Com Sci 330 Assignment 7

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April 1, 2014

Problem 1

(a) **Base Case:** $n = 1$

$6 \times 1 = 6$. $6 \in S$ by the base definition of S.

Inductive Step: Assume $6k \in S$. Prove $6(k+1) \in S$.

$6(k+1) = 6k + 6$. $6k \in S$ by the Inductive Hypothesis; $6 \in S$ by the base definition of S. Therefore $6(k+1) \in S$ by the inductive step of the definition of S.

Therefore $A \subseteq S$

(b) **Basis:** $6 \in S$. $6 \times 1 = 6$, $1 \in \mathbb{Z}^+$, therefore $6 \in A$

Inductive Step: Consider $x, y \in S$. Assume $x, y \in A$ prove $x + y \in A$

$$\begin{aligned} x, y &= 6a, 6b \text{ where } a, b \in \mathbb{Z} && \text{Inductive Hypothesis} \\ x + y &= 6a + 6b \\ &= 6(a+b) \end{aligned}$$

$a + b \in \mathbb{Z}$, so $6(a+b) \in A$. $6(a+b) = x + y$; therefore $x + y \in A$, concluding the inductive step.

Problem 2

(a) (1,3), (3,1)
(2, 6), (4,4), (6,2)
(3,9), (5, 7), (7, 5), (9, 3)
(4, 12), (6, 10) (8, 8), (10, 6), (12, 4)
(5, 15), (7, 13), (9, 11), (11, 9), (13, 7), (15, 5)

(b) **Base Case:** Zero applications of the inductive step of the definition.

Zero applications of the inductive step gives us the base case, $(0,\ 0)$. $0 + 0 = 0$; $4 \mid 0$, proving the base case.

Inductive Step: Assume that with k or fewer applications of the recursive definition of S we have $(x,\ y) \in S$, $4 \mid x + y$. Let $(x',\ y')$ denote the result of applying the recursive step one more time. Prove $4 \mid x' + y'$.

There are two cases. Either $(x',\ y') = (x + 1,\ y + 3)$ or $(x',\ y') = (x + 3,\ y + 1)$. In both cases, $x' + y' = x + y + 4$. By the inductive hypothesis, $x + y = 4a, a \in \mathbb{Z}$; therefore $x' + y'$ can be written as $4(a + 1)$. Since $a + 1 \in \mathbb{Z}$, $4 \mid x' + y'$, concluding the inductive step.

(c) **Base Case:** $(0,\ 0)$. $0 + 0 = 0$ and $4 \mid 0$, proving the base case.

Inductive Step: Assume $(x,\ y) \in S$ and $4 \mid x + y$. Let $(x',\ y')$ denote the result of applying the recursive step to $(x,\ y)$. Prove $4 \mid x' + y'$. There are two cases. Either $(x',\ y') = (x + 1,\ y + 3)$ or $(x',\ y') = (x + 3,\ y + 1)$. In both cases, $x' + y' = x + y + 4$. By the inductive hypothesis, $x + y = 4a, a \in \mathbb{Z}$; therefore $x' + y'$ can be written as $4(a + 1)$. Since $a + 1 \in \mathbb{Z}$, $4 \mid x' + y'$, concluding the inductive step.

(d) Let a be the sum of $(x,\ y) \in S$. With each application of the recursive step a increases. Since by the second application of the recursive step $a = 8$. Since $2 + 2 = 4 < 8$ and $(2,\ 2)$ is not in the base case or produced by the first application of the recursive step, $(2,\ 2) \notin S$.

To change the recursive definition to include all positive integers divisible by 4, add the following: if $(x,\ y) \in S$, $(x + 2,\ y + 2) \in S$, $(x + 4,\ y) \in S$, and $(x,\ y + 4) \in S$.

Problem 3

Base Case: Consider the full binary tree consisting of a single vertex. This tree has one leaf and zero internal nodes. Therefore the number of leaves is one more than the number of internal nodes, proving the base case.

Inductive Step: Let T_1 and T_2 be left and right subtrees of T. By the inductive hypothesis, for both T_1 and T_2 the number of leaves is one greater than the number of internal nodes. T combines T_1 and T_2 with an additional internal node (the root node); therefore the number of leaves is $2 - 1 = 1$ greater than the number of internal nodes, concluding the inductive step.

