ELE-E4760
Modeling and Simulation
Exercise 2

Adam Ilyas 725819

October 10, 2018

1. Let X be a random variable that obeys the following the Pareto distribution with shape parameter $\beta = 2$ and scale parameter $b = 1$. Thus, the cumulative distribution function of X is given by

$$F(x) = 1 - \frac{1}{(1+x)^2}$$

Give an inverse transformation method for generating samples of X

We have to find the inverse function F^{-1} such that $X \sim F^{-1}(U)$:

Let $y = F(x)$

$$\begin{aligned} y &= 1 - \frac{1}{(1+x)^2} \\ (1+x)^2 &= \frac{1}{1-y} \\ x &= -1 + \sqrt{\frac{1}{1-y}} \end{aligned}$$

where $y \geq 0$

Thus the inverse function: $F^{-1}(x) = -1 + \sqrt{\frac{1}{1-x}}$

We can generate samples of X using $-1 + \sqrt{\frac{1}{1-U}}$ where $U \sim U(0,1)$

2. Apply the inverse transform method and give an algorithm to generate samples of X with the pdf

$$f(x) = \begin{cases} \frac{x-2}{2}, & 2 \leq x \leq 3 \\ 1 - \frac{x}{6}, & 3 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} \frac{x^2}{4} - x + c_1, & 2 \leq x \leq 3 \\ x - \frac{x^2}{12} + c_2, & 3 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

We have to choose c_1, c_2 such that $F(x)$ is continuous and monotonically increasing from $[0, 1]$.

The current range is expressed as

$$[c_1 - 1, c_1 - \frac{3}{4}], \quad 2 \leq x \leq 3$$
$$[c_2 + \frac{9}{4}, c_2 + 3], \quad 3 \leq x \leq 6$$

Thus, by solving $c_1 - 1 = 0$ and $c_2 + 3 = 1$, we know that $c_1 = 1, c_2 = -2$

$$F(x) = \begin{cases} \frac{x^2}{4} - x + 1, & 2 \leq x \leq 3 \\ x - \frac{x^2}{12} - 2, & 3 \leq x \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

When we apply the inverse transform method, we can generate X from $U \sim U(0, 1)$. We start by letting $y = F(x)$

For $2 \leq x \leq 3$,

$$y = \frac{x^2}{4} - x + 1$$
$$= x^2 - 4x + 4$$

$$x = 2 \pm 2\sqrt{y}$$

$$F^{-1}(x) = 2 - 2\sqrt{x}$$

For $3 \leq x \leq 6$,

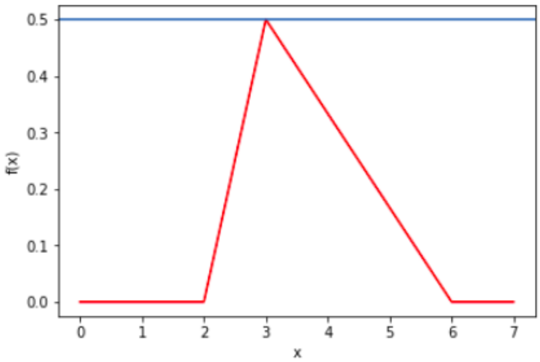
$$y = x - \frac{x^2}{12} - 2$$
$$12y = 12x - x^2 - 24$$
$$x^2 - 12x + (12y - 24) = 0$$

$$x = 6 \pm 2\sqrt{3 - 3y}$$
$$F^{-1}(x) = 6 - 2\sqrt{3 - 3x}$$

$$X = \begin{cases} 2 - 2\sqrt{U}, & 0 \leq U \leq \frac{1}{4} \\ 6 - 2\sqrt{3 - 3U}, & \frac{1}{4} \leq U \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where $U \sim U(0, 1)$

3. Apply the acceptance rejection method in problem 2 and give an algorithm for generating the samples.



   - The function $f(x)$ is limited in a rectangle with height 0.5
   - We can choose $c = 0.5$ for $g(x) = 1$ (A uniform r.v. denoted by the blue line)

   Algorithm

   - Generate X from Uniform Distribution with pdf $g(x)$
   - Generate Y from Uniform Distribution $U(0, 0.5)$
   - If $Y \leq f(X)$, accept X and stop
   - Else if $Y > f(X)$ continue from beginning until an acceptable pair is found

4. Given a uniformly distributed sample U, i.e., $U \sim U(0,1)$, samples of X are generated with the following inverse transformation method:

$$X = \begin{cases} \sqrt{2U}, & 0 \leq U \leq \frac{1}{2} \\ 2 - \sqrt{2 - 2U}, & \frac{1}{2} < U \leq 1 \end{cases}$$

   What is the probability density function of X? Draw a picture.

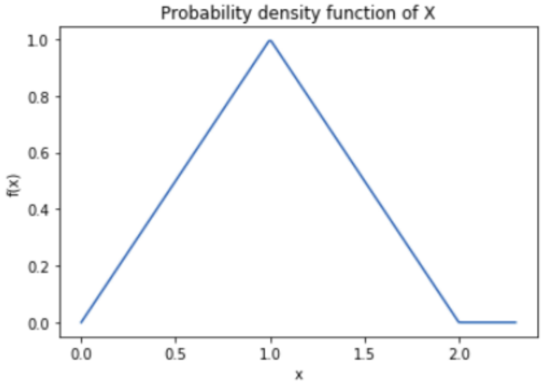
   For $0 \leq U \leq \frac{1}{2}$,

$$\begin{aligned} F(x) &= \frac{x^2}{2} \\ f(x) &= x \\ x &\in [0, 1] \end{aligned}$$

   For $\frac{1}{2} \leq U \leq 1$,

$$\begin{aligned} F(x) &= 1 - \frac{(2-x)^2}{2} \\ f(x) &= 2 - x \\ x &\in [1, 2] \end{aligned}$$

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2 - x, & 2 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$



5. A worker has to go through $k = 4$ stages in order to complete a job. The time to complete stage i, X_i , is an exponentially distributed random variable with mean $1/\lambda$. However, after completing stage i, the worker will go to stage i+1 only with probability α_i , i = 1, . . . , k 1, and with probability $(1 - \alpha_i)$ he will stop working. Denote by X the total time the worker spends on the job. Device an algorithm based on the composite method for generating random samples of X.

Time Spent	Probability (p_i)	
X_1	$(1 - \alpha_i)$	p_1
$X_1 + X_2$	$(1 - \alpha_2)\alpha_1$	p_2
$X_1 + X_2 + X_3$	$(1 - \alpha_3)\alpha_2\alpha_1$	p_3
$X_1 + X_2 + X_3 + X_4$	$(1 - \alpha_4)\alpha_3\alpha_2\alpha_1$	p_4

Algorithm:

- Find the Conditional pdf $f_i(x)$ (probability of picking x given the interval i): $f(x)/p_i$
- draw I from Distribution $p_i, i = 1, 2, 3, 4$
- draw Y with the Conditional pdf
- $X = I + Y$