Generating the Mandelbrot set in 3D

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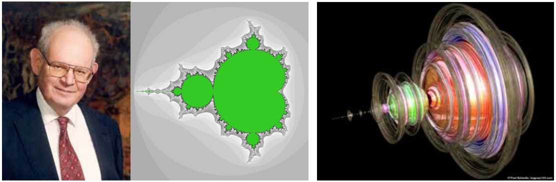
Beginning and further

- Benoît Mandelbrot (1924 - 2010) : his 80's set.
- Further explorations done on the basis of Mandelbrot set.
- People tried the third dimension with quaternions.



(a) $\mathcal{M}^2$ set from $z^2 + c$ (b) Quaternionic $\mathcal{M}^2$ set

FIGURE: Complex and quaternionic versions

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Bicomplex numbers

Definition 1 ($\mathbb{M}(2)$ or $\mathbb{C}_2$ space)

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{M}(1) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A bicomplex number ζ is defined as

$$\zeta = z_1 + z_2\mathbf{i_2} \tag{1}$$

where $\mathbf{i_2}^2 = -1$.

Various representations :

- In terms of four real numbers : $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements :

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

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Operations on Bicomplex numbers

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality : $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$.
2) Addition : $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$.
3) Multiplication : $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$.
4) Modulus : $\|\zeta_1\|_2 := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^{4} x_i^2}$
5) Cartesian subset : if $X \subset \mathbb{M}(2)$, $X_1 \subset \mathbb{M}(1)$ and $X_2 \subset \mathbb{M}(1)$, then

$$X = X_1 \times_{\gamma_1} X_2 := \{\zeta \in X \mid \zeta = u_1\gamma_1 + u_2\overline{\gamma}_1\,,\ u_1 \in X_1 \text{ and } u_2 \in X_2\}\,.$$

Remark :

- $(\mathbb{M}(2)\,,\,+\,,\,\cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2)\,,\,+\,,\,\cdot\,,\,\|\cdot\|_2)$ forms a Banach space.

Tricomplex numbers

Definition 2 ($\mathbb{M}(3)$ or $\mathbb{C}_3$ space)

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A tricomplex number η is defined as

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3} \tag{2}$$

where $\mathbf{i_3}^2 = -1$.

Various representations :

- In terms of four complex numbers : $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers :

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

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Tricomplex numbers

Various representations (continuing) :

- In terms of two idempotent elements :

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_2 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_2$$

where $\gamma_2 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_2 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements :

$$\eta = w_1 \cdot \gamma_1\gamma_2 + w_2 \cdot \gamma_1\overline{\gamma}_2 + w_3 \cdot \overline{\gamma}_1\gamma_2 + w_4 \cdot \overline{\gamma}_1\overline{\gamma}_2$$

where $w_i \in \mathbb{M}(1)$ for $i = 1, 2, 3, 4$ and

$$\begin{aligned} w_1 &= (z_1 + z_4) - (z_2 - z_3)\mathbf{i_1} & \quad w_3 &= (z_1 - z_4) - (z_2 + z_3)\mathbf{i_1} \\ w_2 &= (z_1 + z_4) + (z_2 - z_3)\mathbf{i_1} & \quad w_4 &= (z_1 - z_4) + (z_2 + z_3)\mathbf{i_1}. \end{aligned}$$

Operations on Tricomplex Numbers

Let $\eta_1 = \zeta_1 + \zeta_2 \mathbf{i_2}$ and $\eta_2 = \zeta_3 + \zeta_4 \mathbf{i_2}$.

1) Equality : $\eta_1 = \zeta_2 \iff \zeta_1 = \zeta_3$ and $\zeta_2 = \zeta_4$.
2) Addition : $\eta_1 + \zeta_2 := (\zeta_1 + \zeta_3) + (\zeta_2 + \zeta_4)\mathbf{i_2}$.
3) Multiplication : $\eta_1 \cdot \eta_2 := (\zeta_1\zeta_3 - \zeta_2\zeta_4) + (\zeta_2\zeta_3 + \zeta_1\zeta_4)\mathbf{i_2}$.
4) Modulus : $\|\eta_1\|_3 := \sqrt{\|\zeta_1\|_2^2 + \|\zeta_2\|_2^2} = \sqrt{\sum_{i=1}^4 |z_i|^2} = \sqrt{\sum_{i=1}^8 x_i^2}$
5) Cartesian subset : if $X \subset \mathbb{M}(3)$, $X_1 \subset \mathbb{M}(2)$ and $X_2 \subset \mathbb{M}(2)$, then

$$X = X_1 \times_{\gamma_2} X_2 := \{\eta \in X \,|\, \eta = u_1\gamma_2 + u_2\overline{\gamma}_2\,,\ u_1 \in X_1 \text{ and } u_2 \in X_2\}\,.$$

Remark :

- $(\mathbb{M}(3)\,,+\,,\cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(3)\,,+\,,\cdot\,,\|\cdot\|_3)$ forms a Banach space.
- Since $X_1 \subset \mathbb{M}(2)$ and $X_2 \subset \mathbb{M}(2)$, X can be expressed as a cartesian product of four subsets of $\mathbb{M}(1) \simeq \mathbb{C}$.

