Dynamic Volatility Modelling of Bitcoin using Time-varying Transition Probability Markov-switching GARCH Model

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Abstract

Bitcoin (BTC), as the dominant cryptocurrency, has attracted tremendous attention lately due to its excessive volatility. This paper proposes the time-varying transition probability Markov-switching GARCH (TV-MSGARCH) models incorporated BTC logarithmic daily trading volume or Google daily searches as exogenous factors to model the volatility dynamics of BTC return series. Extensive comparisons were carried out to evaluate the modelling and forecasting performances of TV-MSGARCH models with the benchmark models such as GARCH, GJRGARCH, threshold GARCH and constant transition probability MSGARCH. Results reveal that the TV-MSGARCH models predominate other models for the in-sample model fitting based on log-likelihood function and other benchmark criteria. Furthermore, the TV-MSGARCH model with Google daily searches offers the best out-of-sample forecast evaluated based on quasi-likelihood loss function and also using Hansen's model confidence set. In addition, the Filardo's weighted transition probabilities is computed and the results show the existence of time-varying effect on transition probabilities. Lastly, new approach for computing different levels of long and short positions of value-at-risk forecasts based on TV-MSGARCH models are also provided and tested.

Keywords: Bitcoin, Volatility, Time-varying transition probability, Markov-switching, GARCH model.

JEL classifications: C01, C58, G10

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1. Introduction

Volatility, in general, refers to the statistical deviation from return series and is used as a risk measure to evaluate the uncertainty of asset prices of the financial market. It is commonly used in financial strategies such as risk management and portfolio optimisation and thus has to be monitored closely in order for practitioners to select the appropriate strategies responded to the latest market conditions.

Presently, the cryptocurrency market has attracted significant attention from all around the world due to its intractable behaviour and unpredicted price movements. To date, there are more than two thousand types of cryptocurrencies circulating in the market (www.coinmarketcap.com), with Bitcoin (BTC) dominating the market. One of the interesting aspects of the cryptocurrency market is its radical and frequent changes in prices observed in a very short span of times which are different from other traditional financial markets. The unpredictability and high volatility of Bitcoin are always the central issues of many studies. Williams (2014) contrasted the volatility of BTC with other conventional financial assets and showed that the volatility of BTC is relatively larger than other commodities such as precious metal, Standard and Poor 500, and the U.S. dollar. Dwyer (2015) reported that the volatility of BTC is higher than gold and various types of foreign currencies measured in U.S. dollar. Bariviera et al. (2017) pointed out that the large volatility of BTC was mainly due to the presence of great swings observed in BTC return series. Balcilar et al. (2017) showed that the trading volume can be used to predict BTC return when the market is fluctuated around its median but cannot be used to forecast BTC volatility. Urquhart (2018) utilised Google Trends as a proxy for attention of Bitcoin and found that previous day volatility and volume are significant drivers that attract investors to BTC. Similar conclusion was drawn by Aalborg et al. (2019) in which they discovered that the trading volume can improve the volatility model for BTC when the trading volume is predicted from Google Trends. More specifically, Corbet et al. (2019) and Panagiotidis (2019) provide a more comprehensive and systematic review of cryptocurrency market on major academic research.

Many studies have emerged recently in the study of volatility dynamics of cryptocurrency data which were mostly worked on the basis of autoregressive conditional heteroscedasticity (ARCH) model by Engle (1982) and general autoregressive conditional heteroscedasticity (GARCH) model by Bollerslev (1986). Among those attempts, Dyhrberg (2016) investigated the ability of BTC to be used for hedging purpose by applying asymmetric GARCH model. Lahmiri et al. (2018) studied the long-range memory in seven BTC markets by using fractionally integrated GARCH (FIGARCH). Meanwhile, Katsiampa (2017), incorporating the autoregressive component in the return series, reported that the component GARCH is the best fitted model to estimate the volatility of BTC. Chu et al. (2017) extended the work by considering a wide variety of GARCH models under different error distribution assumptions which were applied to seven cryptocurrencies including the BTC. They noted that the integration GARCH (IGARCH) with normal distribution is the best fitted volatility model for BTC return series. However, they also argued that the existence of structural break in cryptocurrency series, which is not taken into account in the modelling process, might be the reason why IGARCH model appeared to be the best fitted model. A similar argument was raised by Caporale et al. (2003) claiming that the IGARCH model may be the optimal model if structural breaks exhibited by the series are not treated.

In addition, the stochastic volatility (SV) model is another popular model for volatility modelling introduced by Taylor (1986). Philip et al. (2018) fitted 224 cryptocurrencies return series to

SV model incorporated with a long memory, leverage and heavy tails properties and suggested that the cryptocurrencies, in general, exhibit long memory, leverage effects and heavy tailedness. On the other hand, Chaim and Laurini (2018) integrated discontinuous jumps into the SV model to study the volatility and return series behaviour of BTC. Prominent evidence of jumps was found in both return and volatility series, and two high volatility periods were observed from late 2013 to early 2014 and also in December 2017. Nevertheless, it is always the case that the return-based models tend to demonstrate high persistence conditional volatility (Hamilton and Susmel, 1994; Gray, 1996) which may indicate biasness if the series displays structural break according to Bauwens et al. (2014).

In view of the prominent statistical change observed in cryptocurrency data, several studies have been made to partition the data series into segments in volatility modelling and forecasting to account for the possible structural breaks over time. Bouoiyour and Selmi (2016) discovered the pronounced BTC volatility change in January 2015 in which case the volatility starts to decline at a rapid pace, hence, they proposed to separate the BTC return series and fitted nine GARCH models to the series. The presented best fitted models included the component with multiple thresholds (CMTGARCH) (prior-2015) and asymmetric power GARCH (APGARCH) (post-2015). Moreover, Bouri et al. (2017) also examined the effect of breaks at BTC price crash 2013 to derive the asymmetric return-volatility relation before and after the price crash. Results show that BTC possesses the inverse asymmetric property (safe heaven) in the pre-crash period and inclines to diminish after the price crash. All these studies mentioned above again validate the frequent change of BTC statistical properties over time that needs further treatment in financial econometrics studies. Meanwhile, Thies and Mólnar (2018) investigated the presence of a structural break in the BTC return series by using Bayesian change point model. Forty eight change points in BTC return series were estimated and the segments were combined into 7 regimes based on the estimated volatility that exhibited the analogous properties. Mensi et al. (2018) also argued the existence of structural break in mean and variance of BTC series and highlighted the persistence overestimation problem in GARCH volatility if structural breaks were not dealt with in proper manner.

Bouri et al. (2019) studied the impact of structural breaks in the stagnancy of BTC price level and its volatility by using logarithmic price series, squared return series and absolute return series of two BTC markets, namely, Bitstamp and Coindesk. They reported four and five structural breaks respectively in Bitstamp and Coindesk markets adopting Bai and Perron (2003) approach under the maximum of five structural breaks. A more comprehensive study on structural breaks in cryptocurrency data was also discussed by Tan et al. (2019), without imposing restrictions on the number of structural breaks, a couple of structural breaks in price, return and squared return series of top 10 cryptocurrencies data based on market capitalisation and two cryptocurrency indices were discovered. Their findings suggested that the BTC appears to have 72 structural breaks in its price series, 22 in its return series and 56 in its volatility proxied by squared return series. All these provide strong evidence for us to address the unexpected change in BTC series due to its complexity, fast-changing and instability over time and it is indeed the research gap to be filled in.

