Solutions of Higher Spin Gravity and Connections to Integrable Systems

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Abstract

We first examine integrable N-body systems of the Calogero and Calogero-Sutherland type, focusing on the equilibrium points of the former and the energy levels of the latter. Next, gauge solutions of (2+1) dimensional gravity are explored, and are then generalized to include higher spin fields. We then analyze the smooth conical solutions of such systems, with a focus on the energy of these solutions. In conclusion, we note a correlation between the energy of the smooth conical solutions and the energy levels in the Calogero-Sutherland Model.

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Introduction

0.1 General Relativity

Einstein's equations are a set of equations that relate the curvature of space-time with the movement of particles due to gravity. To understand what "curvature" means, let us think about two pen-pals, named Nick and Susie. Nick lives at the North Pole, and Susie lives on the South Pole. One day, Nick decides he wants to visit Susie, but to plan properly, he needs to know how far apart they live. Nick finds out that the diameter of the Earth is 7,918 miles, so this is the distance between his house and Susie's. However, when he travels to meet Susie, he finds that the trip takes much longer than he had thought. In fact, after checking his odometer at the end of his trip, he finds that he has traveled 12,438 miles.

Nick was wrong in his estimation because he forgot to take into account the curvature of the Earth. Nick needs to travel on the surface of the planet, which changes the effective distance between points on the globe. To relate this idea of curvature with the curvature in General Relativity, we need to first recall from Special Relativity, Einstein's earlier theory, that space and time are related very closely, and together comprise what physicists call "space-time." So now that we have a time coordinate to consider, we no longer care about the distance between to physical places, like the North and South Pole, but instead we care about the distance between events.

An event is something that has spatial and temporal coordinates. So for instance, in General Relativity, we might be concerned with the "distance" between Nick's birth and Susie's wedding day. Now curvature is something that changes the distance between these two events. If you have trouble imagining this, that is perfectly fine; this is a very abstract and truly strange concept that physicists have been grappling with for the better part of the past century. According to Einstein's Equations, the position, energy, and momentum of particles change the way that the universe is curved, which affects the way things move. This effect is the force that we call gravity.

Einstein's equations read as:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu} \Lambda = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric, R is the Ricci scalar, Λ is the cosmological constant, G is Newton's gravitational constant, c is the speed of light, and $T_{\mu\nu}$ is the stress-energy tensor. The equations are organized such that on the left-hand side are the elements dealing with the curvature of space-time, and on the right hand side (most notably the stress-energy tensor $T_{\mu\nu}$) are the elements dealing with matter.

In this work we will mainly be dealing with vacuum solutions, in which case the right-hand side of Einstein's equations are zero. Additionally, we will be considering the the case where $\Lambda < 0$, which is known as Anti-de Sitter (AdS) space-time. This negative cosmological constant implies that the energy density of empty space is negative[1]. Although negative vacuum energy has not been observed, AdS space-time has a nice connection with String Theory through what is called the AdS/CFT correspondence.

0.2 Gauges and Gravity

Gauge theory is a concept that appears in many fields of physics. The basic definition of a gauge theory is a system in which the Lagrangian of the system, and therefore the equations of motion, are invariant under a local coordinate transformation. In other words, in a gauge theory we can take the coordinates of our system and apply some transformation:

$$x_i \to x_i + \xi_i \tag{2}$$

and as long as the ξ_i have some property specific to the theory. For an intuitive example of this, we use the basic physics problem of Newtonian gravity.

Consider a man on the top of a 100 m tall building, who drops a ball which lands on a ledge 20 m below. Using the pedagogical methods of solving this problem, we calculate the ball's potential at the beginning and compare it to the end using the equation

$$\phi = -\frac{Gm}{R} \tag{3}$$

where G is again Newton's constant, m is the mass of the Earth, and R is distance from the ball to the center of the Earth. For problems such as this, we often use the approximation

$$\phi \approx gh \tag{4}$$

where g is the acceleration due to gravity on Earth ($g \approx 9.8$ m/s) and h is the "height" of the ball.

This notion of "height" is one that is used in many introductory level classes, but it is a somewhat confusing notion. For instance, we could say that because the building is 100 m tall, the gravitational potential of the ball at the beginning of the experiment is $\phi = g \times (100\text{ m}) \approx 9800$ J. However, we could have just as easily said that because the ball only falls 20 m, the "height" of the ball is 20 m, giving us the starting potential $\phi = g \times (20\text{ m}) \approx 196$ J. We could also take $h = R$, which would lead to a gravitational potential of $\phi = g \times (1.2742 \times 10^7\text{ m}) \approx 1.2487 \times 10^8$ J.

This freedom in defining our "height" is in fact a gauge freedom. We can change our coordinates without changing the equations of motion of the system, as the force on the ball is given by

$$F = -m\nabla\phi \approx mg \tag{5}$$

regardless of how the height is defined.

[1] Although negative energy and negative mass are foreign and seemingly impossible concepts, they are not forbidden in general relativity. The mathematics behind GR remains consistent even when negative mass and energy are introduced.

So we can say that there is a gauge for gravity. However, it turns out that this gauge only works in what is known as the *weak limit* of gravity. The weak limit of gravity is the form of gravity that we are familiar with in our day-to-day lives, where nothing locally strongly distorts the fabric of space-time. Mathematically, this means that we can take the metric of space-time to be

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{6}$$

where $\eta_{\mu\nu}$ is the Minkowski metric (which is simply $\text{diag}(-1, 1, 1, 1)$) and $h_{\mu\nu}$ is a small perturbation. Mapping this perturbation to the classical theory of gravity, we define

$$h_{\mu\nu} = (-2\phi)1_{4\times4} \tag{7}$$

where ϕ is again the gravitational potential and $1_{4\times4}$ is the identity.

Although there exists a simple gauge formulation for gravity in the weak field limit (as well as a few more special cases), the gauge formulation of the general case is very complicated. However, when we eliminate one spatial dimension and consider gravity in 2 spatial and 1 temporal dimension, denoted as (2+1), then one can use the mathematical techniques of Chern-Simons theory to find a relatively simple gauge solution to gravity.

0.3 Chern-Simons Theory

Chern-Simons Theory is a mathematical theory developed by Edward Witten. It is what is called a *topological quantum field theory* which means that it defines a field which has local excitations that are physically thought of as particles, and also contains physical objects that do not change with a metric, which are called topological invariants.

A Chern-Simons Theory must be defined on a three-dimensional manifold, which explains why we can use it in (2+1) dimensions but not the standard (3+1). There is a *connection one-form* $\mathcal{A}$ and a *field curvature two-form* $\mathcal{F}$ which are related by

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \tag{8}$$

We can apply our gauge transformations to the connection:

$$A_\mu \to U^{-1}(A_\mu + \partial_\mu)U \tag{9}$$

where the connection is written in tensorial form (A_μ) and the gauge is transformed by some element of the symmetry group U. This yields a change in the field curvature:

$$F_{\mu\nu} \to U^{-1}(F_{\mu\nu})U \tag{10}$$

As one might expect, the action of the Chern-Simon theory must be gauge invariant:

$$S = \frac{k}{4\pi}\int_M \text{Tr}\left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3}\mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A}\right) \tag{11}$$

When one takes this action to find the equations of motion by imposing

$$\frac{\delta S}{\delta \mathcal{A}} = 0 \tag{12}$$

This turns out to be equivalent to setting $\mathcal{F} = 0$ (which makes the field curvature trivially gauge invariant). The $\mathcal{A}$ for which this is true are referred to as the *flat connections.*

In [1], Witten uses a Chern-Simons theory to find a gauge solution of gravity in (2+1) dimensions. He begins with the case of flat Minkowski space, in which the cosmological constant is zero. In this case, the important symmetries are the Poincaré transformations which consist of translations and Lorentz rotations. This corresponds with the symmetry group ISO(2,1), so the connection $\mathcal{A}$ is constructed such that it spans the space of the generators of this group. A cosmological constant can then be introduced, which changes the symmetry group to SO(3,1) if the constant is positive (de Sitter space) or to SO(2,2) if the constant is negative (anti-de Sitter space).

0.4 Solvable N-particle systems

In general, interactions between N particles are not exactly solvable. Situations such as this are not uncommon in physics. Even fundamental and seemingly simple problems, such as the energy levels of Helium, cannot be found exactly; they are calculated through a series of approximations in a method known as perturbation theory. So the fact that some systems are not exactly solvable should not be concerning, but should instead emphasize the importance of those systems in which an exact solution exists. As [2] write:

> The interest for models of n interacting particles with known exact solutions, is associated with the fact that, while both classical and quantum problems with three and more particles with a realistic interaction acting between them (Coulomb or nuclear) do not allow one to obtain any exact solution, it can be expected that some quantitative properties of these models can be expected to be maintained in the real case. Besides these models may be useful to estimate the accuracy of different approximate methods.

Chapter 1

Properties of Integrable N-Body Systems

We will begin by considering 2 formulations of Calogero N-body systems, one with the Hamiltonian

$$H(p,x) = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + x_j^2) + \sum_{j>k=1}^{N}(x_j - x_k)^{-2} \tag{1.1}$$

and a second with the Hamiltonian

$$H(p,x) = \frac{1}{2}\sum_{j=1}^{N}(p_j^2 + x_j^2) + \sum_{j>k=1}^{N}\log|x_j - x_k| \tag{1.2}$$

We claim that the equilibrium positions for both of these systems are given by the zeros of the Hermite polynomial of degree N, in other words the $\bar{x}_j$ that satisfy

$$H_N(\bar{x}_j) = 0 \tag{1.3}$$

1.1 General Formulation for Motion of Poles

The Hermite Differential Equation is a second order ordinary differential equation of the form

$$y'' - 2xy' + 2\lambda y = 0 \tag{1.4}$$

For integer values of λ, this equation has the property that $y = H_\lambda(x)$ is a solution.

