In WSS the evolution equation is,

$$\partial_t(\langle\lambda\rangle r^2\langle\Omega\rangle) + \partial_r(-\dot{M}\langle r^2\Omega\rangle - \nu\langle\lambda\rangle r^2\partial_r\langle\Omega\rangle) = 2\pi r\Lambda_{ex} - 2\pi\partial_r(F_w) \quad (1)$$

where $\lambda = 2\pi r\Sigma$ and $\dot{M} = -2\pi r\langle\Sigma v_r\rangle$. Setting $\Lambda_d = \Lambda_{ex} - \partial_r F_w$ and $\Omega = \Omega_K$, $\partial_t\Omega = 0$, and dropping the angle brackets gives,

$$\partial_t\lambda = \partial_r\dot{M} \quad (2)$$

$$\dot{M} = \left(\partial_r(r^2\Omega)\right)^{-1}\left[\partial_r(-\nu\lambda r^2\partial_r\Omega) - 2\pi r\Lambda_d\right] \quad (3)$$

If we have a model for the wave deposition, then we can specify Λ_d in terms of λ.

We integrate (2) by integrating over a cell,

$$\frac{d}{dt}\int dr\,\lambda = \dot{M}_+ - \dot{M}_- \quad (4)$$

Where we can calculate the mass flux at the edges of the cell using (3) (using $\frac{d\ln\Omega}{d\ln r} = -3/2$ and $\frac{d\ln\nu}{d\ln r} = \gamma$),

$$\dot{M}_\pm = 3\nu_\pm\,\partial_r\lambda|_\pm + \frac{3\nu_\pm}{r_\pm}(\gamma - 1/2)\,\lambda_\pm - 2\sqrt{r_\pm}2\pi r\Lambda_{d,\pm} \quad (5)$$

The gradients in λ are given by,

$$\partial_r\lambda|_+ = \frac{\lambda_{i+1} - \lambda_i}{r_{i+1} - r_i} \qquad \partial_r\lambda|_- = \frac{\lambda_i - \lambda_{i-1}}{r_i - r_{i-1}} \quad (6)$$

And λ at each edge is,

$$\lambda_+ = \left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right)\lambda_i + \left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\lambda_{i+1} \quad (7)$$

$$\lambda_- = \left(\frac{r_i - r_-}{r_i - r_{i-1}}\right)\lambda_{i-1} + \left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\lambda_i \quad (8)$$

The mass flux (neglecting the planet) at each edge,

$$\begin{aligned}\dot{M}_+ &= \left(\frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right) - \frac{3\nu_+}{r_{i+1} - r_i}\right)\lambda_i \\ &+ \left(\frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\right)\lambda_{i+1}\end{aligned} \quad (9)$$

$$\begin{aligned}\dot{M}_- &= \left(\frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_i - r_-}{r_i - r_{i-1}}\right) - \frac{3\nu_-}{r_i - r_{i-1}}\right)\lambda_{i-1} \\ &+ \left(\frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\right)\lambda_i\end{aligned} \quad (10)$$

Approximating $\int dr\, \lambda = (r_+ - r_-)\lambda_i = \Delta r_i \lambda_i$ across a cell gives an update equation for λ,

$$\Delta r_i \frac{d\lambda_i}{dt} = L_i \lambda_{i-1} + M_i \lambda_i + U_i \lambda_{i+1} \tag{11}$$

$$L_i = -\left(\frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_i - r_-}{r_i - r_{i-1}}\right) - \frac{3\nu_-}{r_i - r_{i-1}}\right) \tag{12}$$

$$\begin{aligned} M_i &= \left(\frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right) - \frac{3\nu_+}{r_{i+1} - r_i}\right) \\ &- \left(\frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\right) \end{aligned} \tag{13}$$

$$U_i = \left(\frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\right) \tag{14}$$

Setting $\mathbf{I}_{ij} = \Delta r_i \delta_{ij}$ and $\mathbf{A}_{ij} = M_i \delta_{ij} + L_i \delta_{i,j-1} + U_i \delta_{i,j+1}$,

$$\mathbf{I}\frac{d}{dt}\lambda = \mathbf{A}\lambda + \mathbf{F} \tag{15}$$

Stepping in time with Crank-Nicholson gives,

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\lambda^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\lambda^n + \Delta t \mathbf{F} \tag{16}$$

For boundary conditions we can use a general mixed b.c

$$\partial_r \lambda\big|_B + \alpha \lambda_B = \beta \tag{17}$$

So that $\dot{M}$ at either boundary is, (setting $A = 3\nu$, $B = 3\nu(\gamma - 1/2)/r$)

$$\dot{M}_\pm = \beta A_\pm \left(\frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)}\right) + \left[\frac{B_\pm}{A_\pm} - \alpha\left(\frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)}\right)\right] A_\pm \lambda_j \tag{18}$$

where $\pm$ refers to the outer and inner boundaries and $j = N-1, 0$ denotes the last and first grid points inside the computational domain. Different choices of α and β give some familiar boundary conditions:

1. Fixed λ : $\alpha = 1/(r_g - r_B)$ $\beta = \lambda_g/(r_g - r_B)$
2. Fixed $\partial_r \lambda$: $\alpha = 0$ $\beta = \lambda'$
3. Fixed $\dot{M}$: $\alpha = B_\pm/A_\pm$ $\beta = \dot{M}_0/A_\pm$

0.1 Adding Planet

Adding the planet in modifies the mass flux,

$$\dot{M}_+ - \dot{M}_- = 2\sqrt{r_+}\left(-2\pi r_+ \Lambda_{d,+}\right) - 2\sqrt{r_-}\left(-2\pi r_- \Lambda_{d,-}\right) \tag{19}$$

Kanagawa's model has,

$$2\pi r \Lambda_d = \begin{cases} f(r)\int_{r_p}^{\infty} dr\, \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r - r_p < r_d + \frac{w_d}{2} \\ f(r)\int_0^{r_p} dr\, \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r_p - r < r_d + \frac{w_d}{2} \end{cases} \tag{20}$$

where,

$$\frac{dT_p}{dr} = \begin{cases} \pm .4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r - r_p}\right)^4 \Sigma & |r - a| > 1.3H(r_p) \\ 0 & \text{else} \end{cases} \tag{21}$$

and

$$f(r) = \frac{1}{w_d} \qquad \text{or} \qquad f(r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((r-(r_p \pm x_d))^2/(2\sigma^2))} \tag{22}$$

On the grid, the deposited torque matrix looks like,

$$(2\pi r \Lambda_d)_{ij} = f_i^L \sum_{j=0}^{r_p - \Delta} w_j^L \Sigma_j + f_i^R \sum_{j=r_p+\Delta}^{N} w_j^R \Sigma_j \tag{23}$$

$$w_j^L = -.4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r_j - r_p}\right)^4 \Delta r_j \tag{24}$$

$$w_j^R = .4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r_j - r_p}\right)^4 \Delta r_j \tag{25}$$

$$(2\pi r \Lambda_d) = (\mathbf{f}_L \mathbf{w}_L^T + \mathbf{f}_R \mathbf{w}_R^T)\boldsymbol{\Sigma} \tag{26}$$