Inversion of Laplace transform of a linear combination of point masses

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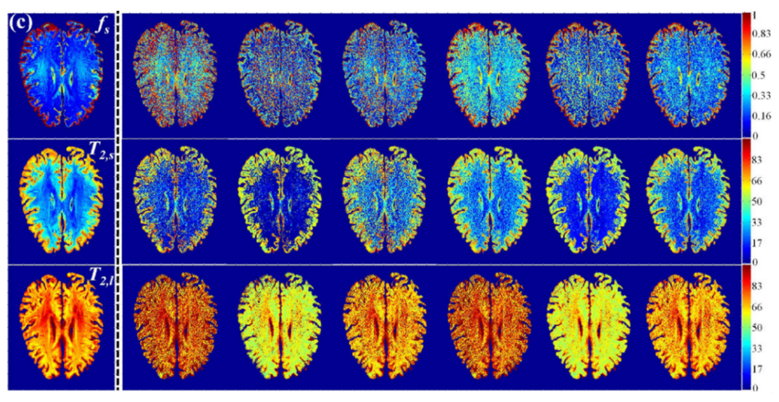
Introduction

The objective of this research project is to investigate a new approach for the inverse problem of finding a finitely supported measure on the real line given samples of its Laplace transform. The problem appears, for example, in the analysis of magnetic resonance relaxometry data. Sabett et al. show that determining the distribution of magnetic resonance transverse relaxation times based on the inverse Laplace transform of decay data is important for assessing the relaxation of tissue in biomedicine [1], which Bouhrara et al. apply to the imaging the human brain [2]. This is a notoriously ill-posed problem. Existing algorithms in the literature require a prior knowledge of the number of points in the support of the measure. We propose an alternative method to determine this number as well as to determine the measure itself, based on the work on Mhaskar and Chui [3].



Applications to brain MR imagining to estimate parameters characterising transverse relaxation models [2], described by the signal decay function:

$$A(\theta; \mathbf{TE_n}) = A_0(f_n \exp(-\frac{TE_n}{T_{2,s}}) + (1-f_s)\exp(-\frac{TE_n}{T_{2,l}}))$$

Theory

The central problem is that given observations of the form:

$$f(t) = \sum_{j=1}^{K} a_j e^{2tt_j}$$

we want to recover the values:

$$K \in \mathbb{Z}^+, t_1, \ldots, t_K \in \mathbb{R}, a_1, \ldots, a_K \geq 0$$

By taking advantage of the Mehler identity, we can write:

$$\begin{aligned} F(t) &= (\pi w)^{-1/2} \exp(\sqrt{5}t^2/2) f(t) \\ &= \sum_{k=0}^{\infty} \mu_k w^k \psi_k(t) \\ \text{where } \mu_k &= \sum_{j=1}^{K} a_j \exp(-\sqrt{5}t_j^2/2)\psi_k(t_j) \end{aligned}$$

This form allows us to compute μ_k using Gauss-Hermite quadrature formulas:

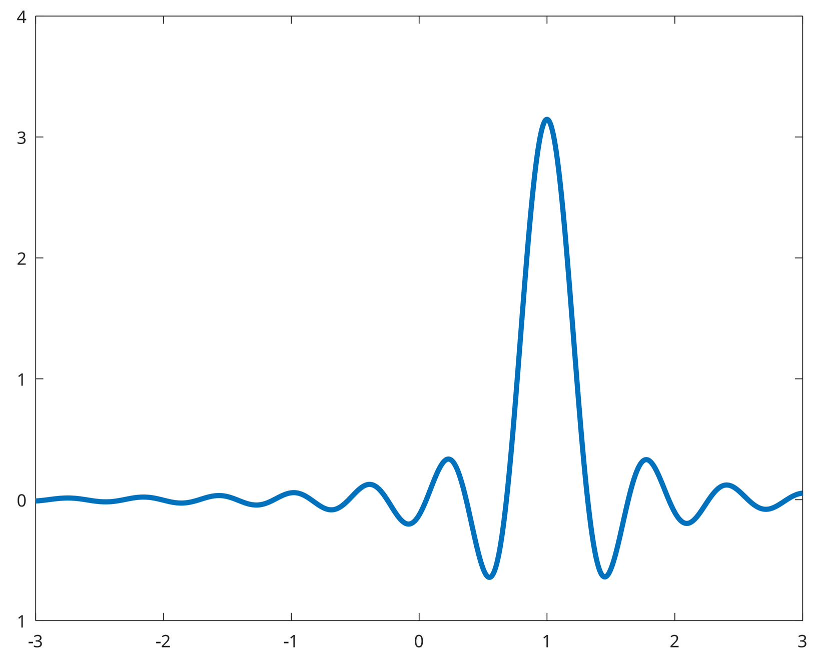
$$\begin{aligned} \mu_k &= w^{-k} \int F(t)\psi_k(t)dt \\ &\approx w^{-k} \sum \lambda_{j,2n} F(x_{j,2n}) \end{aligned}$$

