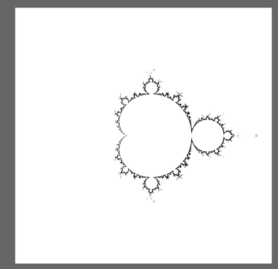
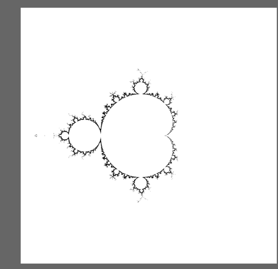
MANDELBROT SETS and ITS THREE-DIMENSIONAL SLICES





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Introduction

Experiments have been done in [1] for the Mandelbrot set of an integer order $p \geq 2$ in the complex plane. These sets are generated by iterating the polynomial $Q_{p,c}(z) = z^p + c$ from the starting point $z = 0$. Other authors generalized the classical Mandelbrot set to higher dimensions.

In 2000, Dominic Rochon [2] used the algebra of bicomplex numbers to generate the classical Mandelbrot set in the three-dimensinal (3D) space. He obtained what he called the Tetrabrot.

Then, in 2009, V. Garant-Pelletier, and Dominic Rochon [3] used the algebra of tricomplex numbers to generalized the Mandelbrot set in eight dimensions. This generalization allowed to do more 3D slices in the Mandelbrot set. They showed, up to conjugacy classes, that there are only eight principal classes.

Goals

- To establish a theory of 3D slices of the Mandelbrot sets.
- To classify the 3D principal slices of the Mandelbrot sets.

Tricomplexes numbers

A tricomplex number η has the following form

$$\eta := x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

where $x_i \in \mathbb{R}$. Figure 1 shows the relations between each units.

$\cdot$	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
1	1	$\mathbf{i_1}$	$\mathbf{i_2}$	$\mathbf{i_3}$	$\mathbf{i_4}$	$\mathbf{j_1}$	$\mathbf{j_2}$	$\mathbf{j_3}$
$\mathbf{i_1}$	$\mathbf{i_1}$	-1	$\mathbf{j_1}$	$\mathbf{j_2}$	$-\mathbf{j_3}$	$-\mathbf{i_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$
$\mathbf{i_2}$	$\mathbf{i_2}$	$\mathbf{j_1}$	-1	$\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$-\mathbf{i_3}$
$\mathbf{i_3}$	$\mathbf{i_3}$	$\mathbf{j_2}$	$\mathbf{j_3}$	-1	$-\mathbf{j_1}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$-\mathbf{i_2}$
$\mathbf{i_4}$	$\mathbf{i_4}$	$-\mathbf{j_3}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	-1	$\mathbf{i_3}$	$\mathbf{i_2}$	$\mathbf{i_1}$
$\mathbf{j_1}$	$\mathbf{j_1}$	$-\mathbf{i_2}$	$-\mathbf{i_1}$	$\mathbf{i_4}$	$\mathbf{i_3}$	1	$-\mathbf{j_3}$	$-\mathbf{j_2}$
$\mathbf{j_2}$	$\mathbf{j_2}$	$-\mathbf{i_3}$	$\mathbf{i_4}$	$-\mathbf{i_1}$	$\mathbf{i_2}$	$-\mathbf{j_3}$	1	$-\mathbf{j_1}$
$\mathbf{j_3}$	$\mathbf{j_3}$	$\mathbf{i_4}$	$-\mathbf{i_3}$	$-\mathbf{i_2}$	$\mathbf{i_1}$	$-\mathbf{j_2}$	$-\mathbf{j_1}$	1

Fig. 1: Relations between each units

The set of tricomplex numbers is denoted by $\mathbb{M}(3)$. It can also have the following form

$$\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$$

where $z_1 := x_1 + x_2\mathbf{i_1}$, $z_2 := x_3 + x_6\mathbf{i_1}$, $z_3 := x_4 + x_7\mathbf{i_1}$, and $z_4 := x_8 + x_5\mathbf{i_1}$ are complex numbers $\mathbb{M}(1)$. Finally, there is another form

$$\eta = w_1 + w_2\mathbf{i_3}$$

where $w_1 := z_1 + z_2\mathbf{i_2}$, and $w_2 := z_3 + z_4\mathbf{i_2}$ are bicomplex numbers $\mathbb{M}(2)$ (see [4]).

