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Unit 4 Homework : Random Variables.

1a)

$$\begin{cases} \$0, & 0 \text{ heads} \\ \$2, & 1 \text{ head} \\ \$4, & 2 \text{ heads} \\ \$?, & 3 \text{ heads} \end{cases} \qquad E(x) = \$6$$

$P(0 \text{ heads}) = \frac{1}{8}$

$P(1 \text{ head}) = \frac{3}{8}$ $\qquad E(x) = \$0 \cdot (\frac{1}{8}) + \$2(\frac{3}{8}) + \$4(\frac{3}{8}) + ?(\frac{1}{8}) = \$6$

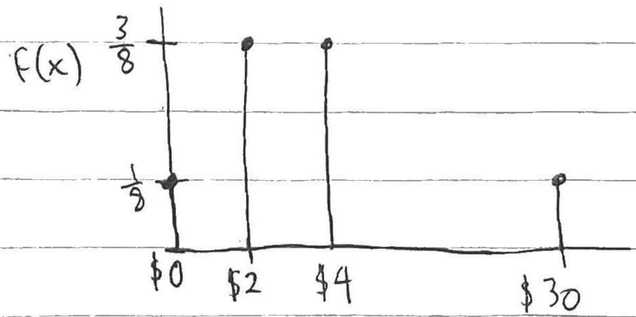
$P(2 \text{ head}) = \frac{3}{8}$ $\qquad \Rightarrow \frac{6}{8} + \frac{12}{8} + ?\frac{1}{8} = 6$

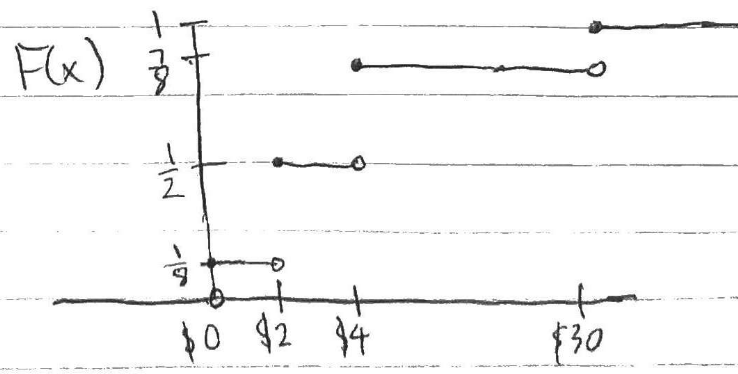
$P(3 \text{ head}) = \frac{1}{8}$ $\qquad 18 + ? = 48 \qquad ? = \$30$

We will get paid \$30 if the coin comes up with 3 heads.

1b)

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \le x < 2 \\ \frac{1}{2}, & 2 \le x < 4 \\ \frac{7}{8}, & 4 \le x < 30 \\ 1, & x \ge 30 \end{cases}$$





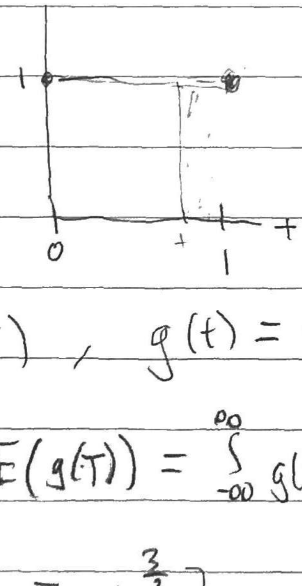
2a) $f(l) = \begin{cases} 0, & l \leq 0 \\ 1/2, & 0 < l \leq 2 \\ 0, & l \geqslant 2 \end{cases}$ $\qquad F(x) = \int_{-\infty}^{x} f(y)\,dy$

$$F(L) = \int_0^2 \frac{l}{2}\,dl = \left[\frac{1}{4}l^2\right]_0^2 = 1$$

$$F(L) = \begin{cases} 0, & l \leq 0 \\ \frac{1}{4}l^2, & 0 < l \leq 2 \\ 1, & l > 2 \end{cases}$$

b) $E(L) = \int_{-\infty}^{\infty} l f(l)\,dl = \int_0^2 \frac{l^2}{2}\,dl = \left[\frac{l^3}{6}\right]_0^2 = \frac{8}{6} = \frac{4}{3}$

$$\boxed{E(L) = \frac{4}{3}}$$

3a) $f(T) =$  $= \begin{cases} 1, & 0 \leq t \leq 1 \text{ years} \\ 0, & \text{otherwise} \end{cases}$

