UNSTABLE $m = 1$ ECCENTRIC MODES IN THERMALLY COOLING, SELF GRAVITATING DISKS

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ABSTRACT

1. INTRODUCTION

2. EQUATIONS OF MOTION

The equations of motion for a two dimensional, viscous disk are

$$(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{\nabla P}{\Sigma} - \nabla\Phi + \frac{1}{\Sigma}\nabla\cdot(\nu\Sigma\mathbf{D}) \quad (1)$$

$$(\partial_t + \mathbf{v}\cdot\nabla)\Sigma = -\Sigma\nabla\cdot\mathbf{v} \quad (2)$$

$$T(\partial_t + \mathbf{v}\cdot\nabla)s = -\frac{T-\bar{T}}{t_{\rm cool}} \quad (3)$$

Where all quantities are assumed to be a vertical average of their three dimensional counterparts. Here, Σ is the surface density, s is the entropy of the fluid, T is the disk temperature, and $\mathbf{D}$ is the viscous stress tensor. These equations describe a viscous fluid that is thermally heated and or cooled to some "background" temperature profile $\bar{T}$ over a timescale $t_{\rm cool}$. To close the system of equations we assume the fluid is ideal and perfect, so that $P = \mathcal{R}\Sigma T$ and $s \equiv \frac{\mathcal{R}}{\gamma-1}\ln(P\Sigma^{-\gamma})$; hereafter we set $\mathcal{R} = 1$ without loss of generality. Following **?**, we can connect the two dimension adiabatic index γ to the true three dimensional adiabatic index Γ by assuming that the gas is vertically isothermal and gaussian distributed.

We consider a disk with a prescribed azimuthally averaged background profile given by $(\bar{v}_r, \bar{v}_\phi, \bar{\Sigma}, \bar{T}) = (0, r\Omega(r), \bar{\Sigma}\propto r^\mu, \bar{T}\propto r^\delta)$. We look at linear perturbations to this background state. Denoted linear quantities with a prime and background quantities with a bar, the linearized equations of motion become

$$(\partial_t + \Omega\partial_\phi)v_r' - 2\Omega v_\phi' = -\frac{1}{\bar{\Sigma}}\partial_r P' + \frac{\Sigma'}{\bar{\Sigma}^2}\frac{d\bar{P}}{dr} - \partial_r\Phi' + \text{visc.} \quad (4)$$

$$(\partial_t + \Omega\partial_\phi)v_\phi' + \frac{\kappa^2}{2\Omega}v_r' = -\frac{1}{r\bar{\Sigma}}\partial_\phi P' - \frac{1}{r}\partial_\phi\Phi' + \text{visc.} \quad (5)$$

$$(\partial_t + \Omega\partial_\phi)\Sigma' + v_r'\frac{d\bar{\Sigma}}{dr} = -\bar{\Sigma}\left(\frac{1}{r}\partial_r(rv_r') + \frac{1}{r}\partial_\phi(v_\phi')\right) \quad (6)$$

$$(\partial_t + \Omega\partial_\phi)s' + v_r'\frac{d\bar{s}}{dr} = -\beta\Omega\frac{T'}{\bar{T}} \quad (7)$$

Where we've parametrized the cooling time as $t_{\rm cool}^{-1} = \beta\Omega$ (**?**). We now Fourier Transform the linear variables in the azimuthal direction as well as in time and write $(v_r', v_\phi', \Sigma', P', s', T', \Phi') = \sum_m (u, v, \sigma, p, s, T, \phi)e^{im(\phi-\Omega_p t)}$. We further work under the assumption that the different m modes only couple to the $m = 0$ mode (i.e the background) and not to other $m \neq 0$ modes. The linearized equations of motion are now

$$im(\Omega-\Omega_p)u - 2\Omega v = -\frac{1}{\bar{\Sigma}}\frac{dp}{dr} + \frac{\sigma}{\bar{\Sigma}^2}\frac{d\bar{P}}{dr} - \frac{d\phi}{dr} + \text{visc.} \quad (8)$$

$$im(\Omega-\Omega_p)v + \frac{\kappa^2}{2\Omega}u = -\frac{imp}{r\bar{\Sigma}} - \frac{im\phi}{r} + \text{visc.} \quad (9)$$

$$im(\Omega-\Omega_p)\frac{\sigma}{\bar{\Sigma}} + u\frac{d\ln\bar{\Sigma}}{dr} = -\left(\frac{1}{r}\frac{d}{dr}(ru) + \frac{imv}{r}\right) \quad (10)$$

$$im\left[\left(1 - i\frac{\beta(\gamma-1)}{m}\right)\Omega - \Omega_p\right]\frac{T'}{\bar{T}} + u\frac{d\ln\bar{T}}{dr} = -(\gamma-1)\left(\frac{1}{r}\frac{d}{dr}(ru) + \frac{imv}{r}\right) \quad (11)$$

