Shreve – Stochastic Calculus for Finance, Vol. 1
Chapter 2 Solutions

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Problem 2.1

(i) We have:

$$\mathbb{P}(A) + \mathbb{P}(A^C) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in A^C} \mathbb{P}(\omega) = \sum_{\omega \in A} \mathbb{P}(\omega) + \sum_{\omega \in \Omega \setminus A} \mathbb{P}(\omega) = \sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$$

The result follows.

$\square$

(ii) We have:

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{\omega \in \bigcup A_n} \mathbb{P}(\omega) = \sum_{\omega \in \tilde{A}_1} \mathbb{P}(\omega) + \cdots + \sum_{\omega \in \tilde{A}_n} \mathbb{P}(\omega) = \mathbb{P}(\tilde{A}_1) + \cdots + \mathbb{P}(\tilde{A}_n)$$

where $\tilde{A}_n = A_n \setminus \left(\bigcup_{1 \leq i < n} A_i\right)$. We then see that:

$$\mathbb{P}(\tilde{A}_n) = \mathbb{P}(A_n) - \mathbb{P}\left(A_n \cap \left(\bigcup_{i=1}^{n-1} A_i\right)\right) \leq \mathbb{P}(A_n)$$

by non-negativity of $\mathbb{P}$. Thus $\mathbb{P}(\tilde{A}_n) \leq \mathbb{P}(A_n)$ for all $1 \leq n \leq n$, so:

$$\mathbb{P}\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \mathbb{P}(\tilde{A}_n) \leq \sum_{n=1}^{N} \mathbb{P}(A_n).$$

Furthermore, if all A_n are disjoint, then $\tilde{A}_n = A_n$, which makes equality hold. $\square$

Problem 2.2

(i) We have:

$$\tilde{\mathbb{P}}\{S_3 = 32\} = 0.125 \qquad \tilde{\mathbb{P}}\{S_3 = 8\} = 0.375$$
$$\tilde{\mathbb{P}}\{S_3 = 2\} = 0.375 \qquad \tilde{\mathbb{P}}\{S_3 = 0.5\} = 0.125$$

$\square$

(ii) First we compute the distributions of S_1 and S_2:

$$\tilde{\mathbb{P}}\{S_1 = 8\} = 0.5 \qquad \tilde{\mathbb{P}}\{S_1 = 2\} = 0.5$$

$$\tilde{\mathbb{P}}\{S_2 = 16\} = 0.25 \qquad \tilde{\mathbb{P}}\{S_2 = 4\} = 0.5 \qquad \tilde{\mathbb{P}}\{S_2 = 1\} = 0.25$$

Then:

$$\begin{aligned}
\tilde{\mathbb{E}}S_1 &= (8)(0.5) + (2)(0.5) = 5 \\
\tilde{\mathbb{E}}S_2 &= (16)(0.25) + (4)(0.5) + 1(0.25) = 6.25 \\
\tilde{\mathbb{E}}S_3 &= (32)(0.125) + (8)(0.375) + (2)(0.375) + (0.5)(0.125) = 7.8125
\end{aligned}$$

i.e: $\tilde{\mathbb{E}}S_n = (1+r)S_{n-1}$.

$\square$

(iii) The distributions are:

$$\mathbb{P}\{S_1 = 8\} = \tfrac{2}{3} \qquad \mathbb{P}\{S_1 = 2\} = \tfrac{1}{3}$$

$$\mathbb{P}\{S_2 = 16\} = \tfrac{4}{9} \qquad \mathbb{P}\{S_2 = 4\} = \tfrac{4}{9} \qquad \mathbb{P}\{S_2 = 1\} = \tfrac{1}{9}$$

$$\mathbb{P}\{S_3 = 32\} = \tfrac{8}{27} \qquad \mathbb{P}\{S_3 = 8\} = \tfrac{4}{9}$$
$$\mathbb{P}\{S_3 = 2\} = \tfrac{2}{9} \qquad \mathbb{P}\{S_3 = 0.5\} = \tfrac{1}{27}$$

And so:

$$\begin{aligned}
\mathbb{E}S_1 &= (8)\left(\tfrac{2}{3}\right) + (2)\left(\tfrac{1}{3}\right) = 6 \\
\mathbb{E}S_2 &= (16)\left(\tfrac{4}{9}\right) + (4)\left(\tfrac{4}{9}\right) + 1\left(\tfrac{1}{9}\right) = 9 \\
\mathbb{E}S_3 &= (32)\left(\tfrac{8}{27}\right) + (8)\left(\tfrac{4}{9}\right) + (2)\left(\tfrac{2}{9}\right) + (0.5)\left(\tfrac{1}{27}\right) = 13.5
\end{aligned}$$

Problem 2.3 Let $M_0, M_1, \ldots, M_n$ be a martingale and let φ be a convex function.

$$\begin{aligned}
&\overset{\text{Martingale}}{\Longrightarrow} && M_n = \mathbb{E}_n M_{n+1} \\
&\overset{\varphi(\cdots)}{\Longrightarrow} && \varphi\left(M_n\right) = \varphi\left(\mathbb{E}_n M_{n+1}\right) \\
&\overset{\text{Jensen's Ineq.}}{\Longrightarrow} && \varphi\left(M_n\right) = \varphi\left(\mathbb{E}_n M_{n+1}\right) \leq \mathbb{E}_n\left[\varphi(M_{n+1})\right]
\end{aligned}$$

Thus for $n = 0, 1, \ldots, N-1$, we have $\varphi\left(M_n\right) \leq \mathbb{E}_n\left[\varphi(M_{n+1})\right]$, so that $\varphi(M_0), \varphi(M_1), \ldots, \varphi(M_n)$ is a submartingale.

