

Arrival/Detection Time of Dirac Particles in One Space Dimension

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Abstract

In this paper we study the arrival/detection time of Dirac particles in one space dimension. We consider particles emanating from a source point inside an interval in space and passing through detectors situated at the endpoints of the interval that register their arrival time. Unambiguous measurements of “arrival time” or “detection time” are problematic in the orthodox narratives of quantum mechanics, since time is not a self-adjoint operator. We instead use an absorbing boundary condition proposed by Tumulka for Dirac’s equation for the particle, which is meant to simulate the interaction of the particle with the detectors. By finding an explicit solution, we prove that the initial-boundary value problem for Dirac’s equation satisfied by the wave function is globally well-posed, the solution is smooth, and depends smoothly on the initial data. We verify that the absorbing boundary condition gives rise to a non-negative probability density function for arrival/detection time computed from the flux of the conserved Dirac current. By contrast, the free evolution of the wave function (i.e., if no boundary condition is assumed) will not in general give rise to a nonnegative density, while Wigner’s proposal for arrival time distribution fails to give a normalized density when no boundary condition is assumed. As a consistency check, we verify numerically that the arrival time statistics of Bohmian trajectories match the probability distribution for particle detection time derived from the absorbing boundary condition.

1 Introduction

In orthodox quantum mechanics, speaking of arrival/detection time of quantum particles is fraught with problems, since time does not lend itself to the self-adjoint operator formalism required for that approach¹. Nevertheless, the arrival/detection time of a particle is something that is routinely measured in time-of-flight (TOF) experiments performed in labs [8, 9, 11]. There are numerous competing recipes in the literature for what the distribution of arrival/detection times should be (see [3, 5, 6, 4] for a critique of some of these approaches, and the possibility of using experiments to distinguish them.) Many of these approaches ignore the presence of detection devices and consider the

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¹It is possible to define time more generally as a *positive operator-valued measure* [13], but there is no unique way of doing that [16].

wave function of the particle to evolve unitarily, either under the free evolution or in presence of an external potential. An alternative approach was taken by Tumulka [14], based on an idea of Werner [17], in which the presence of the detector is modeled through the imposition of an absorbing boundary condition on the Schrödinger flow (in the non-relativistic case) or the Dirac flow (in the relativistic case) of the wave function of the particle. Tumulka showed that under such a boundary condition, his candidate for the probability density of the particle's arrival/detection time is always non-negative. Subsequently, Teufel and Tumulka succeeded in showing [12] that the corresponding boundary value problem for the wave function has a unique normalizable solution, in both the relativistic and the non-relativistic cases. Their existence proof uses techniques from functional analysis, and does not yield an explicit formula for the solution in either case.

In this note we show that in one space dimension, the initial-boundary-value problem for the Dirac equation satisfied by the wave function of a single spin-half particle, with absorbing boundary conditions corresponding to a pair of *ideal* detectors² placed at the two endpoints of an interval containing the particle source and equidistant from it, is exactly solvable, and that the solution inherits the regularity of the initial data. This in particular allows for actual particle trajectories to be computed from any starting position for the particle that is distributed randomly according to any given initial wave function, thereby setting the stage for comparisons to be made with other proposals for arrival/detection time distribution, and the possibility of experimental testing of this theory.

2 Finding the wave function

Using the proposed equations in [14], we let Ω be an open interval in \mathbb{R} and let $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} : [0, \infty) \times \Omega \rightarrow \mathbb{C}^2$ be the unique solution of the initial-boundary value problem (IBVP)

$$\begin{cases} i\hbar\gamma^\mu\partial_\mu\psi &= mc^2\psi \\ \psi(0, \mathbf{s}) &= \psi_0(\mathbf{s}); \quad \mathbf{s} \in \Omega, \quad \psi_0 \in C_c^\infty(\Omega) \\ \vec{n}(\mathbf{s}) \cdot \vec{\alpha} \psi(t, \mathbf{s}) &= \psi(t, \mathbf{s}); \quad t \geq 0, \mathbf{s} \in \partial\Omega \end{cases} \quad (1)$$

Here $\{\gamma^0, \gamma^1\}$ are Dirac gamma matrices, m is the rest mass of the spin-1/2 particle, c is the speed of light in vacuum, \hbar is Planck's constant, \vec{n} is the normal to $\partial\Omega$, and $\vec{\alpha} = \alpha^1 := \gamma^0\gamma^1$.

