Differential Calculus

Gradient:

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

Example: $T = \sqrt{x^2 + y^2 + z^2}$

$$T = \left(x^2 + y^2 + z^2\right)^{1/2}$$

$$\frac{\partial T}{\partial x} = \left(\frac{1}{2}\right)\left(2x\right)\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial T}{\partial y} = \left(\frac{1}{2}\right)(2y)\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial T}{\partial z} = \left(\frac{1}{2}\right)(2z)\left(x^2 + y^2 + z^2\right)^{-1/2}$$

$$= \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\nabla T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

$$= \frac{\wedge}{\sqrt{x^2+y^2+z^2}}\hat{x} + \frac{y}{\sqrt{x^2+y^2+z^2}}\hat{y} + \frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$

$$= \frac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{\vec{r}}{r} = \boxed{\hat{r}}$$

Problem 1.11: find grad of

$$f(x,y,z) = x^2 + y^3 + z^4$$

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial f}{\partial y} = 3y^2 \qquad \frac{\partial f}{\partial z} = 4z^3$$

$$\nabla f = 2x\,\hat{x} + 3y^2\,\hat{y} + 4z^3\,\hat{z}$$

$$\nabla f = (2x, 3y^2, 4z^3)$$

Problem 1.12

$$f(x, y, z) = x^2 y^3 z^4$$

find ∇f.

$$\frac{\partial f}{\partial x} = (2x)(y^3)(z^4)$$

$$\frac{\partial f}{\partial y} = (3y^2)(x^2)(z^4)$$

$$\frac{\partial f}{\partial z} = (4z^3)(x^2)(y^3)$$

$$\nabla f = \left(2x y^3 z^4, \; 3x^2 y^2 z^4, \; 4x^2 y^3 z^3\right)$$

Problem 1.13

$$f(x,y,z) = e^x \sin(y) \ln(z)$$

$$\frac{\partial f}{\partial x} = e^x \qquad \frac{\partial f}{\partial y} = \cos(y) \qquad \frac{\partial f}{\partial z} = \frac{1}{z}$$

Divergence

$$\operatorname{div}(T) = \nabla T \cdot T$$

$$= \frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + \frac{\partial T_z}{\partial z}$$

PROBLEM 1.15

$$V_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$$

$$\nabla \cdot V_a = \frac{\partial}{\partial x} x^2 + \frac{\partial}{\partial y} 3xz^2 - \frac{\partial}{\partial z} 2xz$$

$$= 2x + \frac{\partial}{\partial y} 3xz^2 y^0 - 2x(1)$$

$$= 2x + 0 - 2x$$

$$= 0 \quad : \boxed{\text{zero}}$$

$$V_b = xy\hat{x} + 2yz\hat{y} + 3zx\hat{z}$$

$$\nabla \cdot V_b = \frac{\partial}{\partial x} xy + \frac{\partial}{\partial y} 2yz + \frac{\partial}{\partial z} 3zx$$

$$= (1)y + (2)(1)(z) + 3(1)(x)$$

$$= 3x + y + 2z$$

$$V_c = y^2\,\hat{x} + (2xy + z^2)\,\hat{y} + 2yz\,\hat{z}$$

$$\nabla \cdot V_c = \frac{\partial}{\partial x} y^2 + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z} yz$$

$$= \frac{\partial}{\partial x} y^2 x^0 + \frac{\partial}{\partial y} 2xy + \frac{\partial}{\partial y} z^2 y^0 + \frac{\partial}{\partial z} yz$$

$$= 0 + 2x(1) + 0 + y(1)$$

$$= \boxed{2x + y}$$

Curl

$$\nabla \times v = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_x & v_y & v_z \end{bmatrix}$$

$$= \hat{x}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)$$

$$+ \hat{y}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)$$

$$+ \hat{z}\left(\frac{\partial v_y}{\cdot} - \frac{\partial v_x}{\cdot}\right)$$

$$\left(\frac{}{\partial x} - \frac{}{\partial y} \right)$$