Shreve – Stochastic Calculus for Finance, Vol. 1
Chapter 1 Solutions

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Problem 1.1 We start with the assumption that $X_0 = 0$ and define:

$$\begin{aligned} & X_1 \stackrel{\text{def}}{=} \Delta_0 S_1 + (1+r)(X_0 - \Delta_0 S_0) \\ \implies \quad & X_1 = (S_1 - (1+r)S_0)\,\Delta_0 \end{aligned}$$

Depending on the outcome of the first toss, we have:

$$X_1(H) = (u - (1+r))\,S_0\Delta_0, \quad X_1(T) = (d - (1+r))\,S_0\Delta_0$$

Condition (1.1.2) states that $0 < d < 1 + r < u$, thus we have $u - (1+r) > 0$ and $d - (1+r) < 0$. Therefore:

$$\text{sign}\,(X_1(H)) = \text{sign}\,(S_0\Delta_0) = -\text{sign}\,(X_1(T))$$

If an outcome of the toss ω gives $X_1(\omega) > 0$, then $X_1(\bar{\omega}) < 0$, where $\bar{\omega}$ is the opposite outcome to ω. However H and T are assumed to have a positive probability, so if the probability that $X_1(\omega) > 0$ is positive, then the probability that $X_1(\bar{\omega}) < 0$ is also positive. Thus Condition (1.1.2) precludes arbitrage.

□

Problem 1.2 We compute $X_1(H), X_1(T)$ using $S_1(H) = 8, S_1(T) = 2$:

$$\begin{aligned} X_1(H) &= 8\Delta_0 + 3\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = 3\Delta_0 + 1.5\Gamma_0 \\ X_1(T) &= 2\Delta_0 + (0)\Gamma_0 - \tfrac{5}{4}(4\Delta_0 + 1.20\Gamma_0) = -3\Delta_0 - 1.5\Gamma_0 \end{aligned}$$

Therefore $X_1(H) = -X_1(T)$. By the same argument as in Problem 1.1, if there is a positive probability that $X_1 > 0$, then there is a positive probability that $X_1 < 0$ (assuming that both H and T have a positive probability of occuring).

$\square$

Problem 1.3 We compute V_0 under (1.1.10) using $V_1(H) = S_1(H) = uS_0$ and $V_1(T) = S_1(T) = dS_0$:

$$\begin{aligned}
V_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
&= \frac{1}{1+r}\left[\tilde{p}(uS_0) + \tilde{q}(dS_0)\right] \\
&= \frac{S_0}{1+r}\left[\left(\frac{1+r-d}{u-d}\right)u + \left(\frac{u-1-r}{u-d}\right)d\right] \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - ud + ud - d - dr\right) \\
&= \frac{S_0}{(1+r)(u-d)}\left(u + ur - d - dr\right) \\
&= S_0
\end{aligned}$$

$\square$

Problem 1.4 Let $\omega_1\omega_2\ldots\omega_n$ be fixed. We compute:

$$\begin{aligned}
X_{n+1}(\omega_1\omega_2\ldots\omega_n T) &= \Delta_n(\omega_1\omega_2\ldots\omega_n)S_{n+1}(\omega_1\omega_2\ldots\omega_n T) \\
&\quad + (1+r)\left(X_n(\omega_1\omega_2\ldots\omega_n) - \Delta_n(\omega_1\omega_2\ldots\omega_n)S_n(\omega_1\omega_2\ldots\omega_n)\right)
\end{aligned}$$

We surpress the $\omega_1\omega_2\ldots\omega_n$ and re-write as:

$$\begin{aligned}
X_{n+1}(T) &= \Delta_n S_{n+1}(T) + (1+r)\left(X_n - \Delta_n S_n\right) \\
&= d\Delta_n S_n + (1+r)\left(X_n - \Delta_n S_n\right) \\
&= \left(d - (1+r)\right)\Delta_n S_n + (1+r)X_n
\end{aligned}$$

By the induction hypothesis, $X_n = V_n$. We use the definition of Δ_n and V_n:

$$\Delta_n = \frac{V_{n+1}(H) - V_{n+1}(T)}{S_{n+1}(H) - S_{n+1}(T)} = \frac{V_{n+1}(H) - V_{n+1}(T)}{(u-d)S_n}$$

$$X_n = V_n = \frac{1}{1+r}\left[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right]$$

$$\begin{aligned}
\implies \qquad X_{n+1}(T) &= \left(d - (1+r)\right)\left(\frac{V_{n+1}(H) - V_{n+1}(T)}{u-d}\right) + \left(\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right) \\
&= -\tilde{p}\left(V_{n+1}(H) - V_{n+1}(T)\right) + \tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T) \\
&= (\tilde{p} + \tilde{q})V_{n+1}(T) = V_{n+1}(T)
\end{aligned}$$