The boundary of the spacetime domain for ψ is the set of points $\{(t, -L), (t, L)\}$. So we have $\vec{n} = 1$ at (t, L) , $\vec{n} = -1$ at $(t, -L)$. Choosing $\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, we have $\vec{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Plugging these into the boundary condition, we get $\psi_+(t, L) = 0$ and $\psi_-(t, -L) = 0$ for all $t \geq 0$. Now plugging these two boundary conditions into the Dirac equation, we get the additional boundary condition $m\psi_\mp(t, \pm L) = \mp i\partial_s\psi_\pm(t, \pm L)$.

²See [14] for the definition of ideal vs. non-ideal detectors.

This gives us the the IBVP

$$\left\{ \begin{array}{l} \partial_t^2 \psi_{\pm} - \partial_s^2 \psi_{\pm} + m^2 \psi_{\pm} = 0 \\ \psi_{\pm}(0, s) = \dot{\psi}_{\pm}(s) \\ \partial_t \psi_{\pm}(0, s) = -im\dot{\psi}_{\mp}(s) \pm \partial_s \dot{\psi}_{\pm}(s) \\ \psi_{\pm}(t, \pm L) = 0 \\ m\psi_{\mp}(t, \pm L) \pm i\partial_s \psi_{\pm}(t, \pm L) = 0 \end{array} \right. \quad (2)$$

(We have set $c = \hbar = 1$.)

The following theorem shows that (2) has an explicit solution by splitting the IBVP into its initial and boundary value problem parts and finding the solutions to each, then adding these solutions together to get the solution for ψ . Although this solution is in the form of an infinite series, we shall see that at any fixed time there are only finitely many nonzero terms in it, so that there are no convergence issues.

Theorem 1. *The IBVP in (2) has a unique solution that is smooth everywhere and depends smoothly on the initial data.*

Proof. In order to solve these equations we can set $\psi_{\pm} = \Phi_{\pm} + \chi_{\pm}$, where Φ_{\pm} are the solutions to

$$\left\{ \begin{array}{l} \partial_t^2 \Phi_{\pm} - \partial_s^2 \Phi_{\pm} + m^2 \Phi_{\pm} = 0 \\ \Phi_{\pm}(0, s) = \dot{\psi}_{\pm}(s) \\ \partial_t \Phi_{\pm}(0, s) = -im\dot{\psi}_{\mp}(s) \pm \partial_s \dot{\psi}_{\pm}(s) \end{array} \right. \quad (3)$$

and where χ_{\pm} are functions defined on $[0, \infty) \times (-\infty, L)$, resp. $[0, \infty) \times (-L, \infty)$ that satisfy

$$\left\{ \begin{array}{l} \partial_t^2 \chi_{\pm} - \partial_s^2 \chi_{\pm} + m^2 \chi_{\pm} = 0 \\ \chi_{\pm}(0, s) = 0 \\ \partial_t \chi_{\pm}(0, s) = 0 \\ \chi_{\pm}(t, \pm L) = f_{\pm}(t) \\ m\chi_{\mp}(t, \pm L) \pm i\partial_s \chi_{\pm}(t, \pm L) = i\partial_t f_{\pm}(t) \end{array} \right. \quad (4)$$

where $f_{\pm}(t) := -\Phi_{\pm}(t, \pm L)$.

(3) is the Cauchy problem for the one-dimensional Klein-Gordon equation, in older literature called the *telegraph* equation (e.g. [2], p.544), whose solution is well-known (see e.g. [15]):

$$\begin{aligned} \Phi_{\pm}(t, s) = & \frac{1}{2}[\dot{\psi}_{\pm}(s-t) + \dot{\psi}_{\pm}(s+t)] - \frac{tm}{2} \int_{s-t}^{s+t} \frac{J_1(m\sqrt{t^2 - (s-\sigma)^2})}{\sqrt{t^2 - (s-\sigma)^2}} \dot{\psi}_{\pm}(\sigma) d\sigma \\ & - \frac{1}{2} \int_{s-t}^{s+t} J_0(m\sqrt{t^2 - (s-\sigma)^2}) (im\dot{\psi}_{\mp}(\sigma) \mp \partial_{\sigma} \dot{\psi}_{\pm}(\sigma)) d\sigma \end{aligned} \quad (5)$$

where J_n is the Bessel function of order n .