Where we've replaced the entropy in favor of the temperature through the relations

$$(\gamma-1)s' = \frac{T'}{\bar{T}} - (\gamma-1)\frac{\sigma}{\bar{\Sigma}} \quad (12)$$

$$(\gamma-1)\frac{d\bar{s}}{dr} = \frac{d\ln\bar{T}}{dr} - (\gamma-1)\frac{d\ln\bar{\Sigma}}{dr} \quad (13)$$

3. $m = 1$ SLOW MODES

We specialize to $m = 1$ modes and to the low pattern speed limit $\Omega_p \ll \Omega$. We can simplify the 4 dimensional set of equations to just one equation describing the global structure and evolution of an eccentricity vector, $e(r)$ (**???**). The eccentricity is defined through the Lagrangian displacement ξ_r as

$$e \equiv \frac{\xi_r}{r} \quad (14)$$

Following **?** we can write an evolution equation for the disk eccentricity by an appropriate linear combination of the two momenta equations and using

$$2\Omega r^3(\Omega_p - \omega_p)e = -\frac{d}{dr}\left(r^2\frac{p}{\bar{\Sigma}} + r^2\phi\right) + \frac{r^2}{\bar{\Sigma}^2}\left[\sigma\frac{d\bar{P}}{dr} - p\frac{d\bar{\Sigma}}{dr}\right] \quad (15)$$

$$2\Omega r^3\omega_p = -r\frac{d}{dr}\left(\frac{r^2}{\bar{\Sigma}}\frac{d\bar{P}}{dr}\right) - r\frac{d}{dr}\left(r^2\frac{d\bar{\Phi}}{dr}\right) \quad (16)$$

$$p = \sigma\bar{T} + \bar{\Sigma}T' \quad (17)$$

$$\frac{\sigma}{\bar{\Sigma}} = -\frac{d\ln\bar{\Sigma}}{d\ln r}e - r\frac{de}{dr} \quad (18)$$

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$$(1-i\tilde{\beta})\frac{T'}{\bar{T}} = -\frac{d\ln\bar{T}}{d\ln r}e - (\gamma-1)r\frac{de}{dr} \tag{19}$$

$$(1-i\tilde{\beta})\frac{p}{\bar{P}} = -\left[\left(\frac{d\ln\bar{T}}{d\ln r} + (1-i\tilde{\beta})\frac{d\ln\bar{\Sigma}}{d\ln r}\right)e + \left(\gamma - i\tilde{\beta}\right)r\frac{de}{dr}\right] \tag{20}$$

3.1. *No Cooling*

We define the adiabatic limit as when $\beta = 0$, the eccentricity equation in this limit reduces to,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\frac{d}{dr}\left(\frac{r^2 p}{\bar{\Sigma}} + r^2\phi\right) + \frac{r^2}{\bar{\Sigma}}\left(\sigma\frac{d\bar{P}}{dr} - p\frac{d\bar{\Sigma}}{dr}\right) \tag{21}$$

$$\frac{p}{\bar{P}} = -\frac{d\ln\bar{P}}{d\ln r}e - \gamma r\frac{de}{dr} \tag{22}$$

This is equivalent to the eccentricity equation of e.g **?** (their Eq. 21), when we use (**??**) to replace ω_p.

We define the barotropic limit as when $\gamma \to 1$ and we can write the total pressure as $P = P(\Sigma)$. We then define the sound speed c^2, such that, $p = c^2\sigma$ and $\frac{d\bar{P}}{d\bar{\Sigma}} = c^2$. The eccentricity equation simplifies in this case to,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\frac{d}{dr}\left(r^2\frac{c^2\sigma}{\bar{\Sigma}} + r^2\phi\right) \tag{23}$$

This is equivalent to the eccentricity equation of e.g **?** (his Eq. 18).

The locally isothermal limit is defined as $\gamma \to 1$ and $P = \bar{T}(r)\Sigma$. The eccentricty equation in the locally isothermal limit is then,

$$2\Omega r^3(\Omega_p - \omega_p)e = -\bar{T}\frac{d}{dr}\left(r^2\frac{\sigma}{\bar{\Sigma}}\right) - \frac{d}{dr}\left(r^2\phi\right) \tag{24}$$

This is equivalent to the linear system that e.g, **?** studied.

