On rank-one unitary perturbations of the shift operator over a de Branges-Rovnyak space

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In 1949, A. Beurling characterizes all the closed invariant subspaces of the shift operator $S : H^2 \to H^2$:

Theorem (Beurling's characterization)

If $V \subset H^2$ is a closed invariant subspace of the shift operator S, that is $SV \subset V$, then there exists a inner function $\theta : \mathbb{D} \to \mathbb{C}$ such that

$$V = \theta H^2.$$

Remarks:

- H^2 is the space of holomorphic functions defined on the unit disk $\mathbb{D}$ such that $\langle f, f \rangle_2 := \sum_{n=0}^{\infty} |\frac{f^{(n)}(0)}{n!}|^2 < +\infty$.
- $S : H^2 \to H^2$ is the shift operator defined as $Sf(z) := zf(z)$. In other words, it shifts the sequence of Taylor coefficients by one place to the right.
- A function θ is inner if θ is bounded on $\mathbb{D}$ and extends to the unit circle $\mathbb{T}$ such that $|\theta| = 1$ a.e.

As a consequence of A. Beurling's characterization, we have a characterization of the closed invariant subspaces of the backward shift operator $S^* : H^2 \to H^2$:

Corollary

If $V \subset H^2$ is a closed invariant subspace of the backward shift operator S^*, then there exists an inner function $\theta : \mathbb{D} \to \mathbb{C}$ such that

$$V = K_\theta := H^2 \ominus \theta H^2.$$

Remark:

- The backward shift operator is the adjoint of S and is expressed as

$$S^* f(z) = \frac{f(z) - f(0)}{z}.$$

  It shifts the sequence of Taylor coefficients by one place to the left.
- K_θ is the so-called model space associated to θ.

Definition

We define $S_\theta : K_\theta \to K_\theta$ to be the compression of the shift operator S to K_θ:

$$S_\theta := P_\theta \, S|_{K_\theta} .$$

Remarks:

- The operator $P_\theta : H^2 \to K_\theta$ is the orthogonal projection onto K_θ.
- $S|_{K_\theta}$ is the shift operator restricted to the model space.
- K_θ is a Hilbert space that inherits the Hilbert space structure of H^2.

Why are we interesting in the operator S_θ?

Theorem

If T is a contraction ($\|T\| \leq 1$) on a Hilbert space such that

- $\lim_{n\to\infty} T^{*n} = 0$ *in the operator-norm;*
- *the rank of $I - T^*T$ is one;*
- *the rank of $I - TT^*$ is one;*

then T is unitarily equivalent to S_θ for some inner function θ.

Remark:

- In other words, studying the operator S_θ is the same thing as studying the operator T in the above Theorem.
- The specialists called the model operator in the Sz.-Nagy-Foiaș functional model.

In 1972, Clark was interested in the operators of the form $U = T + S_\theta$ where T has rank equals to 1 and U is unitary.

Theorem

If $\theta(0) = 0$ and U is a rank-one perturbation of S_θ that is also unitary, then there exists $\alpha \in \mathbb{T}$ such that

$$U = U_\alpha := S_\theta + \alpha \otimes S^*\theta.$$

Remarks:

- We already know that T must be a tensor product. Now we know exactly the form of it.
- The tensor product (in the case of Hilbert spaces) is the operator defined on H^2 in the following way: given $f, g \in K_\theta$, $f \otimes g$ is defined as

$$(f \otimes g)h = \langle h, g \rangle_2 f \quad h \in K_\theta.$$

Let H be a Hilbert space of Holomorphic functions equiped with an inner product. H is a RKHS (Reproducing kernel Hilbert space) if it satisfies the following properties:

- $f \mapsto f(w)$ is continuous for all $w \in \mathbb{D}$;
- the family $\{f \mapsto f(w) \, : \, w \in \mathbb{D}\}$ separates points.

By the Riesz-Fischer's Theorem, the first property implies that there exists functions k_w such that $f(z) = \langle f, k_z \rangle$. These functions k_w are called the reproducing kernels of the space.

Example

1. The space H^2 is a RKHS with $k_w(z) := (1 - \overline{w}z)^{-1}$.
2. The model space K_θ is a RKHS with

$$k_w^\theta(z) := (1 - \overline{\theta(w)}\theta(z))k_w(z) = \frac{1-\overline{\theta(w)}\theta(z)}{1-\overline{w}z}.$$

Let L^∞ be the set of bounded functions on $\mathbb{T}$. Let L^2 be the square summable functions on $\mathbb{T}$. Then, we know that

$$L^2 = H^2 \oplus (H_0^2)^\perp.$$

We denote by $P_+ : L^2 \to H^2$ the orthogonal projection onto the space H^2.

