The Impossibility of Polynomial Approximations in Some de Branges-Rovnyak Spaces

Pierre-Olivier Parisé (Université Laval)
Advisor, T. Ransford

AMS/CMS Winter meeting 2018
Vancouver

7^{th}-10^{th}, December, 2018

Spaces of continuous functions

Theorem (1885, Weierstrass)

Let $f \in \mathcal{C}[a, b]$ be a continuous function on the closed interval $[a, b]$. Then, there exists a sequence of polynomials (p_n) such that

$$\|f - p_n\|_\infty \underset{n \to +\infty}{\longrightarrow} 0.$$

Spaces of continuous functions

Theorem (1885, Weierstrass)

Let $f \in \mathcal{C}[a,b]$ be a continuous function on the closed interval $[a,b]$. Then, there exists a sequence of polynomials (p_n) such that

$$\|f - p_n\|_\infty \underset{n \to +\infty}{\longrightarrow} 0.$$

Theorem (1912, Berstein)

If $f \in \mathcal{C}[0,1]$ and $p_n(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$, then

$$\|f - p_n\|_\infty \underset{n \to +\infty}{\longrightarrow} 0.$$

Hardy space: Definition

Let $\mathrm{Hol}(\mathbb{D}) := \{\sum_{n=0}^{\infty} a_n z^n \,:\, a_n \in \mathbb{C}\ \forall n \geq 0\}$ where $\mathbb{D} := \{|z| < 1\}$.

Definition

$H^2 := \left\{ f \sim \sum_{n \geq 0} a_n z^n \,:\, \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\}$. We define:

- A norm: $\|f\|_2^2 := \sum_{n=0}^{\infty} |a_n|^2$;
- The partial sums: $s_n(f)(z) := \sum_{k=0}^{n} a_k z^k$.

Hardy space: Definition

Let $\mathrm{Hol}(\mathbb{D}) := \{\sum_{n=0}^{\infty} a_n z^n \,:\, a_n \in \mathbb{C}\ \forall n \geq 0\}$ where $\mathbb{D} := \{|z| < 1\}$.

Definition

$H^2 := \left\{ f \sim \sum_{n\geq 0} a_n z^n \,:\, \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\}$. We define:

- A norm: $\|f\|_2^2 := \sum_{n=0}^{\infty} |a_n|^2$;
- The partial sums: $s_n(f)(z) := \sum_{k=0}^{n} a_k z^k$.

Theorem

If $f \in H^2$, then $\|f - s_n(f)\|_2 \to 0$ when $n \to +\infty$.

Disk Algebra: Definition

Definition

$\mathcal{A}(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. For $f \in \mathcal{A}(\mathbb{D})$, we define

- the supremum norm:

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

- the Cesáro mean:

$$\sigma_n(f) := \frac{\sum_{k=0}^n s_k(f)}{n+1}.$$

Disk Algebra: Definition

Definition

$\mathcal{A}(\mathbb{D}) := \mathrm{Hol}(\mathbb{D}) \cap \mathcal{C}(\overline{\mathbb{D}})$. For $f \in \mathcal{A}(\mathbb{D})$, we define

- the supremum norm:

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

- the Cesáro mean:

$$\sigma_n(f) := \frac{\sum_{k=0}^{n} s_k(f)}{n+1}.$$

A suprising fact (du Bois Reymond):

$$\exists f \in \mathcal{A}(\mathbb{D}) \text{ s.t. } \sup_{n \geq 0} \|s_n f\|_\infty = +\infty.$$

Theorem

If $f \in \mathcal{A}(\mathbb{D})$, then $\|\sigma_n(f) - f\|_\infty \to 0$ as $n \to +\infty$.

Theorem

If $f \in \mathcal{A}(\mathbb{D})$, then $\|\sigma_n(f) - f\|_\infty \to 0$ as $n \to +\infty$.

Proof.

