# Tricomplex Dynamics Related to Monic Polynomials

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*XIX*e Colloque panquébécois des étudiants de l'ISM 2016, UQÀM
May, 2016

[1]Research supported by CRSNG (Canada).

# Introduction and mind map

Mandelbrot set

1st extension: $z^p + c$

$p \in \mathbb{N} \setminus \{0, 1\}$

2nd extension: numbers

**hyperbolic numbers** P. Senn (1990)

**bicomplex numbers** D. Rochon (2000)

**tricomplex numbers** V. Garrant P. D. Rochon (2009)

$\boxed{\text{FUSION}}$ in this talk

# Bicomplex Numbers: definition

**Definition 1 ($\mathbb{M}(2)$ or $\mathbb{BC}$ space)**

Let $z_1 = x_1 + x_2\mathbf{i_1}$, $z_2 = x_3 + x_4\mathbf{i_1}$ be two complex numbers $\mathbb{C}(\mathbf{i_1}) \simeq \mathbb{C}$ with $\mathbf{i_1}^2 = -1$. A bicomplex number $\zeta$ is defined as:

$$\zeta = z_1 + z_2\mathbf{i_2} \tag{1}$$

where $\mathbf{i_2}^2 = -1$ and $\mathbf{i_2} \neq \mathbf{i_1}$.

Various representations:

- In terms of four real numbers: $\zeta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{j_1}$
- In terms of two idempotent elements:

$$\zeta = (z_1 - z_2\mathbf{i_1})\gamma_1 + (z_1 + z_2\mathbf{i_1})\overline{\gamma}_1$$

where $\gamma_1 = \frac{1+\mathbf{j_1}}{2}$ and $\overline{\gamma}_1 = \frac{1-\mathbf{j_1}}{2}$.

# Bicomplex numbers: operations

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$ and $\zeta_2 = z_3 + z_4\mathbf{i_2}$.

1) Equality : $\zeta_1 = \zeta_2 \iff z_1 = z_3$ and $z_2 = z_4$.
2) Addition : $\zeta_1 + \zeta_2 := (z_1 + z_3) + (z_2 + z_4)\mathbf{i_2}$.
3) Multiplication : $\zeta_1 \cdot \zeta_2 := (z_1 z_3 - z_2 z_4) + (z_2 z_3 + z_1 z_4)\mathbf{i_2}$.
4) Modulus : $\|\zeta_1\|_2 := \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\sum_{i=1}^4 x_i^2}$

Remark:

- $(\mathbb{M}(2)\,,+\,,\cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(2)\,,+\,,\cdot\,,\|\cdot\|_2)$ forms a **Banach space**.

# Tricomplex Numbers: definition

**Definition 2 ($\mathbb{M}(3)$ or $\mathbb{TC}$ space)**

Let $\zeta_1 = z_1 + z_2\mathbf{i_2}$, $\zeta_2 = z_3 + z_4\mathbf{i_2}$ be two bicomplex numbers. A tricomplex number $\eta$ is defined as:

$$\eta = \zeta_1 + \zeta_2\mathbf{i_3} \tag{2}$$

where $\mathbf{i_3}^2 = -1$.

Various representations:

- In terms of four complex numbers: $\eta = z_1 + z_2\mathbf{i_2} + z_3\mathbf{i_3} + z_4\mathbf{j_3}$
- In terms of eight real numbers:

$$\eta = x_1 + x_2\mathbf{i_1} + x_3\mathbf{i_2} + x_4\mathbf{i_3} + x_5\mathbf{i_4} + x_6\mathbf{j_1} + x_7\mathbf{j_2} + x_8\mathbf{j_3}$$

Go to Table

# Tricomplex Numbers: definition

Various representations (continuing):

- In terms of two idempotent elements:

$$\eta = (\zeta_1 - \zeta_2 \mathbf{i_2})\gamma_2 + (\zeta_1 + \zeta_2 \mathbf{i_2})\overline{\gamma}_2$$

where $\gamma_2 = \frac{1+\mathbf{j_3}}{2}$ and $\overline{\gamma}_2 = \frac{1-\mathbf{j_3}}{2}$.