To prove this, we follow the steps outlined in [3]. We begin with a general N-degree polynomial with zeros at the time-dependent $x_j(t)$:

$$\psi(x,t) = \prod_{j=1}^{N}(x - x_j(t)) \tag{1.5}$$

By taking several derivatives of ψ, we find that the linear partial differential equation

$$
\begin{aligned}
(A_0 + A_1 x + A_3 x^2 + A_3 x^3)\psi_{xx} + (B_0 + B_1 x - 2(N-1)A_3 x^2)\psi_x + \\
C\psi_{tt} + (E - (N-1)D_2)\psi_t + (D_0 + D_1 x + D_2 x^2)\psi_{xt} \\
-(N(N-1)(A_2 - A_3 x) + NB_1)\psi = 0
\end{aligned} \tag{1.6}
$$

implies that

$$
\begin{aligned}
C\ddot{x}_j + E\dot{x}_j = B_0 + B_1 x_j + \sum_{k=1}^{N}{}' (x_j - x_k)^{-1}[2(A_0 + A_1 x_j + A_2 x_j^2 + A_3 x_j x_k) \\
+2C\dot{x}_j\dot{x}_k - (\dot{x}_j + \dot{x}_k)(D_0 + D_1 x_j) - D_2 x_j(\dot{x}_j x_k + x_j \dot{x}_k)]
\end{aligned} \tag{1.7}
$$

Where the A's, B's, C's D's and E's are all arbitrary constants. The prime ($'$) indicates that the sum does not include $j = k$. Although this does not look especially meaningful, consider the implications of these equations. In (1.6), we have a statement regarding the polynomial ψ and its derivatives, whereas in (1.7) we have a statement about the zeros of ψ.

Let us consider the relatively simple case in which all of the coefficients are zero, except for $A_0 = \frac{1}{2}$, $B_1 = -1$, and $E = i$. Let us first consider what this does to (1.7). Plugging in these coefficients, we get:

$$
i\dot{x}_j = -x_j + \sum_{k=1}^{N}{}' (x_j - x_k)^{-1} \tag{1.8}
$$

Differentiating with respect to time, this yields

$$
i\ddot{x}_j = -\dot{x}_j + \sum_{k=1}^{N}{}' - (\dot{x}_j - \dot{x}_k)(x_j - x_k)^{-2} \tag{1.9}
$$

Plugging (1.8) into the equation, we can eliminate the first time derivatives:

$$
i\ddot{x}_j = -ix_j + \sum_{k=1}^{N}{}' i(x_j - x_k)^{-1} + \sum_{k=1}^{N}{}' \frac{ix_k - \sum_{m=1}^{N}{}' i(x_k - x_m)^{-1} - ix_j + \sum_{m=1}^{N}{}' i(x_j - x_m)^{-1}}{(x_j - x_k)^2} \tag{1.10}
$$

which simplifies to

$$
\ddot{x}_j = -x_j + 2\sum_{k=1}^{N}{}' (x_j - x_k)^{-3} \tag{1.11}
$$

We now turn to (1.6). Plugging in the coefficients as before, we get the partial differential equation

$$
\frac{1}{2}\psi_{xx} - x\psi_x + i\psi_t + N\psi = 0 \tag{1.12}
$$

As in [3], we use an ansatz to solve this equation. The proposed function is

$$\psi(x,t) = 2^{-N} \sum_{m=0}^{N} c_m(t) H_{N-m}(x) \tag{1.13}$$

where $H_m(x)$ is the Hermite polynomial of degree m. Plugging this in, we get

$$2^{-N-1} \sum_{m=0}^{N} c_m H''_{N-m} - x 2^{-N} \sum_{m=0}^{N} c_m H'_{N-m} + i 2^{-N} \sum_{m=0}^{N} \dot{c}_m H_{N-m} + N 2^{-N} \sum_{m=0}^{N} c_m H_{N-m} = 0$$

$$\sum_{m=0}^{N} (H''_{N-m} - 2x H'_{N-m} + 2i \frac{\dot{c}_m}{c_m} H_{N-m} + 2N H_{N-m}) = 0$$

$$\sum_{m=0}^{N} (H''_{N-m} - 2x H'_{N-m} + 2(i \frac{\dot{c}_m}{c_m} + N) H_{N-m}) = 0 \tag{1.14}$$

By comparing this to (1.4), we can see that this equation is satisfied if, for all m,

$$i \frac{\dot{c}_m}{c_m} + N = N - m \tag{1.15}$$

which simplifies to

$$\dot{c}_m(t) = i m c_m(t) \tag{1.16}$$

which implies

$$c_m(t) = c_m(0) e^{imt} \tag{1.17}$$

Thus, we have found that the zeros of the function

$$\psi(x,t) = 2^{-N} \sum_{m=0}^{N} c_m(0) e^{imt} H_{N-m}(x) \tag{1.18}$$

obey (1.11). It is important to note here a correspondence between the degrees of freedom on either side of this relation. There are N $x_j(t)$, and N $c_m(t)$ (excluding $m = 0$, in which we set $c_0 = 1$ due to its trivial time-dependence). This implies that there exists some non-linear relationship between the x_j and c_m, which will be explored later.

1.2 Equilibrium Points

Let us consider a special case of the function described in (1.18), namely that in which the time-dependence is eliminated. To do this, we set $c_m(0) = 0$ for all m except for $m = 0$. Thus we are left with

$$\psi(x) = 2^{-N} c_0(0) H_N(x) \tag{1.19}$$

Eliminating the time-dependence also affects the equations governing the zeros of the polynomial. Namely, the left-hand sides of (1.8) and (1.11) both become zero. This implies that for these static zeros (denoted by $\bar{x}_j$):

$$\bar{x}_j = 2\sum_{k=1}^{N}{}' (\bar{x}_j - \bar{x}_k)^{-3} \tag{1.20}$$

and

$$\bar{x}_j = \sum_{k=1}^{N}{}' (\bar{x}_j - \bar{x}_k)^{-1} \tag{1.21}$$

This is a very powerful result, as these equations describe the equilibrium positions for the Hamiltonians in (1.1) and (1.2), respectively. So not only do the equilibrium positions for these two Hamiltonians coincide, they are both equal to the zeros of the Hermite polynomial of degree N.

1.3 Motion in Neighborhood of Equilibrium

To study the time-dependent case, we consider the case in which the motion is in a neighborhood around the equilibrium points:

$$x_j(t) = \bar{x}_j + \epsilon y_j(t) \tag{1.22}$$

Plugging this into our equation of motion (1.8), we get

$$i\epsilon\dot{y}_j = -\bar{x}_j - \epsilon y_j + \sum_{k=1}^{N}{}' (\bar{x}_j - \bar{x}_k + \epsilon(y_j - y_k))^{-1} \tag{1.23}$$

We take the Taylor expansion of this equation up to linear terms in ϵ, yielding

$$i\epsilon\dot{y}_j = -\bar{x}_j - \epsilon y_j + \sum_{k=1}^{N}{}' \left(\frac{1}{\bar{x}_j - \bar{x}_k} - \frac{\epsilon(y_j - y_k)}{(\bar{x}_j - \bar{x}_k)^2} \right) \tag{1.24}$$

Plugging in (1.21), we get

$$\dot{y}_j = i\left(y_j + \sum_{k=1}^{N}{}' \frac{(y_j - y_k)}{(\bar{x}_j - \bar{x}_k)^2} \right) \tag{1.25}$$

Reorganizing the terms,

$$\dot{y}_j = i\left(y_j \left(1 + \sum_{k=1}^{N}{}' (\bar{x}_j - \bar{x}_k)^{-2} \right) - \sum_{k=1}^{N}{}' \frac{y_k}{(\bar{x}_j - \bar{x}_k)^2} \right) \tag{1.26}$$

We can write this more compactly as

$$\dot{y}(t) = i(A + 1_{N\times N})y(t) \tag{1.27}$$

where $1_{N\times N}$ is the identity matrix, and A is a matrix defined as

$$A_{jk} = \delta_{jk} \sum_{k=1}^{N}{}' (\bar{x}_j - \bar{x}_k)^{-2} - (1 - \delta_{jk})(\bar{x}_j - \bar{x}_k)^{-2} \tag{1.28}$$

We can now compare (1.27) to (1.16), bearing in mind the relationship between these values. We earlier explained th connection between the x_j and c_m for $m = 1\ldots, n$, which implies that the two equations are the same up to a basis change. Therefore, we can say that the eigenvalues of $(A + 1_{N\times N})$ are the integers $1, \ldots N$, which implies that the eigenvalues of A are the integers $0, \ldots, (N-1)$.

This is a highly non-trivial statement about the zeros of the Hermite polynomial. For instance, consider the matrix's dependence on n. Explicitly, n only determines the rank of the matrix. However, the $\bar{x}_j$ within the matrix have a strong dependence on n through the Hermite polynomial.

1.4 Extension to Calogero-Sutherland Model

We now wish to extend our study to include the Calogero-Sutherland Model (CSM). The Hamiltonian for a general CSM is given by

$$H = \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \frac{1}{2}\left(\frac{\pi}{L}\right)^2 \sum_{i\neq j} \frac{\beta(\beta-1)}{\sin^2 \frac{\pi}{L}(x_i - x_j)} \tag{1.29}$$

It is of interest to note that for some value of β, this model will have the same equilibrium points as the Calogero model, and the same behavior around these points.

We can also consider the non-classical extension of this Hamiltonian:

$$H = \frac{1}{2}\sum_{j=1}^{N} \left(\frac{1}{i}\frac{\partial}{\partial q_j}\right)^2 + \frac{1}{2}\left(\frac{\pi}{L}\right)^2 \sum_{i\neq j} \frac{\beta(\beta-1)}{\sin^2 \frac{\pi}{L}(x_i - x_j)} \tag{1.30}$$

As stated in [4], the eigenfunctions of the Hamiltonian are what are called the Jack symmetric polynomials $J_\lambda(z)$, where $z_j \equiv e^{2\pi i x_j/L}$ and $\lambda = (\lambda_1, \ldots, \lambda_N)$ parameterizes the equation. It is important to note that these λ's are in order, namely

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \tag{1.31}$$

The energy associated with $J_\lambda(z)$ is

$$E_\lambda = \sum_{j=1}^{N} (\lambda_j^2 + \beta(N + 1 - 2j)\lambda_j) \tag{1.32}$$

Considering the $\beta = 1$ case, we can see that the potential of the system becomes 0, meaning that the energies of (1.32) should correspond with N free particles (specifically fermions). To picture this, imagine the energy of each of the N particles separately. Due to the Pauli Exclusion principle, the ground state energy of the system corresponds with the first particle in the 0th level, the second in the 1st, et cetera. This corresponds with the case in which $\lambda_j = 0$ for all j.

To bring the system to a higher energy state, we want to raise one of the particles to a higher energy state. However, the only way we can do this is by exciting the particle with the highest energy. We say that the number of states that we raise this particle is denoted by λ_1. Once we have raised the energy level of the highest energy particle, we are allowed to excite the particle with the second-highest energy. However, we are limited by the fact that we can only raise the energy of this particle up to the level directly below the new energy of the first particle. Continuing with this trend, we find that we are limited by (1.31), which is what is to be expected.