Operations on tricomplex numbers

Let $\zeta = w_1 + w_2\mathbf{i_3}$, and $\eta = s_1 + s_2\mathbf{i_3}$ be two tricomplex numbers.
Addition of two tricomplex numbers is defined as

$$\zeta + \eta := (w_1 + s_1) + (w_2 + s_2)\mathbf{i_3}.$$

Multiplication of two tricomplex numbers is defined as

$$\zeta \cdot \eta := (w_1 s_1 - w_2 s_2) + (w_1 s_2 + w_2 s_1)\mathbf{i_3}.$$

The **norm** of a tricomplex number is defined as

$$\|\eta\|_3 := \sqrt{|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2} = \sqrt{\|w_1\|_2^2 + \|w_2\|_2^2}.$$

With these operations, $\mathbb{M}(3)$ is a commutative unitary ring, but it is not a division ring. Moreover, it is also a Banach space with complex scalar multiplication.

Subsets of tricomplex numbers

The **Tricomplex closed ball** is defined as

$$\overline{B_3}(\zeta, r) := \{\eta \in \mathbb{M}(3) \,:\, \|\eta - \zeta\|_3 \leq r\}.$$

Later, we will need a specific set to make 3D slices in Mandelbrot sets :

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_k} + x_2\mathbf{i_l} + x_3\mathbf{i_m} \,|\, x_1, x_2, x_3 \in \mathbb{R}\}.$$

That is, we choose three different units from the set of the eight units, and we make a linear combination with it. There is one last set that is useful for the classification of the principal 3D slices of the Mandelbrot sets :

$$\mathbb{M}(\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m}) := \{x_1\mathbf{i_k} + x_2\mathbf{i_l} + x_3\mathbf{i_m} + x_4\mathbf{i_k}\mathbf{i_l}\mathbf{i_m} \,:\, x_i \in \mathbb{R}, i = 1, \ldots, 4\}.$$

In this case, we choose three distinct units, but we also consider, in the linear combination, the unit generated by the multiplication of the three others.

Tricomplex Mandelbrot sets

The tricomplex Mandelbrot set of integer order $p \geq 2$ is defined as the set of $c \in \mathbb{M}(3)$ for which the sequence $\{c, c^p + c, (c^p + c)^p + c, \ldots\}$ is bounded, that is

$$\mathcal{M}_3^p := \left\{c \in \mathbb{M}(3) \,:\, \left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty} \text{ is bounded}\right\}$$

where $Q_{p,c}(\zeta) := \zeta^p + c$. Some authors also call these sets *Multibrot* sets. It has been showed in [5] that $\mathcal{M}_3^p \subset \overline{B_3}(0, 2^{1/(p-1)})$, $\mathcal{M}_3^p$ is a compact and connected set. The following theorem helps to generate the Mandelbrot sets in 3D.

Théorème 1. *A number $c \in \mathcal{M}_3^p$ if, and only if, $\|Q_{p,c}^m(0)\|_3 \leq 2^{1/(p-1)}$ for any natural number m.*

A **3D principal slice** of $\mathcal{M}_3^p$ is defined as

$$\mathcal{T}^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \left\{c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \,:\, \left\{Q_{p,c}^m(0)\right\}_{m=1}^{\infty} \text{ is bounded}\right\}.$$

Results

The following figures show some examples of 3D slices.

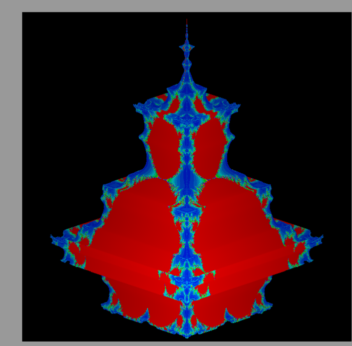

Fig. 2: $\mathcal{T}^2(1, \mathbf{i_1}, \mathbf{i_2})$

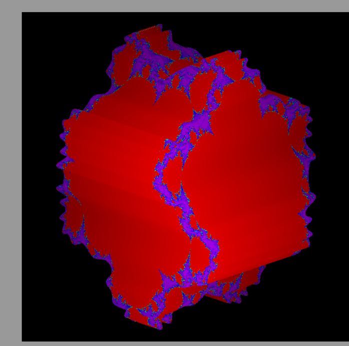

Fig. 3: $\mathcal{T}^6(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_1})$