□

Problem 1.5 We first calculate $\Delta_1(H)$:

$$\Delta_1(H) = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)} = \frac{3.20 - 2.40}{16 - 4} = \frac{0.8}{12} = \frac{1}{30}$$

At time one, the agent has a portfolio valued at $V_1(H) = 2.24$. Using the wealth equation, we calculate:

$$X_2(HH) = (16)\left(\tfrac{1}{15}\right) + \tfrac{5}{4}\left[(2.24) - \left(\tfrac{1}{15}\right)(8)\right] = \tfrac{16}{15} + \tfrac{5}{4} \cdot \tfrac{128}{75} = \tfrac{16}{5} = 3.20 = V_2(HH)$$
$$X_2(HT) = (4)\left(\tfrac{1}{15}\right) + \tfrac{5}{4}\left[(2.24) - \left(\tfrac{1}{15}\right)(8)\right] = \tfrac{4}{15} + \tfrac{5}{4} \cdot \tfrac{128}{75} = \tfrac{12}{5} = 2.40 = V_2(HT)$$

Next we calculate $\Delta_2(HT)$:

$$\Delta_2(HT) = \frac{V_2(HTH) - V_2(HTT)}{S_2(HTH) - S_2(HTT)} = \frac{0 - 6}{8 - 2} = -1$$

We have $X_2(HT) = V_2(HT) = 2.40$:

$$X_3(HTH) = (8)(-1) + \tfrac{5}{4}\left[(2.40) - (-1)(4)\right] = -8 + \tfrac{5}{4} \cdot \tfrac{32}{5} = 0 = V_3(HTH)$$
$$X_3(HTT) = (2)(-1) + \tfrac{5}{4}\left[(2.40) - (-1)(4)\right] = -2 + \tfrac{5}{4} \cdot \tfrac{32}{5} = 6 = V_3(HTT)$$

□

Problem 1.6 If at time zero, the bank purchases Δ_0 shares, they must borrow $\Delta_0 S_0$ from the money market to finance this (if $\Delta_0 < 0$ this represents short position whose initial proceeds are invested in the money market). At time one, the bank then has $X_1 = V_1 + \Delta_0 S_1 - (1+r)\Delta_0 S_0$:

$$X_1(H) = (3) + \Delta_0(8) - \tfrac{5}{4}\Delta_0(4) = 3 + 3\Delta_0$$
$$X_1(T) = (0) + \Delta_0(2) - \tfrac{5}{4}\Delta_0(4) = -3\Delta_0$$

We see that if $\Delta_0 = -0.50$, then $X_1(H) = X_1(T) = 1.50$, as desired. Thus the bank should sell 0.50 shares short at time zero. This will net them proceeds of $(0.50)(4) = 2$, which they should invest into the money market. The above calculations show that regardless of the outcome of the coin toss, the bank will have wealth 1.50.

Alternatively, Theorem 1.2.2 tells us the amount of shares Δ_0' that the *seller* of the option should buy at time zero in order to replicate the option:

$$\Delta_0' = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{3 - 0}{8 - 2} = 0.50$$

It follows, that the *buyer* of the option (in this case the bank) should perform the opposite actions to the seller, in order to replicate the option. Again, we see that the bank should sell 0.50 shares short at time zero.

$\square$

Problem 1.7 The payoff of a lookback option at time three is $V_3 = \max_{0 \leq n \leq 3} S_n - S_3$:

$$\begin{aligned} V_3(HHH) &= 32 - 32 = 0 & V_3(THH) &= 8 - 8 = 0 \\ V_3(HHT) &= 16 - 8 = 8 & V_3(THT) &= 4 - 2 = 2 \\ V_3(HTH) &= 8 - 8 = 0 & V_3(TTH) &= 4 - 2 = 2 \\ V_3(HTT) &= 8 - 2 = 6 & V_3(TTT) &= 4 - 0.50 = 3.50 \end{aligned}$$