Some viewpoints

Bicomplex numbers :

- as a pair of complex variables (z_1, z_2) ;
- as a quadruple of real numbers (x_1, x_2, x_3, x_4).

Tricomplex numbers :

- as a pair of bicomplex numbers (ζ_1, ζ_2) ;
- as a quadruple of complex numbers (z_1, z_2, z_3, z_4) ;
- as a octuple of real numbers $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$.

So, with bicomplex and tricomplex numbers, we can divide vectors in a certain way.

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Multibrot sets : extension of the Mandelbrot set

Definition 3

Set $p \geq 2$ an integer. Let $Q_{p,c}(z) = z^p + c$ where z, c are complex numbers. A *multibrot* set is defined as

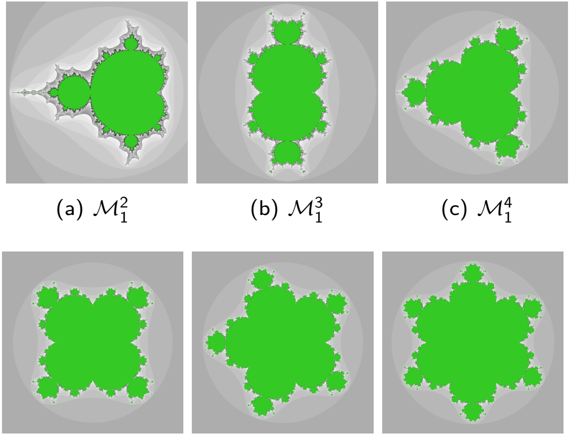
$$\mathcal{M}_1^p := \left\{ c \in \mathbb{C} \mid \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}. \quad (3)$$

Examples :
$c = 0 \Rightarrow \{0,\ 0,\ 0,\ 0,\ \ldots\}$.
$c = 1 \Rightarrow \left\{1,\ 2,\ 2(2^{p-1} + 1),\ \ldots\right\}$.

Remarks :

- $\mathcal{M}_1^p \subset \overline{D}(0, 2^{\frac{1}{p-1}})$.
- Equivalence : $c \in \mathcal{M}_1^p \iff |Q_{p,c}^m(0)| \leq 2^{\frac{1}{p-1}}\ \forall m \in \mathbb{N}$.
- $\mathcal{M}_1^p$ are closed sets.

Some examples of $\mathcal{M}^p$ set



(a) $\mathcal{M}_1^2$ (b) $\mathcal{M}_1^3$ (c) $\mathcal{M}_1^4$

(d) $\mathcal{M}_1^5$ (e) $\mathcal{M}_1^6$ (f) $\mathcal{M}_1^7$

Bicomplex multibrot sets : Definition

D. Rochon [4] used the Bicomplex numbers to give a 4D version of the Mandelbrot set. Here, we give the definition for multibrot sets.

Definition 4

Set $p \geq 2$ an integer. Let $Q_{p,c}(\zeta) = \zeta^p + c$ where $\zeta, c \in \mathbb{M}(2)$. A bicomplex multibrot set is defined as

$$\mathcal{M}_2^p := \left\{ c \in \mathbb{M}(2) \mid \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

Remarks :

- D. Rochon made a 3D slice of the $\mathcal{M}_2^2$ which he had called it the *Tetrabrot*.
- $\mathcal{M}_2^p \subset \overline{B_2}(0, 2^{\frac{1}{p-1}})$.
- Bicomplex cartesian product : $\mathcal{M}_2^p = \mathcal{M}_1^p \times_{\gamma_1} \mathcal{M}_1^p$.