Taken together, the base case and inductive steps prove the proposition that for any full binary tree, the number of leaves is one greater than the number of internal nodes.

Problem 4

Base Definition: $\epsilon,\ a,\ b,\ c \in S$.

Inductive Definition: If $s \in S$,

$$\begin{aligned} \mathrm{a}s\mathrm{a} &\in S \\ \mathrm{b}s\mathrm{b} &\in S \\ \mathrm{c}s\mathrm{c} &\in S \end{aligned}$$

Problem 5

(a) **Base Case:** $1 \in S$

Inductive Cases: $n \in S \to 2n, 3n, 5n \in S$

(b) **Base Case:** $1 \in S$

Inductive Cases: $n \in S \to 5n, 15n, 18n \in S$

Problem 6

(a) **Base Case:** $(0,\ 0) \in S$

Inductive Step: If $(x,\ y) \in S$, then

i $(x + 3,\ y) \in S$
ii $(x - 3,\ y) \in S$
iii $(x,\ y + 3) \in S$
iv $(x,\ y - 3) \in S$
v $(x - 1,\ y - 2) \in S$
vi $(x - 2,\ y - 1) \in S$
vii $(x + 1,\ y + 2) \in S$

viii $(x+2,\ y+1) \in S$

(b) I prove $L' \subseteq L$ using structural induction.

Base Case: $(0,\ 0)$

$0+0=0$; $0 \mod 3 = 0$, proving the base case.

Inductive Step: Consider $(x,\ y) \in S$, with $x+y \mod 3 = 0$. Let (x', y') denote the application of the recursive step of the definition of S on $(x',\ y')$. We prove that for all possible values of $(x',\ y')$ that $3 \mod x'+y' = 0$.

Lemma 1: If $3 \mid a$ and $3 \mid b$, $a+b \mod 3 = 0$. Proof: Since $3 \mid a, b$ then $a+b$ can be written as $3(m+n)$, $m, n \in \mathbb{Z}$. Since $m+n$ is also $\in \mathbb{Z}$, $3 \mid a+b$ and $a+b \mod 3 = 0$.

In all cases, $x'+y' = x+y \pm 3$. Since $3 \mid x+y$ and $3 \mid 3, -3$, by Lemma 1 $x'+y' \mod 3 = 0$, concluding the inductive step.

(c) Not attempted.

Extra Credit

Consider $(3,\ 3)$. $3+3=6$, and $6 \mod 3 = 0$, so $(3,\ 3) \in S$. However, $(3,\ 3)$ can be constructed in multiple ways, two of which I'll enumerate:

$$(0,\ 0) \to (0,\ 3) \to (3,\ 3)$$
$$(0,\ 0) \to (1,\ 2) \to (3,\ 3)$$

The fact that $(3,\ 3)$ can be formed through 2 ways proves that our definition is ambiguous. This definition is not ambiguous:

Base Case: $(0,\ 0) \in S$

Recursive Definition: If $(x,\ y) \in S$, then

$$(x+3,\ y) \in S \leftrightarrow x \geqslant 0 \bigwedge 3 \mid x,\ y$$
$$(x-3,\ y) \in S \leftrightarrow x \leqslant 0 \bigwedge 3 \mid x,\ y$$
$$(x,\ y+3) \in S \leftrightarrow y \geqslant 0 \bigwedge 3 \mid x,\ y$$
$$(x,\ y-3) \in S \leftrightarrow y \leqslant 0 \bigwedge 3 \mid x,\ y$$
$$(x-1,\ y-2) \in S \leftrightarrow x \leqslant 0 \bigwedge x \mod 3 \neq 1$$
$$(x-2,\ y-1) \in S \leftrightarrow x \leqslant 0 \bigwedge x \mod 3 \neq 2$$
$$(x+1,\ y+2) \in S \leftrightarrow x \geqslant 0 \bigwedge x \mod 3 \neq 2$$
$$(x+2,\ y+1) \in S \leftrightarrow x \geqslant 0 \bigwedge x \mod 3 \neq 1$$