0.1 No planet

In WSS the evolution equation is,

$$\partial_t(\langle\lambda\rangle r^2\langle\Omega\rangle) + \partial_r(-\dot{M}\langle r^2\Omega\rangle - \nu\langle\lambda\rangle r^2\partial_r\langle\Omega\rangle) = 2\pi r\Lambda_{ex} - 2\pi\partial_r(F_w) \tag{1}$$

where $\lambda = 2\pi r\Sigma$ and $\dot{M} = -2\pi r\langle\Sigma v_r\rangle$. Setting $\Lambda_d = \Lambda_{ex} - \partial_r F_w$ and $\Omega = \Omega_K$, $\partial_t\Omega = 0$, and dropping the angle brackets gives,

$$\partial_t\lambda = \partial_r\dot{M} \tag{2}$$

$$\dot{M} = \left(\partial_r(r^2\Omega)\right)^{-1}\left[\partial_r(-\nu\lambda r^2\partial_r\Omega) - 2\pi r\Lambda_d\right] \tag{3}$$

If we have a model for the wave deposition, then we can specify Λ_d in terms of λ.

We integrate (2) by integrating over a cell,

$$\frac{d}{dt}\int dr\,\lambda = \dot{M}_+ - \dot{M}_- \tag{4}$$

Where we can calculate the mass flux at the edges of the cell using (3) (using $\frac{d\ln\Omega}{d\ln r} = -3/2$ and $\frac{d\ln\nu}{d\ln r} = \gamma$),

$$\dot{M}_\pm = 3\nu_\pm\,\partial_r\lambda|_\pm + \frac{3\nu_\pm}{r_\pm}(\gamma - 1/2)\,\lambda_\pm - 2\sqrt{r_\pm}2\pi r\Lambda_{d,\pm} \tag{5}$$

The gradients in λ are given by,

$$\partial_r\lambda|_+ = \frac{\lambda_{i+1} - \lambda_i}{r_{i+1} - r_i} \qquad \partial_r\lambda|_- = \frac{\lambda_i - \lambda_{i-1}}{r_i - r_{i-1}} \tag{6}$$

And λ at each edge is,

$$\lambda_+ = \left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right)\lambda_i + \left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\lambda_{i+1} \tag{7}$$

$$\lambda_- = \left(\frac{r_i - r_-}{r_i - r_{i-1}}\right)\lambda_{i-1} + \left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\lambda_i \tag{8}$$

The mass flux (neglecting the planet) at each edge,

$$\begin{aligned}\dot{M}_+ &= \left(\frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right) - \frac{3\nu_+}{r_{i+1} - r_i}\right)\lambda_i \\ &+ \left(\frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\right)\lambda_{i+1}\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{M}_- &= \left(\frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_i - r_-}{r_i - r_{i-1}}\right) - \frac{3\nu_-}{r_i - r_{i-1}}\right)\lambda_{i-1} \\ &+ \left(\frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\right)\lambda_i\end{aligned} \tag{10}$$

Approximating $\int dr\,\lambda = (r_+ - r_-)\lambda_i = \Delta r_i \lambda_i$ across a cell gives an update equation for λ,

$$\Delta r_i \frac{d\lambda_i}{dt} = L_i\lambda_{i-1} + M_i\lambda_i + U_i\lambda_{i+1} \tag{11}$$

$$L_i = -\left(\frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_i - r_-}{r_i - r_{i-1}}\right) - \frac{3\nu_-}{r_i - r_{i-1}}\right) \tag{12}$$

$$\begin{aligned} M_i \quad &= \quad \left(\frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right) - \frac{3\nu_+}{r_{i+1} - r_i}\right) \\ &- \quad \left(\frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-}(\gamma - 1/2)\left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\right) \end{aligned} \tag{13}$$

$$U_i = \left(\frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+}(\gamma - 1/2)\left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\right) \tag{14}$$

Setting $\mathbf{I}_{ij} = \Delta r_i \delta_{ij}$ and $\mathbf{A}_{ij} = M_i\delta_{ij} + L_i\delta_{i,j-1} + U_i\delta_{i,j+1}$,

$$\mathbf{I}\frac{d}{dt}\lambda = \mathbf{A}\lambda + \mathbf{F} \tag{15}$$

Stepping in time with Crank-Nicholson gives,

$$\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)\lambda^{n+1} = \left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)\lambda^n + \Delta t\mathbf{F} \tag{16}$$

For boundary conditions we can use a general mixed b.c

$$\partial_r \lambda|_B + \alpha\lambda_B = \beta \tag{17}$$

So that $\dot{M}$ at either boundary is, (setting $A = 3\nu$, $B = 3\nu(\gamma - 1/2)/r$)

$$\dot{M}_\pm = \beta A_\pm \left(\frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)}\right) + \left[\frac{B_\pm}{A_\pm} - \alpha\left(\frac{1 - \frac{B_\pm}{A_\pm}(r_j - r_\pm)}{1 - \alpha(r_j - r_\pm)}\right)\right] A_\pm \lambda_j \tag{18}$$

where $\pm$ refers to the outer and inner boundaries and $j = N-1, 0$ denotes the last and first grid points inside the computational domain. Different choices of α and β give some familiar boundary conditions:

1. Fixed λ : $\alpha = 1/(r_g - r_B)$ $\beta = \lambda_g/(r_g - r_B)$
2. Fixed $\partial_r\lambda$: $\alpha = 0$ $\beta = \lambda'$
3. Fixed $\dot{M}$: $\alpha = B_\pm/A_\pm$ $\beta = \dot{M}_0/A_\pm$

0.2 Adding Planet

Adding the planet in modifies the mass flux,

$$\dot{M}_+ - \dot{M}_- = 2\sqrt{r_+}\left(-2\pi r_+ \Lambda_{d,+}\right) - 2\sqrt{r_-}\left(-2\pi r_- \Lambda_{d,-}\right) \tag{19}$$

Kanagawa's model has,

$$2\pi r \Lambda_d = \begin{cases} f(r)\int_{r_p}^{\infty} dr\, \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r - r_p < r_d + \frac{w_d}{2} \\ f(r)\int_{0}^{r_p} dr\, \frac{dT_p}{dr} & r_d - \frac{w_d}{2} < r_p - r < r_d + \frac{w_d}{2} \end{cases} \tag{20}$$

where,

$$\frac{dT_p}{dr} = \begin{cases} \pm .4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r - r_p}\right)^4 \Sigma & |r - a| > 1.3H(r_p) \\ 0 & \text{else} \end{cases} \tag{21}$$

and

$$f(r) = \frac{1}{w_d} \qquad \text{or} \qquad f(r) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-((r-(r_p \pm x_d))^2/(2\sigma^2))} \tag{22}$$

On the grid, the deposited torque matrix looks like,

$$(2\pi r \Lambda_d)_{ij} = f_i^L \sum_{j=0}^{r_p - \Delta} w_j^L \Sigma_j + f_i^R \sum_{j=r_p+\Delta}^{N} w_j^R \Sigma_j \tag{23}$$

$$w_j^L = -.4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r_j - r_p}\right)^4 \Delta r_j \tag{24}$$

$$w_j^R = .4 f_{NL} 2\pi r_p^3 \Omega_p^2 q^2 \left(\frac{r_p}{r_j - r_p}\right)^4 \Delta r_j \tag{25}$$

$$(2\pi r \Lambda_d) = (\mathbf{f}_L \mathbf{w}_L^T + \mathbf{f}_R \mathbf{w}_R^T)\boldsymbol{\Sigma} \tag{26}$$

0.3 Density dependent deposition function

To prevent the density from going below zero we can choose our deposition function to be proportional to the density. Without worrying about normalizations for now this looks like,

$$2\pi r \Lambda_d = T_\pm f(r) \frac{\lambda(r)}{\lambda_0} \tag{27}$$

Before the time-step we calculate the total excitation torque, $T_{\pm} = \pm \int_a^{\infty} dr \, \frac{dT_p}{dr}$. The change in cell mass is then,

$$
\begin{aligned}
\dot{M}_+ - \dot{M}_- &= - \quad T\left(2\sqrt{r_+} f(r_+) \frac{\lambda_+}{\lambda_0} - 2\sqrt{r_-} f(r_-) \frac{\lambda_-}{\lambda_0}\right) && (28) \\
&= - \quad T\left[\frac{2\sqrt{r_-} f_-}{\lambda_0} \left(\frac{r_i - r_-}{r_i - r_{i-1}}\right)\right] \lambda_{i-1} && (29) \\
&\quad - \quad T\left[\frac{2\sqrt{r_+} f_+}{\lambda_0} \left(\frac{r_{i+1} - r_+}{r_{i+1} - r_i}\right) - \frac{2\sqrt{r_-} f_-}{\lambda_0} \left(\frac{r_- - r_{i-1}}{r_i - r_{i-1}}\right)\right] \lambda_i && (30) \\
&\quad - \quad T\left[\frac{2\sqrt{r_+} f_+}{\lambda_0} \left(\frac{r_+ - r_i}{r_{i+1} - r_i}\right)\right] \lambda_{i+1} && (31)
\end{aligned}
$$

0.4 Zero torque solution

Inside the inner edge of the computational domain (but still away from the physical inner edge of the disk) we expect the surface density to follow the power law solution, $\dot{M} = 3\pi\nu\Sigma$. This assumes that there is no net torque exerted on the disk in this region. If there was a torque in this region it would offset Σ_i at the inner edge of the domain from $\dot{M}/(3\pi\nu)$, i.e

$$\dot{M}l = 3\pi\nu\Sigma l + \int_{\text{inner}} \Lambda dr \tag{32}$$

If $3\pi\nu_i\Sigma_i \neq \dot{M}$ then there must be some torque injection inside the computational domain. The **zero-torque** solution is the one in which the inner boundary surface density is consistent with both the steady-state $\dot{M}$ and the inner disk power law solution $\dot{M} = 3\pi\nu\Sigma(r)$.

Simulation's which fix Σ to the wrong values at the boundaries are then implicitly injecting torque in the disk.