Cesàro summability of Taylor series in weighted Dirichlet spaces

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Joint work with J. Mashreghi and T. Ransford

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Let f be a holomorphic function on the unit disk $\mathbb{D}$ with Taylor expansion

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The space of such functions is denoted by $\mathrm{Hol}(\mathbb{D})$.

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Given a Banach space $X \subset \mathrm{Hol}(\mathbb{D})$ with norm $\| \cdot \|$ and such that

- the set of polynomials $\mathcal{P} \subset X$;
- $\overline{\mathcal{P}} = X$,

we seek a general procedure that will generate approximations of a function $f \in X$ (in norm) by polymonials.

The easiest procedure is to use the partial sums of the Taylor expansion.

Definition

For $n \geq 0$, the n-th partial sums of a function $f \in \mathrm{Hol}(\mathbb{D})$ with $f(z) = \sum_{n \geq 0} a_n z^n$ is defined by the following formula:

$$s_n[f](z) := \sum_{k=0}^{n} a_k z^k, \quad z \in \mathbb{D}.$$

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Do we have

$$\|s_n[f] - f\| \to 0, \quad n \to \infty$$

for any $f \in X$?

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The answer is yes for H^2 and H^p $(1 < p < +\infty)$.

Let $X = A(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ equipped with the sup norm :

$$\|f\|_\infty := \sup_{z \in \overline{\mathbb{D}}} |f(z)|.$$

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- Due to a slight modification of du Bois-Reymond's construction, there exists a function $f \in A(\mathbb{D})$ such that

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- Moreover, if we view $s_n : A(\mathbb{D}) \to A(\mathbb{D})$ as a bounded linear operator on $A(\mathbb{D})$ for each $n \geq 0$, then

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The answer is no for $A(\mathbb{D})$. However...

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Moreover, consider the Cesàro means of order $\alpha > 0$:

$$\sigma_n^\alpha[f] := \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{k=0}^{n} \binom{n-k+\alpha-1}{\alpha-1} s_k[f].$$

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- By a result attributed to M. Riesz, for any $\alpha > 0$ and $f \in A(\mathbb{D})$,

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$$\|\sigma_n^\alpha[f] - f\|_\infty \to 0, \quad n \to \infty.$$

This last result is also true for the space H^1. The answer is yes, but we've substituted $s_n[f]$ for $\sigma_n^\alpha[f]$.

Definition

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- The case of superharmonic weights was studied by Aleman in his Ph. D. dissertation.

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- $\omega \equiv 1$ on $\mathbb{D}$. Then, $\mathcal{D}_\omega = \mathcal{D}$ the classical Dirichlet space. In this setting, we can show that

$$\|f\|_{\mathcal{D}}^2 = \sum_{n\geq 0}(n+1)|a_n|^2 < +\infty \quad \forall f \in \mathcal{D}.$$

Clearly, we have $\|s_n[f] - f\|_{\mathcal{D}} \to 0$ as $n \to \infty$.

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- $\omega(z) = \frac{1-|z|^2}{|1-z|^2}$, $z \in \mathbb{D}$. Then, $\mathcal{D}_\omega = \mathcal{D}_1$, a local Dirichlet space. In this setting, it is well-known that there is a $f \in \mathcal{D}_1$ such that

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  However, Mashreghi and Ransford proved that

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Actually, this last result is true for any superharmonic weight.

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- *For any superharmonic function* $\omega : \mathbb{D} \to (0, +\infty)$ *and any* $f \in \mathcal{D}_\omega$*, we have*

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$$\|\sigma_n[f] - f\|_{\mathcal{D}_\omega} \to 0, \quad n \to \infty.$$
- *For* $\omega_1(z) := \frac{1-|z|^2}{|1-z|^2}$*, there is a function* $f \in \mathcal{D}_{\omega_1}$ *such that*
$$\|s_n[f] - f\|_{\mathcal{D}_{\omega_1}} \not\to 0, \quad n \to \infty.$$

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The results of Cesàro summability of the Taylor series in the spaces $A(\mathbb{D})$ and H^1 motivate the following question :

Is the Taylor series of a function $f \in \mathcal{D}_\omega$ Cesàro summable for any $\alpha > 0$ with respect to the norm of $\mathcal{D}_\omega$?