- In terms of four idempotent elements:

$$\eta = w_1 \cdot \gamma_1\gamma_2 + w_2 \cdot \gamma_1\overline{\gamma}_2 + w_3 \cdot \overline{\gamma}_1\gamma_2 + w_4 \cdot \overline{\gamma}_1\overline{\gamma}_2$$

where $w_i \in \mathbb{C}(\mathbf{i_1})$ for $i = 1, 2, 3, 4$.

# Table of imaginary units

| $\cdot$ | $1$ | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ |
|---|---|---|---|---|---|---|---|---|
| $1$ | $1$ | $\mathbf{i_1}$ | $\mathbf{i_2}$ | $\mathbf{i_3}$ | $\mathbf{i_4}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ |
| $\mathbf{i_1}$ | $\mathbf{i_1}$ | $-\mathbf{1}$ | $\mathbf{j_1}$ | $\mathbf{j_2}$ | $-\mathbf{j_3}$ | $-\mathbf{i_2}$ | $-\mathbf{i_3}$ | $\mathbf{i_4}$ |
| $\mathbf{i_2}$ | $\mathbf{i_2}$ | $\mathbf{j_1}$ | $-\mathbf{1}$ | $\mathbf{j_3}$ | $-\mathbf{j_2}$ | $-\mathbf{i_1}$ | $\mathbf{i_4}$ | $-\mathbf{i_3}$ |
| $\mathbf{i_3}$ | $\mathbf{i_3}$ | $\mathbf{j_2}$ | $\mathbf{j_3}$ | $-\mathbf{1}$ | $-\mathbf{j_1}$ | $\mathbf{i_4}$ | $-\mathbf{i_1}$ | $-\mathbf{i_2}$ |
| $\mathbf{i_4}$ | $\mathbf{i_4}$ | $-\mathbf{j_3}$ | $-\mathbf{j_2}$ | $-\mathbf{j_1}$ | $-\mathbf{1}$ | $\mathbf{i_3}$ | $\mathbf{i_2}$ | $\mathbf{i_1}$ |
| $\mathbf{j_1}$ | $\mathbf{j_1}$ | $-\mathbf{i_2}$ | $-\mathbf{i_1}$ | $\mathbf{i_4}$ | $\mathbf{i_3}$ | $\mathbf{1}$ | $-\mathbf{j_3}$ | $-\mathbf{j_2}$ |
| $\mathbf{j_2}$ | $\mathbf{j_2}$ | $-\mathbf{i_3}$ | $\mathbf{i_4}$ | $-\mathbf{i_1}$ | $\mathbf{i_2}$ | $-\mathbf{j_3}$ | $\mathbf{1}$ | $-\mathbf{j_1}$ |
| $\mathbf{j_3}$ | $\mathbf{j_3}$ | $\mathbf{i_4}$ | $-\mathbf{i_3}$ | $-\mathbf{i_2}$ | $\mathbf{i_1}$ | $-\mathbf{j_2}$ | $-\mathbf{j_1}$ | $\mathbf{1}$ |

Table: Product of tricomplex imaginary units

Back

# Tricomplex numbers: operations

Let $\eta_1 = \zeta_1 + \zeta_2\mathbf{i_2}$ and $\eta_2 = \zeta_3 + \zeta_4\mathbf{i_2}$.

1) Equality: $\eta_1 = \eta_2 \iff \zeta_1 = \zeta_3$ and $\zeta_2 = \zeta_4$.
2) Addition: $\eta_1 + \eta_2 := (\zeta_1 + \zeta_3) + (\zeta_2 + \zeta_4)\mathbf{i_2}$.
3) Multiplication: $\eta_1 \cdot \eta_2 := (\zeta_1\zeta_3 - \zeta_2\zeta_4) + (\zeta_2\zeta_3 + \zeta_1\zeta_4)\mathbf{i_2}$.
4) Modulus : $\|\eta_1\|_3 := \sqrt{\|\zeta_1\|_2^2 + \|\zeta_2\|_2^2} = \sqrt{\sum_{i=1}^{4} |z_i|^2} = \sqrt{\sum_{i=1}^{8} x_i^2}$.