Definition

Let $\psi \in H^\infty$. The Toeplitz operator $T_\psi : H^2 \to H^2$ is defined as

$$T_\psi f := P_+(\psi f)$$

Remark:

- $\|T_\phi\| = \|\psi\|_{L^\infty}$.
- $T_\psi^* = T_{\overline{\psi}}$.
- For $\psi, \varphi \in H^\infty$, $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$.

de Branges-Rovnyak spaces $\mathcal{H}(b)$ can be defined in the following way.

Definition

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$. The Hilbert space $\mathcal{H}(b)$ is the range of the operator $(I - T_b T_{\overline{b}})^{1/2}$ endowed with the inner product

$$\left\langle (I - T_b T_{\overline{b}})^{1/2} f, (I - T_b T_{\overline{b}})^{1/2} g \right\rangle_b := \langle f, g \rangle_2$$

for $f, g \in (\ker(I - T_b T_{\overline{b}})^{1/2})^\perp$.

Remark:

- $\mathcal{H}(b)$ can be viewed as a RKHS with the following kernel: $k_w^b := (1 - \overline{b(w)} b) k_w$ for $w \in \mathbb{D}$.
- It is a generelization of the model space.
- We have $h \in \mathcal{H}(b)$ if and only if $T_{\overline{b}} h \in \mathcal{H}(\overline{b})$.

An inner function $\theta : \mathbb{D} \to \mathbb{D}$ is an example of an extreme point of the unit ball of H^∞:

Theorem

Let H^∞ be the bounded holomorphic functions on the unit disk and let $f \in H^\infty$ such that $\|f\|_\infty \leq 1$. f is an extreme point of the unit ball of H^∞ if and only if $\log(1 - |f|) \notin L^1(\mathbb{T})$.

Question: Is Clark's result still true in the case of $\mathcal{H}(b)$ where b is an extreme point?

However, we can't work directly with the compressed shift since $\mathcal{H}(b)$ is not invariant under the shift operator:

Theorem (D. Sarason)

The following conditions are all equivalent:

- $b \in \mathcal{H}(b)$.
- *Polynomials are densed in* $\mathcal{H}(b)$.
- b *is non-extreme point of* H^∞.
- $S\mathcal{H}(b) \subset \mathcal{H}(b)$.

So, to extend Clark's result, we must properly defined the shift operator on the space $\mathcal{H}(b)$ in the extreme case.

A general Theorem on de Branges-Rovnyak spaces is the following:

Theorem

Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. Then,

- *$\mathcal{H}(b)$ and $\mathcal{H}(\overline{b})$ are invariant under the Toeplitz operator $T_{\overline{\varphi}}$ with $\varphi \in H^\infty$;*
- *the operator $T_{\overline{\varphi}} : \mathcal{H}(b) \to \mathcal{H}(b)$ is a bounded operator with $\|T_{\overline{\varphi}}\| \leq \|\varphi\|_\infty$.*

Remarks:

- $S^* = T_{\overline{z}}$. We denote $X_b := S^*|_{\mathcal{H}(b)}$.
- X_b plays the role of the "backward shift" on $\mathcal{H}(b)$.
- Also, $S^* b \in \mathcal{H}(b)$ even if $b \notin \mathcal{H}(b)$. This comes from the fact that

$$T_{\overline{b}} S^* b = S^* T_{\overline{b}} T_b 1 = -S^*(I - T_{\overline{b}} T_b)1 \in \mathcal{H}(\overline{b}).$$

  Moreover, $\|S^* b\|_b^2 = 1 - |b(0)|^2$.
- X_b^* will play the role of the "shift" on $\mathcal{H}(b)$.

We have now a formula for the shift X_b^* defined on $\mathcal{H}(b)$.

Theorem

For any $h \in \mathcal{H}(b)$, we have

$$X_b^* h = Sh - \langle h, S^* b \rangle_b \, b.$$

Remark:

- In other words, $X_b^* = S - b \otimes S^* b$. So, we see that X_b^* is the shift operator perturbated by a rank-one operator.
- Moreover, $X_b X_b^* = I - S^* b \otimes S^* b$.
- Also, if $b(0) = 0$, then $X_b^* X_b = I - k_0^b \otimes k_0^b$.

We are now able to define U_α for some $\alpha \in \mathbb{T}$. We suppose for now on that $b(0) = 0$.

Definition

Let $\alpha \in \mathbb{T}$. Then, for $f \in \mathcal{H}(b)$, we define

$$U_\alpha f := X_b^* f + \alpha \langle f, S^* b \rangle_b 1.$$

In other words, we can write U_α as

$$U_\alpha = X_b^* + \alpha \otimes S^* b.$$

Remark:

- If $b(0) = 0$, then $k_0^b = 1$ and so $1 \in \mathcal{H}(b)$. So, U_α is well defined.
- The operator U_α is unitary for each $\alpha \in \mathbb{T}$. That means that

$$U_\alpha U_\alpha^* = U_\alpha^* U_\alpha = I.$$

We have all the tools to state Clark's result.