Using integration by parts, the above can be rewritten as follows:

$$\begin{aligned} \Phi_{\pm}(t, s) = & \dot{\psi}_{\pm}(s \pm t) - \frac{m}{2} \int_{s-t}^{s+t} J_1(m\sqrt{t^2 - (s-\sigma)^2}) \sqrt{\frac{t \mp (s-\sigma)}{t \pm (s-\sigma)}} \dot{\psi}_{\pm}(\sigma) d\sigma \\ & - \frac{im}{2} \int_{s-t}^{s+t} J_0(m\sqrt{t^2 - (s-\sigma)^2}) \dot{\psi}_{\mp}(\sigma) d\sigma. \end{aligned} \quad (6)$$

By using the Laplace transform, we can find the general solution to the first three equations of (4):

$$\begin{aligned}\tilde{\chi}_+(p, s) &= c_1(p)e^{ks} + c_2(p)e^{-ks} \\ \tilde{\chi}_-(p, s) &= c_3(p)e^{ks} + c_4(p)e^{-ks},\end{aligned}\tag{7}$$

where

$$k := \sqrt{p^2 + m^2}, \quad \text{Re}(p) > 0\tag{8}$$

and tilde denotes the Laplace transform, i.e. $\tilde{\Psi}(p) = \int_0^\infty f(t)e^{-pt}dt$. Applying this to the boundary conditions in (4), we get

$$\begin{aligned}\tilde{f}_+(p) &= c_1(p)e^{kL} + c_2(p)e^{-kL} \\ \tilde{f}_-(p) &= c_3(p)e^{-kL} + c_4(p)e^{kL} \\ ip\tilde{f}_+(p) &= m(c_3(p)e^{kL} + c_4(p)e^{-kL}) + i(kc_1(p)e^{kL} - kc_2(p)e^{-kL}) \\ ip\tilde{f}_-(p) &= m(c_1(p)e^{-kL} + c_2(p)e^{kL}) + i(-kc_3(p)e^{-kL} + kc_4(p)e^{kL}).\end{aligned}\tag{9}$$

After solving the system (9) for c_1, c_2, c_3, c_4 and plugging back into (7), we get

$$\begin{aligned}\tilde{\chi}_+(p, s) &= \frac{\tilde{f}_+(m^2e^{6kL} + 2k^2e^{2kL} - m^2e^{2kL} + 2kpe^{2kL}) + i\tilde{f}_-(kme^{4kL} - mpe^{4kL} + km + mp)}{D}e^{k(L+s)} \\ &+ \frac{\tilde{f}_+(2k^2e^{4kL} - 2kpe^{4kL} - m^2e^{4kL} + m^2) + i\tilde{f}_-(mpe^{6kL} - kme^{6kL} - mpe^{2kL} - kme^{2kL})}{D}e^{k(L-s)} \\ \tilde{\chi}_-(p, s) &= \frac{i\tilde{f}_+(mpe^{6kL} - kme^{6kL} - mpe^{2kL} - kme^{2kL}) + \tilde{f}_-(2k^2e^{4kL} - 2kpe^{4kL} - m^2e^{4kL} + m^2)}{D}e^{k(L+s)} \\ &+ \frac{i\tilde{f}_+(kme^{4kL} - mpe^{4kL} + km + mp) + \tilde{f}_-(m^2e^{6kL} + 2k^2e^{2kL} - m^2e^{2kL} + 2kpe^{2kL})}{D}e^{k(L-s)},\end{aligned}$$

where $D := m^2(e^{2kL} + i\frac{k+p}{m})(e^{2kL} - i\frac{k+p}{m})(e^{2kL} + i\frac{k-p}{m})(e^{2kL} - i\frac{k-p}{m})$.

After doing a partial fraction decomposition, the above can be rewritten as

$$\begin{aligned}\tilde{\chi}_\pm(p, s) &= \frac{1}{2}\left[e^{-k(L\mp s)}\left(\pm \frac{\tilde{f}_+(p) - \tilde{f}_-(p)}{1 + i\frac{k-p}{m}e^{-2kL}} + \frac{\tilde{f}_+(p) + \tilde{f}_-(p)}{1 - i\frac{k-p}{m}e^{-2kL}}\right)\right. \\ &\quad \left.\pm i\frac{k-p}{m}e^{-k(L\pm s)}\left(\frac{\tilde{f}_+(p) - \tilde{f}_-(p)}{1 + i\frac{k-p}{m}e^{-2kL}} \mp \frac{\tilde{f}_+(p) + \tilde{f}_-(p)}{1 - i\frac{k-p}{m}e^{-2kL}}\right)\right].\end{aligned}\tag{10}$$