Remark:

- $(\mathbb{M}(3)\,,+\,,\cdot)$ forms a commutative ring with unity and zero divisors.
- $(\mathbb{M}(3)\,,+\,,\cdot\,,\|\cdot\|_3)$ forms a **Banach space**.

# Remarks: some Viewpoints

Bicomplex numbers:

- as a pair of complex variables $(z_1, z_2)$;
- as a quadruple of real numbers $(x_1, x_2, x_3, x_4)$.

Tricomplex numbers:

- as a pair of bicomplex numbers $(\zeta_1, \zeta_2)$;
- as a quadruple of complex numbers $(z_1, z_2, z_3, z_4)$;
- as a octuple of real numbers $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$.

# Remarks: subsets of $\mathbb{M}(3)$

**Definition 3**

Let $\mathbf{i_k} \in \{\mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}\}$ and $\mathbf{j_k} \in \{\mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$, where $\mathbf{i_k}^2 = -\mathbf{1}$ and $\mathbf{j_k}^2 = \mathbf{1}$. We define

$$\mathbb{C}(\mathbf{i_k}) := \{\eta = x_0 + x_1\mathbf{i_k} \; : \; x_0, x_1 \in \mathbb{R}\}$$

and

$$\mathbb{D}(\mathbf{j_k}) := \{x_0 + x_1\mathbf{j_k} \; : \; x_0, x_1 \in \mathbb{R}\} .$$

- $\mathbb{C}(\mathbf{i_k})$ is a subset of $\mathbb{M}(3)$ for $k \in \{1, 2, 3, 4\}$. They are all isomorphic to $\mathbb{C}$.
- $\mathbb{D}(\mathbf{j_k})$ is a subset of $\mathbb{M}(3)$ and is isomorphic to the set of hyperbolic numbers $\mathbb{D}$ for $k \in \{1, 2, 3\}$.

# Remarks: subsets of $\mathbb{M}(3)$ (continued)

**Definition 4**

Let $\mathbf{i_k}, \mathbf{i_l}, \mathbf{i_m} \in \{1, \mathbf{i_1}, \mathbf{i_2}, \mathbf{i_3}, \mathbf{i_4}, \mathbf{j_1}, \mathbf{j_2}, \mathbf{j_3}\}$ with $\mathbf{i_k} \neq \mathbf{i_l}$, $\mathbf{i_k} \neq \mathbf{i_m}$ and $\mathbf{i_l} \neq \mathbf{i_m}$. We define

$$\mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) := \{x_1\mathbf{i_k} + x_2\mathbf{i_l} + x_3\mathbf{i_m} \ : \ x_1, x_2, x_3 \in \mathbb{R}\}. \tag{3}$$

- This set is used to make 3D slices in the Tricomplex Multibrot sets.

# General definition: Tricomplex Multibrot sets

## Definition 5

Let $Q_{p,c}(\eta) = \eta^p + c$ where $\eta, c \in \mathbb{M}(3)$ and $p \geq 2$ an integer. The tricomplex *Multibrot* set is defined as the set

$$\mathcal{M}_3^p := \left\{ c \in \mathbb{M}(3) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded} \right\}. \qquad (4)$$

## Properties

1) $c \in \mathcal{M}_3^p \iff \|Q_{p,c}^m(0)\|_3 \leq 2^{1/(p-1)}$ for all $m \in \mathbb{N}$.
2) $c \in \mathcal{M}_3^p \implies \|c\|_3 \leq 2^{1/(p-1)}$.
3) $\mathcal{M}_3^p$ is a connected set.

# General definition: Tricomplex Multibrot sets

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3) $\mathcal{M}_3^p$ is a connected set.

