Com Sci 330 Assignment 9

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Problem 1

(a) $$f(n) = \begin{cases} \dfrac{-5n}{2}, & \text{if } n \text{ is even} \\ \dfrac{5(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

where $n \in \mathbb{N}$.

(b) $$f(n) = \begin{cases} \dfrac{-5n + 2\lfloor \frac{n}{14} \rfloor}{2}, & \text{if } n \text{ is even} \\ \dfrac{5(n+1+2\lfloor \frac{n+1}{14} \rfloor)}{2} & \text{if } n \text{ is odd} \end{cases}$$

where $n \in \mathbb{N}$.

(c) $$f(n) = \begin{cases} \left(-\dfrac{n}{4},\ 0\right) & \text{if } n \mod 4 = 0 \\ \left(-\dfrac{n+3}{4},\ 1\right) & \text{if } n \mod 4 = 1 \\ \left(\dfrac{n+2}{4},\ 0\right) & \text{if } n \mod 4 = 2 \\ \left(\dfrac{n+1}{4},\ 1\right) & \text{if } n \mod 4 = 3 \end{cases}$$

where $n \in \mathbb{N}$.

Problem 2

Let A and $A \subseteq B$. By definition of countability, there is no function $f : \mathbb{N} \to A$ such that f is one-to-one. This can be restated as $\forall f\ \exists a \in A\ \forall\ n \in$

$N\ f(n) \neq a$. Furthermore, since $A \subseteq B,\ a \in B$, there also exists no one-to-one function from $\mathbb{N}$ to B, proving that B is uncountable.

Problem 3

Assume for contradiction there exists an enumeration of containing all the functions from $\mathbb{N}$ to $\{0,\ 1, 2,\ \ldots\ ,\ 9\}$, denoted as $f_0,\ f_1,\ f_2,\ \ldots\ , f_i$. Let G be defined as the following:

$$G(i) = \begin{cases} 1 \text{ if } f_i(i) \geqslant 5 \\ 9 \text{ if } f_i(i) \leqslant 5 \end{cases} \quad , G :\ \mathbb{N} \to \{0,\ 1, 2,\ \ldots\ ,\ 9\}$$

$\forall n \in N,\ G(n) \neq f_n(n)$, so G is not in the enumeration, a contradiction. Therefore the set is uncountable.

Problem 4

For both parts, let S be the set described in the problem.

(a) Countable. Proof:

Let each number in the set be represented by an ordered pair $(x,\ y)$ where x is the number of ones before the decimal point and y the number after. Since $x,\ y$ are non-negative integers, $(x,\ y) \in \mathbb{N} \times \mathbb{N}$, which we proved is countable in class; therefore the set is countable.

An example enumeration to further prove countability (note that a 0 is the absence of a 1):

0.0, 1.0, 0.1, 11.0, 1.1, 0.11, 111.0, 11.1, ...

(b) Uncountable. Proof:

Let 0 symbolize 1 and and 1 symbolize 9. Through a combination of 0, 1 and a single decimal point we can represent any real number with an infinite length binary decimal; therefore the set is one-to-one with $\mathbb{R}$ and by definition of cardinality has the same size. Since $\mathbb{R}$ is uncountable (proved in class) and $|S| = |\mathbb{R}|$, S is uncountable.

Problem 5

(a) $A = \mathbb{R},\ B = \mathbb{R}$

(b) $A = \mathbb{R},\ B = \mathbb{R} - \mathbb{Z}$

(c) $A = \mathbb{R} \times \{0,\ 1\},\ B = \mathbb{R} \times \{0\}$

Extra Credit:

The set S is countable because it can be enumerated: $\{0,\ 1,\ 2,\ \ldots,\ 9,\ /,\ 00,\ 01,\ 02,\ \ldots\}$

Positive rationals are defined as $\{a/b \mid a,\ b \in \mathbb{Z}^+\}$. $a,\ b$ can be represented by a string of decimal digits, which are in the set, and the division symbol in the set also lets us represent the quotient of those two numbers. Since there exists a string in S that represent each positive rational, and S is countable, positive rationals are also countable.