Noting that $\left|i\frac{k-p}{m}e^{-2kL}\right| < 1$, we can view (10) as the sum of four convergent geometric series, so that

$$\begin{aligned}\tilde{\chi}_\pm(p, s) &= \frac{1}{2}\left[\pm \sum_{n=0}^{\infty} \left(-i\frac{k-p}{m}\right)^n e^{-k[(2n+1)L\mp s]}(\tilde{f}_+(p) - \tilde{f}_-(p))\right. \\ &\quad + \sum_{n=0}^{\infty} \left(i\frac{k-p}{m}\right)^n e^{-k[(2n+1)L\mp s]}(\tilde{f}_+(p) + \tilde{f}_-(p)) \\ &\quad \pm \sum_{n=0}^{\infty} \left(-i\frac{k-p}{m}\right)^{n+1} e^{-k[(2n+1)L\pm s]}(\tilde{f}_+(p) - \tilde{f}_-(p)) \\ &\quad \left.- \sum_{n=0}^{\infty} \left(i\frac{k-p}{m}\right)^{n+1} e^{-k[(2n+1)L\pm s]}(\tilde{f}_+(p) + \tilde{f}_-(p))\right]\end{aligned}\tag{11}$$

From a table of inverse Laplace transforms (e.g. [10], formula 14.52) we find

$$\mathcal{L}^{-1}\left[\left(\frac{k-p}{k}\right)^\nu e^{-kx}\right](\tau, x) = m^\nu \left(\frac{\tau-x}{\tau+x}\right)^{\frac{1}{2}\nu} J_\nu(m\sqrt{\tau^2-x^2}) H(\tau-x)$$

where H is the Heaviside function $H(t) = 1$ for $t > 0$ and 0 otherwise.

We therefore have the following explicit solution for χ_\pm :

$$\begin{aligned} \chi_\pm(t, s) = \frac{1}{2} \sum_{n=0}^{\infty} i^n \Bigg[& H(t - (2n+1)L \pm s) \left((-1)^n \frac{d}{ds} \int_{(2n+1)L \mp s}^t d\xi F_-(t-\xi) Z_n(0, \pm s, \xi, m, L) \right. \\ & \left. \pm \frac{d}{ds} \int_{(2n+1)L \mp s}^t d\xi F_+(t-\xi) Z_n(0, \pm s, \xi, m, L) \right) \\ & - iH(t - (2n+1)L \mp s) \left((-1)^{n+1} \frac{d}{ds} \int_{(2n+1)L \pm s}^t d\xi F_-(t-\xi) Z_n(1, \mp s, \xi, m, L) \right. \\ & \left. \mp \frac{d}{ds} \int_{(2n+1)L \pm s}^t d\xi F_+(t-\xi) Z_n(1, \mp s, \xi, m, L) \right) \Bigg], \end{aligned} \quad (12)$$

where

$$Z_n(j, s, \xi, m, L) := \frac{J_{n+j}(m\sqrt{\xi^2 - ((2n+1)L - s)^2})}{(\xi^2 - ((2n+1)L - s)^2)^{\frac{n+j}{2}}} (\xi - ((2n+1)L - s))^{n+j},$$

and $F_\pm := f_+ \pm f_-$.

Then rewriting the derivative in (12) the χ solution is

$$\begin{aligned} \chi_\sigma(t, s) = & H(t - L + \sigma s) f_\sigma(t - L + \sigma s) \\ & + \sum_{n=0}^{\infty} i^n \left(H(t - (2n+1)L + \sigma s) \int_{(2n+1)L - \sigma s}^t f_{(-1)^n \sigma}(t-\xi) R_n(0, s, \xi, m, L) d\xi \right. \\ & \left. - iH(t - (2n+1)L - \sigma s) \int_{(2n+1)L + \sigma s}^t f_{(-1)^n \sigma}(t-\xi) R_n(1, -s, \xi, m, L) d\xi \right), \end{aligned} \quad (13)$$

where $\sigma \in \{+, -\}$, and

$$\begin{aligned} R_n(j, s, \xi, m, L) := & -m(\xi - ((2n+1)L - s))^{n+j} \frac{J_{n+j+1}(m\sqrt{\xi^2 - ((2n+1)L - s)^2})}{(\xi^2 - ((2n+1)L - s)^2)^{\frac{n+j+1}{2}}} ((2n+1)L - s) \\ & + (n+j) \frac{J_{n+j}(m\sqrt{\xi^2 - ((2n+1)L - s)^2})}{(\xi^2 - ((2n+1)L - s)^2)^{\frac{n+j}{2}}} (\xi - ((2n+1)L - s))^{n+j-1}. \end{aligned}$$

Adding χ_\pm to Φ_\pm , we get the solution for ψ_\pm .

