# Tunneling through a thin potential barrier in free fall

Siddhant Das
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Consider Schrödinger's equation

$$i\hbar\frac{\partial}{\partial t}\psi(z,t) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2}\psi(z,t) + \big(\gamma\,\delta(z) + mgz\big)\,\psi(z,t), \tag{1}$$

for a quantum particle of mass $m$ falling freely under gravity that encounters a narrow potential barrier of strength $\gamma$ placed at $z=0$. Well-known length and energy scales associated with the problem, *viz.*,

$$\ell_0 = \sqrt[3]{\frac{\hbar^2}{2m^2 g}}, \qquad E_0 = mg\ell_0, \tag{2}$$

allow one to define the following dimensionless quantities:

$$z' = \frac{z}{\ell_0}, \quad t' = \frac{E_0}{\hbar}t, \quad \gamma' = \frac{\gamma}{E_0\,\ell_0}. \tag{3}$$

Equation (1) expressed in terms of the primed quantities reads

$$i\frac{\partial}{\partial t'}\psi'(z',t') = -\frac{\partial^2}{\partial z'^2}\psi'(z',t') + \big(\gamma'\,\delta(z') + z'\big)\,\psi'(z',t'), \tag{4}$$

where

$$\psi'(z',t') = \sqrt{\ell_0}\,\psi\left(\ell_0\,z', \frac{\hbar}{E_0}t'\right) \tag{5}$$

is the non-dimensionalized wave function. In what follows, we drop the primes for brevity. (This is equivalent to setting $\hbar = 2m = g/2 = 1$.) Equation (4) therefore becomes

$$i\frac{\partial}{\partial t}\psi(z,t) = -\frac{\partial^2}{\partial z^2}\psi(z,t) + \big[\gamma\,\delta(z) + z\big]\,\psi(z,t). \tag{6}$$

The solution can be written as

$$\psi(z,t) = \int_{-\infty}^{\infty}dz'\;K_\gamma\big(z,z';t\big)\psi\big(z',0\big), \tag{7}$$

where $K_\gamma\big(z,z';t\big)$ is the propagator and $\psi(z,0)$ the specified initial condition. The Green's function,

$$G\big(z,z';E\big) = i\int_0^{\infty}dt\;K\big(z,z';t\big)\,e^{iEt}, \tag{8}$$

can be written as

$$G_\gamma\big(z,z';E\big) = G_0\big(z,z';E\big) - \frac{G_0\big(z,0;E\big)\,G_0\big(0,z';E\big)}{G_0\big(0,0;E\big) - \gamma^{-1}}, \tag{9}$$

where $G_0\big(z,z';E\big)$ is the Green's function in the absence of the $\delta$-barrier ($\gamma = 0$). In the present case, $G_0$ is known exactly:

$$G_0\big(z,z';E\big) = \pi\,\mathrm{Ai}\big(z_> - E\big)\,\mathrm{Ci}\big(z_< - E\big), \tag{10}$$

where Ai, Ci are the Airy functions, and

$$z_{\gtrless} = \begin{matrix}\max\\ \min\end{matrix}\{z,z'\} = \frac{1}{2}\Big(z + z' \pm |z-z'|\Big). \tag{11}$$

Since we are considering initial conditions $\psi(z\le 0,0)\approx 0$ and are only interested in the wave function values at the detection point $z=-L<0$, it suffices to set

$$z_> = z' \;\text{ and }\; z_< = z \tag{12}$$

in what follows. Incorporating this into (9), we obtain

$$G_\gamma\big(z,z';E\big) = \frac{\pi\,\mathrm{Ai}\big(z'-E\big)\,\mathrm{Ci}\big(z-E\big)}{1 - \pi\gamma\,\mathrm{Ai}(-E)\,\mathrm{Ci}(-E)}. \tag{13}$$

The propagator can be recovered from Green's function via the inverse transform formula:

$$K\big(z,z';t\big) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}dE\;G\big(z,z';E\big)\,e^{-iEt}. \tag{14}$$

In view of (13), it follows that

$$K_\gamma\big(z,z';t\big) = \pi\int_{-\infty}^{\infty}dE\;\frac{\mathrm{Ci}\big(z-E\big)\,\mathrm{Ai}\big(z'-E\big)}{1 - \pi\gamma\,\mathrm{Ai}(-E)\,\mathrm{Ci}(-E)}\,e^{-iEt}. \tag{15}$$

Introducing the function

$$\hat{\psi}(E) = \int_{-\infty}^{\infty}dz'\;\underline{\mathrm{Ai}\big(z'-E\big)}\,\underline{\psi\big(z',0\big)}, \tag{16}$$

referred to as the Airy transform of $\psi(z,0)$, the time-dependent wave function, Eq. (7), can be expressed as follows

$$\psi(z,t) = \pi\int_{-\infty}^{\infty}dE\;\frac{\hat{\psi}(E)\,\mathrm{Ci}\big(z-E\big)\,e^{-iEt}}{1 - \pi\gamma\,\mathrm{Ai}(-E)\,\mathrm{Ci}(-E)}. \tag{17}$$

