ELE-E4760
Modeling and Simulation
Exercise 4

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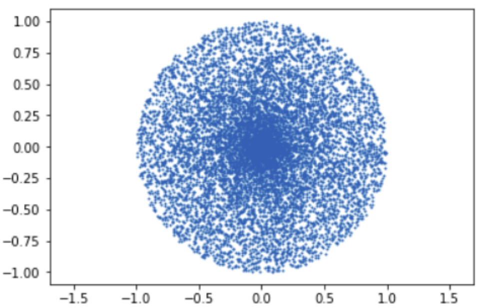
October 19, 2018

1. Assume that we want to generate uniformly distributed points within the unit circle $x^2 + y^2 = 1$. Give an algorithm based on the inverse transformation technique to directly generate samples of the x- and y-coordinates.

   Hint: consider x and y in the polar coordinates (r, ϕ), justify that r and ϕ are independent, derive the marginal distribution of ϕ-component and consider the probability of points falling inside a circle with radius r.

   Ans We can use the following equations $x = r\cos\phi,\ y = r\sin\phi$ And generate x, y accordingly by generating ϕ and r

   Naively, I will draw ϕ from $U(0, 2\pi)$ and r from $U(0, R)$ so that the points will be bounded by the unit circle. However, the resulting plot is as such:

   We observe that the distribution of points nearer to the center is more dense. Hence, we have to choose r properly.

If we consider the probability of x, y falling inside a circle with r

$$\frac{\text{Area of circle with radius r}}{\text{Total area of circle}} = \frac{\pi r^2}{\pi (R)^2} = \frac{r^2}{R^2}$$

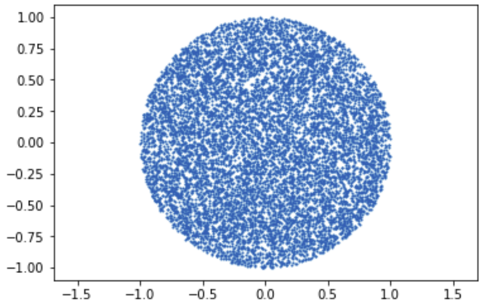
$$\Pr\left(x^2 + y^2 \leq r^2\right) = \frac{r^2}{R^2}$$

Using inverse transformation, $F^{-1}(r) = R\sqrt{r}$, hence $r = \sqrt{U}$ where $U \sim U(0,1)$, since $R = 1$

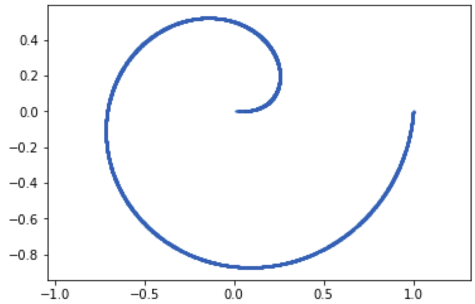
Hence, we can generate x, y by drawing from two independent random variable U, V where $U \sim Uniform(0,1)$ and $V \sim Uniform(0,1)$

$$x = \sqrt{U}\cos(2\pi V)$$
$$y = \sqrt{U}\sin(2\pi V)$$

And the corresponding plot is as such.



Additional. If we draw U and V from a single variable, we will end up with this



2. Let us model a cell being served by a base station in a wireless network as a circle with radius $R_{max} = 100$ m. Inside the cell a user, if he is at a distance r from the base station, can achieve a transmission rate $c(r)$ that decays due to path loss effects as a function of the distance according to

$$c(r) = \begin{cases} C_{max}, & r \leq R_{min}, \\ C_{max}(\frac{R_{min}}{r})^{\alpha}, & R_{min} < r < R_{max} \end{cases}$$