Bicomplex multibrot sets : Definition

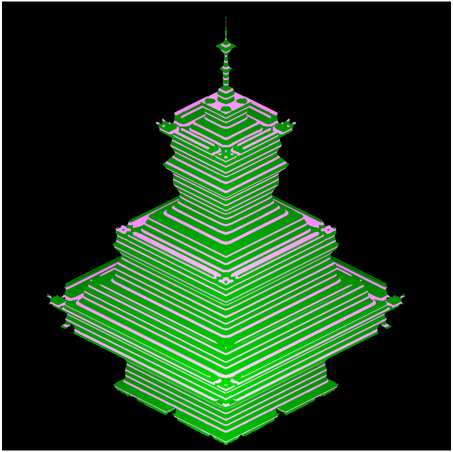


FIGURE: Tetrabrot : $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$

Tricomplex Multribrot sets : Definition

Rochon and Garrant-Pelletier [2] generalized the approach using $\mathbb{M}(3)$. Here we give the analogue definition for multibrot sets.

Definition 5

Set $p \geq 2$ an integer. Let $Q_{p,c}(\eta) = \eta^p + c$ where $\eta, c \in \mathbb{M}(3)$. A Tricomplex multibrot set is defined as

$$\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) \mid \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

Remarks :

- $\mathcal{M}_3^p \subset \overline{B_3}(0, 2^{\frac{1}{p-1}})$.
- Characterization : $c \in \mathcal{M}_3^p \iff \| Q_{p,c}^m(0) \|_3 \leq 2^{\frac{1}{p-1}} \ \forall m \in \mathbb{N}$.
- $\mathcal{M}_3^p$ is connected.

Tricomplex multibrot : Characterizations

Tricomplex multibrots can be characterized as a cartesian product :

- Bicomplex cartesian product : $\mathcal{M}_3^p = \mathcal{M}_2^p \times_{\gamma_2} \mathcal{M}_2^p$.
- Mixed cartesian product : $\mathcal{M}_3^p = (\mathcal{M}_1^p \times_{\gamma_1} \mathcal{M}_1^p) \times_{\gamma_2} (\mathcal{M}_1^p \times_{\gamma_1} \mathcal{M}_1^p)$.

So, $c \in \mathcal{M}_3^p$ if and only if

$$w_1, w_2, w_3 \text{ and } w_4 \text{ are in } \mathcal{M}_1^p$$

where $c = w_1 \cdot \gamma_1\gamma_2 + w_2 \cdot \gamma_1\overline{\gamma}_2 + w_3 \cdot \overline{\gamma}_1\gamma_2 + w_4 \cdot \overline{\gamma}_1\overline{\gamma}_2$.

3D slices : Definition

To visualize $\mathcal{M}_3^p$ in 3D, we need to define a 3D slice :

Definition 6

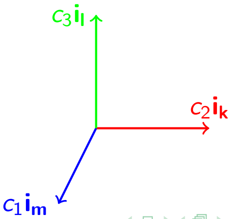
Choose different $\mathbf{i_m}$, $\mathbf{i_k}$, $\mathbf{i_l}$ from the set $\{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$. A 3D slice of $\mathcal{M}_3^p$ is defined as

$$\mathcal{T}_3^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \left\{ c = c_1\mathbf{i_m} + c_2\mathbf{i_k} + c_3\mathbf{i_l} \mid \left\{ Q_{p,c}^m(0)_{m=1} \right\}^{\infty} \text{ is bounded } \right\}.$$

Remark :

In term of the 3D space, we select the vector (c_1, c_2, c_3) where

$$c_1 \longleftrightarrow OX$$
$$c_2 \longleftrightarrow OY$$
$$c_3 \longleftrightarrow OZ.$$



Visualization

The steps :

1) Choose a 3D slice $\mathcal{T}_3^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$.
2) Compute the divergent layers for every c in a cube of $2 \cdot 2^{\frac{1}{p-1}}$ side length and for a maximum number of iterations M.
3) Draw the point on the image in a certain range of iterations (e.g. points that reach between 7 to 14 iterations).

But, we can make the image more attracting and more instructive. Set

$$c = w_1 \cdot \gamma_1\gamma_2 + w_2 \cdot \gamma_1\overline{\gamma}_2 + w_3 \cdot \overline{\gamma}_1\gamma_2 + w_4 \cdot \overline{\gamma}_1\overline{\gamma}_2.$$

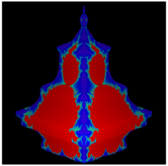
Depending on the number of idempotent components in $\mathcal{M}_1^p$, we associate a color to this number.

Visualization (Continuing)

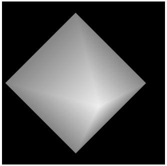
These colors are :

1. Grey : $w_i \in \mathcal{M}_1^p$ for all $i = 1, 2, 3, 4$.
2. Yellow : only one.
3. Red : only two.
4. Orange : only three.
5. Gradient from blue to green : None of all.