This solution can clearly be put in terms of the initial data, as shown for example by the $n = 0$

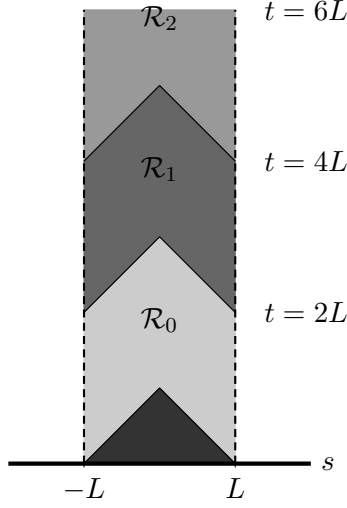


Figure 1: Support of χ_n

term of χ_σ , which we denote by $\chi_{\sigma 0}$:

$$\begin{aligned}
\chi_{\sigma 0}(t, s) = & -H(t - L + \sigma s) \left(\Phi_\sigma(t - L + \sigma s, \sigma L) + \int_{L - \sigma s}^t \dot{\psi}_\sigma(\sigma L + \sigma t - \sigma \xi) R_0(0, s, \xi, m, L) d\xi \right. \\
& - \frac{m}{2} \int_{L - \sigma s}^t \int_{\sigma L - t + \xi}^{\sigma L + t - \xi} Z_0(1, -\sigma \varsigma, \xi, m, \sigma L) \dot{\psi}_\sigma(\varsigma) R_0(0, \sigma s, \xi, m, L) d\varsigma d\xi \\
& \left. - \frac{im}{2} \int_{L - \sigma s}^t \int_{\sigma L - t + \xi}^{\sigma L + t - \xi} J_0(m\sqrt{(t - \xi)^2 - (L - \sigma \varsigma)^2}) \dot{\psi}_{-\sigma}(\varsigma) R_0(0, \sigma s, \xi, m, L) d\varsigma d\xi \right) \\
& + iH(t - L - \sigma s) \left(\int_{L + \sigma s}^t \dot{\psi}_\sigma(\sigma L + \sigma t - \sigma \xi) R_0(1, -s, \xi, m, L) d\xi \right. \\
& - \frac{m}{2} \int_{L + \sigma s}^t \int_{\sigma L - t + \xi}^{\sigma L + t - \xi} Z_0(1, -\sigma \varsigma, \xi, m, \sigma L) \dot{\psi}_\sigma(\varsigma) R_0(1, -\sigma s, \xi, m, L) d\varsigma d\xi \\
& \left. - \frac{im}{2} \int_{L + \sigma s}^t \int_{\sigma L - t + \xi}^{\sigma L + t - \xi} J_0(m\sqrt{(t - \xi)^2 - (L - \sigma \varsigma)^2}) \dot{\psi}_{-\sigma}(\varsigma) R_0(1, -\sigma s, \xi, m, L) d\varsigma d\xi \right)
\end{aligned} \tag{14}$$

□

Furthermore, the Heaviside functions appearing in (13) show that χ_n , the n -th term in the summation, is supported in $\bigcup_{k=0}^n \mathcal{R}_k$, with regions \mathcal{R}_k shown in Fig. 1. This shows that for any fixed $t > 0$ there are only finitely many non-zero terms in (13), so convergence is never an issue.

3 Bohmian Trajectories

We can now generate the statistics for the arrival times of the particles. We take the initial wave function to be Gaussian with mean μ , variance α , and mixing angle θ . For the rest of the paper, unless stated otherwise, the computed statistics will have initial data in the form of a Gaussian wave packet:

$$\begin{cases} \dot{\psi}_+(s, \theta, \mu, \alpha) &= \sin(\theta) \frac{1}{\sqrt[4]{2\pi\alpha}} e^{-\frac{(s-\mu)^2}{4\alpha}} \\ \dot{\psi}_-(s, \theta, \mu, \alpha) &= \cos(\theta) \frac{1}{\sqrt[4]{2\pi\alpha}} e^{-\frac{(s-\mu)^2}{4\alpha}} \end{cases} \tag{15}$$

with parameters $\theta = \frac{\pi}{4}$, $\mu = 0$, and $\alpha = 0.005$. We also set $m = 2$, and $L = 1$. Note that these values are only chosen for ease of computation and visualisation. In particular, in our units the empirical mass of the electron is quite large ($m \approx 777$).

Let $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ be the wave function of a spin-1/2 particle in $\mathbb{R}^{1,1}$. The *Dirac current*

$$j^\mu := \bar{\psi} \gamma^\mu \psi, \quad \bar{\psi} := \psi^\dagger \gamma^0 \quad (16)$$

is the simplest Lorentz vector that can be constructed from the Dirac bispinor ψ . When ψ satisfies the Dirac equation $-i\gamma^\mu \partial_\mu \psi + m\psi = 0$, it follows that the vectorfield j is divergence free, i.e.

$$\partial_\mu j^\mu = 0. \quad (17)$$

Setting $\rho(t, x) := j^0 = \psi^\dagger \psi$ and $J(t, x) := j^1 = \psi^\dagger \alpha^1 \psi$, the above can be written as $\partial_t \rho + \partial_x J = 0$, which has the form of an equation of continuity for a density ρ . The quantity $v(t, s) = J/\rho$ is thus naturally a velocity field defined on the 1-particle configuration space.