We proceed next to obtaining the far-field $|z|\gg 1$ asymptotic form of the wave function. For this, we consider integrals of the form

$$I(z,t) = \int_{-\infty}^{\infty}dE\;f(E)\,\mathrm{Ci}(z-E)\,e^{-iEt}, \tag{18}$$

featuring a complex-valued function $f(E)$ that decays rapidly for $|E|\to\infty$. Invoking the integral representation of the Airy function

$$\mathrm{Ci}(x) = \frac{1}{\pi}\int_0^{\infty}dk\;\exp\!\left[\pm i\frac{k^3}{3} \pm ikx\right] + \exp\!\left[-\frac{k^3}{3} + kx\right], \tag{19}$$

we write $I$ as a double integral, *viz.*,

$$I(z,t) = \frac{1}{2\pi}\iint_{-\infty}^{\infty}dE\,dk\;f(E)\,\exp\big[i\phi(E,k)\big], \tag{20}$$

where

$$\phi(E,k) = \frac{k^3}{3} + (z-E)k - Et. \tag{21}$$

For later convenience, we consider the derivative of $I$ w.r.t. $z$ as well, given by

$$\partial_z I(z,t) = \frac{i}{2\pi} \iint_{-\infty}^{\infty} dE\, dk\, k\, f(E) \exp\big[i\phi(E,k)\big]. \tag{22}$$

We prepare for applying the stationary phase argument to the above integrals. For this, evaluate

$$\partial_E \phi = -k - t \text{ and } \partial_k \phi = k^2 + z - E.$$

It follows that $\nabla\phi(E,k) = \mathbf{0}$ iff $k = -t$ and $E = z + t^2$, i.e., $(z+t^2, -t)$ is a stationary point of the phase $\phi$. The Hessian matrix of $\phi$ and determinant thereof are readily computed:

$$H_\phi(E,k) = \begin{bmatrix} 0 & -1 \\ -1 & 2k \end{bmatrix}, \quad \det\big[H_\phi(E,k)\big] = -1. \tag{23}$$

Applying the stationary phase argument, we are led to the asymptotic expressions

$$I(z,t) \sim f\big(z+t^2\big) \exp\Big[i\phi\big(z+t^2,-t\big) - \frac{i\pi}{4}\Big] \tag{24}$$

$$\partial_z I(z,t) \sim -itf\big(z+t^2\big) \exp\Big[i\phi\big(z+t^2,-t\big) - \frac{i\pi}{4}\Big], \tag{25}$$

which yields

$$\operatorname{Im}\big[I^*(z,t)\,\partial_z I(z,t)\big] = -t\big|f\big(z+t^2\big)\big|^2. \tag{26}$$

Using this result, we can calculate the arrival-time distribution on a plane $z = -L$, *viz.*, $\mathbf{J}\cdot d\mathbf{s}$, where $\mathbf{J} = 2\operatorname{Im}[\psi^*\nabla\psi]$ is the probability current, and $d\mathbf{s} = -dxdy\,\hat{\mathbf{z}}$ the differential area element. All in all, the arrival-time density per unit area is

$$\Pi_\gamma(\tau) = -J_z(-L,\tau) \approx \frac{2\tau\big|\hat{\psi}\big(L+\tau^2\big)\big|^2}{1 + \big(2\pi\gamma\big)^2 \mathrm{Ai}^4\big(-L-\tau^2\big)}. \tag{27}$$

This boils down to the following result

$$\boxed{\Pi_\gamma(\tau) \approx \frac{\Pi_0(\tau)}{1 + \big[2\pi\gamma\,\mathrm{Ai}^2\big(-L-\tau^2\big)\big]^2}} \tag{28}$$

### Appendix A: Gaussian initial condition

Before proceeding further, we write down the Airy transform for a generic (normalized) Gaussian wave packet

$$\psi(z,0) = \frac{1}{\sqrt{\sigma\sqrt{\pi}}} \exp\Big[-\frac{(z-z_0)^2}{2\sigma^2} + iv_0(z-z_0)\Big] \tag{A1}$$

of width $\sigma$, centered a distance $z_0$ to the right of the $\delta$-barrier. Here, $z_0$ and $\sigma$ are expressed in the units of $\ell_0$, while $v_0$, the phase velocity, in the units of $\hbar/(m\ell_0)$. For $v_0 = 0$, the Airy transform (16) of this wave packet is given by

$$\hat{\psi}(E) = \big(2\sigma\sqrt{\pi}\big)^{1/2} \exp\Big[\frac{\sigma^2}{2}\Big(\frac{\sigma^4}{6} + z_0 - E\Big)\Big] \times \mathrm{Ai}\Big(\frac{\sigma^4}{4} + z_0 - E\Big). \tag{A2}$$

For a nonzero $v_0$, the transform is obtained by letting $z_0 \mapsto z_0 + i\sigma^2 v_0$ in the above, and multiplying the result by $\exp\big(-v_0^2\sigma^2/2\big)$.