To capture the effects of a sudden change in time series data, Hamilton and Susmel (1994), and Cai (1994) incorporated Markov-switching process (Hamilton, 1989) with ARCH models to allow more flexibility in volatility modelling that undergo a finite number of changes governed by Markov chain. The methods are ideally reported to be able to account for the high persistence and poor forecasting performance problem when the traditional GARCH models are employed. Several studies employing Markov-switching process with GARCH model for cryptocurrency volatility modelling were carried out, for instance, Ardia et al. (2018) showed that two-regime Markov-switching GARCH (MSGARCH) model outperforms single regime GARCH and Glosten-Jagannathan-Runkle GARCH (GJRGARCH) models in both in-sample and out-sample BTC volatility modelling.

Caporale and Zekokh (2019), applying value-at-risk (VaR) backtests approach to the four main cryptocurrencies including BTC, offered the similar conclusion that two-regime MSGARCH models provide better volatility forecasting estimates than those GARCH models. The papers in Ardia et al. (2018) and Caporale and Zekokh (2019) provide extensive discussion of the application of constant transition probabilities for the entire period. It is plausible that the transition probabilities are affected by some exogenous factors or lagged dependent variables. Diebold et al. (1994) and Filardo (1994) noted that the constant transition probability is too restrictive for some empirical settings and suggested the use of time-varying transition probabilities (TVTP) in which the transition probabilities can evolve as either a logistic function or a probit function of a series of exogenous factors or lagged dependent variables. The advantage of using time-varying transition probabilities MSGARCH (TV-MSGARCH) model in volatility modelling process was also reported by Gray (1996) and Bazzi et al. (2017). In order to improve the flexibility of volatility modelling, the TV-MSGARCH model incorporating exogenous factors e.g. the daily trading volume and Google daily searches is employed to observe the volatility behaviour of BTC which is lacking in previous literature.

Parameter estimation of MSGARCH model using the maximum likelihood (ML) method is another central interest due to the path dependence problem as pointed out by Hamilton and Susmel (1994). Path dependence problem may cause the realisation of log-likelihood (LL) function to be intractable in k^T number of possibilities where k represents the number of regimes and T refers to the length of data. Gray (1996) addressed this issue by collapsing the conditional variances in each regime into a single proxy value at every time point t so that the estimated volatility will not depend on the entire past regime path. Another analytical solution was proposed by Haas et al. (2004) who separated the GARCH dynamics from Markov chain and assumed that each regime is characterised by its own volatility within that regime. In the meantime, Elliot et al. (2012) suggested the use of Viterbi algorithm to identify the current state of volatility before the estimation procedure is carried out so that the process becomes more tractable and the maximisation of LL function becomes feasible. This paper is primarily concerned with the use of the ML method under Gray's framework due to its ease of implementation in the maximisation of LL function through R studio optimisation function optim.

The contribution of this paper is threefold. Firstly, we will build a two-regime TV-MSGARCH model integrating the two main drivers of BTC price as exogenous factors into the model, namely, Google daily searches of the term "Bitcoin" and BTC daily trading volume to estimate BTC volatility. The inclusion of the Google daily searches and trading volume in volatility model estimation appears to be crucial since these factors are known to have a significant impact on BTC prices (Kristoufek, 2015; Parino et al., 2018). To provide a better understanding on the pronounced statistical change in BTC series, we allow that both conditional mean and volatility are subject to regime change where the conditional mean follows an autoregressive process and conditional variance follows GARCH process respectively. This setting enables us to examine the impact of regime change in both mean and volatility levels. Secondly, the BTC return series will be fitted into the GARCH, GJRGARCH and threshold GARCH (TGARCH) models, MSGARCH model and TV-MSGARCH models. Meanwhile, numbers of modelling and forecasting are made with respect to different types of GARCH models, MSGARCH model and, TV-MSGARCH models with and without exogenous factors for comparison purposes. The improvement of TVTP over constant transition probabilities can be accessed via weighted transition probabilities (WTP) as discussed by Filardo (1994). Lastly, BTC volatility forecasting performance of GARCH models, MSGARCH model and TV-MSGARCH models are compared and contrasted by using model confidence set (MCS) proposed by Hansen et al. (2011). We also develop a new approach to calculate the VaR forecast based on the fitted TV-MSGARCH models whereby VaR backtests such as unconditional coverage test (Kupiec, 1995) and conditional coverage test (Christoffersen, 1998) are used for testing the accuracy of these VaR forecasts of the volatility models.

The remainder of this paper is organised as follows. Section 2 reviews some existing GARCH and MSGARCH models. Section 3 outlines the formulation of TV-MSGARCH model together with the discussions on the ML estimation method and the new approach for computing VaR forecasts. Empirical application based on BTC data is provided in Section 4 with a detailed illustration on model fitting and the estimation performance of TV-MSGARCH models accessed via WTP and the forecasted VaR. Moreover, the performance of in-sample estimation and out-of-sample volatility forecasts of these volatility models are also contrasted. Finally, Section 5 concludes the paper.

2. Volatility Models

2.1 GARCH models

Let the daily log return (or return) r_t be defined as

$$r_t = \ln(P_t) - \ln(P_{t-1}),\ t = 1,2,\dots,T,$$

where P_t is the closing price of Bitcoin measured at time t. The r_t can be modelled by the following model

$$r_t = b_t + a_t,\ t = 1,2,\dots,T, \tag{1}$$

in which $b_t = \mathbb{E}(r_t|\ \varphi_{t-1})$ is the conditional mean component where $\varphi_{t-1} = \{r_{t-1}, r_{t-2}, \dots\}$ is information set up at time $(t-1)$. To remove the serial dependence in return series, we use $b_t = \mu + \phi r_{t-1}$. Meanwhile, $a_t = \sigma_t \varepsilon_t$ is the innovation component (or shock) with the volatility term σ_t , and ε_t representing the random noise with zero mean and unit variance. We consider the following three model specifications for the conditional variance at time t, σ_t^2:

(i) GARCH(1,1) model

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \beta \sigma_{t-1}^2,\ t = 1,2,\dots,T, \tag{2}$$

where ω, α and β are model coefficients with $\omega > 0$, $\alpha > 0$ and $\beta > 0$. Note that $(\alpha + \beta)$ is the measure of persistence and the weak stationary condition is given by $(\alpha + \beta) < 1$.

(ii) GJRGARCH(1,1) model

$$\sigma_t^2 = \omega + \alpha a_{t-1}^2 + \gamma a_{t-1}^2 \mathbb{I}\{a_{t-1} < 0\} + \beta \sigma_{t-1}^2,\ t = 1,2,\dots,T, \tag{3}$$

where ω, α, γ and β are model coefficients with $\omega > 0$, $\alpha > 0$, $\gamma \geq 0$, $\beta > 0$, $\mathbb{I}_{t-1}$ is an indicator variable such that $\mathbb{I}\{a_{t-1} < 0\} = 1$ if $a_{t-1} < 0$ and zero otherwise for which γ is the asymmetric component that controls the degree of asymmetry effect corresponding to the past shock in the conditional variance. For the process to be weakly stationary, we have that $[\alpha + \gamma\mathbb{E}(\varepsilon_t^2\ \mathbb{I}\{\varepsilon_t < 0\}) + \beta] < 1$.