$g := f|_{\partial\mathbb{D}}$ is continuous. So, from Fejér theorem, the Cesàro means $\sigma_n(f) := \frac{\sum_{k=0}^{n} s_k(f)}{n+1}$ converge to g uniformly. $\sigma_n(f)$ is the restriction of a polynomial to $\partial\mathbb{D}$ and

$$\|\sigma_n(f) - f\|_\infty = \sup_{z \in \partial\mathbb{D}} |\sigma_n(f)(z) - g(z)| \underset{n \to +\infty}{\longrightarrow} 0.$$

□

De Branges-Rovnyak Spaces

Let $H^\infty := \{f \in \mathrm{Hol}(\mathbb{D}) \, : \, \|f\|_\infty < +\infty\}$.

De Branges-Rovnyak Spaces

Let $H^\infty := \{f \in \mathrm{Hol}(\mathbb{D}) \, : \, \|f\|_\infty < +\infty\}$.

Definition

We define $\mathcal{H}(b) := A_b H^2$ where A_b is a certain operator depending on b.

De Branges-Rovnyak Spaces

Let $H^\infty := \{f \in \mathrm{Hol}(\mathbb{D}) \, : \, \|f\|_\infty < +\infty\}$.

Definition

We define $\mathcal{H}(b) := A_b H^2$ where A_b is a certain operator depending on b.

Example

Let $b_0(z) := \frac{\tau z}{1-\tau^2 z}$ where $\tau := \frac{\sqrt{5}-1}{2}$. If $f \in \mathcal{H}(b_0)$, then

$$\|f_r - f\|_{b_0} \to 0$$

as $r \to 1^-$ where $f_r(z) := f(rz)$.

De Branges-Rovnyak Spaces

Example

Let $b := b_0 B$ where B is a Blaschke product with zeros at $w_n := 1 - 4^{-n}$ for $n \geq 1$.

De Branges-Rovnyak Spaces

Example

Let $b := b_0 B$ where B is a Blaschke product with zeros at $w_n := 1 - 4^{-n}$ for $n \geq 1$. If we let

$$f(z) := \sum_{n\geq 1} \frac{2^{-n}}{1 - w_n z},$$

then it is possible to show that $f \in \mathcal{H}(b)$, but

De Branges-Rovnyak Spaces

Example

Let $b := b_0 B$ where B is a Blaschke product with zeros at $w_n := 1 - 4^{-n}$ for $n \geq 1$. If we let

$$f(z) := \sum_{n\geq 1} \frac{2^{-n}}{1 - w_n z},$$

then it is possible to show that $f \in \mathcal{H}(b)$, but

- $\lim_{r\to 1^-} \|f_r\|_b = +\infty$;
- $\limsup_{n\to\infty} \|s_n(f)\|_b = +\infty$;
- $\limsup_{n\to\infty} \|\sigma_n(f)\|_b = +\infty$.

However, polynomial approximants is obtained by another technic: Toeplitz operators approximations.

De Branges-Rovnyak Spaces

Let $\mathcal{P}_+$ is the set of analytic polynomials $\sum_{k=0}^{n} c_k \chi_k$ where

$$\chi_k(\theta) := e^{\mathbf{i}k\theta}.$$

Example

If b is a special outer function in H^∞, then

$$\mathcal{P}_+ \cap \mathcal{H}(b) = \{0\}.$$

De Branges-Rovnyak Spaces

There exists two different situations:

1. b's where polynomial are dense;
2. b's where polynomial are not dense.

De Branges-Rovnyak Spaces

There exists two different situations:

1. b's where polynomial are dense;
2. b's where polynomial are not dense.

Question: Is there a condition on b that characterized this dichotomy?

definition

H^2 is identified with $f \in L^2$ such that $\hat{f}(n) = 0$ for all $n < 0$.

definition

H^2 is identified with $f \in L^2$ such that $\hat{f}(n) = 0$ for all $n < 0$.

Definition

Let $\varphi \in L^\infty$. The Toeplitz operator $T_\varphi : H^2 \to H^2$ is the operator

$$T_\varphi f := P_+(\varphi f)$$

where $P_+ : L^2 \to H^2$.

definition

H^2 is identified with $f \in L^2$ such that $\hat{f}(n) = 0$ for all $n < 0$.

Definition

Let $\varphi \in L^\infty$. The Toeplitz operator $T_\varphi : H^2 \to H^2$ is the operator

$$T_\varphi f := P_+(\varphi f)$$

where $P_+ : L^2 \to H^2$.