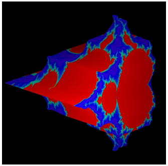
Family shooting : quadratic $\eta^2 + c$



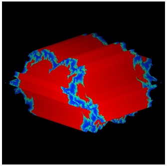
(a) $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$: Tetrabrot



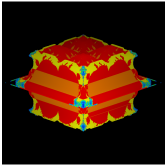
(b) $\mathcal{T}^2(1, \mathbf{j_1}, \mathbf{j_2})$: Perplexbrot



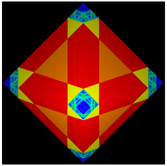
(c) $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{j_1})$: Arrowpitbrot



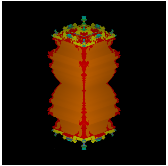
(d) $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$: Mousebrot



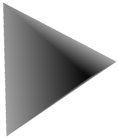
(e) $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$: Turtlebrot



(f) $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$: Metabrot

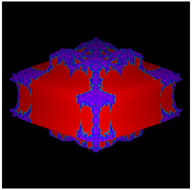


(g) $\mathcal{T}^2(\mathbf{i_1}, \mathbf{j_1}, \mathbf{j_2})$: Kingbrot

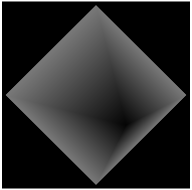


(h) $\mathcal{T}^2(\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3})$: Platonbrot

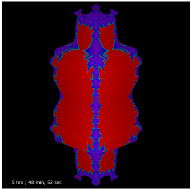
Family shooting : cubic $\eta^3 + c$



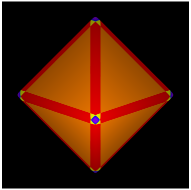
(i) $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{i_2})$: Tetrabric



(j) $\mathcal{T}^3(1, \mathbf{j_1}, \mathbf{j_2})$: Perplexbric



(k) $\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{j_1})$: HourglassBric



(l) $\mathcal{T}^3(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$: Metabric

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Conclusion

What you should know :

- From [3], $\mathcal{M}_1^3 \cap \mathbb{R} = \left[-\frac{2}{3\sqrt{3}}, \frac{2}{3\sqrt{3}}\right]$.
- From [2], there are only eight principal slices for $\mathcal{M}_3^2$.
- From [3], there are only four (!) principal slices for $\mathcal{M}_3^3$ and the Perplexbric is an octahedron.

Further exploration :

- What is the set $\mathcal{M}_1^p \cap \mathbb{R}$ in general ?
- What is the characterization of the slices of $\mathcal{M}_3^p$?

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Table of imaginary units

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
$\mathbf{i_1}$	$\mathbf{i_1}$	$-\mathbf{1}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$-\mathbf{j_3}$	$-\mathbf{i_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$-\mathbf{i_3}$
$\mathbf{i_3}$	$\mathbf{i_3}$	$\mathbf{j_2}$	$\mathbf{j_3}$	$-\mathbf{1}$	$-\mathbf{j_1}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$-\mathbf{i_2}$
$\mathbf{i_4}$	$\mathbf{i_4}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$-\mathbf{1}$	$\mathbf{i_3}$	$\mathbf{i_2}$	$\mathbf{i_1}$
$\mathbf{j_1}$	$\mathbf{j_1}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$\mathbf{i_3}$	$\mathbf{1}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$
$\mathbf{j_2}$	$\mathbf{j_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$\mathbf{i_2}$	$-\mathbf{j_3}$	$\mathbf{1}$	$-\mathbf{j_1}$
$\mathbf{j_3}$	$\mathbf{j_3}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_2}$	$\mathbf{i_1}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	$\mathbf{1}$

TABLE: Product of tricomplex imaginary units

RetourBi RetourTri

References

[1] Baley Price, G. : An Introduction to Multicomplex Spaces and Functions. *Monographs and textbooks on pure and applied mathematics* (1991)

[2] Garant-Pelletier, V. & Rochon, D. : On a Generalized Fatou-Julia Theorem in Multicomplex spaces. *Fractals* **17**(3), 241-255 (2009)

[3] Parisé, P.-O. & Rochon, D. : A Study of The Dynamics of the Tricomplex Polynomial $\eta^p + c$. *Non Linear Dynam.*. (To appear)

[4] Rochon, D. : A Generalized Mandelbrot Set for Bicomplex Numbers. *Fractals*. **8**(4), 355-368 (2000)