Batch means method

- Batch means method is used frequently
- Simulation is done as a single (long) run
  - let the length of simulation be M
    - here we think that we consider the system from a customer point of view; then M may mean the number of interesting observations (as well we may think that M represents time)
  - let the observed variable be X (for instance, waiting time in a queue) and the task is to estimate its expected value $\mu = \mathrm{E}[X]$
- From the beginning of the simulation, the warm-up period of K observations is rejected
- The useful run (of length $M - K$) is divided into N batches; thus in each batch there are

$$n = \frac{M - K}{N}$$

observations

Batch means method (continued)

- In batch i we get for X the sample average (X_{ij} denotes the j^{th} observation in the i^{th} batch)

$$\bar{X}_i = \frac{1}{n} \sum_{j=1}^{n} X_{ij}$$

- The final estimator for the expectation μ is

$$\hat{\mu}_N = \frac{1}{N} \sum_{i=1}^{N} \bar{X}_i = \frac{1}{nN} \sum_{i=1}^{N} \sum_{j=1}^{n} X_{ij}$$

- This is simply the sample average of the whole run (after the warm-up period)
  - the division in batches has no bearing from the point of view of the estimator
  - the sole purpose of the division is to get an idea of the confidence interval of the estimator
- Assuming that the batches are long enough, the sample averages $\bar{X}_i$ of the batches are approximately independent
- Their sample variance then provides an estimate for the variance of a single $\bar{X}_i$

$$S^2 = \frac{1}{N-1} \sum_{i=1}^{N} (\bar{X}_i - \hat{\mu}_N)^2$$

Batch means method (continued)

- The confidence interval of the estimator (at confidence level $1-\beta$) is

$$\hat{\mu}_N \pm z_{1-\beta/2}\frac{S}{\sqrt{N}}$$

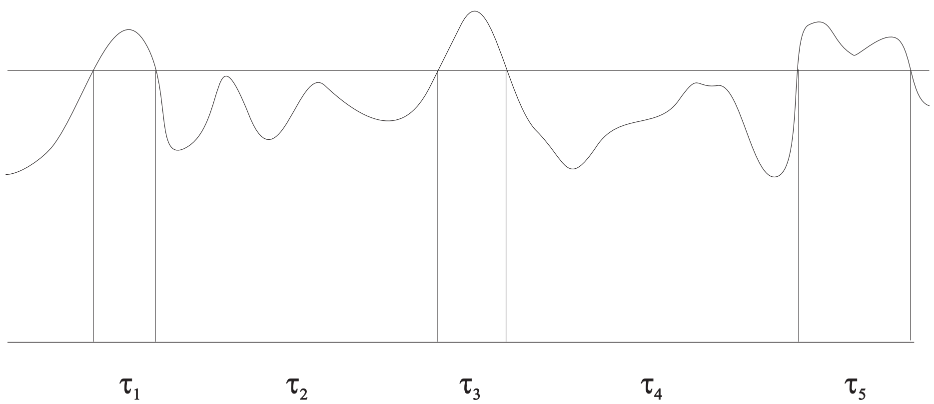
- The advantage of the method is that there is only one warm-up period
- There should be at least 20-30 batches in order to estimate the variance reliably
- The bathes should be long enough (much longer than the duration of the initial transient) to guarantee that the $\bar{X}_i$ are approximately independent
- If there is dependence, the correlation is usually positive
- Then the real confidence interval of $\hat{\mu}$ is larger than the estimate given above based on the assumption of independent batches
  - the dependence does not at all degrade the value of the estimator
  - it only can mislead the user to believe that the accuracy of the estimator is better than it actually is

Regenerative method

- Is applicable in so called regenerative systems
- A regenerative system has at least one regenerative state
  - the stochastic development of the system from that point on does not at all depend on how this state has been reached
  - every state of a Markovian system is regenerative
  - in a G/G/1 queue the state where the system is empty is a regenerative state
- It there are several regenerative states, one of them is chosen as the basis for the data collection method
  - in the sequel, the regenerative state refers to the chosen regenerative state
- Every now and then the system visits the regenerative state or “regenerates itself”
  - this starts “a new life” which does not depend on the past