Remark:

- The operator P_+ is the orthogonal projection of L^2 onto H^2.
- $\|T_\varphi\| = \|\varphi\|_\infty$.
- $T_\varphi^* = T_{\overline{\varphi}}$
- If $\psi \in L^\infty$, then $T_{\overline{\psi}} T_\varphi = T_{\overline{\psi}\varphi}$.

Definition

Definition (Sarason [4])

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$. The space $\mathcal{H}(b)$ is defined as $\mathcal{H}(b) := (I - T_b T_{\overline{b}})^{1/2} H^2$ with the scalar product

$$\left\langle (I - T_b T_{\overline{b}})^{1/2} f, (I - T_b T_{\overline{b}})^{1/2} g \right\rangle_b := \langle f, g \rangle_2$$

where $f, g \in H^2 \ominus \ker(I - T_b T_{\overline{b}})^{1/2}$.

Definition

Definition (Sarason [4])

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$. The space $\mathcal{H}(b)$ is defined as $\mathcal{H}(b) := (I - T_b T_{\overline{b}})^{1/2} H^2$ with the scalar product

$$\left\langle (I - T_b T_{\overline{b}})^{1/2} f, (I - T_b T_{\overline{b}})^{1/2} g \right\rangle_b := \langle f, g \rangle_2$$

where $f, g \in H^2 \ominus \ker(I - T_b T_{\overline{b}})^{1/2}$.

Remark:

- $\mathcal{H}(\overline{b})$ is defined from the operator $(I - T_{\overline{b}} T_b)^{1/2}$. We have

$$\mathcal{H}(\overline{b}) \subset \mathcal{H}(b).$$

- $h \in \mathcal{H}(b)$ if and only if $T_{\overline{b}} h \in \mathcal{H}(\overline{b})$.

Main Theorem

It is possible to prove that $b \in H^\infty$ is extreme if and only if

$$\int_{\mathbb{T}} \log 1 - |b|^2 \, dm = -\infty.$$

Main Theorem

It is possible to prove that $b \in H^\infty$ is extreme if and only if

$$\int_{\mathbb{T}} \log 1 - |b|^2 \, dm = -\infty.$$

Theorem

Let $b \in H^\infty$ such that $\|b\|_\infty \leq 1$. Then,

$$\mathcal{P}_+ \subset \mathcal{H}(b) \iff b \text{ is nonextreme.}$$

Main Theorem: Tools

We need a non-trivial fact about $\mathcal{H}(b)$.

Theorem

Let b be a extreme point of the unit ball of H^∞. If $h \in \mathcal{H}(b) \cap \mathrm{Hol}(\mathrm{clos}(\mathbb{D}))$, then $h \in \ker T_{\overline{b}}$ and h is a rational functions.

Main Theorem:Tools

We need a non-trivial fact about $\mathcal{H}(b)$.

Theorem

Let b be a extreme point of the unit ball of H^∞. If $h \in \mathcal{H}(b) \cap \mathrm{Hol}(\mathrm{clos}(\mathbb{D}))$, then $h \in \ker T_{\overline{b}}$ and h is a rational functions.

Remarks:

- For the space $\mathcal{H}(\overline{b})$, the conclusion is that $h = 0$.
- This tells us the set $\mathrm{Hol}(\mathrm{clos}(\mathbb{D})) \cap \mathcal{H}(b)$ is not a big set.

Main Theorem:Tools

We also need the following Lemma.

Lemma

Let b be a nonextreme point of the unit ball of H^∞. Then, there exists an outer function $a \in H^\infty$ such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$.

Main Theorem: Tools

We also need the following Lemma.

Lemma

Let b be a nonextreme point of the unit ball of H^∞. Then, there exists an outer function $a \in H^\infty$ such that $a(0) > 0$ and $|a|^2 + |b|^2 = 1$ a.e. on $\mathbb{T}$.

This gives another representation for $\mathcal{H}(\overline{b})$.

Theorem

Let b be a nonextreme point of the unit ball of H^∞. Then,

- $T_{\overline{a}} H^2 = \mathcal{H}(\overline{b})$.
- $f \in \mathcal{H}(b)$ *if and only if* $T_{\overline{b}} f = T_{\overline{a}} f_+$ *for a unique* $f^+ \in H^2$.