$Y = g(T)$, $g(t) = \$100\,(1-t)^{\frac{1}{2}}$

$$E(T) = E(g(T)) = \int_{-\infty}^{\infty} g(t) f(t)\,dt = \int_0^1 100(1-t)^{\frac{1}{2}}\,dt$$

$$= \left[\frac{-200}{3}[1-t]^{\frac{3}{2}}\right]_0^1 = 0 + \frac{200}{3}[1]^{\frac{3}{2}} = \$\frac{200}{3} \approx \$66.67$$

$$\boxed{E(T) = \$\frac{200}{3} \approx \$66.67}$$

3b)

i) $X = x$ $\quad \$0 \le x \le \$100$ $\quad X = 100(1-t)^{\frac{1}{2}} \Rightarrow \frac{1}{100}x\,(1-t)^{\frac{1}{2}}$

$$\frac{1}{10{,}000}x^2 = 1-t \Rightarrow t = 1 - \frac{1}{10{,}000}x^2$$

$X = x$ where $x = g(t) = \$100(1-t)^{\frac{1}{2}}$ $\quad 0 \le x \le 100$

$T = t$ where $t = 1 - \frac{1}{10{,}000}x^2$ $\quad 1 \ge t \ge 0$

ii) When $X = 0$, $T = 1$ year and when $X = \$100$, $T = 0$ years.

that means, if $\$0 \le X \le x$, then $t \le T \le 1$ year

Therefore, $T \ge t$ is equivaluent to $X \le x$

iii) $F(X \le x) = F(T \ge t)$ and $f(t) = 1$ when $0 \le t, \le 1$ years

$\therefore F(T \ge t) = 1 - F(T \le t)$:

$$F(T \le t) = \begin{cases} \int_{-\infty}^{t} 0\,dy, & t < 0 \\ \int_{-\infty}^{0} 0\,dy + \int_{0}^{t} 1\,dy, & 0 \le t \le 1 \\ \int_{-\infty}^{0} 0\,dy + \int_{0}^{1} 1\,dy + \int_{1}^{t} 0\,dy, & t > 1 \end{cases} \Rightarrow \begin{cases} 0, & t < 0 \\ t, & 0 \le t \le 1 \\ 1, & t > 0 \end{cases}$$

$$\therefore F(T \ge t) = \begin{cases} 1, & t < 0 \\ 1-t, & 0 \le t \le 1 \\ 0, & t > 1 \end{cases} \Rightarrow F(X \le x) = \begin{cases} 1, & x < 0 \\ 1 - (1 - \frac{1}{10{,}000}x^2), & 0 \le x \le 100 \\ 0, & x > 100 \end{cases}$$

$$\therefore F(X \le x) = \begin{cases} 1, & x < 0 \\ \frac{1}{10{,}000}x^2, & 0 \le x \le 100 \\ 0, & x > 0 \end{cases}$$

iv) $$f(x) = \begin{cases} 0, & x < 0 \\ \frac{2}{10{,}000}x, & 0 \le x \le 100 \\ 0, & x > 100 \end{cases}$$

v) $$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{0}^{100} \frac{2}{10{,}000}x^2 = \left[\frac{2}{30{,}000}x^3\right]_0^{100}$$

$$= \frac{2{,}000{,}000}{30{,}000} = \$66.67$$

$$\boxed{E(x) = \$66.67}$$

4 a) $Y = g(x) = (X - t)^2$ where t is a real number and X is a random variable.

$$E[Y] = E[(x-t)^2] = E[x^2 - 2tx + t^2]$$

$$= E(x^2) - E(2tx) + E(t^2)$$

$$= E(x^2) - 2t\,E(x) + t^2$$

b) $\frac{\partial}{\partial t} E(x^2) - 2t\,E(x) + t^2 = -2E(x) + 2t$

$$-2E(x) + 2t = 0$$

$$2t = 2E(x)$$

$t = E(x)$, minimizes $E(Y)$.

c) $E(Y) = E(x^2) - 2t\,E(X) + t^2$, when $t = E(x)$

$$E(Y) = E(x^2) - 2[E(x)]^2 + [E(x)]^2$$

$$E(Y) = E(x^2) - [E(x)]^2$$

when $t = E(x)$, $E(Y) = E(x^2) - [E(x)]^2 = Var(x)$

5. $Y = h(X)$ prove $g(y) = f(h^{-1}(y)) \cdot \left| \frac{\partial}{\partial y} h^{-1}(y) \right|$

$\downarrow$

$X = h^{-1}(Y)$ $P(Y \leq y) = P(h(X) \leq y) = P(X \leq h^{-1}(y))$

Prob. density function $g(y) = \frac{\partial}{\partial y} P(Y \leq y) = \frac{\partial}{\partial y} P(X \leq h^{-1}(y))$

$P(X \leq x) = \int_{-\infty}^{x} f(x)dx \Rightarrow P(X \leq h^{-1}(y)) = \int_{-\infty}^{x} f(h^{-1}(y))\,dx$

$g(y) = \frac{\partial}{\partial y} \int_{-\infty}^{x} f(h^{-1}(y))\,dx = \left| \frac{\partial}{\partial y} h^{-1}(y) \right| \cdot f(h^{-1}(y))$