstration of how a model of quantum mechanics that assumes "elements of reality" for unmeasured observables must be nonlocal. While it is tempting to try to use this type of nonlocality for superluminal signalling, we discuss the fact that any attempt to do this will result in failure.

I. INTRODUCTION

In 1952, a deterministic, realistic model of quantum mechanics (in which particles always have real trajectories) was proposed by Bohm.[1] In the last decade, there has been much interest in this model, partly due to the widespread availability of computers which have enabled the trajectories predicted by this model to be computed and plotted.[2,3] There have, as a result, been numerous publications investigating, developing, and extending the model, which has resulted in two books devoted specifically to it[4,5] as well as an article in a popular science journal.[6] It has also been suggested that it be more widely taught as an antidote to complacency in asking fundamental questions in quantum mechanics, since unlike the standard approach, Bohm's model is deterministic and references to the observer can be eliminated.[7] One of the above books has also demonstrated how the Bohm model can be presented to students as a natural transition from classical mechanics (in the Hamilton–Jacobi formalism) to quantum mechanics.[5]

One interesting characteristic of the Bohm model of quantum mechanics is its explicit nonlocality for entangled two-particle states. This feature of the Bohm model is what inspired Bell to discover a proof that any deterministic model which assumes realism and reproduces quantum mechanics must be explicitly nonlocal.[8] In previous discussions of the Bohm model of quantum mechanics, the nonlocal aspects of the theory have been demonstrated algebraically.[4,5] In this article it is shown from a general principle which is true in the Bohm model, and the fact that the Bohm model reproduces the predictions of quantum mechanics, that the explicit nonlocality in this model can be demonstrated with a simple geometric method.

II. THE BOHM MODEL OF QUANTUM MECHANICS

To derive the Bohm model of quantum mechanics, first, take the Schrödinger equation for a single particle and make the substitution

$$\Psi = \mathrm{R}e^{iS/\hbar}, \tag{1}$$

where R and S are real functions of space and time for the particle. Then separate the real and imaginary parts of the equation in order to obtain the following two equations for a single-particle state:

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} - \frac{\hbar^2}{2m}\frac{\nabla^2 R}{R} + V = 0, \tag{2}$$

$$\frac{\partial R^2}{\partial t} + \nabla_1\left(R^2\,\frac{\nabla S}{m}\right) = 0. \tag{3}$$

Here, ∇ is the partial derivative with respect to the position space of the particle, and m is the particle's mass.

Bohm noticed the similarity of Eq. (2) to the Hamilton–Jacobi equation for classical mechanics. By analogy with the Hamilton–Jacobi equation, the momentum of the particle is then defined to be

$$\mathbf{p} = \nabla S. \tag{4}$$

Equation (3) then denotes conservation of probability. This is Bohm's model of quantum mechanics for a single particle. It gives the same predictions as quantum mechanics, since it is derived from the Schrödinger equation.

Note that since Ψ is a single-valued function of the position of the particle, therefore by Eq. (1), S is as well. It then follows by Eq. (4) that the momentum $\mathbf{p}$ is also a single-valued function of position. Since the momentum is a single-valued function of position, therefore if a particle described by Ψ is at a particular point in space and time, it has a uniquely defined velocity (where velocity $\mathbf{v}=\mathbf{p}/m$). Because of this, it is impossible for any two possible trajectories of the particle described by Ψ to cross.

In the following demonstration, we will be using the two-particle equation. To derive the two-particle equation in Bohm's model of quantum mechanics, one substitutes Eq. (1) into the two-particle Schrödinger equation, where now Ψ is the two-particle wave function. Then the real and imaginary parts are separated, to obtain the following two two-particle equations:

$$\frac{\partial S}{\partial t}+\frac{(\nabla_1 S)^2}{2m_1}+\frac{(\nabla_2 S)^2}{2m_2}-\frac{\hbar^2}{2m_1}\frac{\nabla_1^2 R}{R}-\frac{\hbar^2}{2m_2}\frac{\nabla_2^2 R}{R}+V=0, \tag{5}$$

$$\frac{\partial R^2}{\partial t}+\nabla_1\left(R^2\frac{\nabla_1 S}{m_1}\right)+\nabla_2\left(R^2\frac{\nabla_2 S}{m_2}\right)=0. \tag{6}$$

Here, ∇_1 and ∇_2 are partial derivatives with respect to the position spaces of particles 1 and 2 respectively, and m_1 and m_2 are the particle masses.