Regenerative method (continued)

- The instant, when the system returns to the regenerative state, is called the regeneration point
- The period between two regeneration points is called the regeneration period
- The developments of different regeneration periods are fully independent of each other
  - this is the “point” of the method



Regenerative method: point estimator

- Let X be the cumulative value of the observed variable during a regenerative period, for instance,
  - the total time the system has spent in a blocking state during a regenerative period
  - the total number of packets overflown from a buffer during the period
- Let τ be the "duration" of the regenerative period
  - this may refer to the real duration (time) of the period
  - it may also refer to e.g. the total number of arrivals during the regenerative period
- The expectation of the observed variable ℓ (for instance, the expectation of time blocking) is

$$\ell = \frac{\mathrm{E}[X]}{\mathrm{E}[\tau]}$$

- In a simulation over n regenerative periods one obtains a (strongly consistent) estimator

$$\bar{\ell}_n = \frac{\bar{X}}{\bar{\tau}}$$

where $\bar{X}$ and $\bar{\tau}$ are the sample averages $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $\bar{\tau} = \frac{1}{n}\sum_{i=1}^{n} \tau_i$

The confidence interval of the estimator

- Consider the variable $Z_i = X_i - \ell\tau_i$
  - the Z_i are independent and identically distributed random variables (with mean 0)
  - so are the X_i and the τ_i
- Denote

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \qquad \bar{\tau} = \frac{1}{n}\sum_{i=1}^{n} \tau_i, \qquad \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i = \bar{X} - \ell\bar{\tau}$$

- By the central limit theorem we have

$$\frac{n^{1/2}\bar{Z}}{\sigma} = \frac{n^{1/2}(\bar{X} - \ell\bar{\tau})}{\sigma} \to \mathrm{N}(0,1), \quad \text{kun } n \to \infty$$

  where σ^2 is the variance of Z

$$\sigma^2 = \mathrm{V}[Z] = \mathrm{V}[X] - 2\ell\mathrm{Cov}[X,\tau] + \ell^2\mathrm{V}[\tau]$$

The confidence interval of the estimator (continued)

- By dividing by $\bar{\tau}$ we get

$$\frac{n^{1/2}(\bar{\ell}_n - \ell)}{\sigma/\bar{\tau}} \to \mathrm{N}(0,1), \quad \text{when } n \to \infty$$

- For the point estimator $\bar{\ell}_n$ based on measurement over n regenerative periods we get the confidence interval (at the confidence level $1-\beta$)

$$\bar{\ell}_n \pm \frac{z_{1-\beta/2} S}{\sqrt{n}\bar{\tau}}$$

where S^2 is the (unbiased) estimator of σ^2 based on the sample

$$S^2 = S_{11} - 2\,\bar{\ell}_n\,S_{12} + \bar{\ell}_n^2\,S_{22}$$

and S_{11}, S_{22} and S_{12} are the sample variances and sample covariance of X and τ

$$S_{11} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2, \quad S_{22} = \frac{1}{n-1}\sum_{i=1}^{n}(\tau_i - \bar{\tau})^2, \quad S_{12} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(\tau_i - \bar{\tau})$$

Regenerative method: discussion

- Advantages
  - separate transient removal is not needed
  - one does not have to fix parameters such as the number of batches in advance
  - asymptotically accurate
  - easy to understand and implement
- There are, however, a few disadvantages
  - it may be difficult to identify regenerative states
  - even if one can be identified
    * the regenerative period may be very long (the user has no control over it)
    * in a complex system the identification of the regenerative state may be computationally expensive
  - with a finite value of n the estimator $\bar{\ell}_n$ is biased
    * in fact, the initial transient problem does exist, though it is somewhat concealed