$c = 1$ throughout. $N = 4$. Metric signature: $(-, +, +, +)$.

Chapter 1

Tensors

Tensors serve to isolate intrinsic geometric and physical properties from those that merely depend on the coordinates. Therefore tensor analysis finds an application in relativity, since the subject is about coordinate transformations.

1.1 Index notation

Index notation is powerful since it encodes complex information about a mathematical object in a compact form.

For instance, a vector is expressed as

$$\mathsf{A} = \begin{pmatrix} A^0 \\ A^1 \\ A^2 \\ A^3 \end{pmatrix} \tag{1.1}$$

but we can refer to a contravariant component of the vector in abstract index notation as A^μ.

A tensor, $A^{i\ldots k}_{j\ldots n}$, with s upper (contravariant) and t lower (covariant) indices has valence (s,t). If two tensors of the same valence have have equal components in any one reference frame, then they are equal. This demonstrates that tensor equations express physical facts that transcend the coordinate system.

Some simple rules allow slick calculations with tensor algebra, avoiding having to write things out in a clunky way.

- Conserve valence differentially $(s-t)$ on either side of equations.

- Sum over repeated indices. To conserve valence, there must be one lower and one upper repeated dummy index.
- Contraction. The sum over an index reduces the rank of the equation. e.g. $A^i_{j\mu}\,B^\mu_k = C^i_{jk}$.
- Substitution. The action of a delta is to contract one index and substitute another: $\delta^\mu_i\,A_{\mu jk} = A_{ijk}$.

1.2 Mathematical treatment of tensors

Consider a vector space, $\mathcal{V}_N$, spanned the set of (real) variables $\{x^i\}$. Any non-singular linear transformation of $\{x^i\}$ may be regarded as a change of basis in $\mathcal{V}_N$.

A tensor is an object in $\mathcal{V}_N$ which that may be described by its components, which form an ordered set of (real) numbers $\{x^0, x^1, \ldots, x^N\}$.

1.2.1 Rank

A tensor of rank r has $M = N^r$ components.

Rank	Components
0 (Scalar)	1
1 (Vector)	4
2	16
Antisymmetric 2	6

What sort of object should be used to describe physical fields? The electromagnetic field may be described in terms of two 3-vector fields, **E** and **B**. We therefore wish to encode six components, so require at least a second-rank tensor (for $N = 4$). With sixteen components to play with, introducing the right degeneracy gives the right object for the job: an antisymmetric second-rank tensor.

$$T^{ab} = -T^{ba} \tag{1.2}$$

1.2.2 Coordinate transformations

The coordinate transformations of interest are non-singular (i.e. reversible) and differentiable.

For simplicity, define the partial derivatives:

$$\frac{\partial x^{i'}}{\partial x^i} \equiv p^{i'}_i\,, \quad \frac{\partial x^{i}}{\partial x^{i'}} \equiv p^{i}_{i'}, \quad \frac{\partial^2 x^{i'}}{\partial x^i\,\partial x^j} \equiv p^{i'}_{ij}, \quad \text{etc.} \tag{1.3}$$

By the chain rule,

$$p^{i}_{i'}\,p^{i'}_{i''} = p^{i'}_{i''} \tag{1.4}$$

and

$$p^{i}_{i'}\,p^{i'}_{j} = \delta^{i}_{j} \tag{1.5}$$

The index notation is a powerful tool that helps navigate coordinate transformations. Contravariant components are described by upper indices and covariant components by lower indices. Using the rules of index notation, the transformation of quantities may be deduced.

For a general mixed tensor,

$$\boxed{A^{i'\ldots k'}_{j'\ldots n'} = A^{i\ldots k}_{j\ldots n}\,p^{i'}_{i}\ \ldots\ p^{k'}_{k}\,p^{j}_{j'}\ \ldots\ p^{n}_{n'}} \tag{1.6}$$

(1.6) is the qualification rule for tensors in certain applications where a coordinate transformation is defined, such as in relativity. Therefore tensors which transform in the way described by (1.6) are qualified (or 'Lorentz') tensors.

Explicit transformations for tensors of rank 0 to 2 are calculated from (1.6). It is useful to write out the Jacobian matrix,

$$J = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \mathcal{L} \tag{1.7}$$

for the Lorentz transformation, where the components are labelled as ${\mathcal{L}^{i'}}_{i}$.

