

In WSS the evolution equation is,

$$\partial_t(\langle\lambda\rangle r^2\langle\Omega\rangle) + \partial_r(-\dot{M}\langle r^2\Omega\rangle - \nu\langle\lambda\rangle r^2\partial_r\langle\Omega\rangle) = 2\pi r\Lambda_{ex} - 2\pi\partial_r(F_w) \quad (1)$$

where  $\lambda = 2\pi r\Sigma$  and  $\dot{M} = -2\pi r\langle\Sigma v_r\rangle$ . Setting  $\Lambda_d = \Lambda_{ex} - \partial_r F_w$  and  $\Omega = \Omega_K$ ,  $\partial_t\Omega = 0$ , and dropping the angle brackets gives,

$$\partial_t\lambda = \partial_r\dot{M} \quad (2)$$

$$\dot{M} = (\partial_r(r^2\Omega))^{-1} [\partial_r(-\nu\lambda r^2\partial_r\Omega) - 2\pi r\Lambda_d] \quad (3)$$

If we have a model for the wave deposition, then we can specify  $\Lambda_d$  in terms of  $\lambda$ .

We integrate (2) by integrating over a cell,

$$\frac{d}{dt} \int dr \lambda = \dot{M}_+ - \dot{M}_- \quad (4)$$

Where we can calculate the mass flux at the edges of the cell using (3) (using  $\frac{d\ln\Omega}{d\ln r} = -3/2$  and  $\frac{d\ln\nu}{d\ln r} = \gamma$ ),

$$\dot{M}_\pm = 3\nu_\pm \partial_r\lambda|_\pm + \frac{3\nu_\pm}{r_\pm} (\gamma - 1/2) \lambda_\pm - 2\sqrt{r_\pm} 2\pi r\Lambda_{d,\pm} \quad (5)$$

The gradients in  $\lambda$  are given by,

$$\partial_r\lambda|_+ = \frac{\lambda_{i+1} - \lambda_i}{r_{i+1} - r_i} \quad \partial_r\lambda|_- = \frac{\lambda_i - \lambda_{i-1}}{r_i - r_{i-1}} \quad (6)$$

And  $\lambda$  at each edge is,

$$\lambda_+ = \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) \lambda_i + \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \lambda_{i+1} \quad (7)$$

$$\lambda_- = \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) \lambda_{i-1} + \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \lambda_i \quad (8)$$

The mass flux (neglecting the planet) at each edge,

$$\begin{aligned} \dot{M}_+ &= \left( \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) - \frac{3\nu_+}{r_{i+1} - r_i} \right) \lambda_i \\ &+ \left( \frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \right) \lambda_{i+1} \end{aligned} \quad (9)$$

$$\begin{aligned} \dot{M}_- &= \left( \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) - \frac{3\nu_-}{r_i - r_{i-1}} \right) \lambda_{i-1} \\ &+ \left( \frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \right) \lambda_i \end{aligned} \quad (10)$$

Approximating  $\int dr \lambda = (r_+ - r_-)\lambda_i = \Delta r_i \lambda_i$  across a cell gives an update equation for  $\lambda$ ,

$$\Delta r_i \frac{d\lambda_i}{dt} = L_i \lambda_{i-1} + M_i \lambda_i + U_i \lambda_{i+1} \quad (11)$$

$$L_i = - \left( \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_i - r_-}{r_i - r_{i-1}} \right) - \frac{3\nu_-}{r_i - r_{i-1}} \right) \quad (12)$$

$$\begin{aligned} M_i &= \left( \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_{i+1} - r_+}{r_{i+1} - r_i} \right) - \frac{3\nu_+}{r_{i+1} - r_i} \right) \\ &- \left( \frac{3\nu_-}{r_i - r_{i-1}} + \frac{3\nu_-}{r_-} (\gamma - 1/2) \left( \frac{r_- - r_{i-1}}{r_i - r_{i-1}} \right) \right) \end{aligned} \quad (13)$$

$$U_i = \left( \frac{3\nu_+}{r_{i+1} - r_i} + \frac{3\nu_+}{r_+} (\gamma - 1/2) \left( \frac{r_+ - r_i}{r_{i+1} - r_i} \right) \right) \quad (14)$$

Setting  $\mathbf{I}_{ij} = \Delta r_i \delta_{ij}$  and  $\mathbf{A}_{ij} = M_i \delta_{ij} + L_i \delta_{i,j-1} + U_i \delta_{i,j+1}$ ,

$$\mathbf{I} \frac{d}{dt} \lambda = \mathbf{A} \lambda + \mathbf{F} \quad (15)$$

Stepping in time with Crank-Nicholson gives,

$$\left( \mathbf{I} - \frac{\Delta t}{2} \mathbf{A} \right) \lambda^{n+1} = \left( \mathbf{I} + \frac{\Delta t}{2} \mathbf{A} \right) \lambda^n + \Delta t \mathbf{F} \quad (16)$$

For boundary conditions we can use a general mixed b.c

$$\partial_r \lambda|_B + \alpha \lambda_B = \beta \quad (17)$$

So that  $\dot{M}$  at either boundary is, (setting  $A = 3\nu$ ,  $B = 3\nu(\gamma - 1/2)/r$ )

$$\dot{M}_{\pm} = \beta A_{\pm} \left( \frac{1 - \frac{B_{\pm}}{A_{\pm}}(r_j - r_{\pm})}{1 - \alpha(r_j - r_{\pm})} \right) + \left[ \frac{B_{\pm}}{A_{\pm}} - \alpha \left( \frac{1 - \frac{B_{\pm}}{A_{\pm}}(r_j - r_{\pm})}{1 - \alpha(r_j - r_{\pm})} \right) \right] A_{\pm} \lambda_j \quad (18)$$

where  $\pm$  refers to the outer and inner boundaries and  $j = N - 1, 0$  denotes the last and first grid points inside the computational domain. Different choices of  $\alpha$  and  $\beta$  give some familiar boundary conditions:

1. Fixed  $\lambda$  :  $\alpha = 1/(r_g - r_B)$   $\beta = \lambda_g/(r_g - r_B)$
2. Fixed  $\partial_r \lambda$ :  $\alpha = 0$   $\beta = \lambda'$
3. Fixed  $\dot{M}$ :  $\alpha = B_{\pm}/A_{\pm}$   $\beta = \dot{M}_0/A_{\pm}$