Chapter 2

Gauge Field of BTZ Black Hole

2.1 Chern-Simons Formulation of Gravity in (2+1) AdS Space-time.

One of the key aspects of (2+1) space-time that set it apart from other dimensional models is the fact that gravity has relatively simply solutions as in [1]. Using a Chern-Simons formulation, we can define a gauge field associated with the symmetry group of (2 + 1) anti-de Sitter space, SO(2,2):

$$\mathcal{A} = e^a P_a + \omega^a J_a \tag{2.1}$$

where e^a is the *Dreibein* (as we are in (2+1)-dimensions), ω^a is the spin connection, and P_a and J_a are the members of the appropriate Lie algebra. We for now use the symmetry group SO(2,2), as in the AdS case in [1], which implies that the Lie algebra is so(2,2), as described in Appendix A. This implies the commutation relations:

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \qquad [J_a, P_b] = \epsilon_{ab}{}^c P_c, \qquad [P_a, P_b] = \frac{1}{l^2}\epsilon_{ab}{}^c J_c \tag{2.2}$$

The equations of motion in the absence of matter are:

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \tag{2.3}$$

This is a desirable equation in that it is very simple. Comparing this to its counterpart in Electricity and Magnetism:

$$\partial_\alpha F^{\alpha\beta} = 0 \qquad \partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = 0 \tag{2.4}$$

We can see that the gauge theory of gravity in (2+1) space-time is, in some senses, much more straight-forward than the gauge theory of E&M.

By plugging (2.1) into (2.3) and using the commutation relations (2.2) we get Einstein's equations of motion with a negative cosmological constant $\Lambda = -l^{-2}$, as explained in [1]. The action associated with the connection is the canonical Chern-Simons Action:

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \tag{2.5}$$

where k is called the *level* of the Chern-Simons theory, and M is the manifold that supports the connection $\mathcal{A}$.

The general definition of the gauge field in terms of the group is

$$\mathcal{A} = U^{-1} A U + U^{-1} dU \tag{2.6}$$

where U and A are both elements of the Lie Group SO(2,2). In this case, we choose the field to be pure gauge, meaning

$$\mathcal{A} = U^{-1} dU \tag{2.7}$$

for some element U of the symmetry group.

2.2 Dreibein and Spin-Connection for BTZ black hole

For this particular example, we will focus on the case of a spinning black hole as in [5]. The metric of this was found in [6] to be

$$ds^2_{\mathrm{BTZ}} = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N^\phi dt + d\phi)^2 \tag{2.8}$$

where N is the lapse function:

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \tag{2.9}$$

and N^ϕ is the angular shift function:

$$N^\phi(r) = -\frac{J}{2r^2} \tag{2.10}$$

where M and J parameterize the black hole.

As shown in [5], we can find an appropriate complex change of variables to change the metric into the form

$$ds^2 = -\left[1 + \left(\frac{r'}{l}\right)^2\right] dt'^2 + \left[1 + \left(\frac{r'}{l}\right)^2\right]^{-1} dr'^2 + r'^2 d\phi'^2 \tag{2.11}$$

To find the *Dreibein* and spin-connections in these coordinates, we first recall that

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = e^a_\mu e^b_\nu \eta_{ab} dx^\mu dx^\nu \tag{2.12}$$

from this it becomes apparent that a logical choice for the *Dreibein* are

$$e^0 = \sqrt{1 + (r'/l)^2} dt' \qquad e^1 = \frac{1}{\sqrt{1 + (r'/l)^2}} dr' \qquad e^2 = r' d\phi' \tag{2.13}$$

To find valid spin-connections, we return to (2.3). By plugging in (2.1), we get

$$P_a de^a + J_a d\omega^a + P_a P_b(e^a \wedge e^b) + P_a J_b(e^a \wedge \omega^b) + J_a P_b(\omega^a \wedge e^b) + J_a J_b(\omega^a \wedge \omega^b) = 0 \tag{2.14}$$

Using the so(2,2) algebra, we note that

$$P_a P_b(e^a \wedge e^b) + P_b P_a(e^b \wedge e^a) = [P_a, P_b](e^a \wedge e^b) = \frac{1}{l^2}\epsilon_{ab}{}^{c} J_c(e^a \wedge e^b) \tag{2.15}$$

$$P_a J_b(e^a \wedge \omega^b) + J_b P_a(\omega^b \wedge e^a) = [J_a, P_b](\omega^a \wedge e^b) = \epsilon_{ab}{}^{c} P_c(\omega^a \wedge e^b) \tag{2.16}$$

$$J_a J_b(\omega^a \wedge \omega^b) + J_b J_a(\omega^b \wedge \omega^a) = [J_a, J_b](\omega^a \wedge \omega^b) = \epsilon_{ab}{}^{c} J_c(\omega^a \wedge \omega^b) \tag{2.17}$$

Using this we find that:

$$J_a\left(d\omega^a + \frac{1}{l^2}(e^{a+1} \wedge e^{a+2}) + (\omega^{a+1} \wedge \omega^{a+2})\right) + P_a\left(de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c\right) = 0 \tag{2.18}$$

The important finding here is that

$$de^a + \epsilon^a{}_{bc}\omega^b \wedge e^c = 0 \tag{2.19}$$

Using this relation, we can confirm the validity of the proposed spin connection:

$$\omega^0 = -\sqrt{1+(r'/l)^2}d\phi' \qquad \omega^1 = 0 \qquad \omega^2 = -r'/l^2 dt' \tag{2.20}$$

By plugging these into (2.19), we get:

$$\begin{gathered}
de^0 + \omega^1 \wedge e^2 - \omega^2 \wedge e^1 \stackrel{?}{=} 0 \\
\frac{\partial}{\partial r'}\left(\sqrt{1+(r'/l)^2}\right)(dr' \wedge dt') + (0 \wedge r'd\phi') - \left(-r/l^2 dt' \wedge \frac{dr'}{\sqrt{1+(r'/l)^2}}\right) \stackrel{?}{=} 0 \\
\frac{r}{l^2\sqrt{1+(r'/l)^2}}(dr' \wedge dt') + \frac{r}{l^2\sqrt{1+(r'/l)^2}}(dt' \wedge dr') \stackrel{\checkmark}{=} 0
\end{gathered} \tag{2.21}$$

$$\begin{gathered}
de^1 - \omega^2 \wedge e^0 + \omega^0 \wedge e^2 \stackrel{?}{=} 0 \\
\frac{\partial}{\partial r'}\left(\frac{1}{\sqrt{1+(r'/l)^2}}\right)(dr' \wedge dr') - \omega^2 \wedge e^0 + \omega^0 \wedge e^2 \stackrel{?}{=} 0 \\
\frac{r}{l^2}\sqrt{1+(r'/l)^2}(dt' \wedge dt') - r\sqrt{1+(r'/l)^2}(d\phi' \wedge d\phi') \stackrel{\checkmark}{=} 0
\end{gathered} \tag{2.22}$$

$$\begin{gathered}
de^2 - \omega^0 \wedge e^1 + \omega^1 \wedge e^0 \stackrel{?}{=} 0 \\
(dr' \wedge d\phi') - \left(-\sqrt{1+(r'/l)^2}d\phi' \wedge \frac{dr'}{\sqrt{1+(r'/l)^2}}\right) \stackrel{?}{=} 0 \\
(dr' \wedge d\phi') + (d\phi' \wedge dr') \stackrel{\checkmark}{=} 0
\end{gathered} \tag{2.23}$$

2.3 Group element associated with gauge field

Now that we have valid equations for the *Dreibein* and spin-connections for the AdS space, we can use (2.7) to find an appropriate group element U associated with our metric. We consider the element proposed in [5]:

$$U = e^{-\phi' J_0} e^{t' P_0} e^{l \sinh^{-1}(r'/l) P_1} \tag{2.24}$$

We can verify the validity of this element by equating (2.1) to (2.7). If our choice of U is a good one, then we should find:

$$\sqrt{1+(r'/l)^2} P_0 - r'/l^2 J_2 = U^{-1} \partial_{t'} U \tag{2.25}$$

$$\frac{1}{\sqrt{1+(r'/l)^2}} P_1 = U^{-1} \partial_{r'} U \tag{2.26}$$

$$r' P_2 - \sqrt{1+(r'/l)^2} J_0 = U^{-1} \partial_{\phi'} U \tag{2.27}$$

Looking first at (2.25), we can use the fact that P_0 commutes with itself and J_0 to say

$$U^{-1} \partial_{t'} U = \exp\left(-l \sinh^{-1}(r'/l) P_1\right) P_0 \exp\left(l \sinh^{-1}(r'/l) P_1\right) \tag{2.28}$$

from here we use the identity

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, b]]] + \cdots \tag{2.29}$$

from which we get

$$U^{-1} \partial_{t'} U = P_0 - \frac{\alpha}{l^2} J_2 + \frac{\alpha^2}{2! l^2} P_0 - \frac{\alpha^3}{3! l^4} J_2 + \cdots \tag{2.30}$$

where $\alpha = l \sinh^{-1}(r'/l)$. From here we say

$$U^{-1} \partial_{t'} U = \cosh\left(\frac{\alpha}{l}\right) P_0 - \frac{1}{l} \sinh\left(\frac{\alpha}{l}\right) \tag{2.31}$$

$$U^{-1} \partial_{t'} U = \sqrt{1+(r'/l)^2} P_0 - r'/l^2 J_2 \tag{2.32}$$

We next consider (2.26). This is very straight-forward, as

$$U^{-1} \partial_{r'} U = e^{-l \sinh^{-1}(r'/l) P_1} \partial_{r'} \left(e^{-l \sinh^{-1}(r'/l) P_1}\right) \tag{2.33}$$

So by the chain rule,

$$\frac{1}{\sqrt{1+(r'/l)^2}} P_1 = U^{-1} \partial_{r'} U \tag{2.34}$$

Finally, we consider (2.27). Similar to (2.25), two of the operators in the exponent are able to commute, leaving

$$U^{-1}\partial_{\phi'}U = \exp\left(-l\sinh^{-1}(r'/l)P_1\right)(-J_0)\exp\left(l\sinh^{-1}(r'/l)P_1\right) \tag{2.35}$$

