

UNSTABLE $m = 1$ ECCENTRIC MODES IN THERMALLY COOLING, SELF GRAVITATING DISKS

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ABSTRACT

1. INTRODUCTION

2. EQUATIONS OF MOTION

The equations of motion for a two dimensional, viscous disk are

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\Sigma} - \nabla \Phi + \frac{1}{\Sigma} \nabla \cdot (\nu \Sigma \mathbf{D}) \quad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \Sigma = -\Sigma \nabla \cdot \mathbf{v} \quad (2)$$

$$T(\partial_t + \mathbf{v} \cdot \nabla) s = -\frac{T - \bar{T}}{t_{\text{cool}}} \quad (3)$$

Where all quantities are assumed to be a vertical average of their three dimensional counterparts. Here, Σ is the surface density, s is the entropy of the fluid, T is the disk temperature, and \mathbf{D} is the viscous stress tensor. These equations describe a viscous fluid that is thermally heated and or cooled to some "background" temperature profile \bar{T} over a timescale t_{cool} . To close the system of equations we assume the fluid is ideal and perfect, so that $P = \mathcal{R} \Sigma T$ and $s \equiv \frac{\mathcal{R}}{\gamma-1} \ln(P \Sigma^{-\gamma})$; hereafter we set $\mathcal{R} = 1$ without loss of generality. Following ?, we can connect the two dimension adiabatic index γ to the true three dimensional adiabatic index Γ by assuming that the gas is vertically isothermal and gaussian distributed.

We consider a disk with a prescribed azimuthally averaged background profile given by $(\bar{v}_r, \bar{v}_\phi, \bar{\Sigma}, \bar{T}) = (0, r\Omega(r), \bar{\Sigma} \propto r^\mu, \bar{T} \propto r^\delta)$. We look at linear perturbations to this background state. Denoted linear quantities with a prime and background quantities with a bar, the linearized equations of motion become

$$(\partial_t + \Omega \partial_\phi) v'_r - 2\Omega v'_\phi = -\frac{1}{\Sigma} \partial_r P' + \frac{\Sigma'}{\bar{\Sigma}^2} \frac{d\bar{P}}{dr} - \partial_r \Phi' + \text{visc.} \quad (4)$$

$$(\partial_t + \Omega \partial_\phi) v'_\phi + \frac{\kappa^2}{2\Omega} v'_r = -\frac{1}{r\bar{\Sigma}} \partial_\phi P' - \frac{1}{r} \partial_\phi \Phi' + \text{visc.} \quad (5)$$

$$(\partial_t + \Omega \partial_\phi) \Sigma' + v'_r \frac{d\bar{\Sigma}}{dr} = -\bar{\Sigma} \left(\frac{1}{r} \partial_r (r v'_r) + \frac{1}{r} \partial_\phi (v'_\phi) \right) \quad (6)$$

$$(\partial_t + \Omega \partial_\phi) s' + v'_r \frac{d\bar{s}}{dr} = -\beta \Omega \frac{T'}{\bar{T}} \quad (7)$$

Where we've parametrized the cooling time as $t_{\text{cool}}^{-1} = \beta \Omega$ (?). We now Fourier Transform the linear variables in the azimuthal direction as well as in time and write $(v'_r, v'_\phi, \Sigma', P', s', T', \Phi') = \sum_m (u, v, \sigma, p, s, T, \phi) e^{im(\phi - \Omega_p t)}$. We further work under

the assumption that the different m modes only couple to the $m = 0$ mode (i.e the background) and not to other $m \neq 0$ modes. The linearized equations of motion are now

$$im(\Omega - \Omega_p)u - 2\Omega v = -\frac{1}{\bar{\Sigma}} \frac{dp}{dr} + \frac{\sigma}{\bar{\Sigma}^2} \frac{d\bar{P}}{dr} - \frac{d\phi}{dr} + \text{visc.} \quad (8)$$

$$im(\Omega - \Omega_p)v + \frac{\kappa^2}{2\Omega} u = -\frac{imp}{r\bar{\Sigma}} - \frac{im\phi}{r} + \text{visc.} \quad (9)$$

$$im(\Omega - \Omega_p) \frac{\sigma}{\bar{\Sigma}} + u \frac{d \ln \bar{\Sigma}}{dr} = -\left(\frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (10)$$

$$im \left[\left(1 - i \frac{\beta(\gamma-1)}{m} \right) \Omega - \Omega_p \right] \frac{T'}{\bar{T}} + u \frac{d \ln \bar{T}}{dr} = -(\gamma-1) \left(\frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (11)$$