3.2. *With Cooling*

Which follow from (**??**) & (**??**) to zeroth order in Ω_p and with $\Omega \approx \Omega_K$. We obtain the evolution equation

$$\Omega_p e = Ae + B\frac{de}{d\ln r} + C\frac{d^2 e}{d\ln r^2} \tag{25}$$

Where the coefficients are,

$$A = \omega_p + \mathcal{H}\left[2\mu + \mu' + \frac{1}{1-i\tilde{\beta}}(\delta' + \delta(2+\mu+\delta)\right] \tag{26}$$

$$B = \mathcal{H}\left[2 + \mu + \frac{1}{1-i\tilde{\beta}}((\gamma\delta + (\gamma-1)(2+\mu))\right] \tag{27}$$

$$C = \mathcal{H}\left[1 + \frac{(\gamma-1)}{1-i\tilde{\beta}}\right] \tag{28}$$

Where,

$$\tilde{\beta} \equiv \beta(\gamma-1) \tag{29}$$

$$\mathcal{H} \equiv \frac{\bar{T}}{2\Omega r^2} \tag{30}$$

$$\mu' = \frac{d^2\bar{\Sigma}}{d\ln r^2} \qquad \delta' = \frac{d^2\bar{T}}{d\ln r^2} \tag{31}$$

To add in the effects of viscosity we first use the alpha parametrization of viscosity and take $\nu = \alpha\frac{T}{\Omega}$ (**?**). The corrections due to shear (α_s) and bulk (α_b) viscosities are,

$$A_\nu = -\frac{i\alpha_s}{6}\mathcal{H}\left(\frac{65}{2} + \delta - 2\mu + 18\mu'\right) \tag{32}$$

$$B_\nu = -\frac{i\alpha_s}{3}\mathcal{H}\left(-\frac{19}{2} - 7\delta + 2\mu\right) + i\alpha_b\mathcal{H}\left(\frac{5}{2} + \delta + \mu\right) \tag{33}$$

$$C_\nu = -\frac{i2\alpha_s}{3}\mathcal{H} - i\alpha_b\mathcal{H} \tag{34}$$

We solve (**??**) as a generalized eigenvalue problem by discretizing the eccentricity on a logarithmic grid in radius.

$$\Omega_p\mathbf{Q}\mathbf{e} = \mathbf{M}\mathbf{e} \tag{35}$$

The matrices $\mathbf{Q}$, and $\mathbf{M}$ are the boundary condition matrix and the coefficient matrix, respectively. We solve for the spectrum of normal eigenmodes, $\mathbf{e}_n$ and their associated (possibly complex) eigenvalues $\Omega_{p,n}$.

3.3. *Boundary Condition*

We adopt a zero Lagrangian pressure boundary condition at both the inner and outer boundaries. Using the equations of motion it can be shown that this is equivalent to satisfying

$$r\frac{de(r)}{dr}\bigg|_{r_i,r_o} = \tilde{\beta}\left[\frac{\tilde{\beta} - i\gamma}{\tilde{\beta}^2 + \gamma^2}\right]\frac{d\ln\bar{T}}{d\ln r}e(r)\bigg|_{r_i,r_o} \tag{36}$$

at each boundary in the slow mode approximation. In the isothermal ($\beta \to \infty$) and adiabatic ($\beta \to 0$) limits the boundary condition reduces to,

$$r\frac{de(r)}{dr}\bigg|_{r_i,r_o} = \begin{cases} \frac{d\ln\bar{T}}{d\ln r}e(r)\Big|_{r_i,r_o} & \text{Isothermal} \\ 0 & \text{Adiabatic} \end{cases} \tag{37}$$

4. METHODS

5. RESULTS

We studied the growth rates of unstable modes in the slow mode ($\Omega_p \ll \Omega_k$) approximation. The classical WKB dispersion relation for a self gravitating and isothermal disk (see e.g **?**) is

$$\kappa^2 - m^2\varpi^2 - 2\pi G\Sigma|k_r| + c^2k_r^2 = 0 \tag{38}$$

Where $\varpi \equiv \Omega - \Omega_p$. In the $m = 1$ slow mode limit this simplifies to (**?**)

$$2\Omega(\Omega_p - \omega_p) = 2\pi G\Sigma|k_r| - c^2k_r^2 \tag{39}$$

Where $\omega_p = \Omega - \kappa$ is the orbital precession frequency of a fluid element. From the dispersion relation it is clear that pressure creates retrograde modes ($\Omega_p < 0$) and self gravity tends to create prograde modes ($\Omega_p > 0$). Additionally, global modes (i.e low k_r) also tend to create prograde modes (**?**).

