Seven Sketches in Compositionality – Exercises

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1 Chapter 1 — Generative Effects: Orders and Galois Connections

(1.1)

(a) *order-preserving* $f : x \mapsto x + 1$
non-order-preserving $f : x \mapsto -x$

(b) *metric-preserving* $x \mapsto x + 2$
non-metric-preserving $x \mapsto 2x$

(c) *addition-preserving* $x \mapsto x$
non-addition-preserving $x \mapsto 2x$

(1.2)

Circle 21, Circle the rest, box around the whole thing. i.e.

$$\{\{21\}, \{11, 12, 13, 22, 23\}\}$$

(1.6)

1. True
2. False
3. True

(1.7)

1. $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$
2. $\{1\} \cup \{1, 3\} = \{1, 3\}$
3. $(h, 1), (h, 2), (h, 3), (1, 1), (1, 2), (1, 3)$
4. $(h, 1), (1, 1), (1, 2), (2, 2), (3, 2)$

5. $A \cup B = \{h, 1, 2, 3\}$

1.11

1. If there were more than one $p' \in P'$ such that $A_p = A'_{p'}$ (i.e. p'_1 and p'_2 such that $A_p = A'_{p'_1}$ and $A_p = A'_{p'_2}$), then $p'_1 \neq p'_2$, so necessarily $A'_{p'_1} \cap A'_{p'_2} = \emptyset$. But then since $A'_{p'_1} = A_p = A'_{p'_2}$, then $A'_{p'_1} = A'_{p'_2}$, thus $A'_{p'_1} \cap A'_{p'_2} = A_p \cap A_p = A_p \neq \emptyset$, by definition of partition, therefore there cannot be more than one $p' \in P'$ such that $A_p = A'_{p'}$. $\blacksquare$

2. Since there exists a $p' \in P'$ such that $A_p = A'_{p'}$, and by 1.11.1, there is at most one such p', it follows that there is a bijection between these $p \in P$ and $p' \in P'$, thus $\forall p' \in P'$ there exists a $p \in P$ such that $A_p = A'_{p'}$. $\blacksquare$

1.12

$(11, 11), (12, 12), (13, 13), (21, 21), (22, 22), (23, 23), (11, 12), (12, 11), (22, 23), (23, 22)$

1.15

1. Each A_p is $(\sim)$-connected, therefore they are nonempty. $\blacksquare$

2. If $A_p \cap A_q \neq \emptyset$, then for any $x \in A_p \cap A_q$, it follows that $x \in A_p$ and $x \in A_q$. Thus for any $x_p \in A_p$, we have $x_p \sim x$, and under $(\sim)$-closure and $x \in A_q$, it follows that $x_p \in A_q$, and the same for $x_q \in A_q$. Therefore, $A_p = A_q \vDash p = q$, which violates the contextual assertion that $p \neq q$, therefore $p \neq q \implies A_p \cap A_q = \emptyset$. $\blacksquare$

3. Each A_p is nonempty and $\sim$ is reflexive, i.e. we have at least for each x that $x \sim x$, therefore we have at least that A is the union of singleton sets that cover A, therefore $A = \bigcup_{p \in P} A_p$. $\blacksquare$

1.19

1. $f : \mathbb{Z} \to \mathbb{R} : x \mapsto x + 1$

2. $f : \mathbb{Q} \to \mathbb{Z} : x \mapsto x - \frac{x}{10}$

3. (1): yes (2): no (3): no (4): yes

4. (1): neither (2): top dot's targets are not unique, bottom dot has no target (3): top dot has no target (partial function?) (4) bijective

1.20

By function definition, each $a \in A$ has a unique $y \in \emptyset$ such that $(a, y) \in f$. But there are no $y \in \emptyset$, therefore there cannot exist any such $a \in A$, therefore

