August 4, 2019

Dear Adam,

Looks like you completely misunderstood me. Here is another attempt.

Suppose $K \ge 1$ is an integer, $t_1, \cdots, t_k \in \mathbb{R}$, and $b_1, \cdots, b_K \ge 0$. The fundamental problem is to find these quantities from observations of the form $\sum_{j=1}^K b_j \exp(2tt_j)$.

My first observation is the following. Writing $a_j = b_j \exp(-(\sqrt{5}/2)t_j^2)$, $w = \dfrac{\sqrt{5}-1}{2}$ we consider

$$F(t) = (\pi w)^{-1/2} \exp(-(\sqrt{5}/2)t^2) \sum_{j=1}^K a_j e^{2tt_j}. \tag{1}$$

From observations of F, my claim is that we can find K, t_j's and a_j's. Knowing t_j's we can then compute $b_j = a_j \exp((\sqrt{5}/2)t_j^2)$.

As we calculated a couple of times,

$$F(t) = \sum_{k=0}^{\infty} \mu_k \psi_k(t) w^k, \tag{2}$$

where

$$\mu_k = \sum_{j=1}^K b_j \psi_k(t_j). \tag{3}$$

Of course, we don't know μ_k's. We can only sample $F(t)$ for whatever values of t. My proposal is to consider $F(x_{\ell,2n})$, $\ell = 1, \cdots, 2n$, where $x_{\ell,2n}$'s are the zeros of the Hermite polynomials of degree $2n$. Using Townsend's code, you have the values of the corresponding Cotes' numbers $\lambda_{\ell,2n}$, so that

$$\sum_{\ell=1}^{2n} \lambda_{\ell,2n} P(x_{\ell,2n}) = \int_{-\infty}^{\infty} P(t) \exp(-t^2) dt$$

for all polynomials of degree at most $4n-1$. With the notation $W_{\ell,2n} = \lambda_{\ell,2n} \exp(x_{\ell,2n}^2)$, this implies in particular that

$$\sum_{\ell=1}^{2n} W_{\ell,2n} \psi_k(x_{\ell,2n}) \psi_m(x_{\ell,2n}) = \delta_{m,k}, \qquad m, k = 0, \cdots, 2n-1. \tag{4}$$

Therefore, my proposal is to compute

$$\mu_k = w^{-k} \sum_{\ell=1}^{2n} W_{\ell,2n} F(x_{\ell,2n}) \psi_k(x_{\ell,2n}), \qquad k = 0, \cdots, n. \tag{5}$$

Please not that F is sampled only at the points $x_{\ell,2n}$.

With μ_k computed in this way, compute

$$\mathcal{T}_n(x) = \sum_{k=1}^{n} H(k/n) \mu_k \psi_k(x), \tag{6}$$

for many many points x on the real line. For $n = 128$, I think say 2048 equidistant points on $[-16, 16]$ are enough. Then plot $|\mathcal{T}_n(x)|$ at these points.

To summarize, with $n = 128$,

1. Construct $F(t)$ by choosing say $K = 1$, $a_1 = 1/2$, $a_2 = 1$ (i.e., write a code for computing this with the input parameters K, $a_1, \cdots, a_K$, $t_1, \cdots, t_k$, and the array t of points at which you wish to compute $F(t)$.

2. Evaluate $F(x_{\ell,2n})$ at the $2n$ Gauss-Hermite quadrature nodes given by Townsend's code.

3. Let $W_{\ell,2n} = \lambda_{\ell,2n} \exp(x_{\ell,2n}^2)$, where $\lambda_{\ell,2n}$ are the quadrature weights given by Townsend's code.

4. Compute for $k = 1, \cdots, n$
$$\mu_k = w^{-k} \sum_{\ell=1}^{2n} W_{\ell,2n} F(x_{\ell,2n}) \psi_k(x_{\ell,2n}).$$
5. Compute
$$\mathcal{T}_n(x) = \sum_{k=1}^{n} H(k/n) \mu_k \psi_k(x),$$
at many many points on $[-n\sqrt{2}, n\sqrt{2}]$.
6. Plot $|\mathcal{T}_n(x)|$ and look where the peaks are.

With best wishes, With best regards,
Hrushikesh