We know $\tilde{p} = \tilde{q} = \frac{1}{2}$, so then:

$$\begin{aligned} V_2(HH) &= \tfrac{4}{5}\left[\tfrac{1}{2}(0) + \tfrac{1}{2}(8)\right] = \tfrac{16}{5} = 3.20 & V_2(TH) &= \tfrac{4}{5}\left[\tfrac{1}{2}(0) + \tfrac{1}{2}(2)\right] = \tfrac{4}{5} = 0.80 \\ V_2(HT) &= \tfrac{4}{5}\left[\tfrac{1}{2}(0) + \tfrac{1}{2}(6)\right] = \tfrac{12}{5} = 2.40 & V_2(TT) &= \tfrac{4}{5}\left[\tfrac{1}{2}(2) + \tfrac{1}{2}(3.50)\right] = \tfrac{11}{5} = 2.20 \end{aligned}$$

$$\begin{aligned} V_1(H) &= \tfrac{4}{5}\left[\tfrac{1}{2}(3.20) + \tfrac{1}{2}(2.40)\right] = \tfrac{56}{25} = 2.24 \\ V_1(T) &= \tfrac{4}{5}\left[\tfrac{1}{2}(0.80) + \tfrac{1}{2}(2.20)\right] = \tfrac{6}{5} = 1.20 \\ V_0 &= \tfrac{4}{5}\left[\tfrac{1}{2}(2.24) + \tfrac{1}{2}(1.20)\right] = \tfrac{172}{125} = 1.376 \end{aligned}$$

We compute the values Δ_i for $i = 0, 1, 2$ according to Theorem 1.2.2:

$$\Delta_0 = \frac{(2.24) - (1.20)}{(8) - (2)} = \frac{13}{75} = 0.173$$

$$\Delta_1(H) = \frac{(3.20) - (2.40)}{(16) - (4)} = \frac{1}{30} = 0.033 \qquad \Delta_1(T) = \frac{(0.80) - (2.20)}{(4) - (1)} = -\frac{7}{15} = -0.467$$

$$\begin{aligned} \Delta_2(HH) &= \frac{(0) - (8)}{(32) - (8)} = -\frac{1}{3} & \Delta_2(TH) &= \frac{(0) - (2)}{(8) - (2)} = -\frac{1}{4} \\ \Delta_2(HT) &= \frac{(0) - (6)}{(8) - (2)} = -1 & \Delta_2(TT) &= \frac{(2) - (3.50)}{(2) - (0.50)} = -1 \end{aligned}$$

As mentioned in Problem 1.6, the owner of the option should carry out the opposite actions determined by Theorem 1.2.2 in order to replicate the option.

The bank should at time zero, sell 0.173 shares short, and invest the proceeds in the money market. We list the actions the bank should take depending on the result of the first two tosses:

Outcomes	Time 1	Time 2
HH	Short 0.033 Shares	Buy 0.333 Shares
HT	Short 0.033 Shares	Buy 1 Shares
TH	Buy 0.467 Shares	Buy 0.25 Shares
TT	Buy 0.467 Shares	Buy 1 Shares

At any period, if the bank is shorting a stock, they should invest the proceeds in the money market, and if they are buying a stock, they should borrow from the money market to fund this. Theorem 1.2.2 guarantees that if the bank follows the above strategy, at time 3, the portfolio will be worth $\left(\frac{5}{4}\right)^3 \cdot 1.376 = 2.6875$, regardless of the outcomes of the tosses.

Problem 1.8

(i) We know that $V_n = \frac{1}{1+r}\left[\tilde{p}V_{n+1}(H) + \tilde{q}V_{n+1}(T)\right]$. Futhermore $Y_{n+1} = Y_n + S_{n+1}$, hence:

$$\begin{aligned} v_n(s,y) &= \tfrac{4}{5}\left[\tfrac{1}{2}v_{n+1}(2s, y+2s) + \tfrac{1}{2}v_{n+1}\left(\tfrac{1}{2}s, y+\tfrac{1}{2}s\right)\right] \\ &= \tfrac{2}{5}\left[v_{n+1}(2s, y+2s) + v_{n+1}\left(\tfrac{1}{2}s, y+\tfrac{1}{2}s\right)\right] \end{aligned}$$

□

(ii) Note that $v_3(s,y) = \left(\frac{1}{4}y - 4\right)^+$ only depends on y. The possible values for y at time three are: 60, 36, 24, 18, 12, 9, 7.5. Also note that $v_3(s,y) = 0$ for $y \leq 16$:

$$\begin{aligned} v_3(s,60) &= \left(\tfrac{1}{4}(60) - 4\right)^+ = 11 \\ v_3(s,36) &= \left(\tfrac{1}{4}(36) - 4\right)^+ = 5 \\ v_3(s,24) &= \left(\tfrac{1}{4}(24) - 4\right)^+ = 2 \\ v_3(s,18) &= \left(\tfrac{1}{4}(18) - 4\right)^+ = \tfrac{1}{2} \\ v_3(s,12) &= v_3(s,9) = v_3(s,7.5) = 0 \end{aligned}$$

