

UNSTABLE $m = 1$ ECCENTRIC MODES IN THERMALLY COOLING, SELF GRAVITATING DISKS

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ABSTRACT

1. INTRODUCTION

2. EQUATIONS OF MOTION

The equations of motion for a two dimensional, viscous disk are

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\Sigma} - \nabla \Phi + \frac{1}{\Sigma} \nabla \cdot (\nu \Sigma \mathbf{D}) \quad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \Sigma = -\Sigma \nabla \cdot \mathbf{v} \quad (2)$$

$$T(\partial_t + \mathbf{v} \cdot \nabla) s = -\frac{T - \bar{T}}{t_{\text{cool}}} \quad (3)$$

Where all quantities are assumed to be a vertical average of their three dimensional counterparts. Here, Σ is the surface density, s is the entropy of the fluid, T is the disk temperature, and \mathbf{D} is the viscous stress tensor. These equations describe a viscous fluid that is thermally heated and or cooled to some "background" temperature profile \bar{T} over a timescale t_{cool} . To close the system of equations we assume the fluid is ideal and perfect, so that $P = \mathcal{R} \Sigma T$ and $s \equiv \frac{\mathcal{R}}{\gamma-1} \ln(P \Sigma^{-\gamma})$; hereafter we set $\mathcal{R} = 1$ without loss of generality. Following Ostriker, Shu, & Adams (5), we can connect the two dimension adiabatic index γ to the true three dimensional adiabatic index Γ by assuming that the gas is vertically isothermal and gaussian distributed.

We consider a disk with a prescribed azimuthally averaged background profile given by $(\bar{v}_r, \bar{v}_\phi, \bar{\Sigma}, \bar{T}) = (0, r\Omega(r), \bar{\Sigma} \propto r^\mu, \bar{T} \propto r^\delta)$. We look at linear perturbations to this background state. Denoted linear quantities with a prime and background quantities with a bar, the linearized equations of motion become

$$(\partial_t + \Omega \partial_\phi) v'_r - 2\Omega v'_\phi = -\frac{1}{\Sigma} \partial_r P' + \frac{\Sigma'}{\Sigma^2} \frac{d\bar{P}}{dr} - \partial_r \Phi' + \text{visc.} \quad (4)$$

$$(\partial_t + \Omega \partial_\phi) v'_\phi + \frac{\kappa^2}{2\Omega} v'_r = -\frac{1}{r\bar{\Sigma}} \partial_\phi P' - \frac{1}{r} \partial_\phi \Phi' + \text{visc.} \quad (5)$$

$$(\partial_t + \Omega \partial_\phi) \Sigma' + v'_r \frac{d\bar{\Sigma}}{dr} = -\bar{\Sigma} \left(\frac{1}{r} \partial_r (r v'_r) + \frac{1}{r} \partial_\phi (v'_\phi) \right) \quad (6)$$

$$(\partial_t + \Omega \partial_\phi) s' + v'_r \frac{d\bar{s}}{dr} = -\beta \Omega \frac{T'}{\bar{T}} \quad (7)$$

Where we've parametrized the cooling time as $t_{\text{cool}}^{-1} = \beta \Omega$ (Gammie 2001). We now Fourier Transform the linear variables in the azimuthal direction as well as in time and write $(v'_r, v'_\phi, \Sigma', P', s', T', \Phi') =$

$\sum_m (u, v, \sigma, p, s, T, \phi) e^{im(\phi - \Omega_p t)}$. We further work under the assumption that the different m modes only couple to the $m = 0$ mode (i.e the background) and not to other $m \neq 0$ modes. The linearized equations of motion are now

$$im(\Omega - \Omega_p)u - 2\Omega v = -\frac{1}{\Sigma} \frac{dp}{dr} + \frac{\sigma}{\Sigma^2} \frac{d\bar{P}}{dr} - \frac{d\phi}{dr} + \text{visc.} \quad (8)$$

$$im(\Omega - \Omega_p)v + \frac{\kappa^2}{2\Omega} u = -\frac{imp}{r\bar{\Sigma}} - \frac{im\phi}{r} + \text{visc.} \quad (9)$$

$$im(\Omega - \Omega_p) \frac{\sigma}{\bar{\Sigma}} + u \frac{d \ln \bar{\Sigma}}{dr} = -\left(\frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (10)$$

$$im \left[\left(1 - i \frac{\beta(\gamma-1)}{m} \right) \Omega - \Omega_p \right] \frac{T'}{\bar{T}} + u \frac{d \ln \bar{T}}{dr} = -(\gamma-1) \left(\frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (11)$$