Applying (2.29) yields

$$U^{-1}\partial_{\phi'}U = -J_0 + \alpha P_2 - \frac{\alpha^2}{2!l^2}J_0 + \frac{\alpha^3}{3!l^2}P_2 - \cdots \tag{2.36}$$

$$= -\cosh\left(\frac{\alpha}{l}\right)J_0 + l\sinh\left(\frac{\alpha}{l}\right)P_2 \tag{2.37}$$

$$= r'P_2 - \sqrt{1+(r'/l)^2}J_0 \tag{2.38}$$

2.4 Group Element around Singular Point

With the group element U found in the coordinates (r',ϕ',t'), it would now be useful to use our original set of coordinates. The transformations used are given by

$$t' = ir_+\left(\frac{t}{l} - \mathcal{J}\phi\right) \qquad r' = -il\sqrt{\nu^2(r)} \qquad \phi' = -i\frac{r_+}{l}\left(\mathcal{J}\frac{t}{l} - \phi\right) \tag{2.39}$$

where

$$r_+ = \frac{Ml^2}{2}\left(1 + \sqrt{1 - \left(\frac{J}{Ml}\right)^2}\right) \tag{2.40}$$

is a zero of the lapse function, $\mathcal{J} = Jl/(2r_+^2)$, and

$$\nu^2(r) = 1 + \frac{(r/r_+)^2 - 1}{1 - \mathcal{J}^2} \tag{2.41}$$

The two zeros of the lapse function r_+ and $r_- = |\mathcal{J}|r_+$ divide the manifold into three distinct sections: $r > r_+$, $r_- < r < r_+$ and $0 < r < r_-$. Because the lapse function changes sign at $r = r_+$ and $r = r_-$, we can treat these as horizons, and therefore we must interchange coordinates at these points.

Because the region $r > r_+$ resembles the purely anti-de Sitter space, it is assumed that the group element of this region can be found by simply applying the transformations to the AdS case in (2.24). To go from here to $r_- < r < r_+$, we must interchange the "0" and "1" *Dreibein* and spin-connections. Then, going from here to the region $0 < r < r_-$, we must exchange the new "0" *Dreibein* and spin-connections for their "2" counterparts. This, in conjunction with the fact that $\nu^2(r)$ is now negative, gives

$$U_{0<r} = e^{-i\phi' J_1}e^{it' P_1}e^{-l\sinh^{-1}(\frac{r'}{l})P_2} \tag{2.42}$$

This is not quite an element of the group; to correct this, we left-multiply by $\exp(-\frac{\pi}{2}lP_0)$, which, in terms of our original coordinates, is

$$U_{0<r} = \exp\left(-\tfrac{r_+}{l}\left(\tfrac{t}{l} - \mathcal{J}\phi\right)J_2\right)\exp\left(-r_+\left(\mathcal{J}\tfrac{t}{l} - \phi\right)P_2\right)\exp(\tfrac{\pi}{2}lP_0)\exp(-l\sinh^{-1}|\nu|P_2) \tag{2.43}$$

where $|\nu| = \sqrt{-\nu^2(r)}$.

2.5 Wilson Loops

As mentioned previously, we were able to set the field curvature $\mathcal{F} = 0$ because we assumed the absence of mater. However, at the singular point $r = 0$ it is possible for $\mathcal{F}$ to be nonzero. To discover the effect of this mass on the equations of motion, we can use a tool useful when dealing with manifolds: the Wilson Loop.

The Wilson loop is defined as

$$W = \mathcal{P} \exp \left(\oint_\gamma \mathcal{A} dx \right) \tag{2.44}$$

where $\mathcal{P}$ is the path-ordering operator and γ is a closed loop on the manifold. This becomes especially powerful in this case when considering a small loop of radius ε around the origin. One can show

$$W_{r=\varepsilon} = 1 + \int_0^\varepsilon dr \int_0^{2\pi} d\phi \mathcal{F}_{r\phi} + \mathcal{O}(\varepsilon^2) \tag{2.45}$$

and thus

$$\log W_{r=\varepsilon} = \int_0^\varepsilon dr \int_0^{2\pi} d\phi \mathcal{F}_{r\phi} + \mathcal{O}(\varepsilon^2) \tag{2.46}$$

$$= 2\pi \int_0^\varepsilon dr \mathcal{F}_{r\phi} + \mathcal{O}(\varepsilon^2) \tag{2.47}$$

So by finding the Wilson loop in other terms, we can find the curvature at $r = 0$. Considering the original formulation, it is clear that for this loop

$$W = U^{-1}(t, r = \varepsilon, \phi = 0) U(t, r = \varepsilon, \phi = 2\pi) \equiv e^w \tag{2.48}$$

up to a gauge transformation. Plugging in the group element for the region $0 < r < r_-$, we find

$$W = V(r)^{-1} \exp\left(2\pi \tfrac{r_+}{l} (\mathcal{J} J_2 + l P_2)\right) V(r) \tag{2.49}$$

We can apply this to (2.47), which after some computation, yields the result

$$\mathcal{F}_{r\phi} = \delta(r) \frac{r_+}{l} \frac{1}{\sqrt{1 - \mathcal{J}^2}} [J_1 - \mathcal{J}^2 J_0 + \mathcal{J} l (P_1 - P_0)] \tag{2.50}$$

meaning at $r = 0$, the Field curvature is non-zero, so the equations of motion at this point are not well defined. This is consistent with most black hole models, and is why this point is called the *singular point*.

Chapter 3

Alternate Gauge Formulation for BTZ Black Hole

3.1 Gauge and Action in SL(2,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$)

It is a well-known fact in Group Theory that SO(2,2) is equivalent to the group SL(2,$\mathbb{R}$)$\times$SL(2,$\mathbb{R}$), as explained in Appendix A. When introducing the higher spin fields, it becomes more convenient to use this second formulation as the symmetry group, which means that there is a slight change in our notation. We now have two Chern-Simons gauge fields, $\mathcal{A}$ and $\bar{\mathcal{A}}$, each associated with the symmetry group SL(2,$\mathbb{R}$).

These gauge fields, instead of being parameterized by the *Dreibein* and spin-connections directly, are instead parameterized by a linear combination of these, such that

$$e = \frac{l}{2}(\mathcal{A} - \bar{\mathcal{A}}) \qquad \omega = \frac{1}{2}(\mathcal{A} + \bar{\mathcal{A}}) \tag{3.1}$$

Both of these fields have the same curvature constraints as before, namely

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} = 0 \qquad \bar{\mathcal{F}} = d\bar{\mathcal{A}} + \bar{\mathcal{A}} \wedge \bar{\mathcal{A}} = 0 \tag{3.2}$$

These fields are connected to the Einstein-Hilbert Action by the equation

$$S = S_{CS}(\mathcal{A}) - S_{CS}(\bar{\mathcal{A}}) \tag{3.3}$$

where S is the Einstein-Hilbert Action from General Relativity, and S_{CS} is the Chern-Simons Action as defined in (2.5).

An additional change to the gauge fields presented here is that it is no longer convenient for us to assume that $\mathcal{A}$ and $\bar{\mathcal{A}}$ are pure gauge. Thus we use the more general representation using elements of the symmetry group:

$$\mathcal{A} = U^{-1}AU + U^{-1}dU \qquad \bar{\mathcal{A}} = \bar{U}^{-1}\bar{A}\bar{U} + \bar{U}^{-1}d\bar{U} \tag{3.4}$$

where U, A, $\bar{U}$ and $\bar{A}$ are all members of the SL(2,$\mathbb{R}$) group.

3.2 General Relationship between Metric and Gauge

In [7], it is claimed that the metric

$$\frac{ds^2}{l^2} = dr^2 - \frac{8\pi G}{l}\left(\mathcal{L}dx^{+2} + \tilde{\mathcal{L}}dx^{-2}\right) - \left(e^{2r} + \frac{64\pi^2 G^2}{l^2}\mathcal{L}\tilde{\mathcal{L}}e^{-2r}\right)dx^+dx^- \tag{3.5}$$

is a solution to Einstein's equations, so long as $\mathcal{L}$ is a function that depends only on x^+ and $\tilde{\mathcal{L}}$ depends only on x^-, where $x^\pm = t/l \pm \phi$. It will be useful for us later to think of $\mathcal{L}$ (and $\tilde{\mathcal{L}}$) as the vacuum energy of the system. As before $l = -\Lambda^{-1/2}$, where Λ is the cosmological constant.

As mentioned in [8], the gauge associated with this metric is one in which $U = e^{rL_0}$, $\bar{U} = e^{-rL_0}$ and

$$A = \left(L_1 + \frac{8G\pi}{l}\mathcal{L}L_{-1}\right)dx^+ \tag{3.6}$$

$$\bar{A} = -\left(\frac{8G\pi}{l}\tilde{\mathcal{L}}L_1 + L_{-1}\right)dx^- \tag{3.7}$$

where L_0 and $L_{\pm 1}$ are members of the SL(2,$\mathbb{R}$) group described in Appendix A. To confirm this relationship, let us first find more explicit equations for the gauges:

$$\begin{aligned}
\mathcal{A} &= e^{-rL_0}\left(L_1 + \frac{8G\pi}{l}\mathcal{L}L_{-1}\right)e^{rL_0}dx^+ + e^{-rL_0}L_0e^{rL_0}dr \\
\mathcal{A} &= \left(e^rL_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)dx^+ + L_0dr
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\bar{\mathcal{A}} &= -e^{rL_0}\left(\frac{8G\pi}{l}\tilde{\mathcal{L}}L_1 + L_{-1}\right)e^{-rL_0}dx^- - e^{rL_0}L_0e^{-rL_0}dr \\
\bar{\mathcal{A}} &= \left(-\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 - e^rL_{-1}\right)dx^- - L_0dr
\end{aligned} \tag{3.9}$$

Next, we must use the relationship between the metric and the *Dreibein* in conjunction with (3.2) to give us

$$\frac{ds^2}{l^2} = \frac{1}{4\epsilon_2}\mathrm{Tr}[(\mathcal{A} - \bar{\mathcal{A}})^2] \tag{3.10}$$

where ϵ_2 is a normalization constant of the Lie algebra given by

$$\mathrm{Tr}(L_aL_b) = \epsilon_2\eta_{ab} \tag{3.11}$$

which is solved by $\epsilon_2 = \frac{1}{2}$.

