Shreve – Stochastic Calculus for Finance, Vol. 1
Chapter 1 Solutions

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Problem 2.1

(i) We have:

$$\mathbb{P}(A) + \mathbb{P}(A^C) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in A^C} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in \Omega \setminus A} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$$

The result follows.

□

(ii) We have:

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{\omega \in \bigcup A_n} \mathbb{P}(\omega) = \sum_{\omega \in \tilde{A}_1} \mathbb{P}(\omega) + \cdots + \sum_{\omega \in \tilde{A}_n} \mathbb{P}(\omega) = \mathbb{P}(\tilde{A}_1) + \cdots + \mathbb{P}(\tilde{A}_n)$$

where $\tilde{A}_n = A_n \setminus \left(\bigcup_{1 \le i < n} A_i\right)$. We then see that:

$$\mathbb{P}(\tilde{A}_n) = \mathbb{P}(A_n) - \mathbb{P}\left(A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)\right) \le \mathbb{P}(A_n)$$

by non-negativity of $\mathbb{P}$. Thus $\mathbb{P}(\tilde{A}_n) \le \mathbb{P}(A_n)$ for all $1 \le n \le n$, so:

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mathbb{P}(\tilde{A}_n) \le \sum_{n=1}^{N} \mathbb{P}(A_n).$$

Furthermore, if all A_n are disjoint, then $\tilde{A}_n = A_n$, which makes equality hold. $\square$

Problem 2.2

(i) We have:

$$\tilde{\mathbb{P}}\{S_3 = 32\} = 0.125 \qquad \tilde{\mathbb{P}}\{S_3 = 8\} = 0.375$$
$$\tilde{\mathbb{P}}\{S_3 = 2\} = 0.375 \qquad \tilde{\mathbb{P}}\{S_3 = 0.5\} = 0.125$$

$\square$

(ii) First we compute the distributions of S_1 and S_2:

$$\tilde{\mathbb{P}}\{S_1 = 8\} = 0.5 \qquad \tilde{\mathbb{P}}\{S_1 = 2\} = 0.5$$

$$\tilde{\mathbb{P}}\{S_2 = 16\} = 0.25 \qquad \tilde{\mathbb{P}}\{S_2 = 4\} = 0.5 \qquad \tilde{\mathbb{P}}\{S_2 = 1\} = 0.25$$

Then:

$$\begin{aligned}
\tilde{\mathbb{E}}S_1 &= (8)(0.5) + (2)(0.5) = 5 \\
\tilde{\mathbb{E}}S_2 &= (16)(0.25) + (4)(0.5) + 1(0.25) = 6.25 \\
\tilde{\mathbb{E}}S_3 &= (32)(0.125) + (8)(0.375) + (2)(0.375) + (0.5)(0.125) = 7.8125
\end{aligned}$$

i.e: $\tilde{\mathbb{E}}S_n = (1+r)S_{n-1}$.

$\square$

(iii) The distributions are:

$$\tilde{\mathbb{P}}\{S_1 = 8\} = \tfrac{2}{3} \qquad \tilde{\mathbb{P}}\{S_1 = 2\} = \tfrac{1}{3}$$

$$\tilde{\mathbb{P}}\{S_2 = 16\} = \tfrac{4}{9} \qquad \tilde{\mathbb{P}}\{S_2 = 4\} = \tfrac{4}{9} \qquad \tilde{\mathbb{P}}\{S_2 = 1\} = \tfrac{1}{9}$$

$$\tilde{\mathbb{P}}\{S_3 = 32\} = \tfrac{8}{27} \qquad \tilde{\mathbb{P}}\{S_3 = 8\} = \tfrac{4}{9}$$
$$\tilde{\mathbb{P}}\{S_3 = 2\} = \tfrac{2}{9} \qquad \tilde{\mathbb{P}}\{S_3 = 0.5\} = \tfrac{1}{27}$$

And so:

$$\begin{aligned}
\tilde{\mathbb{E}}S_1 &= (8)\left(\tfrac{2}{3}\right) + (2)\left(\tfrac{1}{3}\right) = 6 \\
\tilde{\mathbb{E}}S_2 &= (16)\left(\tfrac{4}{9}\right) + (4)\left(\tfrac{4}{9}\right) + 1\left(\tfrac{1}{9}\right) = 9 \\
\tilde{\mathbb{E}}S_3 &= (32)\left(\tfrac{8}{27}\right) + (8)\left(\tfrac{4}{9}\right) + (2)\left(\tfrac{2}{9}\right) + (0.5)\left(\tfrac{1}{27}\right) = 13.5
\end{aligned}$$