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.
2. Let $n \geq 0$. If $k > n$, then

$$\widehat{T_{\overline{a}}\chi_n}(k) = \langle T_{\overline{a}}\chi_n, \chi_k \rangle_2 = \langle \chi_{n-k}, a \rangle_2 = 0.$$

$T_{\overline{a}}\chi_n$ is a polynomial of degree d at most n.

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.
2. Let $n \geq 0$. If $k > n$, then
$$\widehat{T_{\overline{a}}\chi_n}(k) = \langle T_{\overline{a}}\chi_n, \chi_k \rangle_2 = \langle \chi_{n-k}, a \rangle_2 = 0.$$
$T_{\overline{a}}\chi_n$ is a polynomial of degree d at most n.
3. Suppose that $d < n$. Then, $T_{\overline{a}}\chi_n = \sum_{k=0}^{d} c_k \chi_k$ and we have
$$\langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \sum_{k=0}^{d} c_k \langle \chi_k, \chi_n \rangle_2 = 0.$$

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.
2. Let $n \geq 0$. If $k > n$, then
$$\widehat{T_{\overline{a}}\chi_n}(k) = \langle T_{\overline{a}}\chi_n, \chi_k \rangle_2 = \langle \chi_{n-k}, a \rangle_2 = 0.$$
$T_{\overline{a}}\chi_n$ is a polynomial of degree d at most n.
3. Suppose that $d < n$. Then, $T_{\overline{a}}\chi_n = \sum_{k=0}^{d} c_k \chi_k$ and we have
$$\langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \sum_{k=0}^{d} c_k \langle \chi_k, \chi_n \rangle_2 = 0.$$
4. From this, we get
$$0 = \langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \langle \chi_0, a \rangle = \hat{a}(0) = a(0).$$

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.
2. Let $n \geq 0$. If $k > n$, then
$$\widehat{T_{\overline{a}}\chi_n}(k) = \langle T_{\overline{a}}\chi_n, \chi_k \rangle_2 = \langle \chi_{n-k}, a \rangle_2 = 0.$$
$T_{\overline{a}}\chi_n$ is a polynomial of degree d at most n.
3. Suppose that $d < n$. Then, $T_{\overline{a}}\chi_n = \sum_{k=0}^{d} c_k \chi_k$ and we have
$$\langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \sum_{k=0}^{d} c_k \langle \chi_k, \chi_n \rangle_2 = 0.$$
4. From this, we get
$$0 = \langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \langle \chi_0, a \rangle = \hat{a}(0) = a(0).$$
5. This contradicts the Lemma and so $T_{\overline{a}}\chi_n$ is a polynomial of the same degree.

Main Theorem: Proof

1. Suppose that b is nonextreme. Then, $\overline{a}H^2 = \mathcal{H}(\overline{b}) \subset \mathcal{H}(b)$. We will show that $\mathcal{P}_+ \subset \overline{a}H^2$.
2. Let $n \geq 0$. If $k > n$, then
$$\widehat{T_{\overline{a}}\chi_n}(k) = \langle T_{\overline{a}}\chi_n, \chi_k \rangle_2 = \langle \chi_{n-k}, a \rangle_2 = 0.$$
$T_{\overline{a}}\chi_n$ is a polynomial of degree d at most n.
3. Suppose that $d < n$. Then, $T_{\overline{a}}\chi_n = \sum_{k=0}^{d} c_k \chi_k$ and we have
$$\langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \sum_{k=0}^{d} c_k \langle \chi_k, \chi_n \rangle_2 = 0.$$
4. From this, we get
$$0 = \langle T_{\overline{a}}\chi_n, \chi_n \rangle_2 = \langle \chi_0, a \rangle = \hat{a}(0) = a(0).$$
5. This contradicts the Lemma and so $T_{\overline{a}}\chi_n$ is a polynomial of the same degree.
6. The results now follows from linearity.