Plugging the gauges into (3.10), we get

$$\frac{ds^2}{l^2} = \frac{1}{4\epsilon_2}\mathrm{Tr}\left[\alpha^{(++)}dx^{+2} + \alpha^{(--)}dx^{-2} + \alpha^{(rr)}dr^2 + \alpha^{(+-)}dx^+dx^- + \alpha^{(+r)}dx^+dr + \alpha^{(-r)}dx^-dr\right]$$

where

$$\begin{aligned}\alpha^{(++)} &= \left(e^r L_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)^2\\ &= e^{2r}(L_1)^2 + \frac{8G\pi}{l}\mathcal{L}(L_1L_{-1} + L_{-1}L_1) + \left(\frac{8G\pi}{l}\mathcal{L}\right)^2 e^{-2r}(L_{-1})^2\\ &= \frac{8G\pi}{l}\mathcal{L}(L_1L_{-1} + L_{-1}L_1)\end{aligned} \tag{3.12}$$

$$\begin{aligned}\alpha^{(--)} &= \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 + e^r L_{-1}\right)^2\\ &= \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}\right)^2 e^{-2r}(L_1)^2 + \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}\right)(L_1L_{-1} + L_{-1}L_1) + e^{2r}(L_{-1})^2\\ &= \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}\right)(L_1L_{-1} + L_{-1}L_1)\end{aligned} \tag{3.13}$$

$$\begin{aligned}\alpha^{(rr)} &= (2L_0)^2\\ &= 4(L_0)^2\end{aligned} \tag{3.14}$$

$$\begin{aligned}\alpha^{(+-)} &= \left(e^r L_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)\left(\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 + e^r L_{-1}\right)\\ &\quad + \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 + e^r L_{-1}\right)\left(e^r L_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)\\ &= \frac{16G\pi}{l}\tilde{\mathcal{L}}(L_1)^2 + \left(e^{2r} + \left(\frac{8G\pi}{l}\right)^2\mathcal{L}\tilde{\mathcal{L}}e^{-2r}\right)(L_{-1}L_1)\\ &\quad + \left(e^{2r} + \left(\frac{8G\pi}{l}\right)^2\mathcal{L}\tilde{\mathcal{L}}e^{-2r}\right)(L_1L_{-1}) + \frac{16G\pi}{l}\tilde{\mathcal{L}}(L_{-1})^2\\ &= \left(e^{2r} + \left(\frac{8G\pi}{l}\right)^2\mathcal{L}\tilde{\mathcal{L}}e^{-2r}\right)(L_{-1}L_1 + L_1L_{-1})\end{aligned} \tag{3.15}$$

$$\begin{aligned}\alpha^{(+r)} &= \left(e^r L_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)(2L_0) + (2L_0)\left(e^r L_1 + \frac{8G\pi}{l}\mathcal{L}e^{-r}L_{-1}\right)\\ &= 2e^r(L_1L_0 + L_0L_1) + \left(\frac{16G\pi}{l}\mathcal{L}\right)e^{-r}(L_{-1}L_0 + L_0L_{-1})\end{aligned} \tag{3.16}$$

$$\alpha^{(-r)} = \left(\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 + e^r L_{-1}\right)(2L_0) + (2L_0)\left(\frac{8G\pi}{l}\tilde{\mathcal{L}}e^{-r}L_1 + e^r L_{-1}\right)$$
$$= \left(\frac{16G\pi}{l}\tilde{\mathcal{L}}\right)e^{-r}(L_1L_0 + L_0L_1) + 2e^r(L_{-1}L_0 + L_0L_{-1}) \tag{3.17}$$

in which we used the properties of the group outlined in Appendix A.

We can further use these properties to take the trace of these α's, which yields:

$$\mathrm{Tr}(\alpha^{(++)}) = -\frac{16G\pi}{l}\mathcal{L}, \qquad \mathrm{Tr}(\alpha^{(--)}) = -\frac{16G\pi}{l}\tilde{\mathcal{L}}, \qquad \mathrm{Tr}(\alpha^{(rr)}) = 2$$

$$\mathrm{Tr}(\alpha^{(+-)}) = -2\left(e^{2r} + \left(\frac{8G\pi}{l}\right)^2 \mathcal{L}\tilde{\mathcal{L}}e^{-2r}\right) \qquad \mathrm{Tr}(\alpha^{(+r)}) = \mathrm{Tr}(\alpha^{(-r)}) = 0$$

which confirms (3.5).

3.3 BTZ Black Hole Metric

We now know that any metric in the form of (3.5) has a gauge of the form described in the previous section. This means that if we can find the functions $\mathcal{L}$ and $\tilde{\mathcal{L}}$ associated with the BTZ Black Hole, we can easily find the gauge in SL(2,$\mathbb{R}$)×SL(2,$\mathbb{R}$).

The functions as proposed in [7] and used in [8] are

$$\mathcal{L} = -\frac{1}{4\pi}(Ml - J) \qquad \tilde{\mathcal{L}} = -\frac{1}{4\pi}(Ml + J) \tag{3.18}$$

Plugging these into (3.5), we get

$$\frac{ds^2}{l^2} = dr^2 - (e^{2r} - 4GM + 4G^2M^2e^{-2r})\frac{dt^2}{l^2} + (e^{2r} + 4GM + 4G^2M^2e^{-2r})d\phi^2 + \tag{3.19}$$
$$+ \frac{4G^2J^2}{l^2}e^{-2r}\left(\frac{dt^2}{l^2} - d\phi^2\right) - \frac{8GJ}{l^2}dtd\phi$$

As explained in [7], there exists a transformation of coordinates to put the metric in the form as before, namely

$$ds^2_{\mathrm{BTZ}} = -N^2dt^2 + N^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2 \tag{3.20}$$

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \tag{3.21}$$

and

$$N^\phi(r) = -\frac{J}{2r^2} \tag{3.22}$$

Chapter 4

Introducing Higher Spin Fields

4.1 Motivation for Change in Symmetry Group

We now wish to change the symmetry group under which the action is invariant, from $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})$ to $\mathrm{SL}(N,\mathbb{R})\times\mathrm{SL}(N,\mathbb{R})$. The reason for this change is that now not only are massless gauge fields with spin 2 being taken into account (namely gravity), but also a potentially infinite tower of higher spin fields.

In [9], the authors give a description of gravity in Anti-de Sitter space-time that interacts with higher spin gauge fields. The advantage of adding in gauge fields with spins higher than gravity's spin 2 is that it helps form many new connections with string theory. Specifically, it adds new examples of dualities between AdS space-time and Conformal Field Theories, which ties in with the so-called AdS/CFT correspondence. Additionally, as explained in [10], these theories can help test whether properties of gravity that is minimally coupled to matter are affected by the coupling of an infinite tower of higher spin fields to gravity, a topic of importance in models of stringy gravity.

Vasiliev and collaborators have noted that (2+1) dimensions are an especially important example, as there are no propagating degrees of freedom for the higher spin fields (just as in gravity).

4.2 The General metric for Higher Spin Gravity

To find the form of the metric associated with a (2+1) dimensional theory of gravity that takes into account fields of higher spin, we assume as in Chapter 3 that the gauge fields are of the form

$$\mathcal{A} = U^{-1}AU + U^{-1}dU \qquad \bar{\mathcal{A}} = \bar{U}^{-1}\bar{A}\bar{U} + \bar{U}^{-1}d\bar{U} \tag{4.1}$$

where again we choose $U = e^{rL_0}$ and $\bar{U} = e^{-rL_0}$. It is important to note that L_0 is now a member of the $\mathrm{SL}(N,\mathbb{R})$ symmetry group, as described in Appendix A. As a consequence of this choice, A and $\bar{A}$ must be a linear combination of $L_{\pm1}$ and $W^{(s)}_{\pm1}$, which happen to be the generators which in matrix form have non-trivial components immediately above and below the diagonal. Thus, we follow the example of [10], and define A and $\bar{A}$ as:

$$A = \left(\sum_{k=1}^{N-1} B_k(a_k, b_k)\right) dx^{+} \qquad \bar{A} = -\left(\sum_{k=1}^{N-1} B_k(c_k, d_k)\right) dx^{-} \tag{4.2}$$

where as before $x^{\pm} = \frac{t}{l} \pm \phi$, and we define B_k to be a $N \times N$ matrix for which the ijth component is described by

$$[B_k(x,y)]_{ij} = x\delta_{i,k}\delta_{j,k+1} - y\delta_{i,k+1}\delta_{j,k} \tag{4.3}$$

An important property of this matrix is that

$$e^{rL_0} B_k(x,y) e^{-rL_0} = B_k(e^r x, e^{-r} y) \tag{4.4}$$

which can be verified explicitly using the matrix representations of these quantities.

Using this property, we can solve for the metric. Using our equation for the *dreibein* given in Chapter 3:

$$e = \frac{l}{2}(\mathcal{A} - \bar{\mathcal{A}}) \tag{4.5}$$

along with our equation relating the metric to the *dreibein*

$$g_{\mu\nu} = \frac{1}{\epsilon_N}\mathrm{Tr}(e_\mu e_\nu) \tag{4.6}$$

we can find an equation for the metric:

$$\frac{ds^2}{l^2} = \frac{1}{4\epsilon_N}\mathrm{Tr}\left[(\mathcal{A} - \bar{\mathcal{A}})^2\right] \tag{4.7}$$

$$= dr^2 - \frac{1}{2\epsilon_N}\left(\sum_{k=1}^{N-1} a_k b_k\right) dx_+^2 - \frac{1}{2\epsilon_N}\left(\sum_{k=1}^{N-1}(e^{-2r}a_k d_k + e^{2r} b_k c_k)\right) dx^+ dx^- - \tag{4.8}$$

$$- \frac{1}{2\epsilon_N}\left(\sum_{k=1}^{N-1} c_k d_k\right) dx_-^2$$

Because we are interested in static metrics, we can set $g_{++} = g_{--}$, which looking at the metric, implies that

$$\sum_{k=1}^{N-1} a_k b_k = \sum_{k=1}^{N-1} c_k d_k \tag{4.9}$$

Additionally, we make the following definitions to simplify the equations:

$$\beta_N = \frac{1}{2\epsilon_N}\sum_{k=1}^{N-1} b_k c_k \tag{4.10}$$

$$\Lambda_N = \frac{1}{(2\epsilon_N)^2}\left(\sum_{k=1}^{N-1} b_k c_k\right)\left(\sum_{k=1}^{N-1} a_k d_k\right) \tag{4.11}$$

$$M_N = -\frac{1}{2\epsilon_N}\sum_{k=1}^{N-1} a_k b_k \tag{4.12}$$

which makes the metric

$$\frac{ds^2}{l^2} = dr^2 + M_N(dx_+^2 + dx_-^2) - \left(e^{-2r}\frac{\Lambda_N}{\beta_N} + e^{2r}\beta_N\right) dx^+ dx^- \tag{4.13}$$