□

Problem 2.4

(i) If $M_{n+1}(H), M_{n+1}(T)$ are the values of M_{n+1} depending on whether the outcome of the n-th toss is heads or tails respectively, then we have:

$$M_{n+1}(H) = M_n + 1, \qquad M_{n+1}(T) = M_n - 1$$

Thus with the given probabilities:

$$\mathbb{E}_n M_{n+1} = \tfrac{1}{2}(M_n + 1) + \tfrac{1}{2}(M_n - 1) = M_n$$

Therefore $M_0, M_1, M_2, \ldots$ is a martingale.

□

(ii) Again:

$$\begin{aligned}
S_{n+1} = e^{\sigma(M_n \pm 1)}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right)^{n+1} &= e^{\pm\sigma} \cdot e^{\sigma M_n} \cdot \left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right) \cdot \left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right)^{n} \\
&= e^{\pm\sigma}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right) \cdot S_n
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_n S_{n+1} &= \tfrac{1}{2}\left(e^{\sigma}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right) \cdot S_n\right) + \tfrac{1}{2}\left(e^{-\sigma}\left(\frac{2}{e^{\sigma} + e^{-\sigma}}\right) \cdot S_n\right) \\
&= \left(\frac{e^{\sigma}}{e^{\sigma} + e^{-\sigma}}\right) S_n + \left(\frac{e^{-\sigma}}{e^{\sigma} + e^{-\sigma}}\right) S_n = S_n
\end{aligned}$$

Therefore $S_0, S_1, S_2, \ldots$ is a martingale.

$\square$

Problem 2.5

(i) We notice that $M_{j+1} - M_j = X_{j+1}$. Furthermore, $M_0 = 0$ so we can change the starting index of the sum to $j = 1$. Therefore:

$$I_n = \sum_{j=1}^{n-1} M_j X_{j+1} = \sum_{j=1}^{n-1} \left(\left(\sum_{i=1}^{j} X_i \right) X_{j+1} \right) = \sum_{1 \leq i < j}^{n} X_i X_j$$

To see the last equality, notice that X_{j+1} is multiplied by exactly once every by X_i from 1 to j. We have:

$$M_n^2 = \left(\sum_{j=1}^{n} X_j \right)^2 = \sum_{j=1}^{n} X_j^2 + 2 \sum_{1 \leq i < j} X_i X_j = \sum_{j=1}^{n} X_j^2 + 2I_n$$

Since each $X_j = \pm 1$, we have $X_j^2 = 1$. Therefore:

$$M_n^2 = n + 2I_n \qquad \Longrightarrow \qquad I_n = \tfrac{1}{2} M_n^2 - \tfrac{n}{2}$$

$\square$

(ii) We have:

$$I_{n+1} = \tfrac{1}{2} \left(M_n \pm 1 \right)^2 - \tfrac{1}{2}(n+1) = \tfrac{1}{2} M_n^2 \pm M_n + \tfrac{1}{2} - \tfrac{n}{2} - \tfrac{1}{2} = I_n \pm M_n$$

$$\therefore \quad \mathbb{E}_n \left[f \left(I_{n+1} \right) \right] = \tfrac{1}{2} f(I_n + M_n) + \tfrac{1}{2} f(I_n - M_n)$$

We want the LHS to be entirely in terms of I_n so use (i) to re-write $M_n = \sqrt{2I_n + n}$. Note that we can simply let M_n be the positive square root as the above equation is unchanged if we let $M_n' = -M_n$. Therefore our function $g(i)$ is:

$$g(i) = \tfrac{1}{2} f \left(i + \sqrt{2i - n} \right) + \tfrac{1}{2} f \left(i - \sqrt{2i - n} \right)$$

$\square$

Problem 2.6 We calculate:

$$\begin{aligned} \mathbb{E}_n I_{n+1} &= \mathbb{E}_n \left[\sum_{j=0}^{n} \Delta_j (M_{j+1} - M_j) \right] \\ &= \mathbb{E}_n \left[\Delta_n (M_{n+1} - M_n) \right] + \sum_{j=0}^{n-1} \Delta_j (M_{j+1} - M_j) \\ &= \mathbb{E}_n \left[\Delta_n (M_{n+1} - M_n) \right] + I_n \end{aligned}$$

where in the second equality we have used the linearity and the 'taking out what is known' properties of conditional expectations. Then:

$$\mathbb{E}_n\left[\Delta_n(M_{n+1} - M_n)\right] = \mathbb{E}_n\left[\Delta_n M_{n+1}\right] - \mathbb{E}_n\left[\Delta_n M_n\right] = \Delta_n \mathbb{E}_n M_{n+1} - \Delta_n M_n$$

but since $M_0, M_1, \ldots, M_N$ is a martingale, $\mathbb{E}_n M_{n+1} = M_n$, meaning the terms on the RHS cancel. Therefore $\mathbb{E}_n I_{n+1} = I_n$, and $I_0, I_1, \ldots, I_N$ is a martingale.

$\square$