Image Denoising

Total Variation and the ROF Problem

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Problem Setup

Represent images as discretizations of functions $u : \mathbb{R}^2 \to \mathbb{R}$

- $u(x, y) : \Omega \to \mathbb{R}$ for each pixel (x, y) on bounded, closed Ω

Image Noise

$u_0(x, y) = u(x, y) + n(x, y)$

- $u_0(x, y)$: observed image
- $u(x, y)$: desired image
- $n(x, y)$: additive white noise

Total Variation

Continuous:

$$TV(u) = \int_{\Omega} ||\nabla u|| \, dx \, dy$$

Discrete:

$$TV(u) = \sum_{i,j} \sqrt{(u_{i+1,j} - u_{ij})^2 + (u_{i,j+1} - u_{ij})^2}$$

We will use $|| \cdot ||$ as the L2-norm. In 2D:

$$||\nabla u|| = \sqrt{u_x^2 + u_y^2}$$

Rudin, Osher, Fatemi (1992)

Optimization problem:

$$\min_{u} \; TV(u) \quad \text{s.t.} \quad \begin{aligned} \int_\Omega u \, dx \, dy &= \int_\Omega u_0 \, dx \, dy \\ \int_\Omega \frac{1}{2}(u - u_0)^2 \, dx \, dy &= \sigma^2 \end{aligned}$$

Regularization:

$$\min_{u} \; TV(u) + \frac{\lambda}{2} \int_\Omega (u - u_0)^2 \, dx \, dy$$

$$\min_{u} \quad f(u) + \lambda \, g(u)$$

(Derivation)

$$g(u) = \frac{1}{2}||u - u_0||^2_{L^2(\Omega)}$$

$$\partial_u g(u) = u - u_0$$

$$f(u) = \int_\Omega \sqrt{u_x^2 + u_y^2}\, dx\, dy$$

$$\partial_u f(u) = -\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) - \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right)$$

$$= -\nabla \cdot \left(\frac{\nabla u}{||\nabla u||}\right)$$

The PDE

The minimization problem

$$\min_u \; TV(u) + \frac{\lambda}{2}\int_\Omega (u-u_0)^2\,dx\,dy \quad \text{s.t.} \int_\Omega u\,dx\,dy = \int_\Omega u_0\,dx\,dy$$

is equivalent to solving the nonlinear elliptic PDE

$$\nabla \cdot \left(\frac{\nabla u}{||\nabla u||}\right) - \lambda(u-u_0) = 0$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \qquad u \in \partial\Omega$$

Time-Dependent PDE

Now: $u(x, y, t)$ with $u_0 = u(x, y, 0)$ given

Our PDE is:

$$\begin{cases} u_t = \nabla \cdot \left(\frac{\nabla u}{||\nabla u||} \right) - \lambda(u - u_0) & u \in \Omega \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & u \in \partial\Omega \end{cases} \tag{1}$$

Nonlinear Laplacian

$$\begin{aligned}\nabla \cdot \left(\frac{\nabla u}{||\nabla u||} \right) &= \frac{||\nabla u|| \nabla^2 u - \nabla u \cdot \nabla ||\nabla u||}{||\nabla u||^2} \\ &= \frac{1}{||\nabla u||} \nabla^2 u - \frac{\nabla ||\nabla u||}{||\nabla u||^2} \cdot \nabla u \\ &= \underbrace{\frac{1}{||\nabla u||} \nabla^2 u}_{\text{diffusion}} - \frac{\nabla u}{||\nabla u||^3} \cdot \nabla u\end{aligned}$$

ROF Time Marching

$$u_t = \frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{u_x^2 + u_y^2}}\right) - \lambda(u - f)$$

$$\frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t} = \frac{D_x(u_{ij}^n) - D_x(u_{i-1,j}^n)}{\Delta x} + \frac{D_y(u_{ij}^n) - D_y(u_{i,j-1}^n)}{\Delta y} - \lambda(u_{ij}^n - f_{ij}^n)$$

where

$$D_x(u_{ij}) \approx \frac{u_x}{\sqrt{u_x^2 + u_y^2}}$$

ROF Time Marching

$$D_x(u_{ij}) \approx \frac{u_x}{\sqrt{u_x^2 + u_y^2}}$$

$$\approx \frac{u_{i+1,j} - u_{ij}}{\sqrt{(u_{i+1,j} - u_{ij})^2 + (m(u_{i,j+1} - u_{ij}, u_{ij} - u_{i,j-1}))^2}}$$

$$m(a, b) = \mathrm{minmod}(a, b)$$

$$= \left(\frac{\mathrm{sign}(a) + \mathrm{sign}(b)}{2}\right)(\min(|a|, |b|))$$

$$= \begin{cases} 0 & \mathrm{sign}(a) \neq \mathrm{sign}(b) \\ \min(|a|, |b|) & a > 0, b > 0 \\ -\min(|a|, |b|) & a < 0, b < 0 \end{cases}$$

Centered Difference?

Centered differences can miss thin structures

- eg. [0.7, 0.5, 0.3]

Forward difference:

$$|0.5 - 0.7| + |0.3 - 0.5| = 0.4$$

Centered difference:

$$|0.7 - 2 * 0.5 + 0.3| = 0$$

Nondifferentiability

Not differentiable when $||\nabla u|| = 0$:

$$\frac{\nabla u}{||\nabla u||}$$

Add perturbation β:

$$\frac{1}{\sqrt{||\nabla u||^2 + \beta}}$$

Results