Plugging in the definition for $x^{\pm}$, the metric becomes

$$\frac{ds^2}{l^2} = dr^2 - \left(e^{-2r}\frac{\Lambda_N}{\beta_N} - 2M_N + e^{2r}\beta_N\right) dt^2 + \left(e^{-2r}\frac{\Lambda_N}{\beta_N} + 2M_N + e^{2r}\beta_N\right) d\phi^2 \tag{4.14}$$

By transforming $r \to r + \frac{1}{2}\log\beta_N$, we get

$$\frac{ds^2}{l^2} = dr^2 - \left(e^{-2r}\Lambda_N - 2M_N + e^{2r}\right) dt^2 + \left(e^{-2r}\Lambda_N + 2M_N + e^{2r}\right) d\phi^2 \tag{4.15}$$

Because we want this metric to be locally Anti-de Sitter, we further require that $\Lambda_N = M_N^2$. If this is true, then we can write the metric in its simplified form:

$$\frac{ds^2}{l^2} = dr^2 - (e^r - M_N e^{-r})^2 dt^2 + (e^r + M_N e^{-r})^2 d\phi^2 \tag{4.16}$$

Thus we have found the general metric associated with the gauge fields for higher spin gravity. During this derivation, we made two constraints on the values a_k, b_k, c_k, and d_k, namely the static metric assumption (4.9) and the locally AdS assumption, which can be written as

$$\left(\sum_{k=1}^{N-1} b_k c_k\right)\left(\sum_{k=1}^{N-1} a_k d_k\right) = \left(\sum_{k=1}^{N-1} a_k b_k\right)^2 \tag{4.17}$$

Together, these constraints imply that

$$b_k = \alpha a_k, \qquad c_k = \gamma a_k, \qquad d_k = \frac{\alpha}{\gamma} a_k \tag{4.18}$$

where α and γ. We can choose $\alpha = \gamma = 1$, making $A = \bar{A}$ anti-symmetric.

Chapter 5

Conical Defects in Higher Spin Gravity

5.1 Relation between metric and energy

As found in Chapter 4, the general metric for a (2+1) space-time with mass at $r = 0$ that is locally AdS can be described as

$$\frac{ds^2}{l^2} = dr^2 - (e^r - M_N e^{-r})^2 dt^2 + (e^r + M_N e^{-r})^2 d\phi^2 \tag{5.1}$$

It is clear that any choice of M_N will be locally AdS, however the choice of this value will change the physical interpretation of the space-time manifold being described. Specifically, it turns out that the M_N corresponds with the energy of the solution.

We define energy to be

$$\mathcal{L} = \frac{l}{16G\epsilon_N}\mathrm{tr}(A^2) \qquad \tilde{\mathcal{L}} = \frac{l}{16G\epsilon_N}\mathrm{tr}(\bar{A}^2) \tag{5.2}$$

as described in [10], we do this because this gives the stress tensor of the dual CFT for the largest gauge. A comparison with (3.6) and (3.7) shows that these are the same $\mathcal{L}$ and $\tilde{\mathcal{L}}$ from Chapter 3.

By using the definitions of a, $\bar{a}$, and M_N from the previous chapter, we can see that

$$\mathcal{L} = \tilde{\mathcal{L}} = \frac{l}{4G}M_N \tag{5.3}$$

Comparing this to the work done in Chapter 3, we can see that for positive M_N and $N = 2$ (pure gravitational), this describes a non-spinning BTZ black hole with mass

$$M = \frac{M_N}{2G} \tag{5.4}$$

Indeed, by substituting these values of M and J into (3.19), we get (5.1).

However, we have claimed that the metric is locally Anti-de Sitter for *any* value of M_N, including negative ones. We will find that these correspond with conical defects.

5.2 Conical Defects

In Chapter 3, we claimed that the metric

$$\frac{ds^2}{l^2} = dr^2 - (e^{2r} - 4GM + 4G^2M^2e^{-2r})\frac{dt^2}{l^2} + (e^{2r} + 4GM + 4G^2M^2e^{-2r})d\phi^2 + \\ + \frac{4G^2J^2}{l^2}e^{-2r}\left(\frac{dt^2}{l^2} - d\phi^2\right) - \frac{8GJ}{l^2}dtd\phi \tag{5.5}$$

could be transformed into

$$ds^2_{\mathrm{BTZ}} = -N^2dt^2 + N^{-2}dr^2 + r^2(N^\phi dt + d\phi)^2 \tag{5.6}$$

where

$$N^2(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \tag{5.7}$$

and

$$N^\phi(r) = -\frac{J}{2r^2} \tag{5.8}$$

From this, it follows that the metric described in (5.1) can be transformed to

$$ds^2 = -\left(-4M_N + \frac{r^2}{l^2}\right)dt^2 + \left(-4M_N + \frac{r^2}{l^2}\right)^{-1}dr^2 + r^2d\phi^2 \tag{5.9}$$

Considering a small loop around the singular point, we can say that the metric is effectively

$$ds^2 = -\left(-4M_N\right)dt^2 + \left(-4M_N\right)^{-1}dr^2 + r^2d\phi^2 \tag{5.10}$$

which we can transform to

$$ds^2 = -dt^2 + dr^2 + (-4M_N)r^2d\phi^2 \tag{5.11}$$

Now, taking $M_N < 0$, we can physically interpret the manifold described by this metric. Consider the angle

$$\omega = \sqrt{-4M_N}\phi \tag{5.12}$$

for this angle, the metric is completely flat:

$$ds^2 = -dt^2 + dr^2 + r^2d\omega^2 \tag{5.13}$$

however, it is important to note that while ϕ runs from 0 to 2π, ω runs from 0 to $2\pi\sqrt{-4M_N}$. This cuts out an angle $\theta \equiv 2\pi - \omega$ out of a circle, as shown in Figure 5.1.

By connecting the two sides in this figure, we can see that the resulting shape is conical, as shown in Figure 5.2.

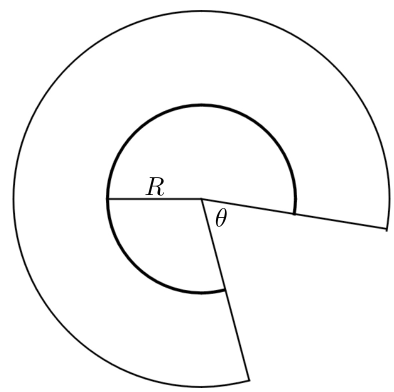

Figure 5.1: A section of angle θ has been removed from the manifold. (Source: [11])

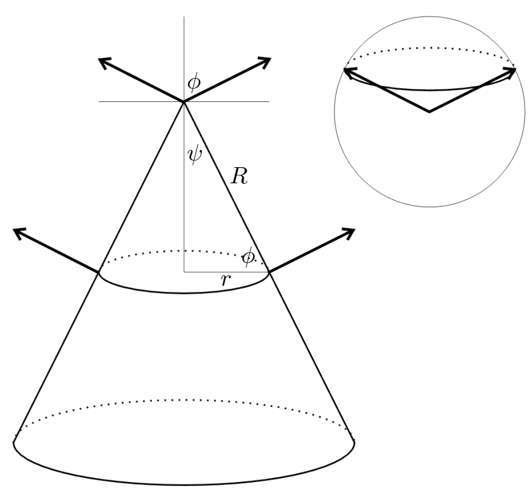

Figure 5.2: By connecting the manifold, a conical defect is formed. (Source: [11])

This shape is described by angle of the removed wedge in Figure 5.1, called the deficit angle δ. By inspection, the deficit angle of the manifolds for which $M_N < 0$ is

$$\theta = 2\pi(1 - 2\sqrt{-M_N}) \tag{5.14}$$

Note that this divides the negative energy solutions into three types: $0 > \mathcal{L} > -\frac{l}{16G}$, $\mathcal{L} = -\frac{l}{16G}$, and $\mathcal{L} < -\frac{l}{16G}$. The first region corresponds with a positive deficit angle, so these solutions are called conical deficits. The second region corresponds with a zero deficit angle, meaning that this is the pure AdS case. The third region has a negative deficit angle, so these solutions are called conical surpluses.

5.3 Smooth Solutions of Underlying Theory

It is of interest which of these solutions are "smooth," that is, for which solutions $\mathcal{F} = 0$ everywhere. In Chapter 2, we noted that for BTZ Black Holes, the Wilson loop around the point $r = 0$ was not the identity, meaning that $\mathcal{F}$ was non-zero at this point. This was in contrast to the pure AdS case, in which the field curvature is zero everywhere.

To apply the idea of Wilson Loops to this new symmetry group, it becomes clear that for the solution to be smooth, W need only be at the centre of the group. So for even N, a "trivial" holonomy can refer to $\pm 1_{N\times N}$, whereas for odd N, a trivial holonomy is only $1_{N\times N}$.

Our Ansatz for smooth connections are $\mathcal{A}$ and $\bar{\mathcal{A}}$ defined in the usual way, only with the additional condition that A and $\bar{A}$ are of the form

$$A = \left(\sum_{j=1}^{\lfloor N/2\rfloor} B_{2j-1}(a_{2j-1}, a_{2j-1})\right) dx^+ \qquad \bar{A} = -\left(\sum_{j=1}^{\lfloor N/2\rfloor} B_{2j-1}(a_{2j-1}, a_{2j-1})\right) dx^- \tag{5.15}$$

in other words, only B_k with odd k contribute. A consequence of this is that A and $\bar{A}$ can be broken into $\lfloor N/2\rfloor$ 2×2 block diagonal matrices with eigenvalues associated with the values of a_{2j-1}. Indeed, we can say that the Wilson loop associated with $\mathcal{A}$[1]

$$W = U^{-1}\exp\left(\oint d\phi A_\phi\right) U \tag{5.16}$$

is proportional to $\exp(2\pi\lambda)$, where λ is a $N\times N$ matrix given by

$$\lambda = i\text{diag}(\pm a_1, \pm a_3, \ldots, \pm a_{N-1}) \qquad \text{if } N \text{ is even} \tag{5.17}$$

$$\lambda = i\text{diag}(\pm a_1, \pm a_3, \ldots, \pm a_{N-1}, 0) \qquad \text{if } N \text{ is odd} \tag{5.18}$$

Thus W is proportional to $1_{N\times N}$ if all of a_{2j-1} are integers, and proportional to $-1_{N\times N}$ if all of a_{2j-1} are half-integers.