# Particular cases: complex numbers

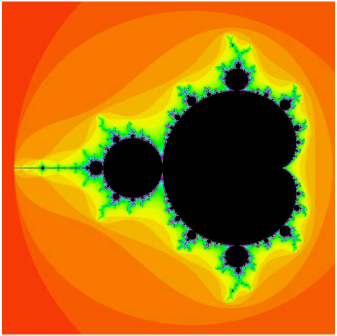
If $c \in \mathbb{C}(\mathbf{i_1})$, then we obtain what we call the Multibrot sets:

$$\mathcal{M}_1^p := \left\{ c = x + y\mathbf{i_1} \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

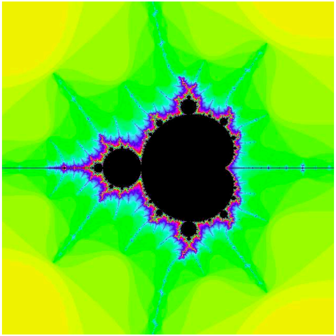
Remark:

- Properties 1 and 2 are used to generate, with a computer, a multibrot set.

# Multibrot sets Pictured

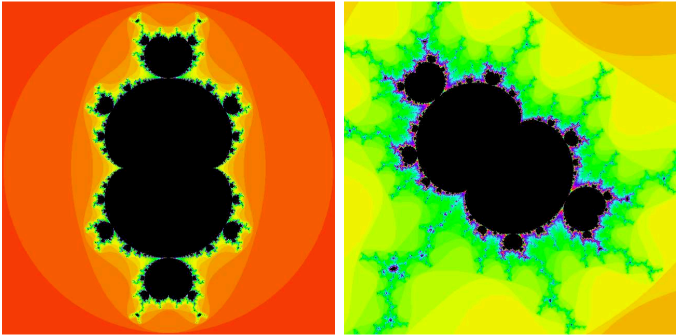


(a) $\mathcal{M}^2$: Mandelbrot set



(b) $\mathcal{M}^2$: Zoom in

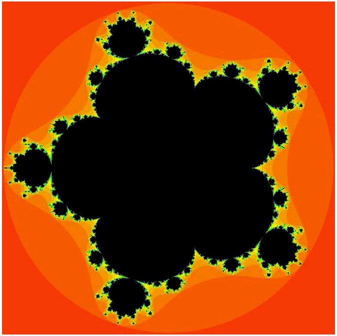
# Multibrot sets Pictured



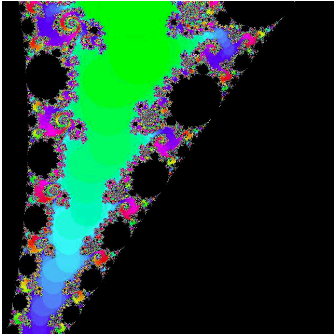
(a) $\mathcal{M}^3$: Mandelbric set

(b) $\mathcal{M}^3$: Zoom in

# Multibrot sets Pictures



(a) $\mathcal{M}^6$



(b) $\mathcal{M}^6$: Zoom in

# Particular cases: complex numbers (continued)

Properties:

- $\mathcal{M}_1^p$ is a connected set.
- For any odd integer $p > 2$,

$$\mathcal{M}_1^p \cap \mathbb{R} = \left[ -\frac{p-1}{p^{p/(p-1)}}, \frac{p-1}{p^{p/(p-1)}} \right].$$

Thus, the real intersection of a multibrot set is an interval.

# Particular cases: complex numbers (continued)

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# Particular cases: hyperbolic numbers

If $c \in \mathbb{D}(\mathbf{j_1})$, then we obtain what we call the Hyperbrot sets:

$$\mathcal{H}^p := \left\{ c = x + y\mathbf{j_1} \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

**Theorem 6**

*Let $p > 2$ be an odd integer. Then,*

$$\mathcal{H}^p = \left\{ c = x + y\mathbf{j_1} \ : \ |x| + |y| \leq \frac{p-1}{p^{p/(p-1)}} \right\}.$$

Remark: In others words, $\mathcal{H}^p$ are squares.

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Remark: In others words, $\mathcal{H}^p$ are squares.

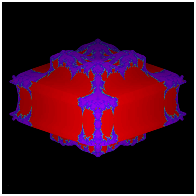
# 3D slices: definition

To visualize the tricomplex multibrot sets, we have to define a **principal 3D slice** of $\mathcal{M}_3^p$.

