

In WSS the evolution equation is,

$$\partial_t(\langle\lambda\rangle r^2\langle\Omega\rangle) + \partial_r(-\dot{M}\langle r^2\Omega\rangle - \nu\langle\lambda\rangle r^2\partial_r\langle\Omega\rangle) = 2\pi r\Lambda_{ex} - 2\pi\partial_r(F_w) \quad (1)$$

where  $\lambda = 2\pi r\Sigma$  and  $\dot{M} = -2\pi r\langle\Sigma v_r\rangle$ . Setting  $\Lambda_d = \Lambda_{ex} - \partial_r F_w$  and  $\Omega = \Omega_K$ ,  $\partial_t\Omega = 0$ , and dropping the angle brackets gives,

$$\partial_t\lambda = \partial_r\dot{M} \quad (2)$$

$$\dot{M} = (\partial_r(r^2\Omega))^{-1} [\partial_r(-\nu\lambda r^2\partial_r\Omega) - 2\pi r\Lambda_d] \quad (3)$$

If we have a model for the wave deposition, then we can specify  $\Lambda_d$  in terms of  $\lambda$ .

We integrate (2) by integrating over a cell,

$$\frac{d}{dt} \int dr \lambda = \dot{M}_+ - \dot{M}_- \quad (4)$$

Where we can calculate the mass flux at the edges of the cell using (3) (using  $\frac{d\ln\Omega}{d\ln r} = -3/2$  and  $\frac{d\ln\nu}{d\ln r} = \gamma$ ),

$$\dot{M}_\pm = 3\nu_\pm \partial_r\lambda|_\pm + \frac{3\nu_\pm}{r_\pm} (\gamma - 1/2) \lambda_\pm - 2\sqrt{r_\pm} 2\pi r\Lambda_{d,\pm} \quad (5)$$

The gradients in  $\lambda$  are given by,

$$\partial_r\lambda|_+ = \frac{\lambda_{i+1} - \lambda_i}{r_{i+1} - r_i} \quad \partial_r\lambda|_- = \frac{\lambda_i - \lambda_{i-1}}{r_i - r_{i-1}} \quad (6)$$

And  $\lambda$  at each edge is,

$$\lambda_+ = \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) \lambda_i + \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \lambda_{i+1} \quad (7)$$

$$\lambda_- = \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) \lambda_{i-1} + \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \lambda_i \quad (8)$$

The mass flux (neglecting the planet) at each edge,

$$\begin{aligned} \dot{M}_+ &= \left( \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) - \frac{3\nu_+}{r_{i+1} - r_i} \right) \lambda_i \\ &+ \left( \frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \right) \lambda_{i+1} \end{aligned} \quad (9)$$

$$\begin{aligned} \dot{M}_- &= \left( \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) - \frac{3\nu_-}{r_i - r_{i-1}} \right) \lambda_{i-1} \\ &+ \left( \frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \right) \lambda_i \end{aligned} \quad (10)$$

Approximating  $\int dr \lambda = (r_+ - r_-)\lambda_i = \Delta r_i \lambda_i$  across a cell gives an update equation for  $\lambda$ ,

$$\Delta r_i \frac{d\lambda_i}{dt} = L_i \lambda_{i-1} + M_i \lambda_i + U_i \lambda_{i+1} \quad (11)$$

$$L_i = - \left( \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) - \frac{3\nu_-}{r_i - r_{i-1}} \right) \quad (12)$$

$$\begin{aligned} M_i &= \left( \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) - \frac{3\nu_+}{r_{i+1} - r_i} \right) \\ &- \left( \frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \right) \end{aligned} \quad (13)$$

$$U_i = \left( \frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \right) \quad (14)$$

