

# UNSTABLE $m = 1$ ECCENTRIC MODES IN THERMALLY COOLING, SELF GRAVITATING DISKS

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## ABSTRACT

### 1. INTRODUCTION

### 2. EQUATIONS OF MOTION

The equations of motion for a two dimensional, viscous disk are

$$(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla P}{\Sigma} - \nabla \Phi + \frac{1}{\Sigma} \nabla \cdot (\nu \Sigma \mathbf{D}) \quad (1)$$

$$(\partial_t + \mathbf{v} \cdot \nabla) \Sigma = -\Sigma \nabla \cdot \mathbf{v} \quad (2)$$

$$T(\partial_t + \mathbf{v} \cdot \nabla) s = -\frac{T - \bar{T}}{t_{\text{cool}}} \quad (3)$$

Where all quantities are assumed to be a vertical average of their three dimensional counterparts. Here,  $\Sigma$  is the surface density,  $s$  is the entropy of the fluid,  $T$  is the disk temperature, and  $\mathbf{D}$  is the viscous stress tensor. These equations describe a viscous fluid that is thermally heated and or cooled to some "background" temperature profile  $\bar{T}$  over a timescale  $t_{\text{cool}}$ . To close the system of equations we assume the fluid is ideal and perfect, so that  $P = \mathcal{R} \Sigma T$  and  $s \equiv \frac{\mathcal{R}}{\gamma-1} \ln(P \Sigma^{-\gamma})$ ; hereafter we set  $\mathcal{R} = 1$  without loss of generality. Following Ostriker, Shu, & Adams (2), we can connect the two dimension adiabatic index  $\gamma$  to the true three dimensional adiabatic index  $\Gamma$  by assuming that the gas is vertically isothermal and gaussian distributed.

We consider a disk with a prescribed azimuthally averaged background profile given by  $(\bar{v}_r, \bar{v}_\phi, \bar{\Sigma}, \bar{T}) = (0, r\Omega(r), \bar{\Sigma} \propto r^\mu, \bar{T} \propto r^\delta)$ . We look at linear perturbations to this background state. Denoted linear quantities with a prime and background quantities with a bar, the linearized equations of motion become

$$(\partial_t + \Omega \partial_\phi) v'_r - 2\Omega v'_\phi = -\frac{1}{\Sigma} \partial_r P' + \frac{\Sigma'}{\bar{\Sigma}^2} \frac{d\bar{P}}{dr} - \partial_r \Phi' + \text{visc.} \quad (4)$$

$$(\partial_t + \Omega \partial_\phi) v'_\phi + \frac{\kappa^2}{2\Omega} v'_r = -\frac{1}{r\bar{\Sigma}} \partial_\phi P' - \frac{1}{r} \partial_\phi \Phi' + \text{visc.} \quad (5)$$

$$(\partial_t + \Omega \partial_\phi) \Sigma' + v'_r \frac{d\bar{\Sigma}}{dr} = -\bar{\Sigma} \left( \frac{1}{r} \partial_r (r v'_r) + \frac{1}{r} \partial_\phi (v'_\phi) \right) \quad (6)$$

$$(\partial_t + \Omega \partial_\phi) s' + v'_r \frac{d\bar{s}}{dr} = -\beta \Omega \frac{T'}{\bar{T}} \quad (7)$$

Where we've parametrized the cooling time as  $t_{\text{cool}}^{-1} = \beta \Omega$  (Gammie 2001). We now Fourier Transform the linear variables in the azimuthal direction as well as in time and write  $(v'_r, v'_\phi, \Sigma', P', s', T', \Phi') =$

$\sum_m (u, v, \sigma, p, s, T, \phi) e^{im(\phi - \Omega_p t)}$ . We further work under the assumption that the different  $m$  modes only couple to the  $m = 0$  mode (i.e the background) and not to other  $m \neq 0$  modes. The linearized equations of motion are now

$$im(\Omega - \Omega_p)u - 2\Omega v = -\frac{1}{\bar{\Sigma}} \frac{dp}{dr} + \frac{\sigma}{\bar{\Sigma}^2} \frac{d\bar{P}}{dr} - \frac{d\phi}{dr} + \text{visc.} \quad (8)$$

$$im(\Omega - \Omega_p)v + \frac{\kappa^2}{2\Omega} u = -\frac{imp}{r\bar{\Sigma}} - \frac{im\phi}{r} + \text{visc.} \quad (9)$$

$$im(\Omega - \Omega_p) \frac{\sigma}{\bar{\Sigma}} + u \frac{d \ln \bar{\Sigma}}{dr} = -\left( \frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (10)$$

$$im \left[ \left( 1 - i \frac{\beta(\gamma-1)}{m} \right) \Omega - \Omega_p \right] \frac{T'}{\bar{T}} + u \frac{d \ln \bar{T}}{dr} = -(\gamma-1) \left( \frac{1}{r} \frac{d}{dr} (ru) + \frac{imv}{r} \right) \quad (11)$$