Let $Q(t)$ denote the actual position of the particle at time t . According to the principles of Bohmian Mechanics (e.g. [7], Chap. 9) the guiding equation for the motion of the particle is

$$\frac{dQ(t)}{dt} = \frac{J(t, Q(t))}{\rho(t, Q(t))}. \quad (18)$$

The above ODE can be uniquely solved given initial data $Q_0 = Q(0) \in \mathbb{R}$ which is assumed to be distributed randomly according to the initial density $\rho_0 := \psi^\dagger \psi$.

Figure 2 shows a sample of 1000 Bohmian trajectories with randomly distributed initial positions guided according to (18) by a wavefunction ψ that solves (2). For a comparison of the trajectories guided by wavefunctions with and without the boundary condition being satisfied, see Figure 8.

Figure 3 shows the computed arrival times of these 1000 trials, from which we can see that particles starting closer to the source point have a later time of detection.

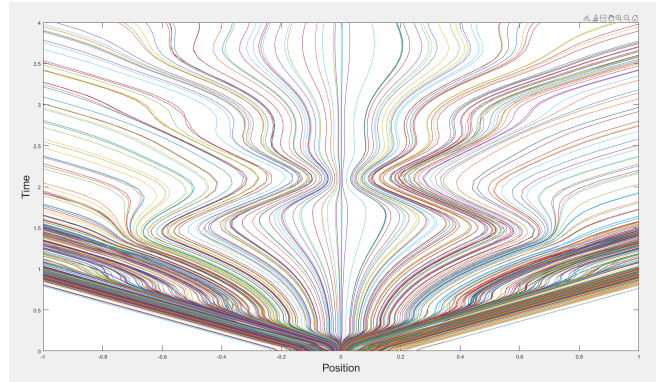


Figure 2: 1000 Bohmian trajectories with the boundary condition satisfied

4 The arrival times probability distribution

Let j^μ be as in (16). Define $Z = (T, \mathbf{X})$ where $T \geq 0$ is the time at which a detector clicks, and \mathbf{X} is the location on $\partial\Omega$ where the particle gets detected. (If the particle never gets detected, we write

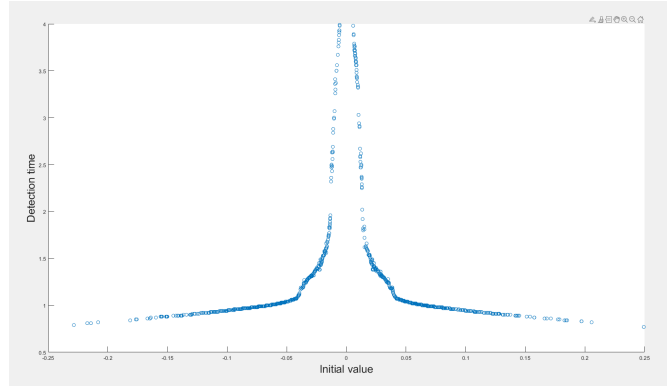


Figure 3: Time of detection versus the initial position

$Z = \infty$.) According to [14] the distribution μ of Z satisfies

$$\begin{aligned} \mu(t_1 \leq T < t_2, \mathbf{X} \in \partial\Omega) &= \int_{t_1}^{t_2} dt \int_{\partial\Omega} \mathbf{n}(\mathbf{x}) \cdot \mathbf{j}^{\psi_t}(\mathbf{x}) d\sigma \\ &= \int_{t_1}^{t_2} (j^1(t, L) - j^1(t, -L)) dt. \end{aligned} \quad (19)$$

Theorem 2. (See [14]) *The integrand in (19) is a probability density function and*

$$\mu(0 \leq T < t, \mathbf{X} \in \{-L, L\}) = \int_0^t (j^0(t, L) + j^0(t, -L)) dt \quad (20)$$

when the absorbing boundary condition is satisfied.