The momentum of each particle is then defined by

$$\mathbf{p}_1=\nabla_1 S, \tag{7}$$

$$\mathbf{p}_2=\nabla_2 S. \tag{8}$$

This is Bohm's model of quantum mechanics for a two-particle system.

From the above, we can denote the system momentum as

$$\mathbf{p}_{12}=\nabla_{12}S=(\nabla_1+\nabla_2)S. \tag{9}$$

Here, $\mathbf{p}_{12}$ is a single-valued function of the positions of both particles 1 and 2, since the wave function Ψ is a single-valued function. Therefore, the possible system trajectories in the configuration space consisting of the combined position space of particles 1 and 2 in the Bohm model of quantum mechanics *do not cross*, since at every position and time in configuration space, there can only be a single value possible for the momentum.[9] We can therefore take as a fundamental principle that:

In the Bohm model, system trajectories in configuration space do not cross.

A clear demonstration of this principle is given in the case of a single particle discussed above. In this situation, the possible system trajectories are actually the possible particle trajectories in position space. Therefore, the single-particle trajectories do not cross. If we prepare a particle in the state $|\Psi\rangle=(1/\sqrt{2})(|A\rangle+|A'\rangle)$, where $|A\rangle$ and $|A'\rangle$ refer to disjoint momentum states, and if then we bring these two states to cross one another as shown in Fig. 1, the particle trajectories do not cross but appear to "bounce" off of one another.

From this principle and the requirement that the Bohm model must reproduce the predictions of quantum mechanics, we can demonstrate nonlocality in the Bohm model in a very simple way.

III. DEMONSTRATING NONLOCALITY

Now, we set up the situation as shown in Fig. 2. In this case, we have the initial state

$$|\Psi\rangle=\frac{1}{\sqrt{2}}(|A\rangle_1|B\rangle_2+|A'\rangle_1|B'\rangle_2). \tag{10}$$

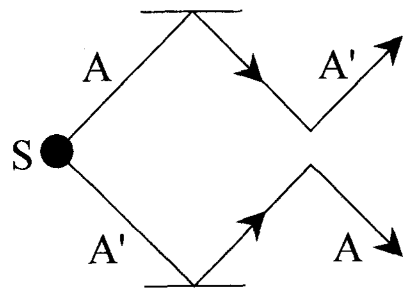

Fig. 1. The trajectories in the Bohm model for a single particle. S is a single particle source, which emits a single particle into the state $(1/\sqrt{2})(|A\rangle+|A'\rangle)$, where A and A' are the possible paths of the particle. After emission, paths A and A' are each reflected from mirrors. The possible paths may not cross, so in the situation pictured here, the paths appear to "bounce" off of one another.

This can be understood to mean either particle 1 travels along path A and particle 2 travels along path B, superposed with the possibility that particle 1 travels along path A' and particle 2 travels along path B'. We then place two mirrors on paths A and A' so that A and A' cross. According to quantum mechanics, if particle 2 is detected at some point along path $B(B')$, then particle 1 must be detected at some point along path $A(A')$, either before or after it crosses path $A'(A)$. Therefore, to reproduce the predictions of quantum mechanics, in this experiment according to the Bohm model of quantum mechanics, the particle 1 trajectories must cross. Note though that in this case, since the particle 2 trajectories do not cross, the system trajectories in configuration space also do not cross.

Now we make a single change to this experiment—we place two mirrors in the paths B and B' of particle 2, so that there is a symmetry in the experiment between what is done to particles 1 and 2, as shown in Fig. 3. To work out the particle 2 trajectories, we must take into account two constraints: the trajectories must reproduce the predictions of quantum mechanics but the system trajectories must not cross. Quantum mechanics predicts that for the initial state $|\Psi\rangle$, if we detect particle 1 in path $A(A')$, then particle 2 must be detected in path $B(B')$.

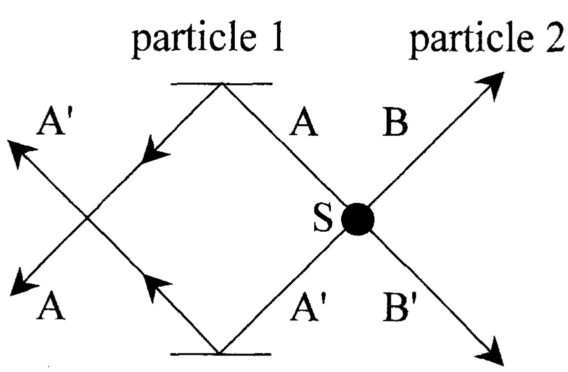

Fig. 2. An example of the trajectories in the Bohm model for two particles. S is a two-particle source, which emits two particles in the state $(1/\sqrt{2})(|A\rangle_1|B\rangle_2+|A'\rangle_1|B'\rangle_2)$, where A and A' are possible paths of particle 1, and B and B' are possible paths of particle 2. After emission, paths A and A' are reflected from mirrors. The system trajectories may not cross; however, the predictions of quantum mechanics must be reproduced. This can be done in the situation pictured here only when paths A and A' cross.