To understand the implications of cooling on the disk, we write the the WKB dispersion relation for a disk which has consonant cooling β and constant background profiles. After some algebra, we obtain

$$\kappa^2 - m^2\varpi^2 - 2\pi G\Sigma|k_r| + k_r^2c_{\mathrm{cool}}^2 = 0 \tag{40}$$

$$c_{\rm cool}^2 = \frac{m\varpi}{m\varpi - i\tilde{\beta}\Omega} c_{\rm adi}^2 - \frac{i\tilde{\beta}\Omega}{m\varpi - i\tilde{\beta}\Omega} c_{\rm iso}^2 \tag{41}$$

In the $m = 1$, slow mode limit we obtain

$$2\Omega(\Omega_p - \omega_p) = 2\pi G\Sigma|k_r| - k_r^2 c_{\rm cool}^2 \tag{42}$$

$$c_{\rm cool}^2 = \frac{1}{1 - i\tilde{\beta}} c_{\rm adi}^2 + \frac{i\tilde{\beta}}{i\tilde{\beta} - 1} c_{\rm iso}^2 \tag{43}$$

Where $c_{\rm adi}^2 = \gamma c_{\rm iso}^2$ is the sound speed in the adiabatic ($\beta = 0$) limit. Figure shows the dependence of the decay rate with β.

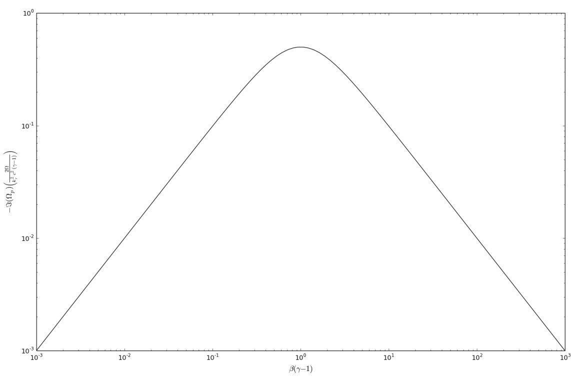

FIG. 1.—

5.1. *Lowest Order Mode with No Self Gravity*

The mode with zero nodes is perhaps the simplest to understand.

5.2. *Self Gravity*

To include the effects of the disk's self gravity, we include the potential due to the perturbed surface density,

$$\phi' = -G \int dr'\, r'\Sigma'(r') \int d\phi\, \frac{\cos\phi}{\sqrt{r^2 + r'^2 - 2rr'\cos\phi}} \tag{44}$$

The ϕ integral is simply the Laplace coefficient, $b_{1/2}^{(1)}$

$$b_{1/2}^{(1)}\left(\frac{r'}{r}\right) = \frac{2}{\pi r}\int_0^\pi d\phi\, \frac{\cos\phi}{\sqrt{1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\phi}} \tag{45}$$

So that the potential is,

$$\phi' = -\frac{G\pi}{r}\int dr'\, r'\Sigma'(r') b_{1/2}^{(1)}\left(\frac{r'}{r}\right) \tag{46}$$

There is also an "edge" contribution to the potential due to there being a non-negligable excess of material outside the outer boundary of the disk and a decificit of disk material inside the inner boundary of the disk. At the outer boundary of the disk we have, $\Sigma'(r)dr = \bar{\Sigma}(r_o)\xi_r(r_o) = \bar{\Sigma}(r_o)r_o e(r_o)$. Similarly at the inner boundary we have, $\Sigma'(r)dr' = -\bar{\Sigma}(r_i)\xi_r(r_i) = \bar{\Sigma}(r_i)r_i e(r_i)$. The "edge" potential is thus,

$$\phi_{\rm edge} = -\frac{G\pi\bar{\Sigma}(r_o)r_o^2 e(r_o)}{r} b_{1/2}^{(1)}\left(\frac{r_o}{r}\right) + \frac{G\pi\bar{\Sigma}(r_i)r_i^2 e(r_i)}{r} b_{1/2}^{(1)}\left(\frac{r_i}{r}\right) \tag{47}$$

The perturbed surface density is related to the eccentricity as $\phi' = -r\frac{d}{dr}\left(\bar{\Sigma}e\right)$. Plugging this in to (**??**) and integrating by parts gives,

$$\phi' = -\phi_{\rm edge} - \frac{G\pi}{r}\int dr'\, \bar{\Sigma}(r')e(r')\partial_{r'}\left(r'^2 b_{1/2}^{(1)}\left(\frac{r'}{r}\right)\right) \tag{48}$$

The "edge" contribution to the potential thus cancels when we take $\phi' \rightarrow \phi' + \phi_{\rm edge}$. The potential enters the eccentricity equation via the term,

$$-\frac{d}{dr}\left(r^2\phi'\right) \tag{49}$$

6. DISCUSSION

7. CONCLUSIONS

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