(iii) Threshold GARCH(1,1) model

$$\sigma_t = \omega + \alpha a_{t-1}\mathbb{I}\{a_{t-1} \geq 0\} + \gamma a_{t-1}\mathbb{I}\{a_{t-1} < 0\} + \beta\sigma_{t-1},\ t = 1,2,\dots,T, \tag{4}$$

where ω, α, γ and β are model coefficients with $\omega > 0, \alpha > 0, \gamma \geq 0$ and $\beta > 0$. Given that $\mathbb{I}\{a_{t-1} \geq 0\} = 1$ if $a_{t-1} \geq 0$ and zero otherwise. Similarly, $\mathbb{I}\{a_{t-1} < 0\} = 1$ if $a_{t-1} < 0$ and zero otherwise. TGARCH(1,1) model takes into account the leverage effect governed by α and γ. In this setting, positive and negative shocks have a different impact on the conditional volatility. Here, weakly stationary requires that $[\alpha^2 + \beta^2 - 2\beta(\alpha + \gamma)\mathbb{E}(\varepsilon_t\ \mathbb{I}\{\varepsilon_t < 0\} - (\alpha^2 - \gamma^2)\mathbb{E}(\varepsilon_t^2\mathbb{I}\{\varepsilon_t < 0\})] < 1$ (see Zakoian, 1994 for details).

2.2 MSGARCH(1,1) model

To allow for the dynamic of parameters in the model, we consider a two-regime Markov-switching model as it is conventional to classify the economic states into high and low volatility states.

Under the regime switching framework, the MSGARCH(1,1) model is given by

$$r_t = b_t(s_{1:t}; \varphi_{t-1}) + a_t(s_{1:t}; \varphi_{t-1}),\ t = 1,2,\dots,T, \tag{5}$$

where the conditional mean component $b_t(s_{1:t}; \varphi_{t-1}) = \mathbb{E}(r_t|\ s_{1:t}; \varphi_{t-1}) = \mu_{s_t} + \phi_{s_t} r_{t-1}$ is assumed to follow a two-regime Markov-switching autoregressive of order 1 (AR(1)) process, $a_t(s_{1:t}; \varphi_{t-1}) = \sigma_t(s_{1:t}; \varphi_{t-1})\varepsilon_t$ is the innovation component with $s_{1:t} = (s_1, s_2, \dots, s_t)$. The latent state process, $s_t \in \{1,2\}$ is governed by a constant transition probability,

$$p_{ij} = \mathbb{P}(s_t = j|s_{t-1} = i, \varphi_{t-1})\ ,\ \text{for } i,j \in \{1,2\},\ t = 1,2,\dots,T,$$

with constraints $0 < p_{ij} < 1$ and $\sum_{j=1}^{2} p_{ij} = 1$ for $i \in \{1,2\}$. The latent state process is assumed to be irreducible and aperiodic Markov chain with stationary probability measure $\pi = (\pi_1, \pi_2)$.

The conditional variance $\sigma_t^2(s_{1:t}; \varphi_{t-1})$ at time *t* is formulated based on GARCH(1,1) model

$$\sigma_t^2(s_{1:t}; \varphi_{t-1}) = \omega_{s_t} + \alpha_{s_t} a_{t-1}^2(s_{1:t-1}; \varphi_{t-2}) + \beta_{s_t}\sigma_{t-1}^2(s_{1:t-1}; \varphi_{t-2}),\ t = 1,2,\dots,T. \tag{6}$$

According to Bauwens et al. (2006), the stationarity of MSGARCH(1,1) model can be guaranteed even if not all of the regimes satisfy the stationary condition. However, this constraint must be satisfied on average with respect to the probability distribution of the regimes. This inference is made on the assumption that the stationary regime dominates the entire process such that it will return to the stationary state after a big shock occurs which is defined as the “relieving effect”. Consequently, to ensure the covariance-stationarity of the two-regime MSGARCH(1,1) model, the constraint of $\mathbb{E}[\alpha_{s_t} + \beta_{s_t}] < 1$ must be satisfied where the expectation is taken with respect to the stationary probability measure, i.e,

$$\sum_{j=1}^{2} \pi_j(\alpha_j + \beta_j) < 1.$$

In particular, for a constant transition probabilities model, the stationary probability can be obtained by

$$\pi_j = \mathbb{P}(s_t = j) = \frac{(1-p_{ii})}{(2-p_{ii}-p_{jj})}, \tag{7}$$

for $i, j = 1,2$ and $i \neq j$.

3. Dynamic Volatility Model

3.1 TV-MSGARCH(1,1) model

For gaining more flexibility, the assumption of constant transition probabilities in MSGARCH(1,1) model can be relaxed in favour of TVTP. Specifications of TV-MSGARCH(1,1) model are similar as presented in Eqs. (5) and (6). We let the transition probabilities of the TVTP model to evolve as a logistic function that depend on an exogenous factor, x_t, at every time t as discussed by Diebold et al. (1994) and Filardo (1994), in which, the extension to multiple exogenous factors is straight forward. More specifically, the TVTP can be expressed as

$$\mathbb{P}(s_t = i | s_{t-1} = i, x_{t-1}, \varphi_{t-1}) = p_{ii,t} = \frac{\exp(c_i + d_i x_{t-1})}{1+\exp(c_i + d_i x_{t-1})}, \tag{8}$$

where c_i is a constant and d_i is the coefficient of exogenous factor in regime i and $\mathbb{P}(s_t = j | s_{t-1} = i, x_{t-1}, \varphi_{t-1}) = p_{ij,t} = 1 - p_{ii,t}$ for $i, j = 1,2$ and $i \neq j$. Note that when d_i is set to zero, the transition probabilities become constant and TV-MSGARCH(1,1) model simply reduces to MSGARCH(1,1) model. In this study, we consider x_t to be (i) log of the daily trading volume; (ii) Google daily searches; and (iii) log difference of Google daily searches. The motivation of log transformation of the difference of Google daily searches is to access the impact of the change of public interest in BTC to volatility. By doing so, we can avoid the biasness in volatility modelling and forecasting procedures since it was discovered that the data of Google daily searches are obviously protruding at the turning points of years 2017 and 2018.

As illustrated in Diebold et al. (1994), the stationary probability of the TVTP model is determined by the coefficient parameter of x_t and to our knowledge, there was no strict analytical formula for its computation. To access the stationarity condition of TV-MSGARCH(1,1), we propose to use the mean of $\mathbb{P}(s_t = j)$ for $j = 1,2$ as the stationary probabilities. To be more concrete, the stationary probabilities are calculated through

$$\pi_j = \frac{1}{T}\sum_{t=1}^{T} \mathbb{P}(s_t = j) = \frac{1}{T}\sum_{t=1}^{T} \frac{(1-p_{ii,t})}{(2-p_{ii,t}-p_{jj,t})}, \tag{9}$$

for $i, j = 1,2$ and $i \neq j$.