Where we've replaced the entropy in favor of the temperature through the relations

$$(\gamma-1)s' = \frac{T'}{\bar{T}} - (\gamma-1) \frac{\sigma}{\bar{\Sigma}} \quad (12)$$

$$(\gamma-1) \frac{d\bar{s}}{dr} = \frac{d \ln \bar{T}}{dr} - (\gamma-1) \frac{d \ln \bar{\Sigma}}{dr} \quad (13)$$

3. $m = 1$ SLOW MODES

We specialize to $m = 1$ modes and to the low pattern speed limit $\Omega_p \ll \Omega$. We can simplify the 4 dimensional set of equations to just one equation describing the global structure and evolution of an eccentricity vector, $e(r)$ (???). The eccentricity is defined through the Lagrangian displacement ξ_r as

$$e \equiv \frac{\xi_r}{r} \quad (14)$$

Following ? we can write an evolution equation for the disk eccentricity by an appropriate linear combination of the two momenta equations and using

$$2\Omega r^3 (\Omega_p - \omega_p) e = -\frac{d}{dr} \left(r^2 \frac{p}{\bar{\Sigma}} + r^2 \phi \right) + \frac{r^2}{\bar{\Sigma}^2} \left[p \frac{d\bar{\Sigma}}{dr} + \sigma \frac{d\bar{P}}{dr} \right] \quad (15)$$

$$p = \sigma \bar{T} + \bar{\Sigma} T' \quad (16)$$

$$\frac{\sigma}{\bar{\Sigma}} = -\frac{d \ln \bar{\Sigma}}{d \ln r} e - r \frac{de}{dr} \quad (17)$$

$$(1 - i\tilde{\beta}) \frac{T'}{\bar{T}} = -\frac{d \ln \bar{T}}{d \ln r} e - (\gamma-1) r \frac{de}{dr} \quad (18)$$

Which follow from (??) & (??) to zeroth order in Ω_p and with $\Omega \approx \Omega_K$. We obtain the evolution equation

$$\Omega_p e = Ae + B \frac{de}{d \ln r} + C \frac{d^2 e}{d \ln r^2} \quad (19)$$

Where the coefficients are,

$$A = \omega_p + \mathcal{H} \left[2\mu + \mu' + \frac{1}{1 - i\tilde{\beta}} (\delta' + \delta(2 + \mu + \delta)) \right] \quad (20)$$

$$B = \mathcal{H} \left[2 + \mu + \frac{1}{1 - i\tilde{\beta}} ((\gamma\delta + (\gamma - 1)(2 + \mu))) \right] \quad (21)$$

$$C = \mathcal{H} \left[1 + \frac{(\gamma - 1)}{1 - i\tilde{\beta}} \right] \quad (22)$$

Where,

$$\tilde{\beta} \equiv \beta(\gamma - 1) \quad (23)$$

$$\mathcal{H} \equiv \frac{\bar{T}}{2\Omega r^2} \quad (24)$$

$$\mu' = \frac{d^2 \bar{\Sigma}}{d \ln r^2} \quad \delta' = \frac{d^2 \bar{T}}{d \ln r^2} \quad (25)$$

To add in the effects of viscosity we first use the alpha parametrization of viscosity and take $\nu = \alpha \frac{T}{\Omega}$ (?). The corrections due to shear (α_s) and bulk (α_b) viscosities are,

$$A_\nu = -\frac{i\alpha_s}{6} \mathcal{H} \left(\frac{65}{2} + \delta - 2\mu + 18\mu' \right) \quad (26)$$

$$B_\nu = -\frac{i\alpha_s}{3} \mathcal{H} \left(-\frac{19}{2} - 7\delta + 2\mu \right) + i\alpha_b \mathcal{H} \left(\frac{5}{2} + \delta + \mu \right) \quad (27)$$

$$C_\nu = -\frac{i2\alpha_s}{3} \mathcal{H} - i\alpha_b \mathcal{H} \quad (28)$$

We solve (??) as a generalized eigenvalue problem by discretizing the eccentricity on a logarithmic grid in radius.

$$\Omega_p \mathbf{Q} \mathbf{e} = \mathbf{M} \mathbf{e} \quad (29)$$

The matrices \mathbf{Q} , and \mathbf{M} are the boundary condition matrix and the coefficient matrix, respectively. We solve for the spectrum of normal eigenmodes, \mathbf{e}_n and their associated (possibly complex) eigenvalues $\Omega_{p,n}$.