[1]The Wilson loop associated with $\bar{\mathcal{A}}$ can be found in the same manner.

Remembering the definition of the energy (5.2), we can find the energy associated with each solution of this form:

$$\mathcal{L} = -\frac{l}{16G\epsilon_N}\sum_{j=1}^{\lfloor N/2 \rfloor} a_{2j-1}^2 \tag{5.19}$$

$$= \frac{l}{32G\epsilon_N}\mathrm{Tr}(\lambda^2) \tag{5.20}$$

By inspection, we can see that smooth solutions of this form will only have negative energy, meaning that the BTZ Black Holes explored earlier will not become smooth by introducing these higher spin fields. However, there are conical defect solutions that do appear to become smooth with the introduction of these fields. For instance, we can compare the $N = 2$ case with $N = 4$.

For the moment, let us only consider the conical deficit region, that is, the region in which $0 > \mathcal{L} > -\frac{l}{16G}$. This implies that in this region

$$0 < \sum_j a_{2j-1}^2 < \frac{1}{2}\epsilon_N = \frac{1}{24}N(N^2 - 1) \tag{5.21}$$

Considering $N = 2$, we can see that there are no integers or half-integers that satisfy this requirement - in other words, no conical deficit solutions are smooth with regards to only gravity. However, when we introduce the spin 3 and spin 4 fields, we find three potentially smooth solutions:

$$\begin{aligned}
(A) \qquad & 1^2 + 1^2 < \frac{5}{2} \\
(B) \qquad & 1^2 + 0^2 < \frac{5}{2} \\
(C) \qquad & \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 < \frac{5}{2}
\end{aligned}$$

each of which is associated with a irrational deficit angle.

5.4 Physical Drawbacks of Solutions

Unfortunately, these solutions may not have the physical interpretation that is desired. To have the correct asymptotic behavior, the connections used in this theory must follow certain falloff conditions:

$$(\mathcal{A} - \mathcal{A}_{AdS})_{r\to\infty} = \mathcal{O}(1), \qquad (\bar{\mathcal{A}} - \bar{\mathcal{A}}_{AdS})_{r\to\infty} = \mathcal{O}(1) \tag{5.22}$$

where

$$\mathcal{A}_{AdS} = U^{-1}\left(L_1 + \frac{1}{4}L_{-1}\right)U dx^+ + U^{-1}dU \tag{5.23}$$

$$\bar{\mathcal{A}}_{AdS} = -U\left(L_{-1} + \frac{1}{4}L_1\right)U^{-1}dx^- + UdU^{-1} \tag{5.24}$$

where L_1 and L_{-1} are generators of SL$(2, \mathbb{R})$.

As explained in [10], this falloff condition is equivalent to saying that A is of the form

$$A = L_1 + C \tag{5.25}$$

where C is an upper triangular matrix, with a similar condition placed on $\bar{A}$. There exists a gauge in which C has non-zero values only in the first row. It can be seen from the characteristic polynomial of this matrix A in this form that its eigenvalues must be non-degenerate.

Thus, for a solution to be asymptotically Anti-de Sitter, the eigenvalues of A must be non-degenerate. Considering the conical defect solutions that were found for $N = 4$, we can see that this is not the case for any of them. For (A) and (C) this is self-evident, and for (B), one must recall that a_{2j-1} and $-a_{2j-1}$ are both eigenvalues, so 0 is a degenerate eigenvalue of A.

Indeed, as proved in [10], all conical deficit solutions have degenerate eigenvalues, meaning that there are no solutions in this region that are consistent with the falloff conditions. This leaves the previously unmentioned conical surplus solutions. These solutions are allowed to have non-degenerate eigenvalues, which means that they can be asymptotically AdS. However, the conical surplus region brings its own problems with physical interpretation: there is no lower limit on the energy of these systems, meaning there could potentially be solutions with negatively infinite energy.

Chapter 6

Conical Solutions and Integrable Systems

6.1 Alternative Form of Smooth Connections

In order to see a potential duality between the smooth connections of the previous chapter, it is convenient to use the formulation of [12] to view these connections. We define

$$A = -i \sum_j m_j P_j + \frac{N^2 - 1}{6} i \mathrm{tr} \left(\sum_j m_j P_j \right) 1_{N \times N} \tag{6.1}$$

where the trace is renormalized to be defined as

$$\mathrm{tr}(M) = \frac{6}{N(N^2 - 1)} \sum_{j=1}^{N} M_{jj} \tag{6.2}$$

and the matrix P_j is simply defined as

$$(P_j)_{kl} = \delta_{jk} \delta_{jl} \tag{6.3}$$

and where m_j are constants. As explained in Chapter 5, this leads to smooth connections if the m_j are unique integers, which we define to be strictly decreasing. By making the shift

$$s_j = m_j + j \tag{6.4}$$

we can redefine the connection to be

$$A = -i \sum_j s_j P_j + \frac{N^2 - 1}{6} i \mathrm{tr} \left(\sum_j s_j P_j \right) 1_{N \times N} - i L_0 \tag{6.5}$$

where L_0 is the generator of $\mathrm{SL}(N, \mathbb{R})$ described in Appendix A. Note that while the m_j were strictly decreasing, the s_j may be equal to each other (but are still decreasing). We make the final shift:

$$r_j = s_j - S \tag{6.6}$$

where S is a constant, which is defined so that after some j all r_j are zero. These conditions on r_j allow us to define a Young Tableau in which the j-th row contains r_j boxes. The connection now becomes

$$A = -i \sum_j r_j P_j + \frac{iB}{N} 1_{N\times N} - iL_0 \tag{6.7}$$

where B is the total number of boxes in the Young Tableau, which can be written as a sum of the rows r_j or the columns c_j:

$$B = \sum_j r_j = \sum_j c_j \tag{6.8}$$

To find the energy associated with the connection, we use the definition from [12], which takes into account the renormalized trace:

$$\mathcal{L} = \frac{c}{12}\left(\mathrm{tr}(A^2) + \frac{1}{2}\right) \tag{6.9}$$

to find the energy, we must undergo some algebra, as laid out in (6.10) - (6.15).

$$(A)_{jj} = -i\left(r_j - \frac{B}{N} + \frac{1}{2}(N+1-2j)\right) \tag{6.10}$$

$$(A^2)_{jj} = -\left(r_j^2 + \frac{B^2}{N^2} + \frac{1}{4}(N+1-2j)^2 - 2\frac{B}{N}r_j + (N+1-2j)r_j - \frac{B}{N}(N+1-2j)\right) \tag{6.11}$$

$$\begin{aligned}\mathrm{tr}(A^2) = -\Big(\sum_j r_j^2 + \frac{B^2}{N} + \sum_j \frac{1}{4}(N+1-2j)^2 - \\ - 2\frac{B^2}{N} + BN + B - 2\sum_j jr_j - BN - B + B(N+1)\Big)\frac{6}{N(N^2-1)}\end{aligned} \tag{6.12}$$

$$\mathrm{tr}(A^2) = -\left(\sum_j r_j^2 - \frac{B^2}{N} + \frac{1}{4}\left(\frac{1}{3}N^3 - \frac{1}{3}N\right) + BN + B - 2\sum_j jr_j\right)\frac{6}{N(N^2-1)} \tag{6.13}$$

$$\mathrm{tr}(A^2) + \frac{1}{2} = -\left(-\frac{B^2}{N} + BN + B - 2\sum_j jr_j + \sum_j r_j^2\right)\frac{6}{N(N^2-1)} \tag{6.14}$$

$$\mathcal{L} = -\left(B^2 - BN^2 + N\left(-B + 2\sum_j jr_j - \sum_j r_j^2\right)\right)\frac{c}{2N^2(1-N^2)} \tag{6.15}$$

Although this is a relatively nice form, we can further simplify this equation by utilizing the relationship between the rows and the columns. If we define a step function

$$H(x) \equiv \begin{cases} 1 & \text{if } x \leq 0 \\ 0 & \text{else} \end{cases} \tag{6.16}$$

we can define the r_i in terms of the columns:

$$r_i = \sum_{j=1}^{N} H(c_j - i) \tag{6.17}$$

this then implies

$$\sum_{i=1}^{N} i r_i = \sum_{i=1}^{N} \sum_{j=1}^{N} i H(c_j - i) \tag{6.18}$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{N} i H(c_j - i) \tag{6.19}$$

$$= \sum_{j=1}^{N} \sum_{i=1}^{c_j} i \tag{6.20}$$

$$= \sum_{j=1}^{N} \frac{1}{2} c_j (c_j + 1) \tag{6.21}$$

$$\tag{6.22}$$

thus

$$-B + 2 \sum_{i=1}^{N} i r_i = \sum_{j=1}^{N} c_j^2 \tag{6.23}$$

By plugging this into (6.15), we get

$$\mathcal{L} = -\left(B^2 - BN^2 + N \sum_j (c_j^2 - r_j^2) \right) \frac{c}{2N^2(1 - N^2)} \tag{6.24}$$

6.2 Duality with $\beta = 1$ CSM

At the end of Chapter 1, we introduced the Calogero-Sutherland Model, which had the Hamiltonian

$$H = \frac{1}{2} \sum_{j=1}^{N} \left(\frac{1}{i} \frac{\partial}{\partial q_j} \right)^2 + \frac{1}{2} \left(\frac{\pi}{L} \right)^2 \sum_{i \neq j} \frac{\beta(\beta - 1)}{\sin^2 \frac{\pi}{L}(x_i - x_j)} \tag{6.25}$$

and energy values

$$E_\lambda = \sum_{j=1}^{N} (\lambda_j^2 + \beta(N+1-2i)\lambda_j) \tag{6.26}$$

which, for $\beta = 1$ naturally becomes

$$E_\lambda = \sum_{j=1}^{N} (\lambda_j^2 + (N+1-2i)\lambda_j) \tag{6.27}$$

this is remarkably similar to the energy of the smooth connections found in the previous section:

$$\mathcal{L} = \sum_{j=1}^{N} \left(r_j^2 + (N+1-2i)r_j - \left(\frac{B}{N}\right)^2 \right) \frac{c}{2N^2(1-N^2)} \tag{6.28}$$

In fact, if we take the large-N limit, they become identical up to a multiplicative constant, with $r_j = \lambda_j$.