where $x_{j,2n}$ are the zeros of the Hermite polynomial of degree 2n, and $\lambda_{j,2n}$ are the corresponding Coates' numbers. It is important that the Gauss-Hermite quadrature rules are precisely constructed to avoid numerical instability problems that are introduced by noise and by high degrees n (>64).

Using Mhaskar's earlier work [4], we compute summability kernels:

$$\Phi_n(x,t) = \sum_{k=1}^{n} H(k/n)\psi_k(x)\psi_k(t)$$

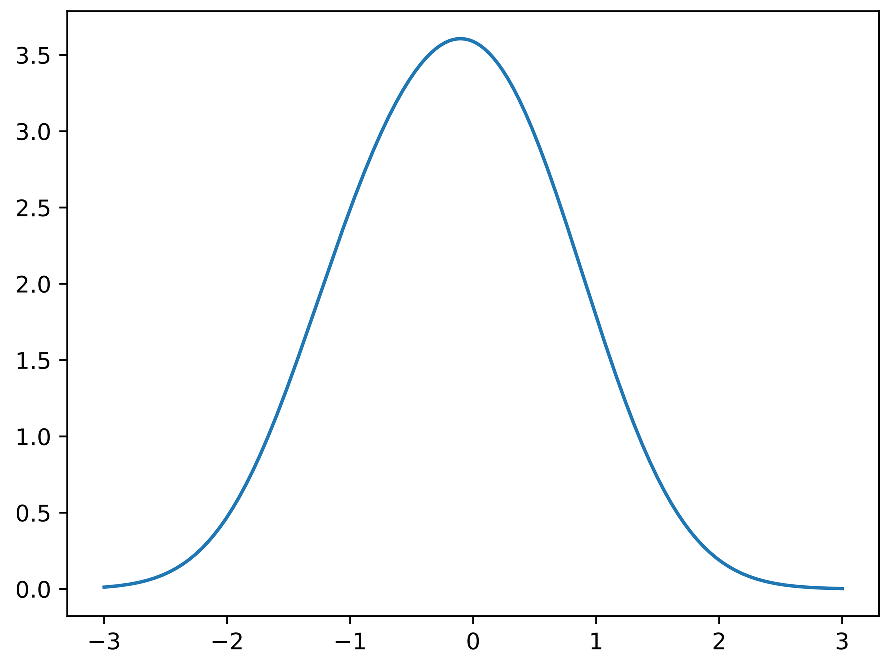
The kernel exhibits highly localised behaviour, allowing peaks to be easily discerned.



