Shreve – Stochastic Calculus for Finance, Vol. 1
Chapter 1 Solutions

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Problem 1.1 We start with the assumption that $X_0 = 0$ and define:

$$\begin{aligned} & X_1 \stackrel{\text{def}}{=} \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ \implies \quad & X_1 = (S_1 - (1+r)S_0)\,\Delta_0 \end{aligned}$$

Depending on the outcome of the first toss, we have:

$$X_1(H) = (u - (1+r))\,S_0\Delta_0, \quad X_1(T) = (d - (1+r))\,S_0\Delta_0$$

Condition (1.1.2) states that $0 < d < 1 + r < u$, thus we have $u - (1+r) > 0$ and $d - (1+r) < 0$. Therefore:

$$\text{sign}\,(X_1(H)) = \text{sign}\,(S_0\Delta_0) = -\text{sign}\,(X_1(T))$$

If an outcome of the toss ω gives $X_1(\omega) > 0$, then $X_1(\bar{\omega}) < 0$, where $\bar{\omega}$ is the opposite outcome to ω. However H and T are assumed to have a positive probability, so if the probability that $X_1(\omega) > 0$ is positive, then the probability that $X_1(\bar{\omega}) < 0$ is also positive. Thus Condition (1.1.2) precludes arbitrage.

Problem 1.2 We compute $X_1(H), X_1(T)$ using $S_1(H) = 8, S_1(T) = 2$:

$$\begin{aligned} X_1(H) &= 8\Delta_0 + 3\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) &= 2\Delta_0 + (0)\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0 \end{aligned}$$

Therefore $X_1(H) = -X_1(T)$. By the same argument as in Problem 1.1, if there is a positive probability that $X_1 > 0$, then there is a positive probability that $X_1 < 0$ (assuming that both H and T have a positive probability of occuring).

Problem 1.3 We compute V_0 using $V_1(H) = S_1(H) = uS_0$ and $V_1(T) = S_1(T) = dS_0$:

$$\begin{aligned}
V_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
&= \frac{1}{1+r}\left[\tilde{p}(uS_0) + \tilde{q}(dS_0)\right] \\
&= \frac{S_0}{1+r}\left[\left(\frac{1+r-d}{u-d}\right)u + \left(\frac{u-1-r}{u-d}\right)d\right] \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - ud + ud - d - dr\right) \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - d - dr\right) \\
&= S_0
\end{aligned}$$

Problem 1.4