1 Lorentz-invariant phase space

From Fermi's Golden Rule

$$\Gamma = 2\pi \left|\mathcal{M}\right|^2 R_n(E) \tag{1}$$

where R_n is the Lorentz-invariant phase space, similar to the density of states, or the number of possible states per energy interval per unit volume, $\mathrm{d}N \,/\, \mathrm{d}E$.

1.1 General case

The coordinate of a motion in phase space is given by a six-dimensional vector.

In a universe with periodic boundary conditions, the elementary volume is $(2\pi)^3$.

Hence, the number of states available to one particle, using a Lorentz-invariant measure, is

$$\begin{aligned} N_1 &= \frac{\text{Total volume}}{\text{Elementary volume}} \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\mathbf{x}\, \frac{\mathrm{d}^3\mathbf{p}}{2E} \\ &= \frac{V}{(2\pi)^3} \int \frac{\mathrm{d}^3\mathbf{p}}{2E} \end{aligned} \tag{2}$$

For n particles,

$$N_n = \frac{V^n}{(2\pi)^{3n}} \int \prod_{i=1}^{n-1} \frac{\mathrm{d}^3\mathbf{p}_i}{2E_i} \tag{3}$$

where the nth momentum is not integrated over due to conservation.

This constraint also be imposed with a Dirac delta function. Use the fact that

$$\int \mathrm{d}\mathbf{p}_n\ \delta^{(3)}\left(\textstyle\sum_{j=1}^n \mathbf{p}_j - \mathbf{p}\right) = \int \mathrm{d}\mathbf{p}_n\ \delta^{(3)}(0) = 1 \tag{4}$$

to write

$$N_n = \frac{V^n}{(2\pi)^{3n}} \int \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_i}{2E_i}\, \delta^{(3)}\left(\textstyle\sum_{j=1}^n \mathbf{p}_j - \mathbf{p}\right) \tag{5}$$

Now the density of states is defined

$$R_n(E) = \left.\frac{\mathrm{d}N_n}{\mathrm{d}E}\right|_{V=1} \tag{6}$$

so we have

$$R_n(E) = \frac{1}{(2\pi)^{3n}} \frac{\mathrm{d}}{\mathrm{d}E} \int \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_i}{2E_i}\, \delta^{(3)}\left(\textstyle\sum_{j=1}^n \mathbf{p}_j - \mathbf{p}\right) \tag{7}$$

Energy is conserved through the statement

$$\int \mathrm{d}E\ \delta\left(\textstyle\sum_{k=1}^{n} E_k - E\right) = 1 \tag{8}$$

inserting this into the density of states

$$\boxed{R_n(E) = \frac{1}{(2\pi)^{3n}} \int \prod_{i=1}^{n} \frac{\mathrm{d}^3\mathbf{p}_i}{2E_i}\ \delta^{(3)}\left(\textstyle\sum_{j=1}^{n} \mathbf{p}_j - \mathbf{p}\right)\ \delta\left(\textstyle\sum_{k=1}^{n} E_k - E\right)} \tag{9}$$

1.2 Two bodies

The expression $R_n(E)$ simplifies for $n = 2$. For the centre-of-mass frame ($\mathbf{p} = 0$), we have

$$\begin{aligned} R_2(E) &= \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3\mathbf{p}_1}{2E_1} \int \frac{\mathrm{d}^3\mathbf{p}_2}{2E_2}\ \delta^{(3)}\left(\mathbf{p}_1 + \mathbf{p}_2\right)\ \delta\left(E_1 + E_2 - E\right) \\ &= \frac{1}{(2\pi)^6} \int \frac{\mathrm{d}^3\mathbf{p}_1}{4E_1E_2}\ \delta(E_1 + E_2 - E) \\ &= \frac{4\pi}{(2\pi)^6} \int \mathrm{d}p\ \frac{p^2}{4E_1E_2} \delta(E_1 + E_2 - E) \end{aligned} \tag{10}$$

where the last step assumes no angular dependance on the momentum.

Now use $E_1\,\mathrm{d}E_1 = p\,\mathrm{d}p$ so

$$R_2(E) = \frac{\pi}{(2\pi)^6} \int \mathrm{d}E_1\ \frac{p}{E_2}\ \delta(E_1 + E_2 - E) \tag{11}$$

We can write E_2 in terms of E_1:

$$E_2 = \sqrt{E_1^2 - m_1^2 + m_2^2} \tag{12}$$

Therefore

$$R_2(E) = \frac{\pi}{(2\pi)^6} \int \mathrm{d}E_1\ \frac{p}{E_2}\ \delta\left[f(E_1)\right] \tag{13}$$

where $f(E_1) = E_1 + \sqrt{E_1^2 - m_1^2 + m_2^2} - E$.

Now use the fact that

$$\boxed{\delta(f(x)) \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=0} = \delta(x)} \tag{14}$$

and $f' = E/E_2$ to get the final result

$$R_2(E) = \frac{\pi}{(2\pi)^6} \int \mathrm{d}E_1\ \frac{p}{E}\ \delta(E_1)$$

$$\boxed{R_2(E) = \frac{\pi}{(2\pi)^6}\ \frac{p}{E}} \tag{15}$$