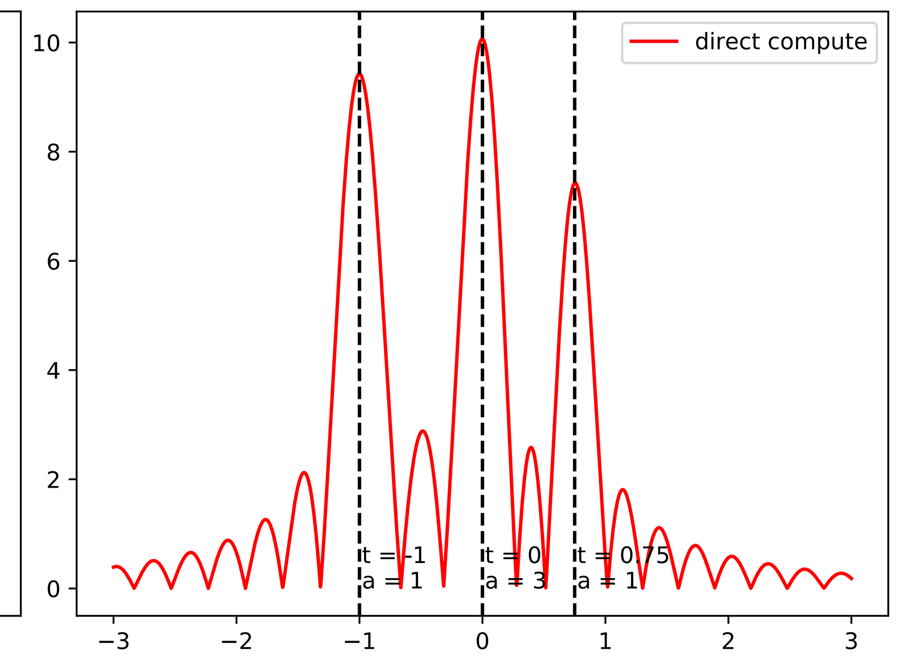
Graph of summability kernel (n=64, peak at 1).

By plotting the sum of kernels, we can identify the peaks and their corresponding magnitudes, thus giving us values of t_j and a_j respectively.

$$\mathcal{T}_n(x) = \sum_{k=1}^{n} H(k/n)\mu_k\psi_k(x)$$



Graph of $F(t)$ for -3 < t < 3



Corresponding graph of $|\mathcal{T}_n(x)|$, n=64

$$K = 3, \{t_j\}_{j=1}^{K} = \{-1, 0, 0.75\}, \{a_j\}_{j=1}^{K} = \{1, 3, 1\}$$

Algorithm

Given observations $F(x_{k,2n})$,

compute $\mu_k \approx w^{-k} \sum_{j=1}^{2n} \lambda_{j,2n} F(x_{j,2n})$ for $k = 1, \ldots, n$

and plot $|\mathcal{T}_n(x)|$ where $\mathcal{T}_n(x) = \sum_{k=1}^{n} H(k/n)\mu_k\psi_k(x)$

Identify peaks and magnitudes to retrieve a_j, t_j

Further research

1) Addressing noisy observations
2) Allowing scattered samples (including equidistant samples)
3) Extending to multiple dimensions
4) Extending beyond point masses

References

[1] Sabett, Christiana, Ariel Hafftka, Kyle Sexton, and Richard G. Spencer. "L 1 , L p , L 2 , and Elastic Net Penalties for Regularization of Gaussian Component Distributions in Magnetic Resonance Relaxometry." Concepts in Magnetic Resonance Part A 46A, no. 2 (March 2017): e21427.

[2] Bouhrara, Mustapha, David A. Reiter, and Richard G. Spencer. "Bayesian Analysis of Transverse Signal Decay with Application to Human Brain: Bayesian Analysis of Transverse Signal Decay with Application to Human Brain." Magnetic Resonance in Medicine 74, no. 3 (September 2015): 785–802.

[3] Chui, Charles K., and H.N. Mhaskar. "A Fourier-Invariant Method for Locating Point-Masses and Computing Their Attributes." Applied and Computational Harmonic Analysis 45, no. 2 (September 2018): 436–52.

[4] Mhaskar, H. N. "Local Approximation Using Hermite Functions." In Progress in Approximation Theory and Applicable Complex Analysis: In Memory of Q.I. Rahman, edited by Narendra Kumar Govil, Ram Mohapatra, Mohammed A. Qazi, and Gerhard Schmeisser, 341–62. Springer Optimization and Its Applications. Cham: Springer International Publishing, 2017.