Rank 0 - Scalars

Since a rank 0 tensor has no indices, its transformation is that of a *Lorentz invariant* quantity,

$$A \to A \tag{1.8}$$

Rank 1 - Covariant vectors

$$A^{\mu} \to A^{\mu'} = {\mathcal{L}^{\mu'}}_{\mu}\,A^{\mu} \tag{1.9}$$

or in expanded notation,

$$\mathsf{A} \to \mathsf{A}' = \mathcal{L}\,\mathsf{A} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{1.10}$$

Rank 2

As with any other tensor, the components transform according to (1.6):

$$A^{\mu\nu} \to A^{\mu'\nu'} = \mathcal{L}^{\mu'}_{\ \ \mu}\,\mathcal{L}^{\nu'}_{\ \ \nu}\,A^{\mu\nu} \tag{1.11a}$$

$$= \mathcal{L}^{\mu'}_{\ \ \mu}\,A^{\mu\nu}\,\mathcal{L}^{\nu'}_{\ \ \nu} \tag{1.11b}$$

$$= \mathcal{L}^{\mu'}_{\ \ \mu}\,A^{\mu\nu}\,(\mathcal{L}^T)_\nu^{\ \ \nu'} \tag{1.11c}$$

or in expanded notation,

$$\mathbb{A} \to \mathbb{A}' = \mathcal{L}\,\mathbb{A}\,\mathcal{L}^T \tag{1.12}$$

1.2.3 Differentiation

A partial derivative is written

$$\frac{\partial}{\partial x^\mu} \equiv \partial_\mu. \tag{1.13}$$

Comma notation may be used to reduce clutter and indicate the result that this derivative raises the valence of a tensor by one covariant index,

$$\partial_a\,\phi \equiv \phi_{,a} \tag{1.14}$$

This is often used in contraction, for example in field equations,

$$\partial_\mu\,J^\mu = 0. \tag{1.15}$$

1.3 Index gymnastics

It is often useful to convert between contravariant and covariant components of a tensor. This is possible in a (pseudo-)Riemannian metric space, where the metric is given by the quadratic differential form

$$\mathbf{ds}^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{1.16}$$

$g_{\mu\nu}$ is the metric tensor and in flat spacetime, the Minkowski metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.17}$$

where the chosen signature is $(-, +, +, +)$.

Define g^{ij} as the inverse of g_{ij}, such that:

$$g^{ij}\, g_{jk} = \delta^i_k \tag{1.18}$$

Now we may define contravariant components of a tensor to be those given by

$$A_i{}^{j\cdots} = g_{i\mu}\, A^{\mu j \cdots} \tag{1.19}$$

So the action of the metric tensor is to raise or lower indices.

Chapter 2

Coordinate transformations

2.1 Non-relativistic boost

In classical physics, the coordinate substitution for a boost along the x-axis is given by the Galilean transformation,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -v & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{2.1}$$

2.2 Postulates of special relativity

Principle of relativity - The laws of physics are the same in all inertial reference frames.

Invariance of c - There is a finite maximum speed for signal propagation, which is the same in all reference frames.

2.3 The Lorentz transformation

The relativistic transformation places space and time on an equal footing:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \tag{2.2}$$

Chapter 3

Transformation of velocity

Consider the 4-velocity, $U^\mu = (\gamma_u c, \gamma_u \mathbf{u})$ in some frame S. Then viewed in some other frame S', moving at velocity $\mathbf{v}$ in the x-direction, U^μ transforms as a four-vector:

$$\begin{pmatrix} \gamma'_u \\ \gamma'_u u'_\parallel \\ \gamma'_u u'_\perp \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u \\ \gamma_u u_\parallel \\ \gamma_u u_\perp \\ 0 \end{pmatrix} \tag{3.1}$$

Which gives

$$\frac{\gamma_{u'}}{\gamma_u \gamma_v} = \frac{1}{1 - \mathbf{u} \cdot \mathbf{v}} \tag{3.2}$$

Therefore, the final answers are

$$\boxed{u'_\parallel = \frac{u_\parallel - v}{1 - \mathbf{u} \cdot \mathbf{v}}} \tag{3.3}$$

and

$$\boxed{u'_\perp = \frac{u_\perp}{\gamma_v \left(1 - \mathbf{u} \cdot \mathbf{v}\right)}} \tag{3.4}$$

Chapter 4

Pure forces

For a pure force

$$\boxed{\frac{\mathrm{d}m}{\mathrm{d}t} = 0} \tag{4.1}$$

$$\begin{aligned}
U_\mu F^\mu &= \frac{\mathrm{d}P^\mu}{\mathrm{d}t} U_\mu && (4.2a)\\
&= mA^\mu U_\mu - \frac{\mathrm{d}m}{\mathrm{d}t} && (4.2b)\\
&= -\frac{\mathrm{d}m}{\mathrm{d}t} \quad \text{since } A^\mu \text{ and } U^\mu \text{ are orthogonal,} && (4.2c)\\
&= 0 \quad \text{for a pure force.} && (4.2d)
\end{aligned}$$

Also,

$$U_\mu F^\mu = \gamma^2 \left(-\frac{\mathrm{d}E}{\mathrm{d}t} + \mathbf{u} \cdot \mathbf{f}\right) \tag{4.3}$$

Therefore

$$\boxed{\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{u} \cdot \mathbf{f}} \tag{4.4}$$

for a pure force.