Theorem A (Mashreghi-Ransford-P., 2020)

For any superharmonic weight $\omega : \mathbb{D} \to (0, +\infty)$, *any* $\alpha > \frac{1}{2}$ *and any* $f \in \mathcal{D}_\omega$, *we have*

$$\|\sigma_n^\alpha[f] - f\|_{\mathcal{D}_\omega} \to 0, \quad n \to \infty.$$

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Theorem B (Mashreghi-Ransford-P., 2020)

There exists a function $f \in \mathcal{D}_{\omega_1}$ *such that*

$$\|\sigma_n^{1/2}[f] - f\|_{\mathcal{D}_{\omega_1}} \not\to 0, \quad n \to +\infty.$$

Go to thanks

Definition

Let $f(z) = \sum_{n \geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$ be two formal power series. Their **Hadamard product** is the following formal power series :

$$(f * g)(z) = \sum_{n \geq 0} a_n b_n z^n.$$

- If f or g is a polynomial, then $f * g$ is a polynomial.
- If both $f, g \in \mathrm{Hol}(\mathbb{D})$, then $f * g \in \mathrm{Hol}(\mathbb{D})$.

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Definition

Let $h \in \mathrm{Hol}(\mathbb{D})$ with Taylor series $h(z) = \sum_{n \geq 0} c_n z^n$ and $\omega : \mathbb{D} \to (0, +\infty)$ be a superharmonic function. The function h is a **Hadamard multiplier** for $\mathcal{D}_\omega$ if

$$h * f \in \mathcal{D}_\omega, \quad \forall f \in \mathcal{D}_\omega.$$

For a function h as in the previous slide, we define

$$T_h := \begin{pmatrix} c_1 & c_2 - c_1 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & c_2 & c_3 - c_2 & c_4 - c_3 & \cdots \\ 0 & 0 & c_3 & c_4 - c_3 & \cdots \\ 0 & 0 & 0 & c_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

Theorem (Mashreghi, Ransford, 2019)

The following assertions are equivalent :

1. *h is a Hadamard multiplier for $\mathcal{D}_\omega$ for any superharmonic weight $\omega : \mathbb{D} \to (0, +\infty)$.*
2. *The matrix T_h acts as a bounded operator from ℓ^2 into ℓ^2.*

In this case, for any superharmonic weight ω and any $f \in \mathcal{D}_\omega$,

$$\mathcal{D}_\omega(h * f) \leq \| T_h : \ell^2 \to \ell^2 \|^2 \mathcal{D}_\omega(f)$$

and the number $\| T_h : \ell^2 \to \ell^2 \|^2$ is the best possible constant.

To a Hadamard multiplier h of $\mathcal{D}_\omega$, we associate the Hadamard multiplication linear operator $M_h : \mathcal{D}_\omega \to \mathcal{D}_\omega$ defined as

$$M_h(f) := h * f, \quad f \in \mathcal{D}_\omega.$$

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Theorem

Let $\omega : \mathbb{D} \to (0, +\infty)$ be a superharmonic weight and h be a polynomial. Then,

- $\|M_h : \mathcal{D}_\omega \to \mathcal{D}_\omega\| \leq \|T_h : \ell^2 \to \ell^2\|$.
- $\|M_h : \mathcal{D}_{\omega_1} \to \mathcal{D}_{\omega_1}\| = \|T_h : \ell^2 \to \ell^2\|$.

There is another summability method which is equivalent to the Cesàro means.

Definition

Let $f(z) = \sum_{n \geq 0} a_n z^n$ belonging to $\mathrm{Hol}(\mathbb{D})$. The **discrete Riesz means** of order $\alpha > 0$ is defined by the following formula

$$\rho_n^\alpha[f](z) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha a_k z^k.$$

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- The means apply directly to each term of the power series $f(z) = \sum_{n\geq 0} a_n z^n$.
- When $\alpha = 1$, we recover the Cesàro means of order 1 if we express

$$\sigma_n[f](z) = \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} a_k z^k$$

M. Riesz (1924) proved that whenever $\alpha \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} x_k = y \iff \lim_{n\to\infty} \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\alpha} x_k = y$$

where $(x_k)_{k\geq 0} \subset \mathbb{C}$.