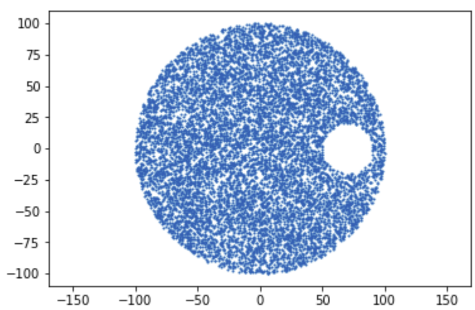
where $C_{max} = 100$ Mbit/s, $R_{min} = 10$ m and $\alpha = 3$. Consider that the base station is located at point (0, 0). At point (70 m, 0) there is another circular cell with radius $R_2 = 20$ m, which represents the service area of a low power base station inside the coverage of the high power base station. Assume that the location of a user is uniformly distributed inside the larger circle (high power base stations coverage area). Let X denote the random variable for the location of the user. What is the mean achievable transmission rate of the user $E[c(X)]$ inside the large cell given that the user is not in the area of the small cell area? **Ans** We want to find $E[c(X)]$ given that is not bounded by the circle $(x - 70)^2 + y^2 = 20^2$

This gives us a rejection range $(x - 70)^2 + y^2 \leq 20^2$

First, we generate uniformly distributed points (x_i, y_i) inside the large cell and not inside the small cell using the rejection method:

$$x = 100\sqrt{U}\cos(2\pi V), \quad y = 100\sqrt{U}\sin(2\pi V)$$

Draw $U \sim Uniform(0, 1)$ and $V \sim Uniform(0, 1)$ to generate points of x, y. We will get the plot below.



Then we apply the function c above to r, which is the distance from each generated points to the origin $(0, 0)$:

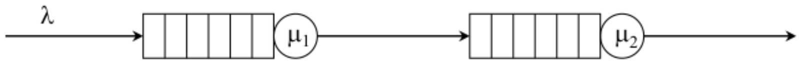
$$c(r) = c(\sqrt{x^2 + y^2})$$

To approximate the expected value, I will generated N points (where N is a large number). such that I have r_i for $i = 1, 2, \ldots, N$

$$E[c(r)] \approx \frac{1}{N} \sum_{i=1}^{N} c(r_i)$$

For calculation, we generate 10^7 points, and the mean transmission rate ≈ 2.91

3. Consider a simple tandem queuing network with two queues as shown in the figure below. Jobs arrive according to a Poisson process with rate λ. The service times of the jobs in the two queues are i.i.d. and obey the log-normal distribution with mean μ_1 and μ_2 , respectively. Hence, the system is a tandem network of M/G/1 queues. We also assume the system is stable, i.e., $\lambda < min(\mu_1, \mu_2)$.



(i) Identify the regeneration points of this system so that one could apply the regenerative method for estimating, e.g., the mean delay of the jobs.

Note Every now and then the system visits the regenerative state or regenerates itself. This starts a new life which does not depend on the past

We can reset state back to zero.

Ans We know that for a general queue G/G/1, the state where the syste is empty is a regenerative state.

(ii) How does the situation change if the service times are i.i.d. and exponentially distributed?

Ans If the service time is exponentially distributed, then the queue will follow a M/M/1 model (a Markovian System). As such, now every state is regenerative.

4. Devise an algorithm by using the composite method to generate rv:s obeying the density

$$f(x) = \frac{1}{2} + x^3 + 2x^7$$

Hint: compute the cumulative distribution.

Ans cdf

$$F(x) = \frac{1}{2}x + \frac{1}{4}x^4 + \frac{1}{4}x^8$$

We can express it as

$$F(x) = p_1 F_1(x) + p_2 F_2(x) + p_3 F_3(x)$$

We let

$$p_1 = \frac{1}{2},\ p_2 = \frac{1}{4},\ p_3 = \frac{1}{4}$$

such that $\sum_i p_i = 1$

Algorithm

- draw index I: $P\{I = 1\} = p_1, P\{I = 2\} = p_2, P\{I = 3\} = p_3$
- draw value from X from distribution $F_I(x)$ for I in [1,2,3]

$$F_1(x) = x,\ F_2(x) = x^4,\ F_3(x) = x^8$$

Let $U \sim U(0,1)$:
if $I = 1$, return U
if $I = 2$, return $U^{1/4}$
if $I = 3$, return $U^{1/8}$