Chapter 1

Fermi theory of β-decay

Fermi's Golden Rule gives the rate in general form,

$$\boxed{\Gamma = 2\pi \left|A_{fi}\right|^2 \frac{\mathrm{d}N}{\mathrm{d}E_f}} \tag{1.1}$$

1.1 The matrix element

$$A_{fi} = \left\langle \Psi_f \middle| A \middle| \Psi_i \right\rangle \tag{1.2}$$

where $\langle x|\Psi_i\rangle = \psi_i$ and $\left\langle x \middle| \Psi_f \right\rangle = \psi_f\,\phi_e\,\phi_\nu$. From the diagram for β-decay $\langle x|A|x\rangle = G_F \equiv g_w^2/M_W^2$. Therefore the matrix element is given by

$$A_{fi} = G_F \int\limits_{\text{nucleus}} \mathrm{d}^3\mathbf{x}\ \phi_e^*\,\phi_\nu^*\,\psi_f^*\,\psi_i \tag{1.3}$$

Next we make the assumptions that the electron and neutrino may be treated as plane waves. That is,

$$\phi_e(\mathbf{x}) = \frac{1}{\sqrt{V}} \exp(-i\,\mathbf{p_e}\cdot\mathbf{x}) \tag{1.4a}$$

$$\phi_\nu(\mathbf{x}) = \frac{1}{\sqrt{V}} \exp(-i\,\mathbf{p}_\nu\cdot\mathbf{x}) \tag{1.4b}$$

Then

$$\phi_e^* \, \phi_\nu^* = \frac{1}{V} \, e^{i \, \mathbf{p} \cdot \mathbf{x}} \tag{1.5a}$$

$$= 1 + i \, \mathbf{p} \cdot \mathbf{x} - (\mathbf{p} \cdot \mathbf{x})^2 + \ldots \tag{1.5b}$$

$$\approx 1 \tag{1.5c}$$

Therefore

$$A_{fi} \approx \frac{G_F}{V} \underbrace{\int_{\text{nucleus}} \mathrm{d}^3\mathbf{x} \; \phi_f^* \, \phi_i}_{M_{fi}} \tag{1.6a}$$

$$= \frac{G_F}{V} \, M_{fi} \tag{1.6b}$$

So the expression for the rate becomes

$$\Gamma = 2\pi \, \frac{G_F^2}{V^2} \left| M_{fi} \right|^2 \frac{\mathrm{d}N}{\mathrm{d}E_f} \tag{1.7}$$

1.2 Density of states

Assume the daughter nucleus is so heavy is doesn't recoil, so we don't need to include a density of states for it.

$$\mathrm{d}N = \left(4\pi \, \frac{V}{(2\pi)^3} \, p_e^2 \; \mathrm{d}p_e \right) \left(4\pi \, \frac{V}{(2\pi)^3} \, p_\nu^2 \; \mathrm{d}p_\nu \right) \tag{1.8a}$$

$$= \frac{(4\pi)^2}{(2\pi)^6} \, V^2 \, p_e^2 \, p_\nu^2 \; \mathrm{d}p_e \; \mathrm{d}p_\nu \tag{1.8b}$$

Now we take $E_f = E_e + E_\nu$, therefore $\mathrm{d}E_f = \mathrm{d}E_\nu = \mathrm{d}p_\nu$ for a fixed E_e (assuming $E_\nu \approx p_\nu$).

$$\frac{\mathrm{d}N}{\mathrm{d}E_f} = \frac{(4\pi)^2}{(2\pi)^6} \, V^2 \, p_e^2 \, (E_f - E_e)^2 \, \mathrm{d}p_e \tag{1.9}$$

so

$$\boxed{\mathrm{d}\Gamma = 2\pi \, G_F^2 \left| M_{fi} \right|^2 \frac{1}{4\pi^4} \, (E_f - E_e)^2 \, p_e^2 \; \mathrm{d}p_e} \tag{1.10}$$

Integrating through possible electron momenta from 0 to p_f (and using $p_e \approx E_e$), we obtain the result

$$\boxed{\Gamma \propto E_f^5} \tag{1.11}$$

1.3 Assumptions

- $|Q| \ll M_W$, so the propagator factor is purely the mass of the exchange boson;
- The wavefunctions for the electron and neutrino may be approximated as plane waves;
- $(\mathbf{p_e} + \mathbf{p}_\nu) \cdot \mathbf{x} \ll 1$, which simplifies the matrix element to some scalar multiple of the nuclear form factor;
- The daughter nucleus is heavy and there is no recoil, so there is no density of states;
- The electron and neutrino are ultrarelativistic, so $E \approx p$.