Where we've replaced the entropy in favor of the temperature through the relations

$$(\gamma-1)s' = \frac{T'}{\bar{T}} - (\gamma-1) \frac{\sigma}{\bar{\Sigma}} \quad (12)$$

$$(\gamma-1) \frac{d\bar{s}}{dr} = \frac{d \ln \bar{T}}{dr} - (\gamma-1) \frac{d \ln \bar{\Sigma}}{dr} \quad (13)$$

3. $m = 1$ SLOW MODES

We specialize to $m = 1$ modes and to the low pattern speed limit $\Omega_p \ll \Omega$. We can simplify the 4 dimensional set of equations to just one equation describing the global structure and evolution of an eccentricity vector, $e(r)$ (? 2; 6). The eccentricity is defined through the Lagrangian displacement ξ_r as

$$e \equiv \frac{\xi_r}{r} \quad (14)$$

Following (6) we can write an evolution equation for the disk eccentricity by an appropriate linear combination of the two momenta equations and using

$$2\Omega r^3 (\Omega_p - \omega_p) e = -\frac{d}{dr} \left(r^2 \frac{p}{\Sigma} + r^2 \phi \right) + \frac{r^2}{\Sigma^2} \left[\sigma \frac{d\bar{P}}{dr} - p \frac{d\bar{\Sigma}}{dr} \right] \quad (15)$$

$$2\Omega r^3 \omega_p = -r \frac{d}{dr} \left(\frac{r^2 d\bar{P}}{\Sigma} \right) - r \frac{d}{dr} \left(r^2 \frac{d\bar{\Phi}}{dr} \right) \quad (16)$$

$$p = \sigma \bar{T} + \bar{\Sigma} T' \quad (17)$$

$$\frac{\sigma}{\bar{\Sigma}} = -\frac{d \ln \bar{\Sigma}}{d \ln r} e - r \frac{de}{dr} \quad (18)$$

$$(1 - i\tilde{\beta})\frac{T'}{\bar{T}} = -\frac{d \ln \bar{T}}{d \ln r}e - (\gamma - 1)r\frac{de}{dr} \quad (19)$$

3.1. No Cooling

We define the adiabatic limit as when $\beta = 0$, the eccentricity equation in this limit reduces to,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\frac{d}{dr}\left(\frac{r^2 p}{\bar{\Sigma}} + r^2 \phi\right) + \frac{r^2}{\bar{\Sigma}}\left(\sigma \frac{d\bar{P}}{dr} - p \frac{d\bar{\Sigma}}{dr}\right) \quad (20)$$

$$\frac{p}{\bar{P}} = -\frac{d \ln \bar{P}}{d \ln r}e - \gamma r \frac{de}{dr} \quad (21)$$

This is equivalent to the eccentricity equation of e.g. Goodchild & Ogilvie (2) (their Eq. 21), when we use (16) to replace ω_p .

We define the barotropic limit as when $\gamma \rightarrow 1$ and we can write the total pressure as $P = P(\Sigma)$. We then define the sound speed c^2 , such that, $p = c^2 \sigma$ and $\frac{d\bar{P}}{d\bar{\Sigma}} = c^2$. The eccentricity equation simplifies in this case to,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\frac{d}{dr}\left(r^2 \frac{c^2 \sigma}{\bar{\Sigma}} + r^2 \phi\right) \quad (22)$$

This is equivalent to the eccentricity equation of e.g. Papaloizou (6) (his Eq. 18).

The locally isothermal limit is defined as $\gamma \rightarrow 1$ and $P = \bar{T}(r)\bar{\Sigma}$. The eccentricity equation in the locally isothermal limit is then,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\bar{T}\frac{d}{dr}\left(r^2 \frac{\sigma}{\bar{\Sigma}}\right) - \frac{d}{dr}(r^2 \phi) \quad (23)$$

This is equivalent to the linear system that e.g. Lin (3) studied.