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Theorem

Let X be a Banach space, $(x_n)_{n\geq 0}$ be a sequence of vectors in X and $y \in X$. Then, for each $\alpha \in (0,1)$, we have

$$\lim_{n\to\infty} \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} x_k = y \iff \lim_{n\to\infty} \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^{\alpha} x_k = y.$$

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In our case, we will take $X = \mathcal{D}_\omega$ and $x_k = a_k z^k$.

It is well-known that if $\alpha < \beta$ and

$$\lim_{n\to\infty} \frac{1}{\binom{n+\alpha}{\alpha}} \sum_{k=0}^{n} \binom{n-k+\alpha}{\alpha} x_k = y$$

for some sequence $(x_k)_{k\geq 0} \subset \mathbb{C}$, then

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We may approach the problem by

- considering the Cesàro means of order $\alpha \in (0,1)$.
- using the discrete Riesz means instead of the Cesàro means.

We have used the following estimate.

Theorem (Mashreghi and Ransford, 2019)

Let $h(z) = \sum_{k=0}^{n} c_k z^k$ *where* $n \geq 0$. *Then we have*

$$\| T_h : \ell^2 \to \ell^2 \|^2 \leq \frac{1}{n+1} \sum_{k=0}^{n} |c_k - c_{k+1}|^2.$$

Consider the function

$$h_n^\alpha(z) := \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right)^\alpha z^k,$$

so that, for any $f \in \mathcal{D}_\omega$, we have

$$\rho_n^\alpha[f] = h_n^\alpha * f = M_{h_n^\alpha}(f).$$

Using this last estimate, we obtain

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Proposition

Let $\alpha > \frac{1}{2}$ and h_n^α be as in the previous slide. Then, we have

$$\|M_{h_n^\alpha} : \mathcal{D}_\omega \to \mathcal{D}_\omega\| \leq \frac{\alpha}{\sqrt{2\alpha - 1}},$$

for any $n \geq 0$ and for any superharmonic function $\omega : \mathbb{D} \to (0, \infty)$.

Proof of Theorem A.

Clearly, we have $\rho_n^\alpha[p] \to p$ in $\mathcal{D}_\omega$ for any polynomial p. Given $f \in \mathcal{D}_\omega$,

$$\|\rho_n^\alpha[f] - f\|_{\mathcal{D}_\omega} \leq \|M_{h_n^\alpha}(f - p)\|_{\mathcal{D}_\omega} + \|M_{h_n^\alpha} p - p\|_{\mathcal{D}_\omega} + \|p - f\|_{\mathcal{D}_\omega}$$

for some polynomial p. We obtain the result from the density of polynomials in $\mathcal{D}_\omega$ and the fact that $M_{h_n^\alpha}$ are uniformly bounded. □

We have used the following estimate:

Theorem (Mashreghi-P.-Ransford, 2020)

Let $h(z) = \sum_{k\geq 0} c_k z^k$. Then, for all integers $1 \leq m \leq n$, we have

$$\| T_h : \ell^2 \to \ell^2 \|^2 \geq m \sum_{k=m}^{n} |c_{k+1} - c_k|^2.$$

As a consequence, we get

Proposition

For $\alpha = \frac{1}{2}$ and $m = \lfloor n/2 \rfloor$ where $n \geq 1$, we have

$$\| T_{h_n^{1/2}} : \ell^2 \to \ell^2 \|^2 \geq \frac{1}{8} \log \left(\frac{n+1}{2} \right).$$

Proof of Theorem B.

Again, we view $\rho_n^\alpha[f]$ as

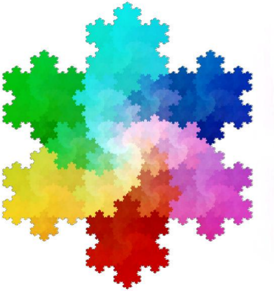
$$\rho_n^\alpha[f] = h_n^\alpha * f = M_{h_n^\alpha}(f).$$

In this particular case, we have

$$\|M_{h_n^{1/2}} : \mathcal{D}_{\omega_1} \to \mathcal{D}_{\omega_1}\| = \|T_{h_n^{1/2}} : \ell^2 \to \ell^2\| \geq \frac{1}{8} \log\left(\frac{n+1}{2}\right).$$

So, the family $\left\{M_{h_n^{1/2}} : \mathcal{D}_{\omega_1} \to \mathcal{D}_{\omega_1} \mid n \geq 0\right\}$ is unbounded. The result follows from the Banach-Steinhauss Theorem. □

Thanks for your attention!



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