Theorem

Let $b(0) = 0$, $\alpha \in \mathbb{T}$ *and let* T *be a rank-one operator on* $\mathcal{H}(b)$*. Suppose that* $U = X_b^* + T$ *is a unitary operator on* $\mathcal{H}(b)$*. Then, there exists a* $\alpha \in \mathbb{T}$ *such that*

$$T = \alpha \otimes S^* b.$$

To prove this Theorem, we need two key lemmas.

Lemma

If $\|h\|_b = \|X_b h\|_b$, *then* $h \perp S^* b$.

Lemma

If $\|h\|_b = \|X_b^* h\|_b$ *and* $b(0) = 0$, *then* $h \perp 1$.

Rank-one perturbations

T is a rank-one operator, so U can be written as

$$U = X_b^* + f \otimes g \quad f, g \in \mathcal{H}(b).$$

Since U is unitary, we must have that

$$\|Uh\|_b = \|h\|_b = \|U^* h\|_b \quad \forall h \in \mathcal{H}(b).$$

The goal is to show that $f = \alpha$ and $g = S^*$.

- The first step will be to show that $\mathbb{C}.S^* b = \mathbb{C}.g$.
- The second step will be to show that $\mathbb{C}.1 = \mathbb{C}f$.

Rank-one perturbations

First step

Suppose that $h \perp g$. Then, we have that

$$(f \otimes g)h = \langle h, g \rangle_b f = 0$$

and so

$$Uh = X_b^* h.$$

From this, we deduce that

$$\|X_b^* h\|_b = \|Uh\|_b = \|h\|_b.$$

Thus, $h \perp S^* b$. We have just shown that $(\mathbb{C}.g)^\perp \subset (\mathbb{C}.S^* b)^\perp$. Thus,

$$\mathbb{C} S^* b = \mathbb{C} g \Rightarrow g = c_1 S^* b.$$

Rank-one perturbations

Second step

Suppose that $h \perp f$. Then, the adjoint of U_α is

$$U_\alpha^* = X_b + g \otimes f.$$

So, we have

$$U_\alpha^* h = X_b h + \langle h, f \rangle_b \, g = X_b h.$$

Since U_α is unitary, we have

$$\|X_b h\|_b = \|U_\alpha h\|_\alpha = \|h\|_b.$$

From the second Lemma, $h \perp 1$. We have just shown that $(\mathbb{C}.f)^\perp \subset (\mathbb{C}.1)^\perp$. Thus,

$$\mathbb{C}.f = \mathbb{C}.1 \Rightarrow f = c_2.$$

Rank-one perturbations

Conclusion

We have $f = c_2$ and $g = c_1 S^* b$. Thus, U_α has now the following form

$$U = X_b^* + c \otimes S^* b \quad \alpha \in \mathbb{C}.$$

It remains to show that $|c| = 1$. Since $b(0) = 0$, we know that $1 \in \mathcal{H}(b)$. So

$$U^* 1 = X_b 1 + (S^* b \otimes c) 1 = \overline{c} \left\langle 1, 1 \right\rangle_b S^* b.$$

By the reproducing kernel property, we have that

$$\left\langle 1, 1 \right\rangle_b = \left\langle 1, k_0^b \right\rangle_b = 1.$$

Thus, $U^* 1 = \overline{c} S^* b$. Now, it remains to remark that

$$1 = \|1\|_b = \|U^* 1\|_b = |c| \|S^* b\|_b = |c|. \qquad \square$$

Play the odds

We could continu to inspect the properties of the operators U_α, like characterizing their spectrum and much more.

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Theorem

Let $U_\alpha := X_b^* + \alpha(1 - \alpha\overline{b(0)})^{-1} k_0^b \otimes S^* b$ with $|\alpha| = 1$. Then, U_α is an isometry of $\mathcal{H}(b)$. Moreover, it is a unitary operator of $\mathcal{H}(b)$ if and only if b is extreme and in that case, the operator U_α are the only one-dimensional perturbation of X_b^* which are unitary.

Remark:

- This covers the case $b(0) \neq 0$.

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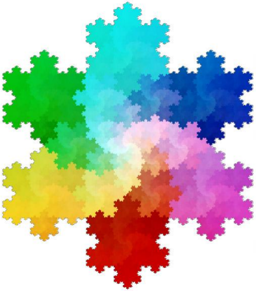
Remark:

- This covers the case $b(0) \neq 0$.
- Moral of the story : always check in the litterature if things have already been done...

Thanks for your attention!



Fricain, E. Bases of reproducing kernels in de Branges-Rovnyak spaces. *J. Funct. Anal. 266*, 2 (2005), 373-405.

Sarason, D. Sub-Hardy Hilbert spaces in the unit disk, vol. 10 of *University of Arkansas Notes in the Mathematical Sciences*. Wiley-Interscience, New-York, 1994.