July 15, 2019

Dear Adam,

Suppose $n \geq 1$ is an integer, $y_M < y_{M-1} < \cdots < y_1$ be points in $\mathbb{R}$ distributed sufficiently densely in an interval of the form $[-c\sqrt{n}, c\sqrt{n}]$. Denoting by h_k the orthonormalized Hermite polynomial of degree k, we wish to find w_j such that for $k = 0, \cdots, n-1$,

$$\sum_{j=1}^{M-1} w_j \sqrt{(y_j - y_{j+1})} \exp(-y_j^2/2)\mathbf{h}_k(y_j) = \begin{cases} 1, & \text{if } k = 0, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Denoting by Λ the diagonal matrix with entries $\sqrt{(y_j - y_{j+1})} \exp(-y_j^2/2)$, by V the matrix whose (j,k)-th entry is $\mathbf{h}_k(y_j)$, and by $\mathbf{e}$ the right hand side of (1), we need to solve

$$(\Lambda V)^T \mathbf{w} = \mathbf{e}. \tag{2}$$

This can be done either by minimal residual method or by finding $\mathbf{w}$ with minimal norm subject to the conditions (2). In the first case, one has to solve

$$\mathbf{w} = (\Lambda V V^T \Lambda)^{-1} \Lambda V \mathbf{e}. \tag{3}$$

In the second case, it is

$$\mathbf{w} = \Lambda V (V^T \Lambda^2 V)^{-1} \mathbf{e}. \tag{4}$$

Clearly, you should not find the inverse but use something like the conjugate residual to solve a square equation indicated by either (3) or (4). I have proved in the paper with Demanet and Batenkov that if the y_j's are sufficiently dense, then the system is well conditioned.

Here, the challenges are to find the right value of n for a given data. Also, the evaluation of $\exp(-y_j^2/2)\mathbf{h}_k(y_j)$ poses numerical challenges. I hope Xiaosheng has done this evaluation somewhere in his code. I am aware that the papers of Gil et. al. https://arxiv.org/pdf/1709.09656.pdf and the paper of Townsend https://arxiv.org/pdf/1410.5286.pdf are useful in this context, especially the paper of Gil. If Xiaosheng has neither done this calculation nor is he willing to explain, then we will need to scratch our heads together.

With best regards,

Hrushikesh