3.2 Estimation

The parameter estimation of MSGARCH and TV-MSGARCH models are obtained via the maximisation of LL function. Gray (1996) approach is applied in which all the possible paths of return series and conditional variance at time t are collapsed into single proxy values to avoid the path dependence problem. Hence the evolution of regimes will depend only on the current regime instead of the entire regime path. This method consequently resolves the path dependence problem of an

MSGARCH(1,1) model and leads to the solution of ML estimation. The collapsed $a_t(s_{1:t};\varphi_{t-1})$ in Eq. (5) can be proxied by $a_c(t;\varphi_{t-1})$ which is described as

$$\begin{aligned}
a_c(t;\varphi_{t-1}) &= r_t - \mathbb{E}[r_t|\varphi_{t-1}] \\
&= r_t - \mathbb{E}[b_t(s_{1:t};\varphi_{t-1}) + a_t(s_{1:t};\varphi_{t-1})|\varphi_{t-1}] \\
&= r_t - \mathbb{E}[b_t(s_{1:t};\varphi_{t-1})|\varphi_{t-1}] \\
&= r_t - \textstyle\sum_{j=1}^{2} b_t(s_t=j;\varphi_{t-1}) \times \mathbb{P}(s_t=j|\varphi_{t-1}).
\end{aligned} \tag{10}$$

Similarly, the path dependent conditional variance, $\sigma_t^2(s_{1:t};\varphi_{t-1})$ of Eq. (6) is proxied by $\sigma_c^2(t;\varphi_{t-1})$ estimated with the following equation

$$\begin{aligned}
\sigma_c^2(t;\varphi_{t-1}) &= \mathbb{E}[r_t^2(s_{1:t};\varphi_{t-1})|\varphi_{t-1}] - \{\mathbb{E}[r_t(s_{1:t};\varphi_{t-1})|\varphi_{t-1}]\}^2 \\
&= \mathbb{E}[b_t^2(s_{1:t};\varphi_{t-1})|\varphi_{t-1}] + \mathbb{E}[a_t^2(s_{1:t};\varphi_{t-1})|\varphi_{t-1}] - \{\mathbb{E}[b_t(s_{1:t};\varphi_{t-1})|\varphi_{t-1}]\}^2 \\
&= \textstyle\sum_{j=1}^{2}[b_t^2(s_t=j;\varphi_{t-1}) + \sigma_t^2(s_t=j;\varphi_{t-1})] \times \mathbb{P}(s_t=j|\varphi_{t-1}) \\
&\quad - \left[\textstyle\sum_{j=1}^{2} b_t(s_t=j;\varphi_{t-1}) \times \mathbb{P}(s_t=j|\varphi_{t-1})\right]^2,
\end{aligned} \tag{11}$$

where

$b_t(s_t=j;\varphi_{t-1}) = \mu_j + \phi_j r_{t-1}$,

and

$\sigma_t^2(s_t=j;\varphi_{t-1}) = \omega_j + \alpha_j a_c^2(t-1;\varphi_{t-2}) + \beta_j \sigma_c^2(t-1;\varphi_{t-2})$.

Assuming the normally distributed error, LL function can be written as follows:

$$\begin{aligned}
\mathrm{LL} &= \textstyle\sum_{t=1}^{T} \ln f(r_t|\varphi_{t-1}) \\
&= \textstyle\sum_{t=1}^{T} \ln\left[\sum_{j=1}^{2} f(r_t, s_t=j|\varphi_{t-1})\right] \\
&= \textstyle\sum_{t=1}^{T} \ln[\sum_{j=1}^{2} f(r_t|s_t=j,\varphi_{t-1}) \times \mathbb{P}(s_t=j|\varphi_{t-1})],
\end{aligned} \tag{12}$$

where $f(r_t|s_t=j,\varphi_{t-1}) = \frac{1}{\sqrt{2\pi\sigma_t^2(s_t=j;\varphi_{t-1})}} \exp\left\{-\frac{[r_t-b_t(s_t=j;\varphi_{t-1})]^2}{2\sigma_t^2(s_t=j;\varphi_{t-1})}\right\}$ denotes the conditional density of r_t on $s_t=j$ and φ_{t-1}. For completeness of procedure for the LL function, prediction probabilities, $\mathbb{P}(s_t=j|\varphi_{t-1})$ are calculated by

$$\mathbb{P}(s_t=j|\varphi_{t-1}) = \textstyle\sum_{i=1}^{2} \mathbb{P}(s_t=j|s_{t-1}=i,\varphi_{t-1}) \times \mathbb{P}(s_{t-1}=i|\varphi_{t-1}),$$

where the filter probability $\mathbb{P}(s_{t-1}=i|\varphi_{t-1})$ is known as the filtered probabilities given by

$$\mathbb{P}(s_{t-1}=i|\varphi_{t-1}) = \frac{f(r_{t-1}|s_{t-1}=i,\varphi_{t-2}) \times \mathbb{P}(s_{t-1}=i|\varphi_{t-2})}{\sum_{i=1}^{2} f(r_{t-1}|s_{t-1}=i,\varphi_{t-2}) \times \mathbb{P}(s_{t-1}=i|\varphi_{t-2})}.$$

The parameter estimates and their corresponding standard errors can be acquired through the R studio functions such as optim and numericHessian and the programme could be provided upon request.

3.3 Risk Measure

VaR is often used as an indicator to estimate the possible risk exposure for a given time horizon and risk level. The $100(\nu_2 - \nu_1)\%$ confidence limits for VaR forecast at time t is defined as

$$(\mathrm{VaR}_{t,\nu_1}, \mathrm{VaR}_{t,\nu_2}) = [b_t + q_{\nu_1}\sigma_t,\ b_t + q_{\nu_2}\sigma_t], \tag{13}$$

where q_{ν_1} and q_{ν_2} respectively are lower and upper quantiles of standard normal distribution, while b_t is the conditional mean and σ_t is the conditional standard deviation at time *t*.

As for Markov-switching models, i.e. MSGARCH(1,1) and TV-MSGARCH(1,1) models, we propose the use of collapsed conditional mean at time t, denoted by $b_c(t; \varphi_{t-1})$, and collapsed conditional standard deviation at time *t*, denoted by $\sigma_c(t; \varphi_{t-1})$, to estimate the VaR forecast. Consequently, the $100(\nu_2 - \nu_1)\%$ confidence limits for VaR forecast at time *t* is

$$(\mathrm{VaR}_{t,\nu_1}, \mathrm{VaR}_{t,\nu_2}) = \left[b_c(t; \varphi_{t-1}) + q_{\nu_1}\sigma_c(t; \varphi_{t-1}),\ b_c(t; \varphi_{t-1}) + q_{\nu_2}\sigma_c(t; \varphi_{t-1})\right], \tag{14}$$

where $b_c(t; \varphi_{t-1}) = \mathbb{E}[b_t(s_{1:t}; \varphi_{t-1})|\varphi_{t-1}] = \sum_{j=1}^{2} b_t(s_t = j; \varphi_{t-1}) \times \mathbb{P}(s_t = j|\varphi_{t-1})$ and $\sigma_c(t; \varphi_{t-1})$ is the collapsed conditional standard deviation at time *t* derived from Eq. (11).

Several backtests procedures were used to evaluate the accuracy of estimated VaR, namely, the unconditional coverage (UC) test (Kupiec, 1995) and conditional coverage (CC) test (Christoffersen, 1998) which can be implemented via R studio package GAS.

4. Data

The historical BTC daily closing prices and daily trading volume were retrieved from CoinMarketCap (www.coinmarketcap.com), the most popular site for cryptocurrency information. The data took place from 1 August 2010 until 30 April 2018, resulting in *T* = 2830 observations. Table 1 provides the summary statistics of the BTC daily return series. It is observed that the largest return recorded is about 147% while the lowest return recorded is about -85%. The data series also displays positive skewness and leptokurtic distribution.