3.2. With Cooling

Which follow from (10) & (11) to zeroth order in Ω_p and with $\Omega \approx \Omega_K$. We obtain the evolution equation

$$\Omega_p e = A e + B \frac{de}{d \ln r} + C \frac{d^2 e}{d \ln r^2} \quad (24)$$

Where the coefficients are,

$$A = \omega_p + \mathcal{H} \left[2\mu + \mu' + \frac{1}{1 - i\tilde{\beta}}(\delta' + \delta(2 + \mu + \delta)) \right] \quad (25)$$

$$B = \mathcal{H} \left[2 + \mu + \frac{1}{1 - i\tilde{\beta}}((\gamma\delta + (\gamma - 1)(2 + \mu))) \right] \quad (26)$$

$$C = \mathcal{H} \left[1 + \frac{(\gamma - 1)}{1 - i\tilde{\beta}} \right] \quad (27)$$

Where,

$$\tilde{\beta} \equiv \beta(\gamma - 1) \quad (28)$$

$$\mathcal{H} \equiv \frac{\bar{T}}{2\Omega r^2} \quad (29)$$

$$\mu' = \frac{d^2 \bar{\Sigma}}{d \ln r^2} \quad \delta' = \frac{d^2 \bar{T}}{d \ln r^2} \quad (30)$$

To add in the effects of viscosity we first use the alpha parametrization of viscosity and take $\nu = \alpha \frac{T}{\Omega}$ (7). The

corrections due to shear (α_s) and bulk (α_b) viscosities are,

$$A_\nu = -\frac{i\alpha_s}{6}\mathcal{H} \left(\frac{65}{2} + \delta - 2\mu + 18\mu' \right) \quad (31)$$

$$B_\nu = -\frac{i\alpha_s}{3}\mathcal{H} \left(-\frac{19}{2} - 7\delta + 2\mu \right) + i\alpha_b \mathcal{H} \left(\frac{5}{2} + \delta + \mu \right) \quad (32)$$

$$C_\nu = -\frac{i2\alpha_s}{3}\mathcal{H} - i\alpha_b \mathcal{H} \quad (33)$$

We solve (24) as a generalized eigenvalue problem by discretizing the eccentricity on a logarithmic grid in radius.

$$\Omega_p \mathbf{Q} \mathbf{e} = \mathbf{M} \mathbf{e} \quad (34)$$

The matrices \mathbf{Q} , and \mathbf{M} are the boundary condition matrix and the coefficient matrix, respectively. We solve for the spectrum of normal eigenmodes, \mathbf{e}_n and their associated (possibly complex) eigenvalues $\Omega_{p,n}$.

3.3. Boundary Condition

We adopt a zero Lagrangian pressure boundary condition at both the inner and outer boundaries. Using the equations of motion it can be shown that this is equivalent to satisfying

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \tilde{\beta} \left[\frac{\tilde{\beta} - i\gamma}{\tilde{\beta}^2 + \gamma^2} \right] \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} \quad (35)$$

at each boundary in the slow mode approximation. In the isothermal ($\beta \rightarrow \infty$) and adiabatic ($\beta \rightarrow 0$) limits the boundary condition reduces to,

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \begin{cases} \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} & \text{Isothermal} \\ 0 & \text{Adiabatic} \end{cases} \quad (36)$$

4. METHODS

5. RESULTS

We studied the growth rates of unstable modes in the slow mode ($\Omega_p \ll \Omega_k$) approximation. The classical WKB dispersion relation for a self gravitating and isothermal disk (see e.g. Binney & Tremaine (1)) is

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + c^2 k_r^2 = 0 \quad (37)$$

Where $\varpi \equiv \Omega - \Omega_p$. In the $m = 1$ slow mode limit this simplifies to (6)

$$2\Omega(\Omega_p - \omega_p) = 2\pi G \Sigma |k_r| - c^2 k_r^2 \quad (38)$$

Where $\omega_p = \Omega - \kappa$ is the orbital precession frequency of a fluid element. From the dispersion relation it is clear that pressure creates retrograde modes ($\Omega_p < 0$) and self gravity tends to create prograde modes ($\Omega_p > 0$). Additionally, global modes (i.e low k_r) also tend to create prograde modes (6).