Proof. Define the rectangle $\mathcal{D} = [-L, L] \times [0, t]$, with $\ell_i, i \in \{1, 2, 3, 4\}$ the sides of the rectangle. By (17) and the Divergence Theorem, we have

$$\begin{aligned} \iint_{\mathcal{D}} \nabla \cdot \mathbf{j} &= \int_{\partial\mathcal{D}} \mathbf{j} \cdot \vec{n} \\ 0 &= \int_{\ell_1} \mathbf{j} \cdot \langle -1, 0 \rangle + \int_{\ell_2} \mathbf{j} \cdot \langle 0, 1 \rangle + \int_{\ell_3} \mathbf{j} \cdot \langle 1, 0 \rangle + \int_{\ell_4} \mathbf{j} \cdot \langle 0, -1 \rangle \\ &= -\int_{\ell_1} j^0 + \int_{\ell_2} j^1 + \int_{\ell_3} j^0 - \int_{\ell_4} j^1. \end{aligned} \quad (21)$$

Thus we obtain

$$\int_{-L}^L |\dot{\psi}(s)|^2 ds - \int_{-L}^L |\psi(t, s)|^2 ds = \int_0^t (j^1(T, L) - j^1(T, -L)) dT. \quad (22)$$

The right side of this equation is

$$\int_0^t (|\psi_{-}(T, L)|^2 - |\psi_{+}(T, L)|^2 - |\psi_{-}(T, -L)|^2 + |\psi_{+}(T, -L)|^2) dT \quad (23)$$

After applying the boundary condition we obtain that it is equal to

$$\int_0^t (|\psi_{-}(T, L)|^2 + |\psi_{+}(T, -L)|^2) dT = \int_0^t (j^0(T, L) + j^0(T, -L)) dT. \quad (24)$$

Since $\dot{\psi}$ is assumed to be compactly supported in $[-L, L]$, the left hand side of (22) is

$$1 - \text{Prob}(T = t, X \in \Omega)$$

which is precisely the left-hand side of (20). \square

Now that we have a probability distribution, we can superimpose the plot of its probability density function with the histogram of the arrival times for the computed trajectories depicted in Fig. 2. This is Fig. 4. We can also make the same comparison between the cumulative distribution function and the cumulative histogram of those arrival times, which gives us Fig. 5. Both indicate excellent agreement, as we expected.

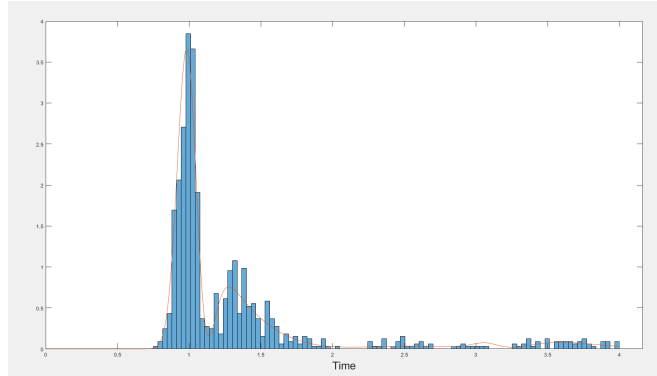


Figure 4: Probability density superimposed on the histogram of arrival times for Bohmian trajectories

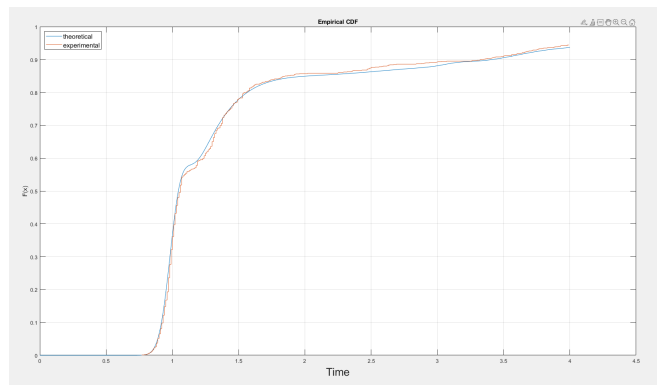


Figure 5: Cumulative distribution superimposed on the cumulative histogram

5 Comparisons of different proposals for arrival times distribution

This section explores three different proposals for the probability density function for the arrival times of the particle. Proposal 1 is the integrand in (19), also known as the quantum flux, using the free evolution. Proposal 2 is the right hand side of (20), again using the free evolution, which was proposed by Wigner [1]. Proposal 3 is the quantum flux computed using the boundary condition, which has been proved to be a probability distribution in Section 4. In this section, we show that proposals 1 and 2 are not viable candidates to be probability distributions. The graphs in this section will be generated using $\alpha = .05$.

The graphs (6) and (7) show the three proposals, plotted on top of each other. From Fig. 6, we can see that the quantum flux in Proposal 1 clearly takes on negative values; thus it cannot be a probability density function.