Additionally, Google daily searches were accessed via https://trends.google.com. Since the data shows only the relative search interest corresponding to the highest points within the given regions and times, the necessary adjustment needs to be made. First, the daily searches series of the term "Bitcoin" in a monthly block was downloaded followed by the monthly search series in a daily block. The daily searches were then relatively rescaled to its respective months so that the compatible daily search for desired sample periods can be obtained. However, it is noticed that only those periods at the turning of year 2017/2018 shows significant search and if no adjustments were made, the impact of search affecting transition probabilities may be biased to that particular period. Hence, we would also consider the log difference of two consecutive searches that indicates the daily change of public interest in BTC. For the sake of providing a visual picture of the distinction, the BTC price series, BTC return series, BTC daily trading volume, BTC log daily trading volume, Google daily searches and log difference of Google daily searches are presented graphically in Figure 1.

4.1 Model Fitting

The results of estimated parameters for the fitted GARCH(1,1) model, TGARCH(1,1) model, GJRGARCH(1,1) model, MSGARCH(1,1) model, TV-MSGARCH(1,1) model with Google daily searches, denoted by TV-MSGARCH$_S$(1,1), TV-MSGARCH(1,1) model with log difference of

Google daily searches, denoted by TV-MSGARCH$_{ldS}$(1,1), and TV-MSGARCH(1,1) model with log daily trading volume, denoted by TV-MSGARCH$_{lnV}$(1,1), are displayed in Table 2.

From Table 2, both TGARCH(1,1) and GJRGARCH(1,1) models show the presence of leverage effect with γ being positive, indicating that negative shocks have a larger impact on conditional volatility. The findings also suggest that the GARCH(1,1), TGARCH(1,1) and GJRGARCH(1,1) models are unstable and explosive.

In this study, when the Markov-switching process is embedded, it is noted that the difference between the two regimes are apparent in the sense that one of the regimes is stable and the other is highly explosive. However, as discussed by Bauwens et al. (2006), the process can possess a non-stationary regime but is still globally stable if the condition for global covariance-stationarity is satisfied. From the result, the global persistence for MSGARCH(1,1), TV-MSGARCH$_S$(1,1), TV-MSGARCH$_{ldS}$(1,1) and TV-MSGARCH$_{lnV}$(1,1) models are 1.0706, 0.9857, 0.8835 and 0.8538 respectively. Thus, with the exception of MSGARCH(1,1) model, the covariance-stationarity condition is satisfied for all TV-MSGARCH(1,1) models. Furthermore, we also notice that there is a "mean-reverting effect" in regime 1 in such a way that today's return relates negatively to yesterday's return. This behaviour will eventually lead the return series in regime 1 to gradually move towards its mean in the long run while the return series in regime 2 react in a reverse way. These findings are in line with the significance of the application of regime switching component in GARCH models which may give rise to the wrong assumption of explosive persistency level if the existence of regime change is not accounted for.

The maximised LL values reported in Table 2 also reflect that the estimation could be improved by considering TVTP since all the TV-MSGARCH(1,1) models give higher values of LL than MSGARCH(1,1) model and other GARCH(1,1) models. As can be seen, the LL value for MSGARCH(1,1) model is 5212.07, and when TVTP are employed, the LL value for TV-MSGARCH$_S$(1,1) model is 5224.08, TV-MSGARCH$_{ldS}$(1,1) model is 5232.75 and TV-MSGARCH$_{lnV}$(1,1) model is 5225.97 respectively.

To assess the modelling performance of each model, six typical model selection benchmark criteria are adopted, namely, LL, Akaike information criterion (AIC), corrected Akaike information criterion (AICC), Bayesian information criterion (BIC), Hannan-Quinn criterion (HQC) and consistent Akaike information criterion (CAIC). The best model is the one with the optimal criteria values. Table 3 highlights the results for each model selection benchmark criteria. It can be observed that TV-MSGARCH(1,1) models tend to give relatively lower values compared to GARCH(1,1) models and MSGARCH(1,1) model. Among all the TV-MSGARCH(1,1) models, TV-MSGARCH$_{ldS}$(1,1) model gives the best fit followed by TV-MSGARCH$_{lnV}$(1,1) and TV-MSGARCH$_S$(1,1) models. On the whole, these results lend support to the idea of the inclusion of TVTP which surpasses the specification with fixed transition probabilities.

Ljung-Box test statistics were computed on the basis of the standardised residuals and squared standardised residuals to examine the capabilities of these models in capturing ARCH effects and the p-values for Ljung-Box test is shown in Table 4. There are some points worth making about Table 4. Consider the first 5 and 10 lags of standardised residual series as well as the first 10 lags of squared standardised residual series, it is noticeable that the GARCH(1,1) models are inadequate in fitting the mean equation of the models since the p-values for both Q(5) and Q(10) are minimal, supported by some serial correlation between the return series that have not been accounted for. Meanwhile, even though the p-values for Q(10) are rejected at 5% significance level for MSGARCH(1,1) model and the entire TV-MSGARCH(1,1) models, yet the p-values for Q(5) are accepted. These results indicate that the regime-switching specification in the mean equation of these models adequately remove the serial dependence in return series described by μ_{s_t} and ϕ_{s_t}. Hence, such findings underscore the importance of integrating the Markov-switching process in the return series in volatility modelling.

To evaluate the improvement of TVTP models against constant transition probabilities model, we utilise the approach from Filardo (1994) to access the marginal contributions of TVTP, estimated by taking the difference between the transition probabilities at time t and the average of transition probabilities over time, then weighted by the filtered probabilities $\mathbb{P}(s_t = j|\varphi_t)$, where $j = 1$ or 2, while φ_t represents all information up to time t. The respective formulas for WTP for regime 1, WTP(p_{11t}) and regime 2, WTP(p_{22t}) are given as follows:

$$\text{WTP}(p_{11t}) = (p_{11t} - \bar{p}_1) \times \mathbb{P}(s_t = 1|\varphi_t), \tag{15}$$

and

$$\text{WTP}(p_{22t}) = (p_{22t} - \bar{p}_2) \times \mathbb{P}(s_t = 2|\varphi_t), \tag{16}$$

where $\bar{p}_i$ is the mean of p_{iit} such that $\bar{p}_i = \frac{1}{T}\sum_{t=1}^{T} p_{iit}$ for $i = 1{,}2$, with T denoting the length of data. As noted by Filardo (1994), the marginal contribution can be examined by its deviations from zero based on Eqs. (15) and (16).

With respect to the marginal contribution of TVTP in TV-MSGARCH(1,1) models illustrated in Figure 2, for TV-MSGARCH$_S$ (1,1) model, it is pronounced that the marginal contribution of TVTP is only significant at the periods in the year 2018 in which the downward spikes can clearly be observed. This is in complete agreement with our proposal of using log difference for Google daily searches. As noted in Figure 2, the WTP deviating from zero substantiates the contribution of trading volume and Google daily searches in the sense that these exogenous factors are capable of providing additional flexibility to the transition probabilities for TV-MSGARCH(1,1) models at each period of time t.

4.2 Forecast and value-at-risk

In terms of forecasting evaluation, one day ahead forecasts from 1 May 2018 to 31 July 2018 were computed, resulting in h = 92 forecasting points. One step ahead rolling window technique with window size T = 2830 was used to predict the volatility of BTC.

One of the criteria to evaluate the performance of forecasts is based on loss functions. Patton (2011) studied the properties of nine loss functions and suggested that only mean square error (MSE) and quasi-likelihood (QLIKE) are robust to the noise of volatility proxy. Hence, in this study, MSE and QLIKE of all volatiliy models are calculated and the forecasting performance of the set of models is accessed on the basis of Hansen's model confidence set (MCS) procedure. Table 5 presents the forecast errors based on the robust loss functions, MSE and QLIKE, of various volatility models.