A is empty. ■

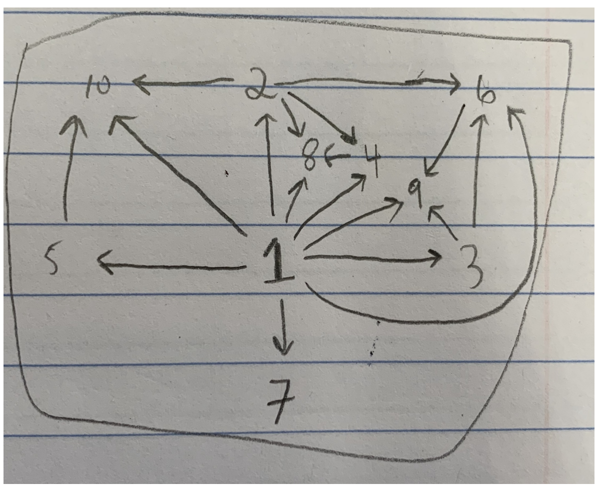
1.33

$$\begin{vmatrix} a & 1 & 2 \\ b & 1 & 3 \\ c & 1 & 3 \\ d & 2 & 2 \\ e & 2 & 3 \end{vmatrix}$$

1.35

$(1,1), (1,2), (1,3), (2,2), (2,3), (3,3), (4,4)$

1.41

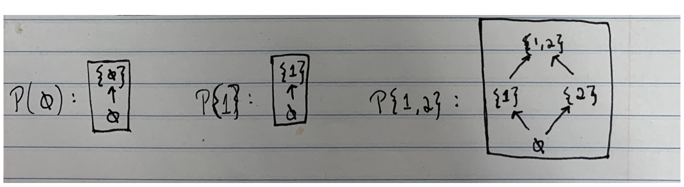


It is not a total order; for instance, neither $7 \leq 9$ nor $9 \leq 7$.

1.43

Yes.

1.46



1.48

Coarsest: identity function (top level of Hasse diagram)
Finest: projection onto singleton sets for each $s \in S$ (second to bottom level of Hasse diagram)

1.50

This is the same as proving that all subsets of X are upper sets of the discrete preorder on X, i.e. that $\forall [x, y \in X] . [x \in U \wedge x \leq y \implies y \in U]$. The empty set is in U via vacuous truth of the antecedent. Also, each $\{x\}$ for $x \in X$ is an upper set, via reflexivity of (discrete) preorder. Then for any $x \in X$, take the union of the singleton set $\{x\}$ and any other subset $Y \subseteq X$ disjoint from this singleton. By the argument from two sentences ago, each such singleton set is an upper set. This union is a subset of X, and furthermore it is an upper set, since the antecedent conjunction is false – it is not the case that $x \leq y$ for any $y \in Y$, since they are not comparable via definition of discrete preorder. Therefore all such Y are upper sets. These three statements (empty set is an upper set, all singletons are upper sets, all other subsets are upper sets) entail that all subsets of X are upper sets, therefore the preorder of upper sets on X is the power set.$\blacksquare$

1.61 *(Yoneda Lemma for Preorders)*

1. We have $p \in \uparrow p$ via reflexivity of $\leq$, and exactly when $p \leq p'$ we have $p' \in \uparrow p$; this is exactly the definition of upper set. $\blacksquare$

2. For $x, y \in P$, if $x \leq y$ then $x \leq^{OP} y$ in P^{OP}. There's an obvious map $\uparrow$ from P^{OP} to the set of upper sets of elements of P, $U(P)$ as defined in (1.61.1). $U(P)$ is preordered via the subset relation. Recalling that $x \leq^{OP} y \iff y \leq x$, then we must show that $x \leq^{OP} y \implies \uparrow(x) \subseteq \uparrow(y)$, i.e. $y \leq x \implies \uparrow(x) \subseteq \uparrow(y)$. $y \leq x$, and due to transitivity of $\leq$, if $x \leq z$, then $y \leq z$, i.e. if $z \in \uparrow(x)$ then $z \in \uparrow(y)$, thus $\uparrow(x) \subseteq \uparrow(y)$. $\blacksquare$

3. Suppose $p \leq p'$ in P. Then $p' \leq^{op} p$, and due to monotonicity of $\uparrow$ on the domain P^{op}, $\uparrow(p') \subseteq \uparrow(p)$.
   Now suppose that $\uparrow(p') \subseteq \uparrow(p)$. Then for every element $x \in P$ such that $x \leq p$, it follows that $x \leq p'$. Then there exists a $y \in P$ such that $p \leq y \leq p'$, therefore $p \leq p'$. $\blacksquare$

