Topic 3, Asymptotic Analysis: Solutions

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1. We can extend asymptotic notation to the case of two parameters n and m that can go to infinity independently at different rates. For example, we denote by **O(g(n,m))** the set of functions:

> O(g(n,m)) = {f(n,m) : there exists positive constants c, n_0 **and m_0** such that 0 ≤ f(n,**m**) ≤ cg(n,**m**) for all n ≥ n_0 **or m ≥ m_o**}

Give a corresponding definition for Θ(g(n,m)).

> Θ(g(n,m)) = {f(n,m) : there exists positive constants **c_1, c_2,** n_0 **and m_0** such that 0 ≤ **c_1g(n,m)** ≤ f(n,**m**) ≤ c_2g(n,**m**) for all n ≥ n_0 **or m ≥ m_o**}

2. Indicate, for each pair of expressions *f*(*n*) and *g*(*n*) in the table below, whether f(n) = __(g(n)), where the __ may be O, o, Ω, ω or Θ. Assume that <u>*k* ≥ 1, *c* > 1, and *d* > 0 are constants</u> and we are <u>analyzing growth rates in terms of the variable *n*</u>. To respond, write "Yes" or "No" in each box. Justifications are also required so we can give better feedback and possibly partial credit in case of wrong answers. <u>Justifications should give formulas used to transform the expressions and/or refer to statements proven in the text.</u>

	f(*n*)	*g*(*n*)	O?	o?	Ω?	ω?	Θ?
a.	$n^{\lg c}$	$c^{\lg n}$	Y	N	Y	N	Y
b.	$\lg^k n$	n^d	Y	Y	N	N	N
c.	2^n	$2^{n/2}$	N	N	Y	Y	N
d.	lg(n!)	$\lg(n^n)$	Y	N	Y	N	Y

Justifications (required):

a. It was sufficient in class to refer to the equivalence 3.16 on page 56 of CLRS, but here is a formal proof:

> Claim: $n^{log\, c} = c^{log\, n}$.
>
> Proof: let $x = n^{log\, c}$, then $log\, x \;=\; log\,(n^{log\, c}) = (log\, c) \cdot (log\, n) = log\,(c^{log\, n})$.

Then $x = 2^{log\,x} = 2^{log\,(c^{log\,n})} = c^{log\,n}$ Q.E.D.

b. It was sufficient in class to refer to the result page 57 CLRS that any polynomial grows larger than any poly-logarithmic. Here is a formal proof:

Claim: $log^k n = o(n^d)$

Proof:

Definition of $f(n) = o(g(n))$ is if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

$f(n) = log^k n = (log\,n)^k$

$g(n) = n^d$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{log^k n}{n^d} = \lim_{n\to\infty} \left(\frac{log\,n}{n^{d/k}}\right)^k.$$

Let $l = d/k$. Obviously l is just another constant. We know that $log\,n = o(n^l)$, i.e. logarithms grow slower than any polynomial.

Thus, $\lim_{n\to\infty} \frac{log\,n}{n^l} = 0$. Then $\lim_{n\to\infty} \left(\frac{log\,n}{n^{d/k}}\right)^k$ is also 0. Q.E.D.

c. Observe that $2^{n/2} = (2^{1/2})^n = \left(\sqrt{2}\right)^n$ and $\sqrt{2} < 2$, or approximately 1.414. No constant c can make $c1.414^n$ keep up with 2^n as n grows. More formally:

Claim: $a^n = \omega(b^n)$ for any constants a and b, such that $0 < b < a$.

Proof: Definition of $f(n) = \omega(g(n))$ is if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$.

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{a^n}{b^n} = \lim_{n\to\infty} \left(\frac{a}{b}\right)^n = \lim_{n\to\infty} c^n$ for some $c > 1$ because $b < a$.

Thus, $\lim_{n\to\infty} \frac{a^n}{b^n} = \infty$. Q.E.D.

d. $log(n!) = \Theta(n\,log\,n)$ (by Stirling's approximation, p. 57-58 CLRS)

$log(n^n) = n\,log\,n$ Q.E.D.

Extra Credit:

	f(*n*)	*g*(*n*)	O?	o?	Ω?	ω?	Θ?
e.	$4n^2$	$4^{\lg n}$	Y	N	Y	N	Y
f.	$2^{\lg n}$	$\lg^2 n$	N	N	Y	Y	N
g.	$\sqrt{n}$	$n^{\sin n}$	N	N	N	N	N

Justifications:

e. $4^{\lg n} = n^{\lg 4} = n^2 = \Theta(n^2)$, and clearly $4n^2 = \Theta(n^2)$

f. $2^{\lg n} = n^{\lg 2} = n$ by definition, and n (a polynomial) grows larger than any poly-logarithmic

g. sin n oscillates between -1 and 1, so $n^{\sin n}$ oscillates between 1/n and n, while for n > 1, $\sqrt{n}$ is always greater than 1/n and less than n: they cannot be compared.