Results in Table 5 show that the GARCH(1,1) model is the best model based on MSE while QLIKE suggests otherwise, that is, the TV-MSGARCH$_S$(1,1) appears to be the best model for forecasting purpose. It is worth noting that MSE is a loss function based on forecast error, $\hat{a}_t^2 - \hat{\sigma}_t^2$ whereas QLIKE is a loss function based on standardised forecast error, $\frac{\hat{a}_t^2}{\hat{\sigma}_t^2}$. Due to this reasoning, Patton (2011) was motivated to use QLIKE rather than MSE in volatility forecasting application especially when extreme observations are observed since MSE is sensitive to extreme observations and the level of volatility return.

To further assessing the forecating performance of the models, MCS procedure by Hansen et al. (2011) was employed. MCS procedure consists of a sequence of statistical tests for constructing a set of "superior models" where the null hypothesis of equal predictive ability (EPA) will not be rejected at a certain confidence level. In other words, MCS procedure accounts for the forecasting performance for a set of models rather than comparing relatively over a benchmark model. Examples of the tests including the Diebold-Mariano test (Diebold and Mariano, 1995) and superior predictive ability test (Hansen, 2005). In this paper, EPA is evaluated on QLIKE and the forecasting performance are ranked as shown in Table 6.

The implementation of MCS procedure was completed via the Rstudio package MCS and evaluated using 5000 bootstrap replications tested on 95% confidence level. The corresponding forecasting performance is ranked in column 2 of Table 6 in such a way that the lower the value of test statistic, the higher the rank. None of the models are eliminated from MCS procedure. Nevertheless, empirical results suggest that TV-MSGARCH(1,1) models gain the predictive superiority than other models. Also, we discover that Google daily searches is a better exogenous variable as compared to the trading volume in BTC volatility forecasting application. Our results bring forth to the discusssion of BTC volatility which is impacted by public's attention as mentioned by Urquhart (2018) and Aalborg et al. (2019).

VaR backtests such as UC and CC tests were also carried out to assess the accuracy of VaR estimates at various risk levels and the p-values for the respective tests are reported in Table 7. Results in Table 7 seem to show that the TV-MSGARCH(1,1) models are capable of providing more precise estimates that is in accordance with the observed and forecasted values. These volatility forecasting results are of great importance as accurate volaility will provide useful information for risk management decisions. For illustration, Figure 3 depicts the VaR plots for TV-MSGARCH(1,1) models.

5. Conclusion

The cryptocurrency market, especially for the dominant cryptocurrency BTC, has attracted worldwide attention for its fast changing phenomena, and many of the traditional GARCH models unable to explain this observed phenomena due to its excessive volatility observed over time. Our findings reveal that the conditional variances of GARCH models (e.g., GARCH, GJRGARCH and TGARCH) constantly show explosive behaviour that will greatly affect the model fitting and forecasting.

To enhance the model flexibility, we extend the two-regime MSGARCH model into two-regime TV-MSGARCH models by incorporating the BTC logarithmic daily trading volume and Google daily searches as exogenous factors. Results seem to warrant the conclusion that the conditional variance processes are globally stationary for TV-MSGARCH and MSGARCH models although one of the regimes shows highly explosive behaviour as indicated by Bauwens et al. (2006). This finding provides strong evidence of the necessity of imposing the regime-components element in both the mean and variance of BTC modelling. Given the exploratory nature of this study, the TV-MSGARCH models appear to be the optimal model throughout most of the evaluation criteria, and both in-sample and out-of-sample forecasting performances suggest that the TV-MSGARCH models outperform the GARCH and MSGARCH models. Demonstration on VaR estimates evaluated using a number of backtests also lend the support that the TV-MSGARCH models are able to produce reliable VaR estimates. Besides, we also discover that Google daily searches is a better indicator for BTC volatility as compared to trading volume.

Despite the encouraging results of this study as to the positive effect in the modelling process, additional future work might usefully be extended to explore other exogenous factors in the presence of dynamic changes of the BTC prices. In so doing it seeks to contribute to our growing understanding of how the approach can be employed in financial strategy planning such as in risk management and portfolio optimisation.

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Table 1. Summary statistics of BTC daily return series

Min	**1st Q**	**Median**	**Mean**	**3rd Q**	**Max**	**Variance**	**Std. Dev.**	**Kurtosis**	**Skewness**
-0.8488	-0.0133	0.0020	0.0042	0.0229	1.4744	0.0048	0.0691	96.9270	3.0231

Table 2. Parameter estimates for the fitted models. Values in parentheses are standard errors of the parameter estimates

	Models										
Parameters	GARCH (1,1)	GJRGARCH (1,1)	TGARCH (1,1)	MSGARCH (1,1)		TV-MSGARCH$_S$ (1,1)		TV-MSGARCH$_{ldS}$ (1,1)		TV-MSGARCH$_{lnV}$ (1,1)	
				Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2	Regime 1	Regime 2
μ / μ_j	0.0024 (0.0006)	0.0024 (0.0006)	0.0033 (0.0003)	0.0019 (0.0004)	0.0028 (0.0016)	0.0019 (0.0004)	0.0021 (0.0015)	0.0021 (0.0005)	0.0009 (0.0023)	0.0020 (0.0004)	0.0016 (0.0015)
ϕ/ ϕ_j	0.0049 (0.0232)	0.0296 (0.0004)	-0.0076 (0.0092)	-0.0499 (0.0539)	0.1035 (0.1835)	-0.1059 (0.0266)	0.2974 (0.0619)	-0.1115 (0.0268)	0.6770 (0.0753)	-0.1085 (0.0227)	0.4047 (0.0646)
ω/ω_j	4e-5 (0.0000)	0.0001 (0.0000)	0.0024 (0.0003)	1e-5 (0.0980)	0.0003 (0.0001)	9e-6 (0.0000)	0.0003 (0.0001)	9e-5 (0.0000	0.0002 (0.0000)	9e-6 (0.0000)	0.0002 (0.0001)
α/α_j	0.2358 (0.0180)	0.2572 (0.0230)	0.2842 (0.0197)	0.0791 (0.0551)	0.8167 (0.5724)	0.0745 (0.0227)	0.5910 (0.2039)	0.0705 (0.0263)	0.6075 (0.1416)	0.0653 (0.0170)	0.5815 (0.1639)
γ/γ_j	-	0.0048 (0.0137)	0.3082 (0.0153)	-	-	-	-	-	-	-	-
β/β_j	0.8045 (0.0116)	0.7876 (0.0251)	0.7623 (0.0026)	0.1146 (0.0154)	1.3985 (0.0989)	0.1129 (0.0170)	1.3318 (0.0967)	0.1376 (0.0135)	1.7587 (0.0549)	0.1196 (0.0154)	1.5147 (0.1011)
p_{11}	-	-	-	0.6100 (0.0405)	-	-	-	-	-	-	-
p_{22}	-	-	-	-	0.4909 (0.1653)	-	-	-	-	-	-
c	-	-	-	-	-	0.4625 (0.1023)	0.1999 (0.2390)	0.7715 (0.1228)	-0.2854 (0.1738)	0.2611 (0.1099)	0.3657 (0.1085)
d	-	-	-	-	-	-0.0172 (0.0070)	0.0056 (0.0017)	-0.1707 (0.0729)	-0.0045 (0.0023)	0.0209 (0.0074)	-0.0191 (0.0050)
LL	4787.84	4789.04	4804.29	5212.07		5224.08		5232.75		5225.97	