Where we've replaced the entropy in favor of the temperature through the relations

$$(\gamma-1)s' = \frac{T'}{\bar{T}} - (\gamma-1) \frac{\sigma}{\bar{\Sigma}} \quad (12)$$

$$(\gamma-1) \frac{d\bar{s}}{dr} = \frac{d \ln \bar{T}}{dr} - (\gamma-1) \frac{d \ln \bar{\Sigma}}{dr} \quad (13)$$

### 3. $m = 1$ SLOW MODES

We specialize to  $m = 1$  modes and to the low pattern speed limit  $\Omega_p \ll \Omega$ . We can simplify the 4 dimensional set of equations to just one equation describing the global structure and evolution of an eccentricity vector,  $e(r)$  (5; 4; 3). The eccentricity is defined through the Lagrangian displacement  $\xi_r$  as

$$e \equiv \frac{\xi_r}{r} \quad (14)$$

Following (3) we can write an evolution equation for the disk eccentricity by an appropriate linear combination of the two momenta equations and using

$$2\Omega r^3 (\Omega_p - \omega_p) e = -\frac{d}{dr} \left( r^2 \frac{p}{\bar{\Sigma}} + r^2 \phi \right) + \frac{r^2}{\bar{\Sigma}^2} \left[ p \frac{d\bar{\Sigma}}{dr} + \sigma \frac{d\bar{P}}{dr} \right] \quad (15)$$

$$p = \sigma \bar{T} + \bar{\Sigma} T' \quad (16)$$

$$\frac{\sigma}{\bar{\Sigma}} = -\frac{d \ln \bar{\Sigma}}{d \ln r} e - r \frac{de}{dr} \quad (17)$$

$$(1 - i\tilde{\beta}) \frac{T'}{\bar{T}} = -\frac{d \ln \bar{T}}{d \ln r} e - (\gamma-1) r \frac{de}{dr} \quad (18)$$

Which follow from (10) & (11) to zeroth order in  $\Omega_p$  and with  $\Omega \approx \Omega_K$ . We obtain the evolution equation

$$\Omega_p e = Ae + Br \frac{de}{dr} + Cr^2 \frac{d^2 e}{dr^2} \quad (19)$$

Where the coefficients are,

$$A = \omega_p + \mathcal{H} \left[ 2\mu + \mu' + \frac{1}{1 - i\tilde{\beta}} (\delta' + \delta(2 + \mu + \delta)) \right] \quad (20)$$

$$B = \mathcal{H} \left[ 3 + \mu + \frac{1}{1 - i\tilde{\beta}} ((\gamma\delta + (\gamma - 1)(3 + \mu))) \right] \quad (21)$$

$$C = \mathcal{H} \left[ 1 + \frac{(\gamma - 1)}{1 - i\tilde{\beta}} \right] \quad (22)$$

Where,

$$\tilde{\beta} \equiv \beta(\gamma - 1) \quad (23)$$

$$\mathcal{H} \equiv \frac{\bar{T}}{2\Omega r^2} \quad (24)$$

$$\mu' = \frac{d^2 \bar{\Sigma}}{d \ln r^2} \quad \delta' = \frac{d^2 \bar{T}}{d \ln r^2} \quad (25)$$

We solve (19) as a generalized eigenvalue problem by discretizing the eccentricity on a logarithmic grid in radius.

$$\Omega_p \mathbf{Q} \mathbf{e} = \mathbf{M} \mathbf{e} \quad (26)$$

The matrices  $\mathbf{Q}$ , and  $\mathbf{M}$  are the boundary condition matrix and the coefficient matrix, respectively. We solve for the spectrum of normal eigenmodes,  $\mathbf{e}_n$  and their associated (possibly complex) eigenvalues  $\Omega_{p,n}$ .

