Shreve – Stochastic Calculus for Finance, Vol. 1
Chapter 1 Solutions

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Problem 1.1 We start with the assumption that $X_0 = 0$ and define:

$$\begin{aligned} & X_1 \stackrel{\text{def}}{=} \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ \implies \quad & X_1 = (S_1 - (1+r)S_0)\,\Delta_0 \end{aligned}$$

Depending on the outcome of the first toss, we have:

$$X_1(H) = (u - (1+r))\, S_0 \Delta_0, \quad X_1(T) = (d - (1+r))\, S_0 \Delta_0$$

Condition (1.1.2) states that $0 < d < 1+r < u$, thus we have $u - (1+r) > 0$ and $d - (1+r) < 0$. Therefore:

$$\text{sign}\,(X_1(H)) = \text{sign}\,(S_0 \Delta_0) = -\text{sign}\,(X_1(T))$$

If an outcome of the toss ω gives $X_1(\omega) > 0$, then $X_1(\bar{\omega}) < 0$, where $\bar{\omega}$ is the opposite outcome to ω. However H and T are assumed to have a positive probability, so if the probability that $X_1(\omega) > 0$ is positive, then the probability that $X_1(\bar{\omega}) < 0$ is also positive. Thus Condition (1.1.2) precludes arbitrage.

□

Problem 1.2 We compute $X_1(H), X_1(T)$ using $S_1(H) = 8, S_1(T) = 2$:

$$\begin{aligned} X_1(H) &= 8\Delta_0 + 3\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) &= 2\Delta_0 + (0)\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0 \end{aligned}$$

Therefore $X_1(H) = -X_1(T)$. By the same argument as in Problem 1.1, if there is a positive probability that $X_1 > 0$, then there is a positive probability that $X_1 < 0$ (assuming that both H and T have a positive probability of occuring).

□

Problem 1.3 We compute V_0 under (1.1.10) using $V_1(H) = S_1(H) = uS_0$ and $V_1(T) = S_1(T) = dS_0$:

$$\begin{aligned}
V_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
&= \frac{1}{1+r}\left[\tilde{p}(uS_0) + \tilde{q}(dS_0)\right] \\
&= \frac{S_0}{1+r}\left[\left(\frac{1+r-d}{u-d}\right)u + \left(\frac{u-1-r}{u-d}\right)d\right] \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - ud + ud - d - dr\right) \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - d - dr\right) \\
&= S_0
\end{aligned}$$

□

Problem 1.4 Let $\omega_1\omega_2\ldots\omega_n$ be fixed. We compute:

$$\begin{aligned}
X_{n+1}(\omega_1\omega_2\ldots\omega_n T) &= \Delta_n(\omega_1\omega_2\ldots\omega_n)S_{n+1}(\omega_1\omega_2\ldots\omega_n T) \\
&\quad + (1+r)\left(X_n(\omega_1\omega_2\ldots\omega_n) - \Delta_n(\omega_1\omega_2\ldots\omega_n)S_n(\omega_1\omega_2\ldots\omega_n)\right)
\end{aligned}$$

We surpress the $\omega_1\omega_2\ldots\omega_n$ and re-write as:

$$\begin{aligned}
X_{n+1}(T) &= \Delta_n S_{n+1}(T) + (1+r)\left(X_n - \Delta_n S_n\right) \\
&= d\Delta_n S_n + (1+r)\left(X_n - \Delta_n S_n\right) \\
&= \left(d - (1+r)\right)\Delta_n S_n + (1+r)X_n
\end{aligned}$$

By the induction hypothesis, $X_n = V_n$. We use the definition of Δ_n and V_n:

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$$

$$X_n = V_n = \frac{1}{1+r}\left[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right]$$

$$\begin{aligned}
\implies \quad X_{n+1}(T) &= \left(d - (1+r)\right)\left(\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\right) + \left(\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right) \\
&= -\tilde{p}\left(V_{n+1}(H) - V_{n+1}(T)\right) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (\tilde{p} + \tilde{q})V_{n+1}(T) = V_{n+1}(T)
\end{aligned}$$

$\square$

Problem 1.5 We first calculate $\Delta_1(H)$:

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.20 - 2.40}{16 - 4} = \frac{0.8}{12} = \frac{1}{30}$$

At time one, the agent has a portfolio valued at $V_1(H) = 2.24$. Using the wealth equation, we calculate:

$$X_2(HH) = (16)\left(\tfrac{1}{15}\right) + \tfrac{5}{4}\left[(2.24) - \left(\tfrac{1}{15}\right)(8)\right] = \tfrac{16}{15} + \tfrac{5}{4} \cdot \tfrac{128}{75} = \tfrac{16}{5} = 3.20 = V_2(HH)$$
$$X_2(HT) = (4)\left(\tfrac{1}{15}\right) + \tfrac{5}{4}\left[(2.24) - \left(\tfrac{1}{15}\right)(8)\right] = \tfrac{4}{15} + \tfrac{5}{4} \cdot \tfrac{128}{75} = \tfrac{12}{5} = 2.40 = V_2(HT)$$

Next we calculate $\Delta_2(HT)$:

$$\Delta_1(H) = \frac{V_2(HTH) - V_2(HTT)}{S_2(HTH) - S_2(HTT)} = \frac{0 - 6}{8 - 2} = -1$$

We have $X_2(HT) = V_2(HT) = 2.40$:

$$X_3(HTH) = (8)(-1) + \tfrac{5}{4}[(2.40) - (-1)(4)] = -8 + \tfrac{5}{4} \cdot \tfrac{32}{5} = 0 = V_3(HTH)$$
$$X_3(HTT) = (2)(-1) + \tfrac{5}{4}[(2.40) - (-1)(4)] = -2 + \tfrac{5}{4} \cdot \tfrac{32}{5} = 6 = V_3(HTT)$$

$\square$

Problem 1.6 If at time zero, the bank purchases Δ_0 shares, they must borrow $\Delta_0 S_0$ from the money market to finance this (if $\Delta_0 < 0$ this represents short position whose initial proceeds are invested in the money market). At time one, the bank then has $X_1 = V_1 + \Delta_0 S_1 - (1+r)\Delta_0 S_0$:

$$X_1(H) = (3) + \Delta_0(8) - \tfrac{5}{4}\Delta_0(4) = 3 + 3\Delta_0$$
$$X_1(T) = (0) + \Delta_0(2) - \tfrac{5}{4}\Delta_0(4) = -3\Delta_0$$

We see that if $\Delta_0 = -0.50$, then $X_1(H) = X_1(T) = 1.50$, as desired. Thus the bank should sell 0.50 shares short at time zero. This will net them proceeds of $(0.50)(4) = 2$, which they should invest into the money market. The above calculations show that regardless of the outcome of the coin toss, the bank will have wealth 1.50.

Alternatively, Theorem 1.2.2 tells us the amount of shares Δ_0' that the *seller* of the option should buy at time zero in order to replicate the option:

$$\Delta_0' = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{3 - 0}{8 - 2} = 0.50$$

It follows, that the *buyer* of the option (in this case the bank) should perform the opposite actions to the seller, in order to replicate the option. Again, we see that the bank should sell 0.50 shares short at time zero.

□