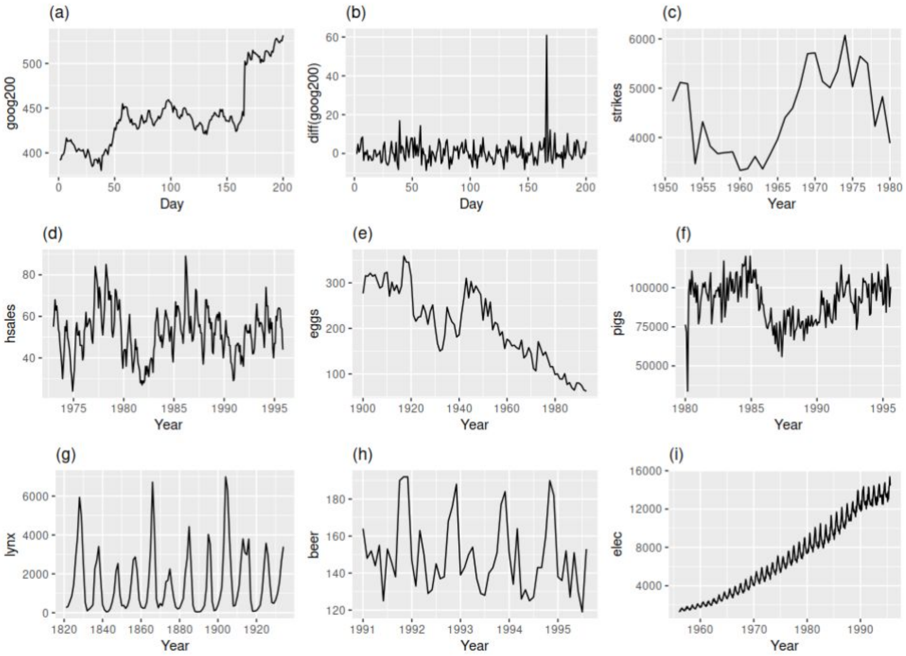
ARIMA Quiz

By Team Fourier!

Q1: Which of the following time series are stationary?



Answer: (B). (B) is stationary because the data is randomly oscillating without any clear pattern. The rest of the plots have some seemly trend such and rising stock prices and periodically fluctuating data points. (G) turns to be stationary, but it's because of the underlying dynamics of Lynx population in Canada, which is stochastic.

Q2: What is the effect of the lag factor and why is it useful?

$$ARIMA(p, d, q) = (1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 ... - \phi_p L^p)(1 - L)^d y_t = c + (1 + \theta_1 L + \theta_2 L^2 ... + \theta_q L^q)\epsilon_t$$

Answer: The effect of the lag factor is that it gives data from farther away in time exponentially less weight. We want this effect because data that are more recent in the time of concern matter more than those that are farther away.

Q3: What are the semantics of the following combos?

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t$$

$\Phi = 0$
$\Phi = 1,\ c = 0$
$\Phi = 1,\ c \neq 0$
$\Phi < 0$

Answer: If phi is zero, that means "y" is just white noise. If phi is one and "c" is zero, this is equivalent of the random walk model. If phi is one and "c" is not zero, this is the random model with drifting. If phi is less than zero, "y" will just oscillate around the mean.

Q4: Prove EMA weighs terms in the near past more than in the far past

Answer:

$$\hat{m}_t = \frac{1-\phi}{\phi} \sum_{i=1}^{\infty} \phi^i X_{t-i}$$

The weight for further away X_{t-i} is phi^i, which is exponentially smaller for the lower $t-i$, which is greater th i.

Q5: What is the process of differencing and how is it used? What is the equation for first order differencing? Second order?

Differencing is the practice of calculating the difference between consecutive time series data.

$$\nabla X_t = X_t - X_{t-1} \qquad \forall t \in 2, \ldots, n.$$

$$\begin{aligned} \nabla^2 X_t &= \nabla(\nabla X_t) \qquad \forall t \in 3, \ldots, n \\ &= \nabla X_t - \nabla X_{t-1} \\ &= X_t - 2X_{t-1} + X_{t-2}. \end{aligned}$$

Q6: Derive an example of a summary statistic that would vary between a weakly stationary and strongly stationary process.

Answer: The covariance matrix of X would be the same if the data is strongly stationary, and different if weak stationary.

Q7: What's causality?

Causality is when the AR time series can be represented as a linear process. Formally, an AR model is causal if and only if it can be written as

$$
\begin{aligned}
X_t &= \psi(B)Z_t \\
&= \sum_{j=0}^{\infty} \psi_j Z_{t-j}
\end{aligned}
$$

$$where\ \psi(B) = \textstyle\sum_{j=0}^{\infty} \psi_j B^j\ \ and\ \textstyle\sum_{j=0}^{\infty} |\psi_j| < \infty\ \ and\ \psi_0 = 1.$$

Q8 Describe the invertibility condition. Why is it useful?

Answer: Consider the acf of the MA(1) model on the right:

$$\rho(1) = \frac{\theta}{1+\theta^2}$$

The invertibility condition is that θ must lie in the unit circle.

$$|\theta| < 1$$

This allows us to write any MA model as an AR model.

Q9: Prove causality for ARIMA.

We can rewrite (23) as:

$$
\begin{aligned}
X_t &= \frac{Z_t}{\phi(B)} \\
&= \frac{1}{1-\phi B} Z_t
\end{aligned}
\tag{23}
$$

Notice we have the sum of a geometric series for $|\phi| < 1$:

$$
\begin{aligned}
X_t &= (1 + \phi B + \phi^2 B^2 + \ldots) Z_t \\
&= \sum_{j=0}^{\infty} \phi^j Z_{t-j}
\end{aligned}
\tag{24}
$$

Consult the proof for $P = 1$ on the right. By induction, X_{i+1} is a linear combination of $X_{\leq i}$'s and X_{i+1} because the ϕ^i are encapsulated in the past $x_{\leq i}$'s.

Q10: Derive the Yule Walker equation for a lag of 3.

Given a zero mean and causal AR(p) process $\{X_t\}$ defined as (19), we can try to solve for its parameters. Let us begin by multiplying (19) by X_{t-1} and taking the expectation to obtain:

$$\begin{aligned}\mathbb{E}[X_t X_{t-1}] &= \sum_{j=1}^{p} \phi_j \mathbb{E}[X_{t-j} X_{t-1}] + \mathbb{E}[Z_t X_{t-1}] \\ &= \sum_{j=1}^{p} \phi_j \mathbb{E}[X_{t-j} X_{t-1}] \end{aligned} \tag{27}$$

where $\mathbb{E}[Z_t X_{t-1}]$ is 0 because Z_t is uncorrelated with previous values of the process. We have just derived the Yule-Walker equation for *lag*. We repeat this derivation for *lag 2, ..., lag p*. As an exercise, derive the Yule-Walker equation for *lag p*. The following is the generalized equation:

$$\begin{aligned}\mathbb{E}[X_t X_{t-k}] &= \sum_{j=0}^{p} \phi_j \mathbb{E}[X_{t-j} X_{t-k}] + \mathbb{E}[Z_t X_{t-k}] \\ &= \sum_{j=0}^{p} \phi_j \mathbb{E}[X_{t-j} X_{t-k}] \end{aligned} \tag{28}$$

for $k \in [1, \ldots, p]$. Note $r_{-l} = r_l$. Using r_l for $\mathbb{E}[X_{t-l} X_{t-k}]$, (28) simplifies to:

$$r_l = \sum_{j=1}^{p} \phi_j r_{j-l} \tag{29}$$