Table 3. Model selection benchmark criteria values for each model considered

Models	*k*	LL	AIC	AICC	BIC	HQC	CAIC
GARCH(1,1)	5	4787.84	-9565.68	-9565.66	-9535.94	-9554.95	-9530.94
GJRGARCH(1,1)	6	4789.04	-9566.08	-9566.05	-9530.39	-9553.21	-9524.39
TGARCH(1,1)	6	4804.29	-9596.58	-9596.55	-9560.89	-9583.71	-9554.89
MSGARCH(1,1)	12	5212.07	-10400.15	-10400.03	-10328.77	-10374.40	-10316.77
TV-MSGARCH$_S$(1,1)	14	5224.08	-10420.16	-10420.01	-10336.88	-10390.12	-10322.88
TV-MSGARCH$_{ldS}$(1,1)	14	**5232.75**	**-10437.50**	**-10437.35**	**-10354.23**	**-10407.46**	**-10340.23**
TV-MSGARCH$_{lnV}$(1,1)	14	5225.97	-10423.94	-10423.79	-10340.66	-10393.90	-10326.66

***** *k* refers to the number of parameters in the model.

Table 4. P-values for Ljung-Box test

Models	GARCH (1,1)	GJRGARCH (1,1)	TGARCH (1,1)	MSGARCH (1,1)	TV-MSGARCH$_S$ (1,1)	TV-MSGARCH$_{ldS}$ (1,1)	TV-MSGARCH$_{lnV}$ (1,1)
Q(5)	3.22e-08*	3.82e-06*	1.27e-08*	0.0690	0.0553	0.1712	0.1565
Q(10)	2.64e-11*	4.33e-09*	1.30e-11*	0.0014*	0.0003*	0.0036*	0.0012*
$Q^2(10)$	0.5156	0.6271	0.7856	0.8033	0.8859	0.5483	0.6280

*significant at 5% significance level.

Table 5. Comparison of forecast errors using MSE and QLIKE

Models	GARCH (1,1)	GJRGARCH (1,1)	TGARCH (1,1)	MSGARCH (1,1)	TV-MSGARCH$_S$ (1,1)	TV-MSGARCH$_{ldS}$ (1,1)	TV-MSGARCH$_{lnV}$ (1,1)
MSE	**2.95e-5**	2.96e-5	3.14e-5	3.27e-5	3.14e-5	3.28e-5	3.11e-5
QLIKE	1.7623	1.7421	1.7597	1.7251	**1.6751**	1.6793	1.7255

Note: $\hat{a}_t$ is the proxy volatility and $\hat{\sigma}_t^2$ is the predicted volatility. $\hat{a}_t$ is estimated from Eq. (1) for GARCH(1,1), GJRGARCH(1,1) and TGARCH(1,1) models while $\hat{a}_t$ is estimated from Eq. (10) for MSGARCH(1,1) and TV-MSGARCH(1,1) models. Given that MSE= $h^{-1}\sum_{t=T+1}^{T+h}\left\{\hat{a}_t^2-\hat{\sigma}_t^2\right)\}^2$ and QLIKE= $h^{-1}\sum_{t=T+1}^{T+h}\left\{\frac{\hat{a}_t^2}{\hat{\sigma}_t^2}-\ln\left(\frac{\hat{a}_t^2}{\hat{\sigma}_t^2}\right)-1\right\}$.

Table 6. Comparison of forecasting performance on MCS procedure

Models	Rank	Test statistic	p-value
GARCH(1,1)	6	0.9724	0.9406
GJRGARCH(1,1)	4	0.8745	1.0000
TGARCH(1,1)	7	1.0882	0.8982
MSGARCH(1,1)	5	0.9590	1.0000
TV-MSGARCH$_S$(1,1)	**1**	**-0.0461**	**1.0000**
TV-MSGARCH$_{ldS}$(1,1)	2	0.0461	1.0000
TV-MSGARCH$_{lnV}$(1,1)	3	0.8614	1.0000

Table 7. P-values for several VaR backtests

	Risk Levels					
Models	ν_1			ν_2		
	0.05	**0.025**	**0.005**	**0.95**	**0.975**	**0.995**
TV-MSGARCH$_S$(1,1)						
UC test	0.4150	0.8377	0.0920	0.1631	0.3290	0.3369
CC test	0.6476	0.9362	0.2329	0.3615	0.6141	0.6306
TV-MSGARCH$_{ldS}$(1,1)						
UC test	0.8502	0.3033	0.0928	0.1631	0.3290	0.3369
CC test	0.7344	0.4898	0.2329	0.3615	0.6141	0.6306
TV-MSGARCH$_{lnV}$(1,1)						
UC test	0.5215	0.1177	0.0928	0.4150	0.8377	0.3369
CC test	0.5329	0.2199	0.2329	0.1270	0.9362	0.6306