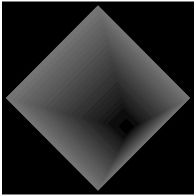
$$\mathcal{T}^p := \mathcal{T}^p(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) = \left\{ c \in \mathbb{T}(\mathbf{i_m}, \mathbf{i_k}, \mathbf{i_l}) \ : \ \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded } \right\}.$$

There are 56 possible 3D principal slices.

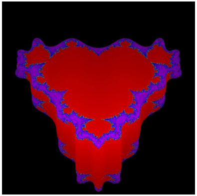
# 3D slices: some photo shooting



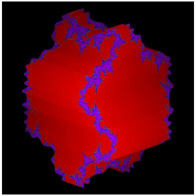
(a) $\mathcal{T}^3(1, \mathbf{i}_1, \mathbf{i}_2)$



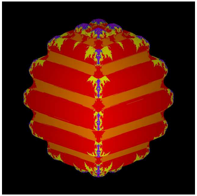
(b) Perplexbrot



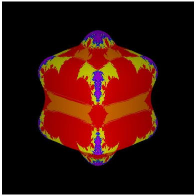
(c) $\mathcal{T}^4(1, \mathbf{i}_1, \mathbf{j}_1)$



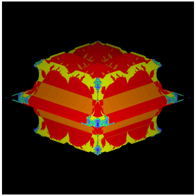
(d) $\mathcal{T}^6(\mathbf{i}_1, \mathbf{i}_2, \mathbf{j}_1)$



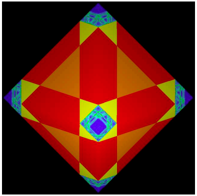
(e) $\mathcal{T}^8(\mathbf{i}_1, \mathbf{i}_2, \mathbf{j}_2)$



(f) $\mathcal{T}^4(\mathbf{i}_1, \mathbf{j}_1, \mathbf{j}_2)$



(g) Turtlebrot



(h) Metabrot

# 3D slices: Perplexbrot

We define the Perplexbrot, denoted by $\mathcal{P}^p$, as the set:

$$\left\{ c = c_1 + c_4\mathbf{j_1} + c_6\mathbf{j_2} \ : \ c_i \in \mathbb{R} \text{ and } \left\{ Q_{p,c}^m(0) \right\}_{m=1}^{\infty} \text{ is bounded} \right\}.$$

**Lemma 7**

*We have the following characterization of the Perplexbrot:*

$$\mathcal{P}^p = \bigcup_{y \in [-m_p, m_p]} \{[(\mathcal{H}^p - y\mathbf{j_1}) \cap (\mathcal{H}^p + y\mathbf{j_1})] + y\mathbf{j_2}\}$$

*where $\mathcal{H}^p$ is the Hyperbrot generated by the polynomial $z^p + c$ where $p$ is an odd integer and $m_p := \frac{p-1}{p^{p/(p-1)}}$.*

Corollary: The set $\mathcal{P}^p$ is a regular octahedron.

# 3D slices: Perplexbrot

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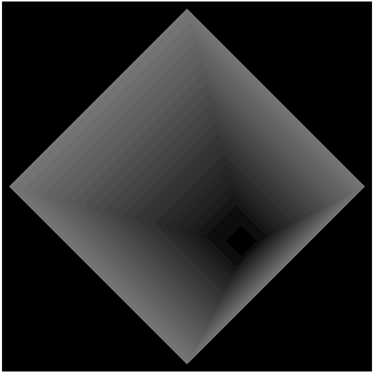
# Platonic Solids

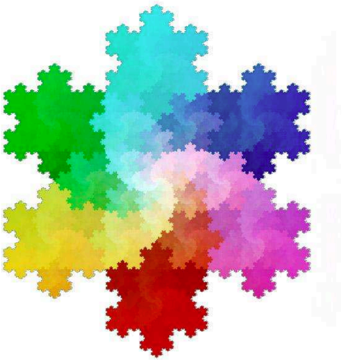
## Conclusion

Tricomplex dynamics is related with the famous Platonic solids.



(a) Octahedron

# Thanks for your attention!



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