4.1 Expression for 3-force

3-force is defined by

$$\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \tag{4.5a}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(m\gamma\mathbf{u}\right) \tag{4.5b}$$

$$= \underbrace{\frac{\mathrm{d}m}{\mathrm{d}t}\gamma\mathbf{u}}_{0} + m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{u} + m\gamma\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} \tag{4.5c}$$

$$= m\frac{\mathrm{d}\gamma}{\mathrm{d}t}\mathbf{u} + m\gamma\mathbf{a} \tag{4.5d}$$

Now use the equation $E = \gamma m$. Therefore,

$$\mathbf{f}\cdot\mathbf{u} = \frac{\mathrm{d}E}{\mathrm{d}t} = m\frac{\mathrm{d}\gamma}{\mathrm{d}t} \quad \text{for pure force.} \tag{4.6}$$

So

$$\boxed{\mathbf{f} = \gamma m\mathbf{a} + (\mathbf{f}\cdot\mathbf{u})\mathbf{u}} \tag{4.7}$$

Chapter 5

Transformation of force

Consider the 4-force,

$$F^\mu = \gamma \begin{pmatrix} \frac{\mathrm{d}E}{\mathrm{d}t} \\ \mathbf{f} \end{pmatrix} \tag{5.1}$$

Transforming gives

$$f'_\parallel = \frac{\gamma_u \gamma_v}{\gamma_{u'}} \left(f_\parallel - v \frac{\mathrm{d}E}{\mathrm{d}t} \right) \tag{5.2a}$$

$$f'_\perp = \frac{\gamma_u}{\gamma_{u'}} f_\perp \tag{5.2b}$$

A quick method to find $\frac{\gamma_u \gamma_v}{\gamma_{u'}}$

Take $U^\mu = \gamma_u(1, \mathbf{u})$ and $V^\mu = \gamma_v(1, \mathbf{v})$ in a general frame. Then

$$U_\mu V^\mu = -\gamma_u \gamma_v (1 - \mathbf{u} \cdot \mathbf{v}) \tag{5.3}$$

But in the frame of a particle moving with V^μ, $V^{\mu'} = (1, 0)$ and $U^{\mu'} = \gamma_{u'}(1, \mathbf{u}')$. Therefore

$$U_{\mu'} V^{\mu'} = -\gamma_{u'} \tag{5.4}$$

is invariant. Equating the expressions for the scalar product of the velocities,

$$\boxed{\frac{\gamma_u \gamma_v}{\gamma_{u'}} = \frac{1}{1 - \mathbf{u} \cdot \mathbf{v}}} \tag{5.5}$$

So we get

$$f'_{\parallel} = \frac{f_{\parallel} - v\frac{\mathrm{d}E}{\mathrm{d}t}}{1 - \mathbf{u} \cdot \mathbf{v}} \tag{5.6a}$$

$$\Rightarrow \quad f'_{\parallel} = \frac{f_{\parallel} - v(\mathbf{f} \cdot \mathbf{u})}{1 - \mathbf{u} \cdot \mathbf{v}} \quad \text{for a pure force.} \tag{5.6b}$$

and

$$f'_{\perp} = \frac{f_{\perp}}{\gamma_v \left(1 - \mathbf{u} \cdot \mathbf{v}\right)} \tag{5.7}$$

Chapter 6

The Doppler effect

Consider the invariant $K_\mu U^\mu$. In the rest frame of the source, $K^a = (\omega_0, 0, 0, 0)^T$ and $U^a = (1, 0, 0, 0)^T$, so

$$K_\mu U^\mu = -\omega_0. \tag{6.1}$$

In the observer frame, $K^a = (\omega, \mathbf{k})^T$ and $U^a = (\gamma, \gamma\mathbf{v})$, so

$$\begin{aligned} K_\mu U^\mu &= \gamma(-\omega + \mathbf{k}\cdot\mathbf{v}) && \text{(6.2a)} \\ &= \gamma\omega(-1 + \frac{kv}{\omega}\cos\theta) && \text{(6.2b)} \\ &= -\gamma\omega(1 - \frac{v}{v_p}\cos\theta) && \text{(6.2c)} \end{aligned}$$

Equating (6.1) and (6.2c),

$$\boxed{\frac{\omega}{\omega_0} = \frac{1}{\gamma\left(1 - (v/v_p)\cos\theta\right)}} \tag{6.3}$$