Characterization of weak-$*$ closed sets by the Krein–Smulian Theorem

Pierre-Olivier Parisé
Ph.D. Student, advisor Thomas J. Ransford

Departement of mathematics and statistic, Laval University
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Abstract

In a normed vector space X, we have the reflex to use sequences to show that a subset is closed. This is a fundamental result in normed vector spaces and more generally, in any metric spaces. However, if X is infinite dimensional, a desadvantage appears: the closed unit ball is no more compact. We can patch this problem by considering the dual space X^* equiped with its weak star (weak-$*$) topology.

In this new context, the Alaoglu Theorem states that the closed unit ball is weak-$*$ compact. This is a significant payout brought to us by the weak-$*$ topology. However, by ruling out this compactness problem, do we lost the advantage that normed spaces had?

In this talk, we will see that the answer to this question is brought to us by the Krein–Smulian Theorem. In particular, with the aim of this Theorem, we will show that if X is a separable Banach space, then a weak-$*$ closed subspace of X is characterized by sequences. So, we can keep our "old reflex" from normed spaces.

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1 Motivations

In a normed vector space X, we have the reflex to use sequences to show that a subset is closed. In fact, this is true for any metric spaces.

Sequence Lemma (Munkres, p.130). *Let (X, d) be a metric space and $A \subset X$. Then, x lies in the closure of A if and only if there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $x_n \to x$.*[1]

However, if X is infinite dimensional, a desadvantage appears: the closed unit ball is no more compact. This is a big lost for the topology induced by the metric (and consequently by the norm). Can we find a topology that unify both properties? The next example shows that we can't search for a general topology.

Let $\mathbb{R}^{\mathbb{N}}$ be the set of sequences indexed by the set of natural numbers $\mathbb{N}$. We equip $\mathbb{R}^{\mathbb{N}}$ with the box-topology denoted by τ_b. The topology τ_b is defined from a collection of basis elements. Those basis elements B are of the following forms:

$$B = (a_1, b_1) \times (a_2, b_2) \times \ldots = \prod_{i=1}^{\infty} (a_i, b_i).$$

This last basis element is simply the cartesian product (infinite) of open intervals in $\mathbb{R}$. We can show that $(\mathbb{R}^{\mathbb{N}}, \tau_b)$ does not satisfy the Sequence Lemma. To show that, consider the set $A := \{(x_i)_{i\geq 1} : x_i > 0 \quad \forall i \geq 1\}$. The element is clearly in the closure of A. However, there is not sequence in A that converges to 0. Let $(a_n)_{n\geq 1}$ be a sequence in A where each a_n is a sequence $a_n = (x_{in})_{i\geq 1}$. Consider the following neighborhood of 0:

$$B = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \cdots = \prod_{n=1}^{\infty} (-x_{nn}, x_{nn}).$$

However, no element $a_n = (x_{in})_{i\geq 1}$ belongs since x_{nn} doesn't belong to $(-x_{nn}, x_{nn})$. So, $(a_n)_{n\geq 1}$ doesn't converge to 0. From this, we can conclude that (1) $(R^{\mathbb{N}}, \tau_b)$ is not metrizable, and (2) we can't use, in general, sequences to characterize the closure of a set. We refer to this last property as "sequential closure".

The new topology that we consider is the weak star (weak-$*$) topology defined on the dual space X^*. In this new context, the Alaoglu's Theorem states that the closed unit ball is weak-$*$ compact. This is a significant payout brought to us by the weak-$*$ topology. However, by ruling out this compactness problem, do we lost the advantage that normed spaces had? In this presentation, we will show that for subspace of the dual space of a separable Banach space, the notions of "closed sets" and "sequentially closed sets" are reconciled. The precise goal is the following result:

Goal. *Let X be a separable Banach space and let X^* be its dual. If a vector subspace $E \subset X^*$ is weak-$*$ sequentially closed, then it is weak-$*$ closed.*

[1] This result is, in fact, true if the topological space satisfy the first countability axiom.

2 Basic notions

Let (X, τ) be a topological space where τ is the family of open sets. We define

Definition 1. *A **basis** for the topology τ is a subcollection $\mathcal{B} \subset \tau$ such that*

1. *for each $x \in X$, there exists a element $B \in \mathcal{B}$ containing x.*
2. *for each $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there exists a third element $B_3 \in \mathcal{B}$ such that $x \in B_3$.*

Example 1. *Disks in the plane.*

For our purposes, we suppose that (X, τ) has a basis of elements for its topology τ.