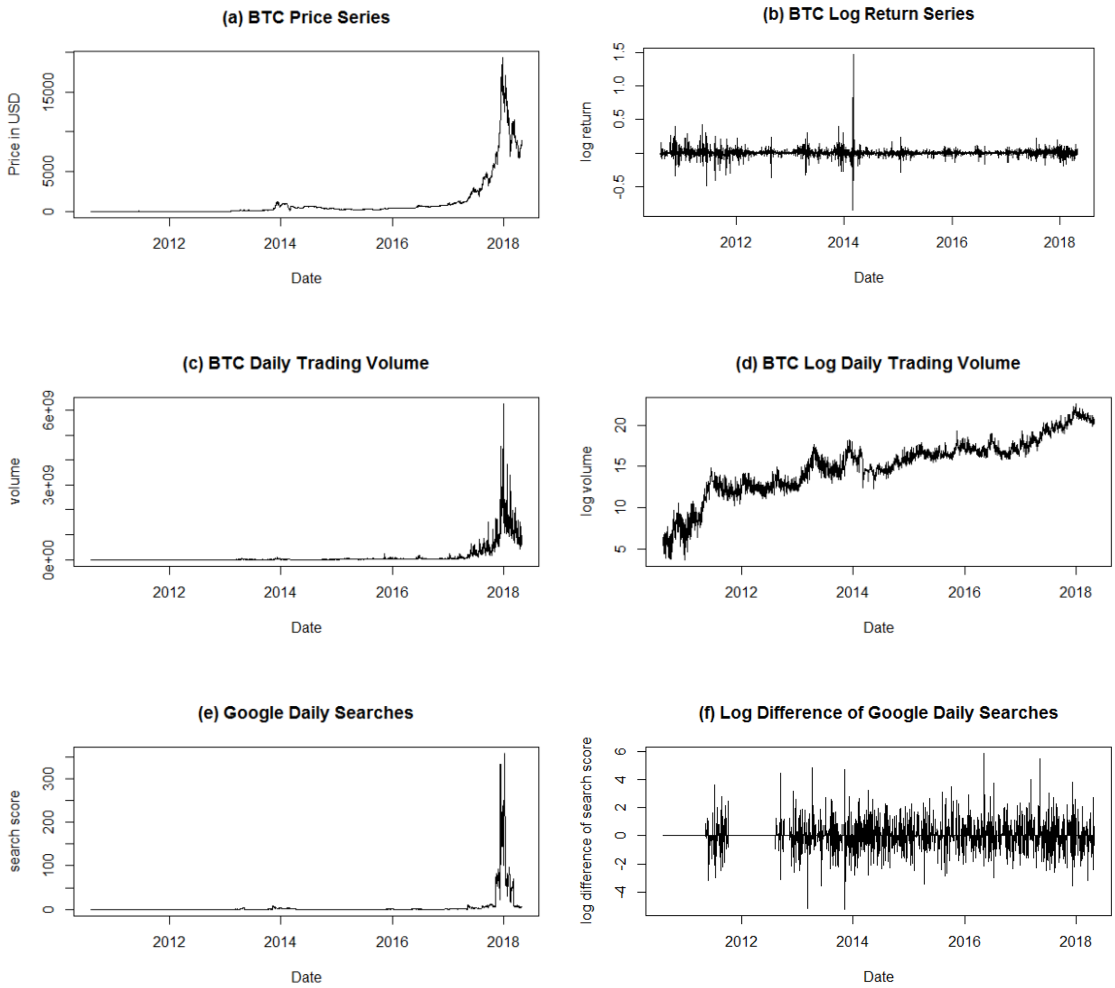

Figure 1. (a) BTC price series (b) BTC return series (c) BTC daily trading volume (d) BTC log daily trading volume (e) Google daily searches (f) Log difference of Google daily searches.

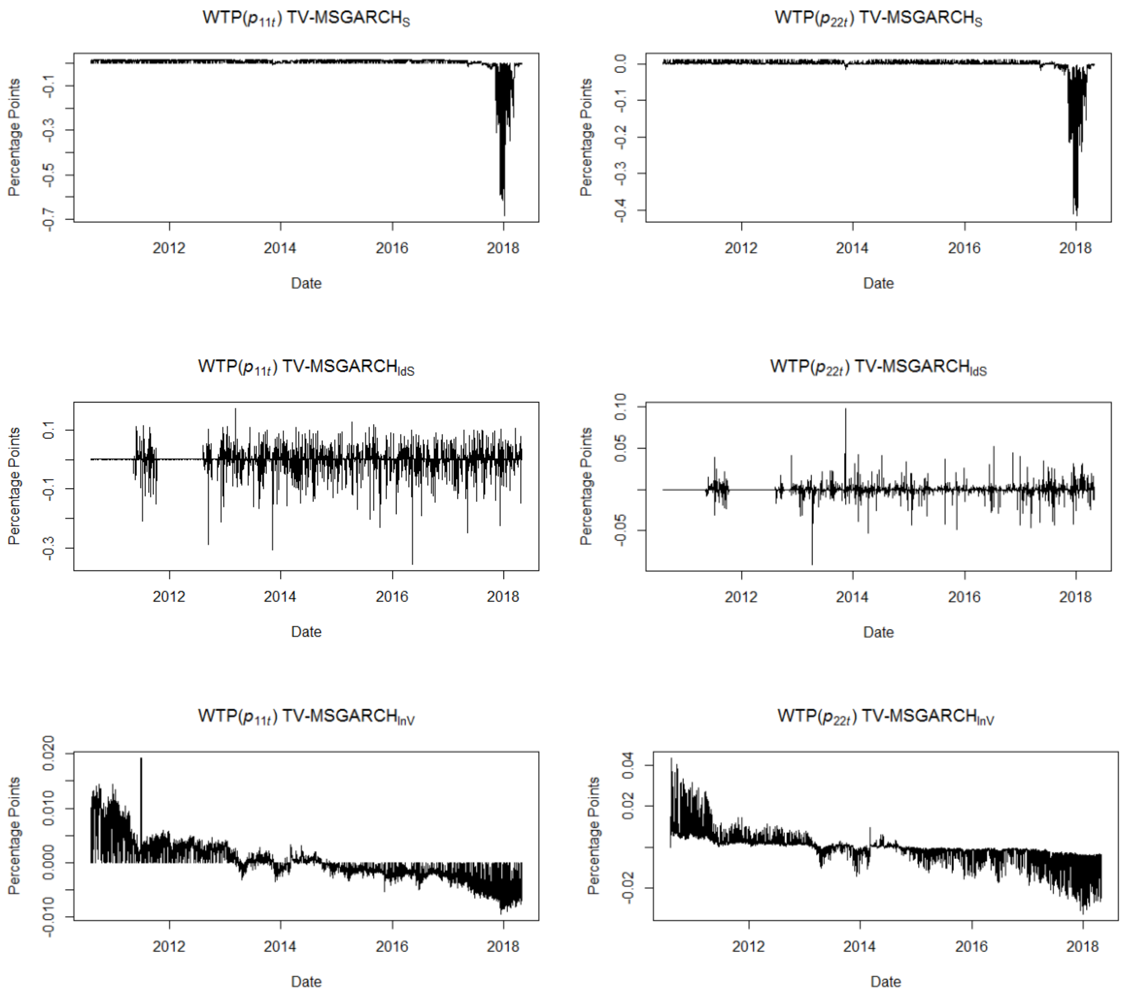

Figure 2. Marginal contributions of TVTP for TV-MSGARCH(1,1) models.

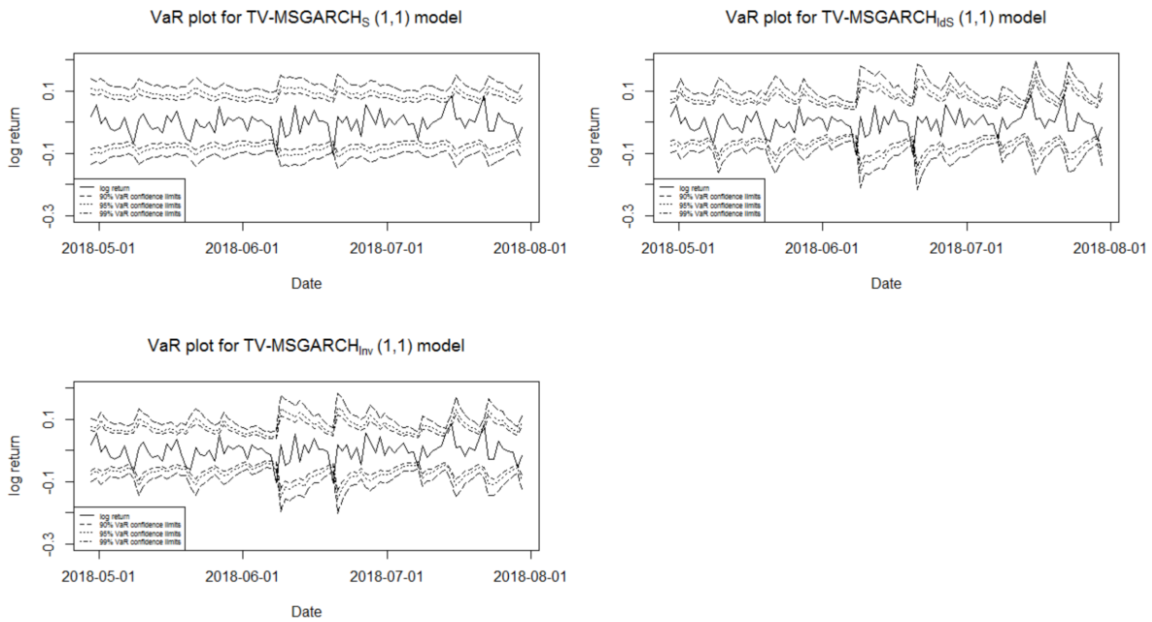

Figure 3: VaR plots for TV-MSGARCH(1,1) models.