### 3.1. Boundary Condition

We adopt a zero Lagrangian pressure boundary condition at both the inner and outer boundaries. Using the equations of motion it can be shown that this is equivalent to satisfying

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \tilde{\beta} \left[ \frac{\tilde{\beta} - i\gamma}{\tilde{\beta}^2 + \gamma^2} \right] \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} \quad (27)$$

at each boundary in the slow mode approximation. In the isothermal ( $\beta \rightarrow \infty$ ) and adiabatic ( $\beta \rightarrow 0$ ) limits the boundary condition reduces to,

$$r \frac{de(r)}{dr} \Big|_{r_i, r_o} = \begin{cases} \frac{d \ln \bar{T}}{d \ln r} e(r) \Big|_{r_i, r_o} & \text{Isothermal} \\ 0 & \text{Adiabatic} \end{cases} \quad (28)$$

## 4. METHODS

## 5. RESULTS

We studied the growth rates of unstable modes in the slow mode ( $\Omega_p \ll \Omega_k$ ) approximation. The classical WKB dispersion relation for a self gravitating and isothermal disk (see e.g Binney & Tremaine (1)) is

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + c^2 k_r^2 = 0 \quad (29)$$

Where  $\varpi \equiv \Omega - \Omega_p$ . In the  $m = 1$  slow mode limit this simplifies to (3)

$$2\Omega(\Omega_p - \omega_p) = 2\pi G \Sigma |k_r| - c^2 k_r^2 \quad (30)$$

Where  $\omega_p = \Omega - \kappa$  is the orbital precession frequency of a fluid element. From the dispersion relation it is clear that pressure creates retrograde modes ( $\Omega_p < 0$ ) and self gravity tends to create prograde modes ( $\Omega_p > 0$ ). Additionally, global modes (i.e low  $k_r$ ) also tend to create prograde modes (3).

To understand the implications of cooling on the disk, we write the the WKB dispersion relation for a disk which has consonant cooling  $\beta$  and constant background profiles. After some algebra, we obtain

$$\kappa^2 - m^2 \varpi^2 - 2\pi G \Sigma |k_r| + k_r^2 c_{\text{cool}}^2 = 0 \quad (31)$$

$$c_{\text{cool}}^2 = \frac{m\varpi}{m\varpi - i\tilde{\beta}\Omega} c_{\text{adi}}^2 - \frac{i\tilde{\beta}\Omega}{m\varpi - i\tilde{\beta}\Omega} c_{\text{iso}}^2 \quad (32)$$

In the  $m = 1$ , slow mode limit we obtain

$$2\Omega(\Omega_p - \omega_p) = 2\pi G \Sigma |k_r| - k_r^2 c_{\text{cool}}^2 \quad (33)$$

$$c_{\text{cool}}^2 = \frac{1}{1 - i\tilde{\beta}} c_{\text{adi}}^2 + \frac{i\tilde{\beta}}{i\tilde{\beta} - 1} c_{\text{iso}}^2 \quad (34)$$

Where  $c_{\text{adi}}^2 = \gamma c_{\text{iso}}^2$  is the sound speed in the adiabatic ( $\beta = 0$ ) limit. Figure shows the dependence of the decay rate with  $\beta$ .

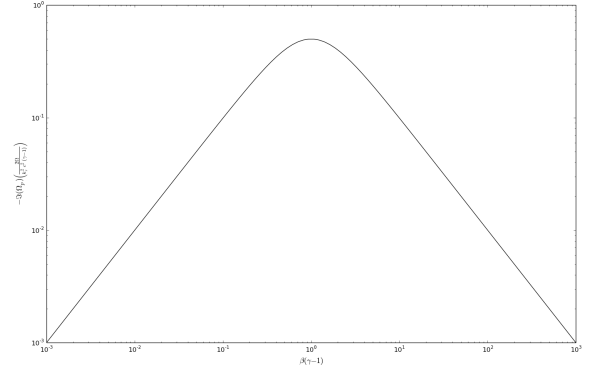


FIG. 1.—

### 5.1. Lowest Order Mode with No Self Gravity

The mode with zero nodes is perhaps the simplest to understand.

### 5.2. Self Gravity

## 6. DISCUSSION

## 7. CONCLUSIONS

## REFERENCES

Binney, J., Tremaine, S. *Galactic Dynamics*. Princeton University Press, 2002.

Gammie, C. F. (2001). *ApJ*, 553(1), 174-183.

- Ostriker, E. C., Shu, F. H., & Adams, F. C. (1992). ApJ, 399, 192-212.
- Papaloizou, J. C. B. (2002). Astronomy and Astrophysics, 388(2), 615-631.
- Goodchild, S., & Ogilvie, G. (2006). MNRAS, 368(3), 1123-1131.
- Ogilvie, G. I. (2001). MNRAS, 325(1), 231-248.