Then:

$$\begin{aligned} v_2(16,28) &= \tfrac{2}{5}\left[v_3(32,60) + v_3\left(8,36\right)\right] = \tfrac{2}{5}((11)+(5)) = \tfrac{32}{5} \\ v_2(4,16) &= \tfrac{2}{5}\left[v_3(8,24) + v_3\left(2,18\right)\right] = \tfrac{2}{5}\left((2)+\left(\tfrac{1}{2}\right)\right) = 1 \\ v_2(4,10) &= \tfrac{2}{5}\left[v_3(8,18) + v_3\left(2,12\right)\right] = \tfrac{2}{5}\left(\left(\tfrac{1}{2}\right)+(0)\right) = \tfrac{1}{5} \\ v_2(1,7) &= \tfrac{2}{5}\left[v_3(2,9) + v_3\left(0.5,7.5\right)\right] = \tfrac{2}{5}((0)+(0)) = 0 \end{aligned}$$

$$v_1(8,12) = \tfrac{2}{5}\left[v_2(16,28) + v_2\left(4,16\right)\right] = \tfrac{2}{5}\left(\left(\tfrac{32}{5}\right) + (1)\right) = \tfrac{74}{25}$$
$$v_1(2,6) = \tfrac{2}{5}\left[v_2(4,10) + v_2\left(1,7\right)\right] = \tfrac{2}{5}\left(\left(\tfrac{1}{5}\right) + (0)\right) = \tfrac{2}{25}$$

$$v_0(4,4) = \tfrac{2}{5}\left[v_1(8,12) + v_1(2,6)\right] = \tfrac{2}{5}\left[\left(\tfrac{74}{25}\right) + \left(\tfrac{2}{25}\right)\right] = \tfrac{152}{125} = 1.216$$

□

(iii)

$$\begin{aligned}\delta_n(s,y) &= \frac{v_{n+1}(2s, y+2s) - v_{n+1}\left(\frac{1}{2}s, y + \frac{1}{2}s\right)}{(2s) - \left(\frac{1}{2}s\right)} \\ &= \frac{2}{3s}\left(v_{n+1}(2s, y+2s) - v_{n+1}\left(\tfrac{1}{2}s, y+\tfrac{1}{2}s\right)\right)\end{aligned}$$

□

Problem 1.9

(i) We surpress $\omega_1\omega_2\ldots\omega_n$ from notation for brevity. For $0 \le n < N$, the wealth equation reads $X_{n+1} = \Delta_n S_{n+1} + (1+r_n)(X_n - \Delta_n S_n)$:

$$X_{n+1}(H) = u_n \Delta_n S_n + (1+r_n)(X_n - \Delta_n S_n) \tag{1}$$
$$X_{n+1}(T) = d_n \Delta_n S_n + (1+r_n)(X_n - \Delta_n S_n) \tag{2}$$

We then define $\tilde{p}_n = \frac{1+r_n-d_n}{u_n-d_n}$ and $\tilde{q}_n = \frac{u_n-1-r_n}{u_n-d_n}$, which give $\tilde{p}_n + \tilde{q}_n = 1$. We multiply (1) by $\tilde{p}_n$ and (2) by $\tilde{q}_n$, then add:

$$\tilde{p}_n X_{n+1}(H) + \tilde{q}_n X_{n+1}(T) = (\tilde{p}_n u_n + \tilde{q}_n d_n)\Delta_n S_n + (1+r_n)(X_n - \Delta_n S_n)$$

Notice that:

$$\tilde{p}_n u_n + \tilde{q}_n d_n = \frac{1}{u_n - d_n}\left((1+r_n)u_n - u_n d_n + u_n d_n - (1+r_n)d_n\right) = 1 + r_n$$

Hence:

$$\begin{aligned} \tilde{p}_n X_{n+1}(H) + \tilde{q}_n X_{n+1}(T) &= (1+r_n)X_n \\ \implies \qquad X_n &= \frac{1}{1+r_n}\left[\tilde{p}_n X_{n+1}(H) + \tilde{q}_n X_{n+1}(T)\right] \end{aligned} \tag{3}$$

Note that both the LHS and RHS are functions of $\omega_1\omega_2\dots\omega_n$, so our result is consistent.