Definition 2. *Let $(x_n)_{n\in\mathbb{N}} \subset X$ be a sequence. We say that (x_n) **converges to** x if for every element B of the basis containing x, there exists an index n_0 such that for all $n \geq n_0$, $x_n \in B$.*

The sequences can be used to introduce the concept of a sequentially closed set in a topological space.

Definition 3. *Let $E \subset X$ be a subset. We say that*

1. *E is **closed** if $E^c = X - E$ is open in X.*
2. *E is **sequentially closed** if the following condition is satisfied: if $(x_n)_{n\in\mathbb{N}} \subset E$ converges to x, then $x \in E$.*

3 Definitions for weak-$*$ topology

Our specific framework is the dual space X^* of a Banach space with the weak-$*$ topology. What is the weak-$*$ topology?

Definition 4. *Let $(X, \|\cdot\|)$ be a Banach space. The **dual space** of X is the set*

$$X^* := \{x^* : X \to \mathbb{C} \, : \, x^* \text{ is linear and bounded.}\}$$

What is meant by "bounded" is that $\sup_{\|x\|\leq 1} |x^*(x)| < +\infty$. In the following, we use the notation $\langle x, x^* \rangle$ to denote $x^*(x)$, the evaluation of x^* at x. We define the weak-$*$ topology on X^* in two steps.

First of all, we define what we call a subbasis $\mathcal{S}$ for the weak-$*$ topology.

Definition 5. *Let X^* be the dual space. A **subbasis** $\mathcal{S}$ for X^* is a collection of subsets of X^* such that $\bigcup_{S\in\mathcal{S}} S = X^*$.*

The family of all intersections of elements of $\mathcal{S}$ forms, in fact, a basis. An element of the subbasis has the following form:

$$S_{x,x_0^*} := \{x^* \in X^* \,:\, |\langle x, x^* - x_0^* \rangle| \le r\}$$

for $x \in X$, $x_0^* \in X^*$ and $r > 0$ being fixed. We put $\mathcal{S} := \left\{S_{x,x_0^*} \,:\, x \in X, x_0^* \in X^*\right\}$.

Second of all, we define the weak-$*$ topology in the following way:

Definition 6. *The* ***weak-$*$ topology*** $\sigma(X^*, X)$ *is the topology generated by the collection of all finite intersections of elements in* $\mathcal{S}$.

What is meant by "generated" is that a member in $\sigma(X^*, X)$ is a union of finite intersections of elements in $\mathcal{S}$. The following proposition is easy to show.

Proposition 1. *Let* $O \subset X^*$. *Then,* $O \in \sigma(X^*, X)$ *if and only if for every* $x_0^* \in O$, *there exist* $x_1, x_2, \ldots, x_n \in X$ *and positive numbers* $r_1, r_2, \ldots, r_n$ *such that*

$$\bigcap_{i=1}^{n} \{x^* \,:\, |\langle x_i, x^* - x_0^* \rangle| < r_i\} \subset O.$$

In the weak-$*$ topology on X^*, we can characterize the sequences that converges to a point belonging to X^*.

Proposition 2. *Let* $(x_n^*)_{n\in\mathbb{N}} \subset X^*$ *be a sequence. Then,* (x_n^*) *converges to* $x^* \in X^*$ *if and only if for every element* $x \in X$, *the complex sequence* $(\langle x, x_n^* \rangle)_{n\in\mathbb{N}}$ *converges to* $\langle x, x^* \rangle$.