To understand the implications of cooling on the disk, we write the the WKB dispersion relation for a disk which has consonant cooling β and constant background profiles. After some algebra, we obtain

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + k_r^2 c_{\text{cool}}^2 = 0 \quad (39)$$

$$c_{\text{cool}}^2 = \frac{m\varpi}{m\varpi - i\tilde{\beta}\Omega} c_{\text{adi}}^2 - \frac{i\tilde{\beta}\Omega}{m\varpi - i\tilde{\beta}\Omega} c_{\text{iso}}^2 \quad (40)$$

In the $m = 1$, slow mode limit we obtain

$$2\Omega(\Omega_p - \omega_p) = 2\pi G\Sigma |k_r| - k_r^2 c_{\text{cool}}^2 \quad (41)$$

$$c_{\text{cool}}^2 = \frac{1}{1 - i\tilde{\beta}} c_{\text{adi}}^2 + \frac{i\tilde{\beta}}{i\tilde{\beta} - 1} c_{\text{iso}}^2 \quad (42)$$

Where $c_{\text{adi}}^2 = \gamma c_{\text{iso}}^2$ is the sound speed in the adiabatic ($\beta = 0$) limit. Figure shows the dependence of the decay rate with β .

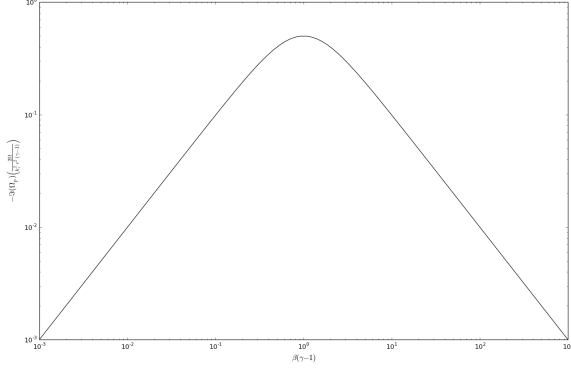


FIG. 1.—

5.1. Lowest Order Mode with No Self Gravity

The mode with zero nodes is perhaps the simplest to understand.

5.2. Self Gravity

To include the effects of the disk's self gravity, we include the potential due to the perturbed surface density,

$$\phi' = -G \int dr' r' \Sigma'(r') \int d\phi \frac{\cos \phi}{\sqrt{r^2 + r'^2 - 2rr' \cos \phi}} \quad (43)$$

The ϕ integral is simply the Laplace coefficient, $b_{1/2}^{(1)}$

$$b_{1/2}^{(1)} \left(\frac{r'}{r} \right) = \frac{2}{\pi r} \int_0^\pi d\phi \frac{\cos \phi}{\sqrt{1 + \left(\frac{r'}{r} \right)^2 - 2 \left(\frac{r'}{r} \right) \cos \phi}} \quad (44)$$

So that the potential is,

$$\phi' = -\frac{G\pi}{r} \int dr' r' \Sigma'(r') b_{1/2}^{(1)} \left(\frac{r'}{r} \right) \quad (45)$$

There is also an "edge" contribution to the potential due to there being a non-negligible excess of material outside the outer boundary of the disk and a deficit of disk material inside the inner boundary of the disk. At the outer boundary of the disk we have, $\Sigma'(r)dr = \bar{\Sigma}(r_o)\xi_r(r_o) = \bar{\Sigma}(r_o)r_o e(r_o)$. Similarly at the inner boundary we have, $\Sigma'(r)dr' = -\bar{\Sigma}(r_i)\xi_r(r_i) = \bar{\Sigma}(r_i)r_i e(r_i)$. The "edge" potential is thus,

$$\phi_{\text{edge}} = -\frac{G\pi\bar{\Sigma}(r_o)r_o^2 e(r_o)}{r} b_{1/2}^{(1)} \left(\frac{r_o}{r} \right) + \frac{G\pi\bar{\Sigma}(r_i)r_i^2 e(r_i)}{r} b_{1/2}^{(1)} \left(\frac{r_i}{r} \right) \quad (46)$$

The perturbed surface density is related to the eccentricity as $\phi' = -r \frac{d}{dr} (\bar{\Sigma}e)$. Plugging this in to (45) and integrating by parts gives,

$$\phi' = -\phi_{\text{edge}} - \frac{G\pi}{r} \int dr' \bar{\Sigma}(r') e(r') \partial_{r'} \left(r'^2 b_{1/2}^{(1)} \left(\frac{r'}{r} \right) \right) \quad (47)$$

The "edge" contribution to the potential thus cancels when we take $\phi' \rightarrow \phi' + \phi_{\text{edge}}$. The potential enters the eccentricity equation via the term,

$$-\frac{d}{dr} (r^2 \phi') \quad (48)$$

6. DISCUSSION

7. CONCLUSIONS

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