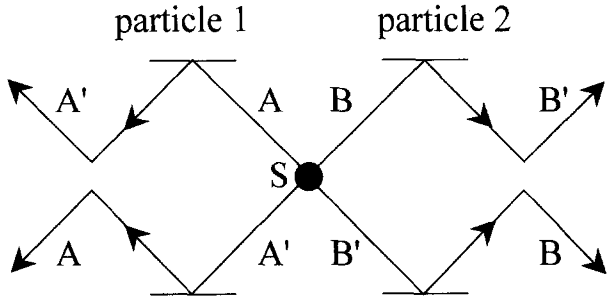

Fig. 3. The situation here is identical to that in Fig. 2, except that now all paths A, A', B, and B' are simultaneously reflected by mirrors. The system paths may not cross, but the predictions of quantum mechanics must be reproduced. This can only be achieved in the situation shown here when paths A and A' "bounce" off of each other, and paths B and B' "bounce" off of each other. The combination of both Figs. 2 and 3 show that an experimenter changing what is done to particle 2 can instantaneously change the trajectory of particle 1.

In the above case, the trajectory for the whole system is situated in the position space of the two particles. This a six-dimensional space, with three dimensions from the position space of particle 1, and three dimensions from the position space of particle 2. We first consider the situation where the possible paths of particle 1 cross, and the possible paths of particle 2 cross. Before and after the paths cross, at any particular time there are two possible positions particles 1 and 2 can be—particle 1 can be in path A and particle 2 in path B, or particle 1 can be in path A' and particle 2 in path B'. This means that in the six-dimensional position space of both particles, there are two possible trajectories for the system. However, at the point at which both paths cross, there is only one possible position particle 1 can be, and one possible position particle 2 can be. This means that in the six-dimensional position space of both particles, in this case there is only one possible point each particle can be, and hence only one possible point the two trajectories can pass through. Therefore, if the paths cross in this experiment, then the system trajectories in the position space of both particles must cross. This situation therefore cannot occur in the Bohm model since the system trajectories cannot cross.

The only other possible alternative which would reproduce the predictions of quantum mechanics is the situation shown in Fig. 3, where the particle 1 trajectories appear to "bounce" off of each other, and similarly the particle 2 trajectories appear to "bounce" off of each other. In this situation, the system trajectories in configuration space do not cross; therefore these are the trajectories taken by the particles according to the Bohm model of quantum mechanics.[10]

Comparing the experiments shown in Figs. 2 and 3, we note that the only experimental difference between them are the two mirrors placed in the possible trajectories of particle 2 in the second experiment (Fig. 3). However, this single change in the experimental setup has caused an instantaneous change in the particle 1 trajectories, that is, instead of crossing, the particle 1 trajectories now seem to "bounce" off of each other. Thus, a simple change in the experimental apparatus, affecting only particle 2, has caused an instantaneous, and thus superluminal, change in the possible paths of particle 1.

IV. CAN WE SEND A SUPERLUMINAL SIGNAL?

The above discussion suggests that it might be possible for one person to send a superluminal signal using this device. If this is true, then this would permit us to violate causality.

Consider now two observers, Aisha and Ben, such that particle 1 is heading toward Aisha, and particle 2 heads toward Ben. Ben is quite attracted to Aisha, and therefore he would like to send her a very special superluminal signal. Let there be mirrors in the possible paths of particle 1, and let Ben be able to make the choice as to whether he places mirrors in the path of particle 2 or not. Assume that Aisha will always make measurements of the path of the particle after the possible paths of particle 1 have crossed. Therefore, the experimental setup will either be that of Fig. 2 or Fig. 3, depending on whether Ben chooses to place mirrors in the possible paths of particle 2 or not.