3.1. Boundary Condition

We adopt a zero Lagrangian pressure boundary condition at both the inner and outer boundaries. Using the equations of motion it can be shown that this is equivalent to satisfying

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \tilde{\beta} \left[\frac{\tilde{\beta} - i\gamma}{\tilde{\beta}^2 + \gamma^2} \right] \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} \quad (30)$$

at each boundary in the slow mode approximation. In the isothermal ($\beta \rightarrow \infty$) and adiabatic ($\beta \rightarrow 0$) limits the boundary condition reduces to,

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \begin{cases} \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} & \text{Isothermal} \\ 0 & \text{Adiabatic} \end{cases} \quad (31)$$

4. METHODS

5. RESULTS

We studied the growth rates of unstable modes in the slow mode ($\Omega_p \ll \Omega_k$) approximation. The classical WKB dispersion relation for a self gravitating and isothermal disk (see e.g ?) is

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + c^2 k_r^2 = 0 \quad (32)$$

Where $\varpi \equiv \Omega - \Omega_p$. In the $m = 1$ slow mode limit this simplifies to (?)

$$2\Omega(\Omega_p - \omega_p) = 2\pi G \Sigma |k_r| - c^2 k_r^2 \quad (33)$$

Where $\omega_p = \Omega - \kappa$ is the orbital precession frequency of a fluid element. From the dispersion relation it is clear that pressure creates retrograde modes ($\Omega_p < 0$) and self gravity tends to create prograde modes ($\Omega_p > 0$). Additionally, global modes (i.e low k_r) also tend to create prograde modes (?).

To understand the implications of cooling on the disk, we write the the WKB dispersion relation for a disk which has consonant cooling β and constant background profiles. After some algebra, we obtain

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + k_r^2 c_{\text{cool}}^2 = 0 \quad (34)$$

$$c_{\text{cool}}^2 = \frac{m\varpi}{m\varpi - i\tilde{\beta}\Omega} c_{\text{adi}}^2 - \frac{i\tilde{\beta}\Omega}{m\varpi - i\tilde{\beta}\Omega} c_{\text{iso}}^2 \quad (35)$$

In the $m = 1$, slow mode limit we obtain

$$2\Omega(\Omega_p - \omega_p) = 2\pi G \Sigma |k_r| - k_r^2 c_{\text{cool}}^2 \quad (36)$$

$$c_{\text{cool}}^2 = \frac{1}{1 - i\tilde{\beta}} c_{\text{adi}}^2 + \frac{i\tilde{\beta}}{i\tilde{\beta} - 1} c_{\text{iso}}^2 \quad (37)$$

Where $c_{\text{adi}}^2 = \gamma c_{\text{iso}}^2$ is the sound speed in the adiabatic ($\beta = 0$) limit. Figure shows the dependence of the decay rate with β .

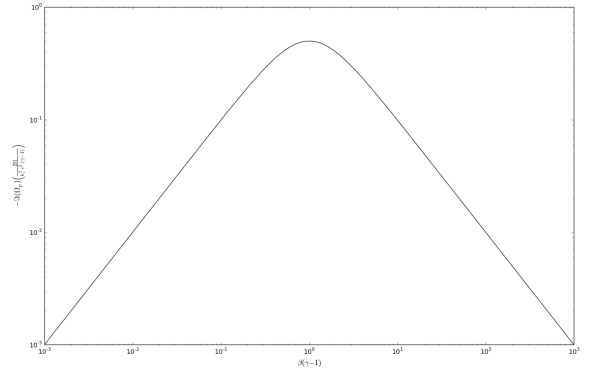


FIG. 1.—

5.1. Lowest Order Mode with No Self Gravity

The mode with zero nodes is perhaps the simplest to understand.

5.2. *Self Gravity*
 6. DISCUSSION

7. CONCLUSIONS

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