1 2-flavour neutrino model

1.1 Mixing

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \tag{1}$$

Start with ν_e:

$$\begin{aligned} |\psi(0)\rangle &= |\nu_e\rangle \\ &= \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle \end{aligned} \tag{2}$$

In contrast to neutral meson mixing, we work in the lab frame and approximate the neutrinos as plane waves. The mass eigenstates evolve with a time-dependent complex phase.

$$|\psi(t)\rangle = \cos\theta\, e^{-i\phi_1}\,|\nu_1\rangle + \sin\theta\, e^{-i\phi_2}\,|\nu_2\rangle \tag{3}$$

So the ν_e survival amplitude is given by

$$\langle\nu_e|\psi t\rangle = \cos^2\theta\, e^{-i\phi_1} + \sin^2\theta\, e^{-i\phi_2} \tag{4}$$

therefore the survival probability is

$$P(\nu_e \to \nu_e) = \cos^4\theta + \sin^4\theta + 2\sin^2\theta\cos^2\theta\,\cos(\Delta\phi) \tag{5}$$

This can be rearranged into a more convenient form. First, complete the square:

$$\cos^4\theta + \sin^4\theta = \underbrace{(\cos^2\theta + \sin^2\theta)^2}_{1} - 2\sin^2\theta\cos^2\theta \tag{6}$$

to give

$$P(\nu_e \to \nu_e) = 1 - 2\sin^2\theta\cos^2\theta\,(1 - \cos(\Delta\phi)) \tag{7}$$

Now use the double-angle formula:

$$\sin\theta\cos\theta = \frac{1}{2}\sin(2\theta) \tag{8}$$

so

$$P(\nu_e \to \nu_e) = 1 - \frac{1}{2}\sin^2(2\theta)\,[1 - \cos(\Delta\phi)] \tag{9}$$

and the half-angle formula

$$\cos(\Delta\phi) = 1 - 2\sin^2\left(\frac{\Delta\phi}{2}\right) \tag{10}$$

giving the final result

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta\phi}{2}\right) \tag{11}$$

1.2 Plane wave approximation

Under the approximation to plane waves, the phase is given by

$$\phi_i = p_i L + Et \tag{12}$$

where we assume all neutrinos have the same energy E.

So the phase difference is $\Delta\phi = (p_1 - p_2)L$.

Expand the momentum in terms of the mass for each mass eigenstate:

$$\begin{aligned} p_i &= (E + m_i^2)^{\frac{1}{2}} \quad (13) \\ &\simeq E + \frac{m_i^2}{2E} \quad (14) \end{aligned}$$

using $m_i \ll E$.

Hence,

$$\Delta\phi = (m_1^2 - m_2^2)\frac{L}{2E} \equiv \frac{\Delta m_{12}^2 L}{2E} \tag{15}$$

Inserting this into (11), we get the final answer

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m_{12}^2 L}{4E}\right) \tag{16}$$