Proof. The "only if" is trivial since we just have to take, as open sets containing x^* in the definition, the sets

$$\left\{y^* \,:\, |\langle x, y^* - x^* \rangle| < \frac{1}{n}\right\}.$$

Suppose that, for every $x \in X$, $\langle x, x_n^* \rangle \to \langle x, x^* \rangle$ as $n \to \infty$. Let O be an open set that contains x^*. Then, from the last proposition, there exist $x_1, x_2, \ldots, x_m \in X$ and $r_1, r_2, \ldots, r_m > 0$ such that

$$\bigcap_{i=1}^{m} \{y^* \,:\, |\langle x_i, y^* - x^* \rangle| < r_i\} \subset O.$$

For each x_i, there exists an $n_i > 0$ such that for all $n \ge n_i$,

$$|\langle x_i, x_n^* - x^* \rangle| = |\langle x_i, x_n^* \rangle - \langle x_i, x^* \rangle| < r_i.$$

Put $n_0 := \max\{n_i \,:\, 1 \le i \le m\}$. Then, for all $n \ge n_0$ and any $1 \le i \le m$, we have that

$$x_n^* \in \{y^* \,:\, |\langle x_i, y^* - x^* \rangle| < r_i\}.$$

Thus, for all $n \geq n_0$, we have that x_n^* belongs to the intersection, from $i = 1$ to $i = m$, of the above sets. So, $x_n^* \in O$ for each $n \geq n_0$. □

4 The Krein-Smulian's Theorem

Now that we defined the weak-$*$ topology on X^*, we are almost ready to attack the proof of the main Theorem. At this level, we don't have enough tools to get the proof right.

Let E be as in the statement of Krein–Smulian's Theorem. Our goal is to show that E is $\sigma(X^*, X)$-closed. However, how can we use the $\sigma(X^*, X)$-sequentially closedness in our advantage? The key point is the Krein-Smulian's Theorem!

Theorem 1 (Krein-Smulian's)**.** *Let X be a Banach space and $E \subset X^*$ be a convex subset. Then, E is $\sigma(X^*, X)$-closed if and only if for each $r > 0$,* $\mathrm{ball}_r(X^*) \cap E$ *is $\sigma(X^*, X)$-closed.*

The set $\mathrm{ball}_r(X^*)$ stands for the set of $x^* \in X^*$ such that $\sup_{x \in X} |\langle x, x^* \rangle| \leq r$.

Proof. Give the main ideas behind the proof. □

So, by the Krein-Smulian Theorem, we can show that $\mathrm{ball}_r(X^*) \cap E$ is $\sigma(X^*, X)$-closed which is probably easier. However, it doesn't seem to be easier. This is where the assumption of separability gets an importance. It is due to the following remarkable Theorem.

Theorem 2. *Suppose that X is a Banach space. Then,* $(\mathrm{ball}(X^*), \sigma(X^*, X))$ *is metrizable if and only if X is separable.*

What is meant by $(X^*, \sigma(X^*, X))$ is metrizable is that the topology τ^* and the topology generated by the open balls defined by the metric are the same. The set $\mathrm{ball}(X^*)$ corresponds to the set $\mathrm{ball}_1(X^*)$. With this theorem, we can prove our main result.

Proof of the Goal. Let $E \subset X^*$ be a $\sigma(X^*, X)$-sequentially closed vector subspace of X^*. Then, from the Krein-Smulian Theorem, E is $\sigma(X^*, X)$-closed if and only if $\mathrm{ball}_r(X^*) \cap E$ is $\sigma(X^*, X)$-closed for any $r > 0$. Let $r > 0$ be arbitrary. Since E is a vector space, we may scaled the set $\mathrm{ball}_r(X^*) \cap E$ to $r = 1$ since this operation is a homeomorphism.

So, it remains to show that $\mathrm{ball}(X^*) \cap E$ is $\sigma(X^*, X)$-closed. However, from the last Theorem, $\mathrm{ball}(X^*)$ is metrizable! So, if E is $\sigma(X^*, X)$-sequentially closed, so is $\mathrm{ball}(X^*) \cap E$. Then, in a metric space, we know that sequentially closedness is equivalent to closedness. Thus, $\mathrm{ball}(X^*) \cap E$ is $\sigma(X^*, X)$-closed. This conclude the proof. □

5 Topological remarks

The definitions related to topological notions are not exactly the more general ones. Those who want to learn the more general definitions in a topological space, the book of J. R. Munkres *Topology* is a good reference for that.

There is a general concept from which the weak-$*$ topology emerges. It is the **locally convex topological** spaces. The topology is generated by a family of semi-norms. For the weak-$*$ topology, the family of semi-norms are defined as $p_x(x^*) := |\langle x, x^* \rangle|$ for some $x \in X$. For more details, see *A Course in Functional analysis*, J. B. Conway.