It is important to note, however, that when $\beta = 1$, the potential of the CSM disappears, yielding the Hamiltonian:

$$H = \frac{1}{2} \sum_{j=1}^{N} \left(\frac{1}{i} \frac{\partial}{\partial q_j} \right)^2 \tag{6.29}$$

which is the Hamiltonian for N free particles. This corresponds with a Fermi gas. This confirms the findings of [13], among others, that the eigenstates of the Hamiltonian associated with free fermions correspond with a Young tableau, and that the energy level of these states is given by (6.27).

6.3 Duality with general CSMs

6.3.1 Formulation 1

Beyond the trivial case of $\beta = 1$, it would be of interest to find some duality between the smooth conical solutions and the more general CSM. One possible way to connect these two is by modifying (6.4) to

$$s_j = m_j + \beta j \tag{6.30}$$

To understand the potential problems with this, we recall the original definition of the connection, in terms of m_j in (6.1). As before, if the connection is smooth, than the Wilson loop should be trivial. Indeed, as shown in [12],

$$e^{2\pi A} = e^{\frac{N^2-1}{6} i \mathrm{tr}(\sum_j m_j P_j)} \left(1_{N\times N} + \sum_j (e^{-2\pi i m_j} - 1) P_j \right) \tag{6.31}$$

which is trivial when all of m_j are integers. Previously, this implied that s_j and therefore r_j were also all integers, which allowed us to set r_j equal to λ_j (as λ_j were defined to be integers).

However, with the use of (6.30), s_j are now no longer necessarily integers, meaning that potentially we cannot equate r_j and λ_j.

Thus, if we are to use this method, we must require that β is an integer. Once we have done this, we find the energy of the smooth conical solutions to be:

$$\mathcal{L} = \sum_j \left(r_j^2 + \beta(N+1-2j)r_j - \left(\frac{B}{N}\right)^2 + \frac{1}{12}(\beta^2-1)(N^2-1) \right) \frac{c}{2N^2(1-N^2)} \tag{6.32}$$

This is slightly less than ideal, as there is the non-vanishing term $\frac{1}{12}(\beta^2-1)(N^2-1)$ inside the sum. However, the energy differences between these states should be the same as in the general CSM for integer β.

6.3.2 Formulation 2

An alternate method of generalizing this duality to more CSM is instead of equating λ_j to r_j, we require

$$\lambda_j = \beta r_j \tag{6.33}$$

For this to be a useful statement, we must again require that β is an integer. In the $\beta = 1$ case, the energy of the smooth solutions (for the large N limit) in terms of the energy of the CSM was

$$\mathcal{L} = E_\lambda \left(\frac{c}{2N^2(1-N^2)} \right) \tag{6.34}$$

but for this new generalization, the relation becomes

$$\mathcal{L} = E_\lambda \left(\frac{c\beta^2}{2N^2(1-N^2)} \right) \tag{6.35}$$

This has the advantage of the energies continuing to be directly related, however, this formulation also leads to new problems. Looking closely at (6.33), λ_j can only be a multiple of β, meaning there are some energy levels of the CSM that do not correspond to smooth conical solutions. This is suboptimal, as ideally we would see a one-to-one correspondence between these two systems.

6.4 Conclusions

Although we have not found a perfect duality between Calogero-Sutherland Models and the smooth connections of (2+1) dimensional higher spin AdS space-times, an interesting connection exists nonetheless. It is likely that the similarities between the allowed energy levels of these systems has to do with the fact that both can be represented in Young Tableaux, which in turn is related to their connection with the integers.

Appendix A

Properties of Specific Lie Groups

A.1 Lie Algebra

The focus of this paper is on three Lie Groups: SO(2,2), SL(2, $\mathbb{R}$), and SL(N, $\mathbb{R}$). As such, the Lie algebra associated with these groups are very important, so they are explicitly stated in this section.

A.1.1 SO(2,2)

Called the Special Orthogonal group, this is a subgroup of the Special Linear group. It contains all 4×4 matrices which are orthogonal with a metric of sign $(+,+,-,-)$ and have a determinant of 1. The algebra of this group, as stated in Chapter 2, is:

$$[J_a, J_b] = \epsilon_{ab}{}^c J_c, \qquad [J_a, P_b] = \epsilon_{ab}{}^c P_c, \qquad [P_a, P_b] = \frac{1}{l^2}\epsilon_{ab}{}^c J_c \tag{A.1}$$

A.1.2 SL(2, $\mathbb{R}$)

This is the Special Linear Group for 2×2 matrices. It is a subgroup of the General Linear Group, for which the determinant of the matrices is always 1. There are three generators of this group: L_0 and $L_{\pm 1}$. They have the commutation relation:

$$[L_i, L_j] = (i-j)L_{i+j} \tag{A.2}$$

It is additionally important for some calculations in this paper that $(L_{-1})^2 = (L_1)^2 = 0$ and $(L_0)^2 = 1_{2\times 2}$.

A.1.3 SL(N, $\mathbb{R}$)

Here we have the Special Linear Group for $N \times N$ matrices, which is a generalization of SL(2, $\mathbb{R}$). There are $N^2 - 1$ generators of this group; we use a basis in which we have a principle embedding of SL(2, $\mathbb{R}$), denoted again by $\{L_0, L_{\pm 1}\}$. Additionally, we introduce generators of the form $W_m^{(s)}$. Together, we have the commutation rules:

$$[L_i, L_j] = (i-j)L_{i+j}$$
$$[L_i, W_m^{(s)}] = (i(s-1)-m)W_{i+m}^{(s)} \tag{A.3}$$

in which s runs from 3 to N, m runs from $(1-s)$ to $(s-1)$, and as before i, j are -1, 0 or 1.

We can express these additional generators in terms of those in the principal embedding as in [10]:

$$W_m^{(s)} = (-1)^{s-m-1}\frac{(s+m-1)!}{(2s-2)!}\underbrace{[L_{-1}, [L_{-1}, \ldots, [L_{-1}}_{s-m-1 \text{ terms}}, L_1^{s-1}]\ldots]] \tag{A.4}$$

A.2 Equivalence of Lie Groups

In transitioning between our two methods of finding a gauge for the BTZ Black Holes, we used the property that SO(2,2) is isomorphic to SL(2, $\mathbb{R}$)$\times$SL(2, $\mathbb{R}$). This is one from a group of isomorphism known as *accidental isomorphisms*. Many simple and semi-simple Lie Groups of lower dimensionality have these isomorphisms, that do not follow a pattern (hence the name). Many of these are listed in [14], among other publications.

A.3 Trace of Special Linear Generators

As made explicit in Section A.4, the generators of SL(2, $\mathbb{R}$) and the principle embedding of this group within SL(N, $\mathbb{R}$) all have trace equalling 0. In fact, all generators of SL(N, $\mathbb{R}$) are traceless. To see this, we follow the proof from [15].

We begin with a definition: Assuming that there exists a function $g : M \to M'$ such that $g(P_0) = P_0'$, where M and M' are Lie Groups with associated Lie Algebra Λ and Λ', respectively. We define $g_* : \Lambda \to \Lambda'$ as a function which satisfies

$$(g_*\lambda)(f') = \lambda(f'g) \tag{A.5}$$

where f' is any function on M' and $\lambda \in \Lambda(P_0)$.

It directly follows from this that

$$g_*\left(\frac{dP}{dt}\bigg|_{t=0}\right) = \frac{d}{dt}(gP)|_{t=0} \tag{A.6}$$

Next, we consider the determinant of the matrix $(\mathbb{1} + tA)$, where A is a general matrix. Calculating the determinant, we find that

$$\det(\mathbb{1} + tA) = 1 + t\mathrm{Tr}(A) + \cdots \tag{A.7}$$

where the ellipses represent terms with higher order of t. Thus:

$$\det{}_*(A) = \det{}_*\left(\frac{d}{dt}(\mathbb{1} + tA)\bigg|_{t=0}\right) = \frac{d}{dt}\det\,(\mathbb{1} + tA)|_{t=0} = \mathrm{Tr}(A) \tag{A.8}$$

Finally, we consider an element in SL(N, $\mathbb{R}$), $P(t)$. We can then say:

$$\mathrm{Tr}\left(\left.\frac{dP}{dt}\right|_{t=0}\right) = \det_*\left(\left.\frac{dP}{dt}\right|_{t=0}\right) = \frac{d}{dt}\det(P|_{t=0}) = \frac{d}{dt}1 = 0 \tag{A.9}$$

Thus, all of the generators of SL(N, $\mathbb{R}$) are traceless.

A.4 Matrix Representations of Generators

It is often helpful, while doing calculations with the generators of Lie Groups, to have an explicit formulation. For this reason, we include here the matrix representations associated with the major generators of two of the groups that we deal heavily with: SL(2, $\mathbb{R}$) and SL(N, $\mathbb{R}$).

A.4.1 SL(2, $\mathbb{R}$)

The three generators of this group can be represented as 2×2 matrices as follows:

$$L_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad L_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad L_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{A.10}$$

These are clearly traceless, and follow the commutation rules of the associated algebra.

A.4.2 SL(N, $\mathbb{R}$)

As defined in [10], the matrix representation of the principal embedding of SL(2, $\mathbb{R}$) within SL(N, $\mathbb{R}$) can be taken as

$$L_{-1} = \begin{pmatrix} 0 & \sqrt{N-1} & & \cdots & & & 0 \\ \vdots & 0 & \sqrt{2(N-2)} & & & & \\ & \vdots & \ddots & \ddots & & & \\ & & & 0 & \sqrt{|i(N-i)|} & & \\ & & & & & 0 & \sqrt{(N-1)} \\ 0 & \cdots & & & & & 0 \end{pmatrix} \tag{A.11}$$

$$L_0 = \frac{1}{2}\begin{pmatrix} (N-1) & 0 & \cdots & & & 0 \\ 0 & (N-3) & & & \vdots & \\ \vdots & & (N+1-2i) & & & \\ & & & \ddots & & \\ & & & & -(N-3) & \\ 0 & \cdots & & & & -(N-1) \end{pmatrix} \tag{A.12}$$

$$L_1 = -\begin{pmatrix} 0 & \cdots & & & & & 0 \\ \sqrt{N-1} & 0 & & & \cdots & & \\ 0 & \sqrt{2(N-2)} & 0 & & & & \\ \vdots & & \ddots & \ddots & & & \\ & & & \sqrt{|i(N-i)|} & 0 & & \\ & & & & \ddots & \ddots & \\ 0 & \cdots & & & & \sqrt{(N-1)} & 0 \end{pmatrix} \tag{A.13}$$

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