Setting  $\mathbf{I}_{ij} = \Delta r_i \delta_{ij}$  and  $\mathbf{A}_{ij} = M_i \delta_{ij} + L_i \delta_{i,j-1} + U_i \delta_{i,j+1}$ ,

$$\mathbf{I} \frac{d}{dt} \lambda = \mathbf{A} \lambda + \mathbf{F} \quad (15)$$

Stepping in time with Crank-Nicholson gives,

$$\left( \mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right) \lambda^{n+1} = \left( \mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \lambda^n + \Delta t \mathbf{F} \quad (16)$$

For boundary conditions we can use a general mixed b.c

$$\partial_r \lambda|_B + \alpha \lambda_B = \beta \quad (17)$$

So that  $\dot{M}$  at either boundary is, (setting  $A = 3\nu$ ,  $B = 3\nu(\gamma - 1/2)/r$ )

$$\dot{M}_\pm = \beta A_\pm \left( \frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)} \right) + \left[ \frac{B_\pm}{A_\pm} - \alpha \left( \frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)} \right) \right] A_\pm \lambda_j \quad (18)$$

where  $\pm$  refers to the outer and inner boundaries and  $j = N - 1, 0$  denotes the last and first grid points inside the computational domain. Different choices of  $\alpha$  and  $\beta$  give some familiar boundary conditions:

1. Fixed  $\lambda$  :  $\alpha = 1/(r_g - r_B)$   $\beta = \lambda_g/(r_g - r_B)$
2. Fixed  $\partial_r \lambda$ :  $\alpha = 0$   $\beta = \lambda'$
3. Fixed  $\dot{M}$ :  $\alpha = B_\pm/A_\pm$   $\beta = \dot{M}_0/A_\pm$

## 0.1 Adding Planet

Adding the planet in modifies the mass flux,

$$\dot{M}_+ - \dot{M}_- = 2\sqrt{r_+}(-2\pi r_+ \Lambda_{d,+}) - 2\sqrt{r_-}(-2\pi r_- \Lambda_{d,-}) \quad (19)$$

Kanagawa's model has,

$$2\pi r \Lambda_d = \begin{cases} f(r) \int_{r_p}^{\infty} dr \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r - r_p < r_d + \frac{w_d}{2} \\ f(r) \int_0^{r_p} dr \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r_p - r < r_d + \frac{w_d}{2} \end{cases} \quad (20)$$

where,

$$\frac{dT_p}{dr} = \begin{cases} \pm 4f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left( \frac{r_p}{r-r_p} \right)^4 \Sigma & |r-a| > 1.3H(r_p) \\ 0 & \text{else} \end{cases} \quad (21)$$

and

$$f(r) = \frac{1}{w_d} \quad \text{or} \quad f(r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((r-(r_p \pm x_d))^2/(2\sigma^2))} \quad (22)$$

On the grid, the deposited torque matrix looks like,

$$(2\pi r \Lambda_d)_{ij} = f_i^L \sum_{j=0}^{r_p-\Delta} w_j^L \Sigma_j + f_i^R \sum_{j=r_p+\Delta}^N w_j^R \Sigma_j \quad (23)$$

$$w_j^L = -.4f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left( \frac{r_p}{r_j - r_p} \right)^4 \Delta r_j \quad (24)$$

$$w_j^R = .4f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left( \frac{r_p}{r_j - r_p} \right)^4 \Delta r_j \quad (25)$$

$$(2\pi r \Lambda_d) = (\mathbf{f}_L \mathbf{w}_L^T + \mathbf{f}_R \mathbf{w}_R^T) \Sigma \quad (26)$$

## 0.2 Zero torque solution

If we fix  $\Sigma$  at both ends of the domain, the disk will approach a constant  $\dot{M}$  profile. The proper solution should have

$$\dot{M} l_i = F_{\nu,i} \quad (27)$$

at the inner boundary, but in general there will be an extra offset,

$$\dot{M} l_i = F_{\nu,i} + \Delta F_i \quad (28)$$

Note that both of these satisfy the EOM

$$\dot{M} \frac{dl}{dr} \Big|_i = \frac{dF_\nu}{dr} \Big|_i \quad (29)$$