Now let us assume that Ben can somehow know which path particle 2 takes without disturbing the wave function of particles 1 and 2. In this situation, if particle 2 is in path B, Ben can remove any mirrors in the path of particle 2, ensuring that Aisha will detect particle 1 in path A. If particle 2 is in path B', then Ben can choose to put the mirrors in place, again ensuring that the Aisha will detect particle 1 in path A. By exchanging his decisions, Ben could also cause particle 1 to be detected instead in path A'. Therefore, with this knowledge of which path particle 2 takes, Ben can instantaneously control which outcome Aisha sees. This can thus be used as a superluminal signaling device.

The mistake in the above reasoning is the assumption that Ben can detect which path particle 2 is in without somehow affecting the two-particle wave function. In reality, detecting which path particle 2 takes will alter the wave function, entangling the 2-particle wave function with Ben. Say Ben has a detector for the path of particle 2, and without affecting the path of particle 2, this detector causes a third particle to travel a path $C(C')$ if particle 2 is in path $B(B')$. In this case, the system trajectories now not only include particles 1 and 2, but also the path of the particle 3, and the state of the particles will be evolve to become

$$|\Psi\rangle \rightarrow \frac{1}{\sqrt{2}}(|A\rangle_1|B\rangle_2|C\rangle_3+|A'\rangle_1|B'\rangle_2|C'\rangle_3). \tag{11}$$

Paths C and C' do not cross; this is necessary for Ben to be able to use particle 3 to obtain path-knowledge. In order for particle 1 to be in path $A(A')$ and particle 2 to be in path $B(B')$ when particle 3 is in path $C(C')$, at all possible path positions (including before and after paths A and A' cross), the possible paths of particle 1 are required to cross. Therefore, if Ben obtains path knowledge, he loses his control over whether the paths of particle 1 cross, and thus also his ability to control what Aisha sees and thus to send a superluminal signal. Since the Bohm model reproduces the experimental results of quantum mechanics, this argument then also extends to the inability to send a superluminal signal using this technique in quantum mechanics. Ben cannot send the superluminal signal he hoped he could; instead, he will have to use the telephone or the internet like everyone else.

Although quantum mechanics has been shown (using some reasonable assumptions) by Bell to exhibit a type of nonlocality,[8] it has also been demonstrated that nevertheless, it is impossible in general to use this nonlocality to send a superluminal signal.[11] This particular type of nonlocality is called "passion at a distance." [12] Ben, however, personally did not find this name very appropriate for his intended applications.

V. CONCLUSION

By using only the principle that system trajectories in the Bohm model cannot cross, and that the Bohm model reproduces the predictions of quantum mechanics, we have demonstrated geometrically that the Bohm model is explicitly nonlocal. This serves as a simple illustration of how a model of quantum mechanics which assumes that elements of reality exist for unmeasured observables must be nonlocal. Despite this nonlocality, quantum mechanics does not permit the sending of superluminal signals. Nature manages to thwart any attempts to violate causality.

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The quasilinear generator and $\int \mathbf{B}\cdot d\mathbf{S}=0$

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At the undergraduate level, it is often easier to measure electrical magnitudes than magnetic ones. Working within the framework of classical electromagnetism (i.e., without referring to massive photons or to hypothetical magnetic monopoles) we develop an inexpensive and easy-to-use device capable of verifying a fundamental property of the magnetic flux density as applied to free space, $\int \mathbf{B}\cdot d\mathbf{S}=0$, to better than 5 parts per million accuracy. In order to achieve the above purpose, we make a slight variant of our former quasilinear generator. Gauss's theorem ensures that div $\mathbf{B}=0$ for empty space to the same accuracy as above.

I. INTRODUCTION

In a former paper we described the behavior of a quasilinear generator in which a conducting cylinder rotates while immersed in a radial (uniform and static) magnetic field $\mathbf{B}$.[1]

Promptly after the beginning of the study of electromagnetic induction, we realized that ordinary (varying flux) generators are additive, i.e., the electromotive force produced (emf) by N loops connected in a series turns out be N times the emf due to a single wire.

Unlike ordinary generators, unipolar devices cannot be additive. It is easy to show experimentally that the fundamental law div $\mathbf{B}=0$ precludes the existence of additive unipolar devices.

Our present aim is to check a well-known law of magnetism with the aid of electrical measurements. Employing the same source of $\mathbf{B}$ as in Ref. 1, we replace the above cylinder with a toroidal coil ($N=1000$ loops, inner radius $R_I=30.0\pm0.2$ mm; outer radius $R_O=40.0\pm0.2$ mm) in order

