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Unit 5 HW: Joint Distributions

1. W: wingspan V: velocity

$\downarrow$

Normal distribution w/ $\mu_W = 10$, $\sigma_W = 4$

U = R.V. w/ normal distribution $\mu_u = 0$, $\sigma_u = 1$

W and U are independent $\Rightarrow f_{W,U}(w,u) = f_W(w) f_U(u)$

$f_{W|U}(w,u) = f_W(w)$

$f_{U|W}(w,u) = f_U(u)$

$V = 0.5W + U$

variance/covariance matrix $= \begin{bmatrix} Var(W) & Cov(W,V) \\ Cov(V,W) & Var(V) \end{bmatrix}$

$Var(W) = \sigma_W^2 = 16$ W and U are independent $\rightarrow 0$

$Var(V) = Var(0.5W + U) = Var(0.5W) + Var(U) + 2\cancel{Cov(W,U)}$

$Var(V) = 0.25\,Var(W) + Var(U) = \frac{16}{4} + 1 = 5$

$Cov(W,V) = Cov(V,W) = E(WV) - E(W)E(V)$

$= E(0.5W^2 + WU) - E(W)\,E(0.5W + U)$

$= 0.5\,E(W^2) + E(WU) - E(W)[0.5\,E(W) + E(U)]$

$= 0.5\,E(W^2) + \cancel{E(W)E(U)} - 0.5\,E(W)^2 - \cancel{E(W)E(U)}$

because W and U are indep.

$= 0.5\left(E(W^2) - E(W)^2\right) = 0.5\,Var(W) = \frac{16}{2} = 8$

Covariance Matrix of W, V $= \begin{bmatrix} 16 & 8 \\ 8 & 5 \end{bmatrix}$

2. $f_x(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$ $\quad f_{Y|X}(x,y) = \begin{cases} \frac{1}{x} & 0 \le y \le x \\ 0 & \text{otherwise.} \end{cases}$

a)

$$E(X) = \int_0^1 x\, dx = \left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{1}{2}$$

$$E(Y|x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dy = \int_0^x \frac{y}{x}\, dy = \left[\frac{1}{2}\frac{y^2}{x}\right]_0^x$$

$$= \frac{1}{2}\frac{x^2}{x} = \frac{x}{2}, \quad \text{if } x = E(X) \text{ then } E(Y|X=E(x)) = \frac{1}{4}$$

$$\boxed{\therefore E(Y|X=x) = \frac{x}{2} \text{ and } E(Y|X=E(x)) = \frac{1}{4}}$$

b) $E(Y) = E(E(Y|X=x)) = E\left(\frac{x}{2}\right) = \frac{1}{2}E(X) = \frac{1}{4}.$

$E(Y) = \frac{1}{4}$

c) $E(XY) = E(E(XY|X)) = E(XE(Y|X)) = E\left(\frac{x^2}{2}\right)$

$E\left(\frac{x^2}{2}\right) = \int_0^1 \frac{x^2}{2}\, dx = \left[\frac{x^3}{6}\right]_0^1 = \frac{1}{6}$

d) $\text{cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{24}$

$\text{Cov}(X,Y) = \frac{1}{24}$

3a) M = wait time in morning
N = wait time in evening

M and N are independent to each other

$$f_M(m) = \begin{cases} \frac{1}{5} & 0 \le m \le 5 \\ 0 & \text{otherwise} \end{cases}$$

$$f_N(n) = \begin{cases} \frac{1}{10} & 0 \le n \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Total Expected wait time $= E(5M + 5N) = 5[E(M) + E(N)]$

$$E(M) = \int_0^5 \frac{1}{5} m \, dm = \left[\frac{m^2}{10}\right]_0^5 = 2.5$$

$$E(N) = \int_0^{10} \frac{1}{10} n \, dn = \left[\frac{n^2}{20}\right]_0^{10} = 5$$

Total expected wait time = 37.5 minutes

b) $Var(5M + 5N) = 25 Var(M) + 25 Var(N)$

$$Var(M) = E(m^2) - (E(m))^2 = \int_0^5 f(m) g(m) \, dm - 2.5^2$$

$$= \int_0^5 \frac{1}{5} m^2 \, dm - 6.25 = \left[\frac{m^3}{15}\right]_0^5 - 6.25 = \frac{25}{12}$$

$$Var(N) = \int_0^{10} \frac{1}{10} n^2 \, dn - 25 = \left[\frac{n^3}{30}\right]_0^{10} - 25 = \frac{25}{3}$$

$$Var(5M + 5N) = 25\left[\frac{25}{12} + \frac{25}{3}\right] = \frac{3125}{12} \approx 260.417$$

c) $E(5N - 5M) = 5E(N) - 5E(M)$
$= 5(5) - 5(2.5) = \underline{12.5}$

d) $Var(5N - 5M) = 5^2 Var(N) + (-5)^2 Var(M)$.
$= 25\left(\frac{25}{3}\right) + 25\left(\frac{25}{12}\right) = \underline{\frac{3125}{12} \approx 260.417}$

4. $Y = aX + b$ where X, Y are random variables and $a \neq 0$, $corr(X,Y) = -1$ or $+1$.

$$corr(X,Y) = \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = E[X(aX+b)] - E(X)E(aX+b)$$
$$= E(aX^2 + bX) - E(X)[aE(X) + b]$$
$$= aE(X^2) + \cancel{bE(X)} - aE(X)^2 - \cancel{bE(X)}$$
$$= a[E(X^2) - E(X)^2] = a\,Var(X)$$

$$\therefore \rho_{X,Y} = \frac{a\,Var(X)}{\sigma_X \sigma_Y} = \frac{a\sigma_X}{\sigma_Y}$$

$$Var(Y) = E(Y^2) - E(Y)^2 = E[(aX+b)^2] - [E(aX+b)]^2$$
$$= E[a^2X^2 + 2abX + b^2] - [aE(X) + b]^2$$
$$= a^2E(X^2) + \cancel{2abE(X)} + \cancel{b^2} - (a^2E(X)^2 + \cancel{2abE(X)} + \cancel{b^2})$$
$$= a^2[E(X^2) - E(X)^2] = a^2\,Var(X)$$
$$\sigma_Y = \sqrt{a^2 Var(X)} = \pm a\sigma_X$$
$$\therefore \rho_{X,Y} = \frac{a\sigma_X}{\sigma_Y} = \frac{\cancel{a\sigma_X}}{\pm \cancel{a\sigma_X}} = \underline{\pm 1}$$