1.66

1. $x \leq y \implies id(x) \leq id(y) \implies x \leq y$

2. for $p_1, p_2 \in P$, $p_1 \leq p_2 \implies f(p_1) \leq f(p_2)$, and for $q_1, q_2 \in Q$, $q_1 \leq q_2 \implies g(q_1) \leq g(q_2)$. Let $q_1 = f(p_1)$, $q_2 = f(p_2)$. Then $p_1 \leq p_2 \implies g(f(p_1)) \leq g(f(p_2))$, i.e. $(f; g)$ is monotone. $\blacksquare$

1.80

1. Check that p is $\leq$ each element: $p \leq p$; and that each least element is $\leq$ to p: $p \leq p \implies p \leq p$. ■

2. If P is a partial order, then if $x \equiv y \iff x = y$. It follows that if $\bigwedge A \equiv p$, then $\bigwedge A = p$. ■

3. Yes, these are still true if we replace meets with joins. This can be checked simply by replacing $\leq$ with $\leq^{op}$, and that $p \leq^{op} p$ entails $p \leq p$. ■

1.92

A right adjoint for $(3 \times -)$ is $\lfloor -/3 \rfloor$. We check that $x \leq \lfloor y/3 \rfloor \iff 3x \leq y$.
$(\Rightarrow)$ Suppose $x \leq \lfloor y/3 \rfloor$. Since $\lfloor y/3 \rfloor \leq y/3$, it follows that $x \leq y/3$, thus $3x \leq y$.
$(\Leftarrow)$ Suppose now that $3x \leq y$. Then $x \leq y/3 \implies \lfloor x \rfloor \leq \lfloor y/3 \rfloor$, and since $x \in \mathbb{Z}$, it follows that $\lfloor x \rfloor = x$, therefore $x \leq \lfloor y/3 \rfloor$. ■

1.93

1. $$f(1) = 1 \implies \left| \begin{array}{c} 1 \leq 1 \implies g(1) = 2 \implies 1 \leq 2 \\ 1 \leq 2 \implies g(2) = 2 \implies 1 \leq 2 \\ 1 \leq 3 \implies g(3) = 3 \implies 1 \leq 3 \end{array} \right|$$
$$f(2) = 1 \implies \left| \begin{array}{c} 1 \leq 1 \implies g(1) = 2 \implies 2 \leq 2 \\ 1 \leq 2 \implies g(2) = 2 \implies 2 \leq 2 \\ 1 \leq 3 \implies g(3) = 3 \implies 2 \leq 3 \end{array} \right|$$
$$f(3) = 3 \implies [3 \leq 3 \implies g(3) = 3 \implies 3 \leq 3]$$

   therefore f is left-adjoint to g. ■

2. $$f(1) = 1 \implies \left| \begin{array}{c} 1 \leq 1 \implies g(1) = 2 \implies 1 \leq 2 \\ 1 \leq 2 \implies g(2) = 2 \implies 1 \leq 2 \\ 1 \leq 3 \implies g(3) = 3 \implies 1 \leq 3 \end{array} \right|$$
$$f(2) = 2 \implies \left| \begin{array}{c} 2 \leq 2 \implies g(2) = 2 \implies 2 \leq 2 \\ 2 \leq 3 \implies g(3) = 3 \implies 3 \leq 3 \end{array} \right|$$
$$f(3) = 3 \implies [3 \leq 3 \implies g(3) = 3 \implies 3 \leq 3]$$

   therefore f is left-adjoint to g. ■

2 Chapter 2 – Resource Theories: Monoidal Preorders and Enrichment

(2.4)

This fails due to condition *(a)* of Definition (2.1) –
$\forall\, [x_1, x_2, y_1, y_2 \in X]\, .\, [(x_1 \leq y_1 \wedge x_2 \leq y_2) \implies x_1 \otimes x_2 \leq y_1 \otimes y_2]$
for instance, any model in which x_1, x_2 are both negative real numbers larger in magnitude than positive reals y_1, y_2 will not yield $x_1 \otimes x_2 \leq y_1 \otimes y_2$. ■

(2.6)

Yes, $\mathbf{Disc}_M$ is indeed a symmetric monoidal preorder. It satisfies conditions (b) and (c) of Definition (2.1) via equation (2.2). It satisfies condition (d) of Definition (2.1) via the asserted commutativity of $*$. Finally, this induces satsifaction of condition (a) of Definition (2.1), via renaming of identical variables: if $m = n$ and $s = t$, then $m * s = n * s = n * t$. ■