Main Theorem: Proof

1. Suppose that $\mathcal{P}_+ \subset \mathcal{H}(b)$ and b is extreme.

Main Theorem: Proof

1. Suppose that $\mathcal{P}_+ \subset \mathcal{H}(b)$ and b is extreme.
2. From the Theorem, $\chi_n \in \ker T_{\overline{b}}$ for every $n \geq 0$.

Main Theorem: Proof

1. Suppose that $\mathcal{P}_+ \subset \mathcal{H}(b)$ and b is extreme.
2. From the Theorem, $\chi_n \in \ker T_{\overline{b}}$ for every $n \geq 0$.
3. If $k \geq 0$, then

$$0 = \left\langle T_{\overline{b}}\chi_n, \chi_k \right\rangle_2 = \langle \chi_{n-k}, b \rangle_2 = \hat{b}(n-k).$$

Main Theorem: Proof

1. Suppose that $\mathcal{P}_+ \subset \mathcal{H}(b)$ and b is extreme.
2. From the Theorem, $\chi_n \in \ker T_{\overline{b}}$ for every $n \geq 0$.
3. If $k \geq 0$, then

$$0 = \left\langle T_{\overline{b}}\chi_n, \chi_k \right\rangle_2 = \langle \chi_{n-k}, b \rangle_2 = \hat{b}(n-k).$$

4. In other words, $\hat{b}(m) = 0$ for all $m \geq 0$, so $b = 0$.

Main Theorem: Proof

1. Suppose that $\mathcal{P}_+ \subset \mathcal{H}(b)$ and b is extreme.
2. From the Theorem, $\chi_n \in \ker T_{\overline{b}}$ for every $n \geq 0$.
3. If $k \geq 0$, then

$$0 = \left\langle T_{\overline{b}} \chi_n, \chi_k \right\rangle_2 = \langle \chi_{n-k}, b \rangle_2 = \hat{b}(n-k).$$

4. In other words, $\hat{b}(m) = 0$ for all $m \geq 0$, so $b = 0$.
5. This contradicts our assumption that b is not zero.

Conclusion

This last Theorem tells:

- No approximation is possible by polynomials.
- Moreover, no approximation is possible by rational functions.
- Finally, no approximation is possible by elements of $\ker T_{\overline{b}}$.

Conclusion

This last Theorem tells:

- No approximation is possible by polynomials.
- Moreover, no approximation is possible by rational functions.
- Finally, no approximation is possible by elements of $\ker T_{\overline{b}}$.

Huston, we got a problem...

Conclusion

However, recently

- Aleman and Malman [1] showed that $\mathcal{A} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.
- The authors of [3] showed that

$$\| T_{\overline{\varphi}_n} f - f \|_b \to 0 \quad n \to \infty$$

for a suitable choice of $(\varphi_n)_{n \geq 1} \subset H^\infty$.

Conclusion

However, recently

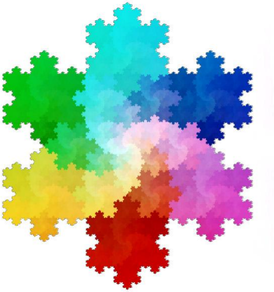
- Aleman and Malman [1] showed that $\mathcal{A} \cap \mathcal{H}(b)$ is dense in $\mathcal{H}(b)$.
- The authors of [3] showed that

$$\| T_{\overline{\varphi}_n} f - f \|_b \to 0 \quad n \to \infty$$

for a suitable choice of $(\varphi_n)_{n \geq 1} \subset H^\infty$.

The goal is to give a constructive proof of the density of $\mathcal{A} \cap \mathcal{H}(b)$.

Thanks for your attention!



Bibliography I

A. Aleman and B. Malman.
Hilbert spaces of analytic functions with a contractive backward shift.
Journal of Functional Analysis, 2018.

L. De Branges.
Square summable power series.
Toronto : Holt-Rinehart and Winston, 1966.

O. El-Fallah, E. Fricain, K. Kellay, J. Mashreghi, and T. Ransford.
Constructive approximation in de branges–rovnyak spaces.
Constructive Approximation, 44(2):269–281, Oct 2016.

D. Sarason.
Sub-Hardy Hilbert Spaces in the Unit Disk.
The University of Arkensas Lecture Notes in the Mathematical Sciences. Wiley-Interscience, 1994.