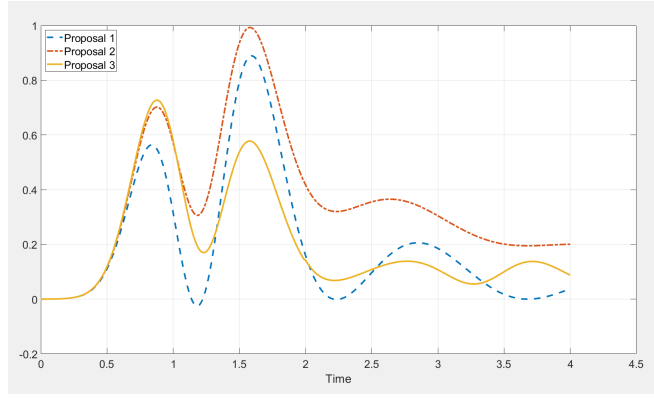


Figure 6: Comparison of the pdfs of the three proposals

From Fig. 7, we see that Proposal 2 cannot be a probability distribution either, since the distribution is clearly not normalized (recall that the initial wavefunction is already normalized.)

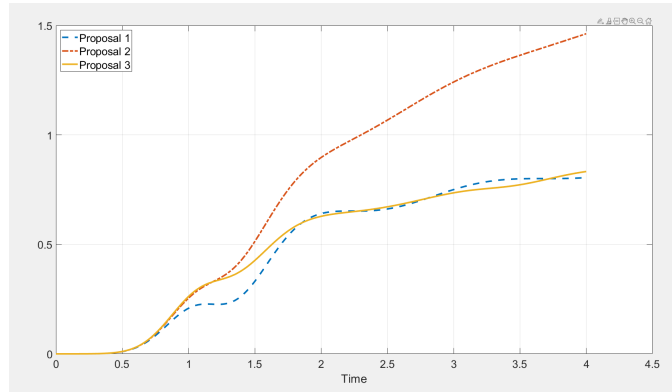


Figure 7: Comparison of the cdfs of the three proposals

It thus follows that proposals 1 and 2, which use the free evolution, will not produce probability distributions. The only viable candidate among these three for a probability distribution for arrival times of a Dirac particle is Proposal 3, and the absorbing boundary condition is sufficient for providing it.

Finally, Fig. 8 show a handful of trajectories for the particle guided by ψ that satisfies the boundary condition (here continued outside the interval,) plotted with a solid line, versus those guided by the free evolution, i.e. by Φ , plotted with a dotted line. The plots seem to indicate that the absorbing boundary condition generally speaking hastens the arrival of particles at the boundary points ($x = \pm 1$ in the picture). We plan to examine this phenomenon more carefully in the near future.

6 Summary and Outlook

We have demonstrated that there exists an explicit, unique, and smooth solution to the initial-boundary-value problem for the Dirac equation for a single spin-1/2 particle with the absorbing boundary condition, and that this boundary condition is sufficient to produce a non-negative probability density function. We have also shown that the statistics of arrival times of Bohmian trajectories

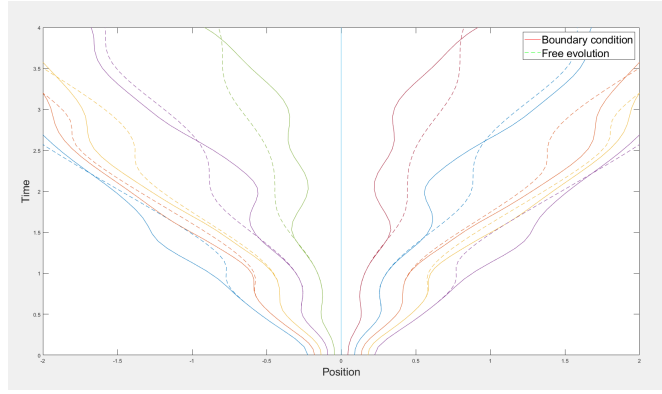


Figure 8: Comparison of trajectories guided by the free evolution with those under the absorbing boundary condition

are consistent with this probability distribution.

Next we plan to consider the arrival/detection times for two entangled particles that fly apart from the same source, and study detection correlations as the detector positions are varied. This amounts to generalizing the IBVP to allow for an arbitrary source point in the interior of the interval, not just at the center. We expect that the two entangled particles equidistant from the source point will have nearly perfectly correlated arrival times; which would allow us to set up a Bell-type experiment to see whether or not Bell’s inequality is violated in arrival time measurements of Dirac particles.

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