Density Theorems in Analytic function spaces

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Spaces of continuous functions

Theorem (1885, Weierstrass)

Let $f \in \mathcal{C}[a, b]$ be a continuous function on the closed interval $[a, b]$. Then, there exists a sequence of polynomials (p_n) such that

$$\|f - p_n\|_\infty \underset{n \to +\infty}{\longrightarrow} 0.$$

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Several proofs:

1. Weierstrass by using convolution with the Heat Kernel;
2. Berstein, with the Berstein polynomials;
3. Stone, with the Stone-Weierstrass theorem;
4. With the Hahn-Banach theorem.

Spaces of continuous functions

Theorem (1912, Berstein)

If $f \in \mathcal{C}[0,1]$ *and* $p_n(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}$, *then*

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Remark. Exact constructive proof method!

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In general, given a space of functions V, one can ask many questions about the structure of V:

1. Can we endow V with a norm that makes it a Banach space?
2. With a given norm, can we find approximants in a "nice" subset of V for an $f \in V$?

Hardy space: Definition

Let $f(z) := \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D} := \{|z| < 1\}$.

Definition

$H^2 := \left\{ f \in \mathcal{H}(\mathbb{D}) \, : \, \sum_{n=0}^{\infty} |a_n|^2 < +\infty \right\}.$

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We define:

- A norm: $\|f\|_2^2 := \sum_{n=0}^{\infty} |a_n|^2$;
- The partial sums: $s_n(f)(z) := \sum_{k=0}^{n} a_k z^k$.

Hardy space: Polynomial approximants

Theorem

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Theorem

If $f \in H^2$, *then* $\|f - s_n(f)\|_2 \to 0$ *when* $n \to +\infty$.

Proof.

We have $f - s_n(f) = \sum_{k=n+1}^{\infty} a_k z^k$ and so

$$\|f - s_n(f)\|_2^2 = \sum_{k=n+1}^{\infty} |a_k|^2 \xrightarrow[n \to +\infty]{} 0.$$

$\square$

Disk Algebra: Definition

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A suprising fact (du Bois Reymond):

$$\exists f \in \mathcal{A}(\mathbb{D}) \text{ s.t. } \sup_{n \geq 0} \|s_n f\|_\infty = +\infty.$$

Theorem

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Proof.

$g := f|_{\partial\mathbb{D}}$ is continuous. So, from Fejér theorem, the Cesàro means $\sigma_n(f) := \frac{\sum_{k=0}^{n} s_k(f)}{n+1}$ converge to g uniformly. $\sigma_n(f)$ is the restriction of a polynomial to $\partial\mathbb{D}$ and

$$\|\sigma_n(f) - f\|_\infty = \sup_{z \in \partial\mathbb{D}} |\sigma_n(f)(z) - g(z)| \underset{n \to +\infty}{\longrightarrow} 0.$$

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Definition

Let $b \in H^\infty$ with $\|b\|_\infty \leq 1$. Then,

$$\mathcal{H}(b) := \left\{ f \in H^2 \, : \, \sup_{g \in H^2} \left(\|f + gb\|_2^2 - \|g\|_2^2 \right) < +\infty \right\}.$$

We define a norm:

$$\|f\|_b^2 := \sup_{g \in H^2} \left(\|f + gb\|_2^2 - \|g\|_2^2 \right).$$

Example 1.

Let $b_0(z) := \frac{\tau z}{1-\tau^2 z}$ where $\tau := \frac{\sqrt{5}-1}{2}$. If $f \in \mathcal{H}(b_0)$, then

$$\|f_r - f\|_{b_0} \to 0$$

as $r \to 1^-$ where $f_r(z) := f(rz)$.

Example 2.

However, let $b := b_0 B$ where B is a Blaschke product with zeros at $w_n := 1 - 4^{-n}$ for $n \geq 1$.

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- $\lim_{r \to 1^-} \|f_r\|_b = +\infty$;
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We are in trouble...

Characterization

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When b is a non-extreme point of B_{H^∞}, then

Theorem (1994, Sarason[6])

Let $b \in H^\infty$ and $\|b\|_\infty \leq 1$. Then, the following statements are equivalent:

1. *b is non-extreme;*
2. $\log(1 - |b|^2) \in L^1(\partial\mathbb{D})$;
3. *$\mathcal{H}(b)$ contains all holomorphic functions in $\overline{\mathbb{D}}$;*
4. *The polynomials form a dense subset of $\mathcal{H}(b)$.*

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- From Sarason, by showing that $\mathcal{P}^{\perp}$ is trivial in $\mathcal{H}(b)$.
- From O. El-Fallah, E. Fricain, K. Kellay, J. Mashreghi, and T. Ransford by using Toeplitz operators and building up polynomials to approach a giving function in $\mathcal{H}(b)$.

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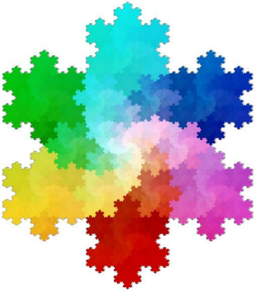
Remark.

- $\mathcal{H}(b)$ may contain no function in $\mathcal{H}(\overline{\mathbb{D}})$.
- The proof is not constructive.
- There is no known constructive proof of this fact!!

Thanks for your attention!



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