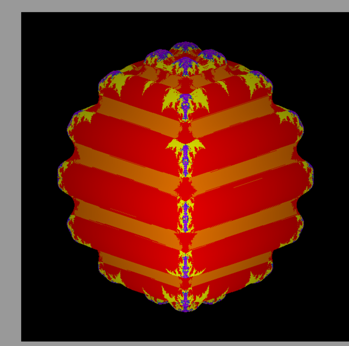

Fig. 4: $\mathcal{T}^8(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$

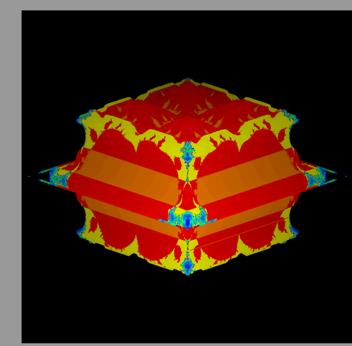

Fig. 5: $\mathcal{T}^2(\mathbf{i_1}, \mathbf{i_2}, \mathbf{j_2})$

We say that two slices $\mathcal{T}_1^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$, and $\mathcal{T}_2^p(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$ are equivalent if there exists a bijective linear map ϕ from $\text{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$ to $\text{span}_{\mathbb{R}}\{\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s}\}$ that conjugates Q_{p,c_1} with Q_{p,c_2}, that is $\phi \circ Q_{p,c_1} \circ \phi^{-1} = Q_{p,c_2}$ for any $c_1 \in \mathcal{T}_1^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$, and $c_2 \in \mathcal{T}_2^p(\mathbf{i_n}, \mathbf{i_q}, \mathbf{i_s})$. This is an equivalence relation, and we have the following result.

Théorème 2. *For $p = 3$, the Mandelbrot set $\mathcal{M}_3^3$ has only four principal slices:*

1. *$\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{i_2})$ called Tetrabric;*
2. *$\mathcal{T}^3(1, \mathbf{j_1}, \mathbf{j_2})$ called Perplexbric;*
3. *$\mathcal{T}^3(1, \mathbf{i_1}, \mathbf{j_1})$ called Hourglassbric;*
4. *$\mathcal{T}^3(\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3})$ called Metabric.*

The reduction of the number of 3D slices is caused by the particularity of the degree of the polynomial $\zeta^3 + c$. Indeed, for any $\zeta \in \text{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$, $\zeta^3 \in \mathbb{M}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l})$. On the other hand, for $p = 2$, the square of a number $\zeta \in \text{span}_{\mathbb{R}}\{\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}\}$ may spread overall $\mathbb{M}(3)$. Thus, the degree $p = 3$ has a special feature.

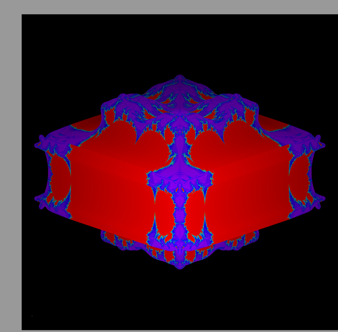

Fig. 6: Tetrabric

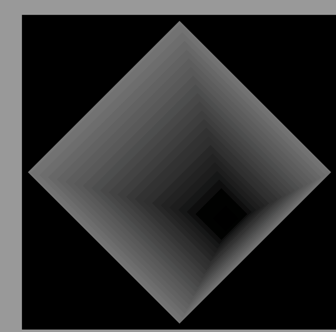

Fig. 7: Perplexbric

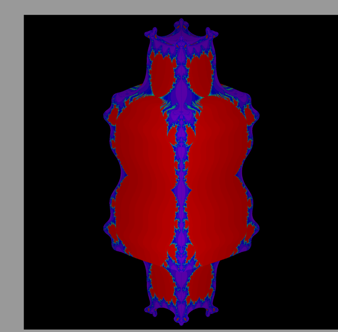

Fig. 8: Hourglassbric

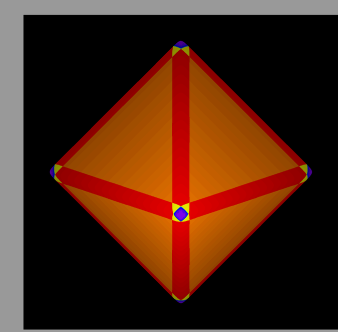

Fig. 9: Metabric

Conclusion

The nature of the integer p seems to influence the number of principal 3D slices. We have a conjecture, based on computer experiments, that for any odd integer p, there are four principal 3D slices and for any even integer p, there are eight principal 3D slices. One way to approach the problem would consist of using the Coxeter's theory. Indeed, the bijective linear maps are signed permutation matrices. A second approach would use the idempotent representation, a very useful representation of tricomplex numbers.

Références

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