Thus an agent's wealth at time n can be determined recursively backwards (3). If V_n, the price of a derivative security at time n (prior to the expiration at time N), was greater than X_n, then the agent could sell the derivative security, replicate the portfolio, and have a surplus of cash to invest in the money market. In this case, an arbitrage exists. Similarly, if $V_n < X_n$, then one could buy the derivative security for less than their portfolio, and again have a surplus of cash. Thus it follows that $V_n = X_n$ for all $0 \leq n \leq N$. Applying this to (3):

$$\therefore \quad V_n = \frac{1}{1+r_n}\left[\tilde{p}_n V_{n+1}(H) + \tilde{q}_n V_{n+1}(T)\right]$$

This determines a backwards recursive relation for determining V_0, given V_N, the payoffs at time N.

$\square$

(ii) Subtracting gives (2) from (1) above gives $(u_n - d_n)\Delta_n S_n = X_{n+1}(H) - X_{n+1}(T)$, hence:

$$\Delta_n = \frac{X_{n+1}(H) - X_{n+1}(T)}{(u_n - d_n)S_n}$$

$\square$

(iii) We first derive a formula for $v_n(s)$. The possible prices at time five are: 130, 110, 90, 70, 50, and 30. Since $v_5(s) = (s-80)^+$ we have that $v_5(s) = 0$ for $s \leq 80$.

$$v_5(130) = 50, \quad v_5(110) = 30, \quad v_5(90) = 10, \quad v_5(70) = v_5(50) = v_5(30) = 0$$

We have $u_n = \frac{S_{n+1}(H)}{S_n} = \frac{S_n+10}{S_n} = 1 + \frac{10}{S_n}$ and $d_n = \frac{S_{n+1}(T)}{S_n} = \frac{S_n-10}{S_n} = 1 - \frac{10}{S_n}$ Furthermore $r_n = 0$ for all n. We calculate:

$$\tilde{p}_n = \frac{1+(0)-\left(1-\frac{10}{S_n}\right)}{\left(1+\frac{10}{S_n}\right)-\left(1-\frac{10}{S_n}\right)} = \frac{\left(\frac{10}{S_n}\right)}{\left(\frac{20}{S_n}\right)} = \frac{1}{2} \qquad \Longrightarrow \qquad \tilde{q}_n = \frac{1}{2}$$

Therefore the recursive formula for v_n is:

$$\begin{aligned} v_n(s) &= \frac{1}{1+(0)}\left[\left(\tfrac{1}{2}\right)v_{n+1}(s+10) + \left(\tfrac{1}{2}\right)v_{n+1}(s-10)\right] \\ &= \tfrac{1}{2}\left[v_{n+1}(s+10) + v_{n+1}(s-10)\right] \end{aligned}$$

Therefore:

$$v_4(120) = \tfrac{1}{2}\left[(50) + (30)\right] = 40 \qquad v_4(100) = \tfrac{1}{2}\left[(30) + (10)\right] = 20$$
$$v_4(80) = \tfrac{1}{2}\left[(10) + (0)\right] = 5 \qquad v_4(60) = v_4(40) = 0$$

$$v_3(110) = \tfrac{1}{2}\left[(40) + (20)\right] = 30 \qquad v_3(90) = \tfrac{1}{2}\left[(20) + (5)\right] = \tfrac{25}{2}$$
$$v_3(70) = \tfrac{1}{2}\left[(5) + (0)\right] = \tfrac{5}{2} \qquad v_3(50) = 0$$

$$v_2(100) = \tfrac{1}{2}\left[(30) + \left(\tfrac{25}{2}\right)\right] = \tfrac{85}{4} \qquad v_2(80) = \tfrac{1}{2}\left[\left(\tfrac{25}{2}\right) + \left(\tfrac{5}{2}\right)\right] = \tfrac{15}{2}$$
$$v_2(60) = \tfrac{1}{2}\left[\left(\tfrac{5}{2}\right) + (0)\right] = \tfrac{5}{4}$$

$$v_1(90) = \tfrac{1}{2}\left[\left(\tfrac{85}{4}\right) + \left(\tfrac{15}{2}\right)\right] = \tfrac{115}{8} \qquad v_1(70) = \tfrac{1}{2}\left[\left(\tfrac{15}{2}\right) + \left(\tfrac{5}{4}\right)\right] = \tfrac{35}{8}$$

$$\therefore \quad v_0(80) = \tfrac{1}{2}\left[\left(\tfrac{115}{8}\right) + \left(\tfrac{35}{8}\right)\right] = \tfrac{150}{16} = 9.375$$

The price at time zero of the option is $V_0 = 9.375$

$\square$