ELE-E4760
Modeling and Simulation
Exercise 1

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1. Generate, by using the linear congruential method (LCG), a sequence of four pseudo random numbers. Use initial values $X_0 = 11$, $a = 5$, $c = 3$, and $m = 61$

   Using the following formula

   $$Z_{i+1} = (aZ_i + c) \mod m$$

   We can generate the following sequence: $\{58, 49, 4, 23\}$

2. Some further questions:

   a) What is the length of the random number sequence generated by the LCG algorithm with the above parameters?

      The first 60 numbers generated with the above parameters are as follows:

      11, 58, 49, 4, 23, 57, 44, 40, 20, 42, 30, 31, 36, 0, 3,
      18, 32, 41, 25, 6, 33, 46, 50, 9, 48, 60, 59, 54, 29, 26,
      11, 58, 49, 4, 23, 57, 44, 40, 20, 42, 30, 31, 36, 0, 3,
      18, 32, 41, 25, 6, 33, 46, 50, 9, 48, 60, 59, 54, 29, 26,
      11, 58, 49..

      Hence by observation, the length is 30.

   b) Does the sequence length depend on X_0 ?
      Yes, some numbers do not appear. Hence by choosing these numbers that do not appear as our seed value, by default we would generate a different sequence.

c) How would you change the value of m to obtain a full length period from the generator? (i.e., a sequence of length m different numbers)?
   We can obtain the full length if the following is fulfilled:
   m of the form 2^b, c to be odd, a of the form $4 * k + 1$ (k is any integer > 0)
   Thus, I will change m (61) to 64.

3. What is the sequence length of the multiplicative congruential generator (MCG) with parameters a = 7 and m = 61? (Again, use some software to simulate the MCG generator.)

   For MCG we use the following formula:

   $$Z_{i+1} = (aZ_i) \mod m$$

   If we choose Z_i to be prime (such as 7) and a is chosen appropriately, the sequence length will be $m - 1$. Let us examine the sequence generated:

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   7, 49, 38, 22, 32, 41, 43, 57, 33, 48, 31, 34, 55, 19,
   11, 16, 51, 52, 59, 47,24, 46, 17, 58, 40, 36, 8, 56,
   26, 60, 54, 12, 23, 39, 29, 20, 18, 4, 28, 13, 30, 27,
   6, 42, 50, 45, 10, 9, 2, 14, 37, 15, 44, 3, 21, 25, 53,
   5, 35, 1,
   7, 49, 38, 22, 32, 41, 43, 57, 33, 48, 31, 34, 55, 19,
   11, 16, 51, 52, 59,47, 24, 46, 17, 58, 40, 36, 8, 56,
   26, 60, 54, 12, 23, 39, 29, 20, 18, 4, 28, 13, 30 ..
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   Hence by observation, the length is 60. (which corresponds to m - 1, this case 60 - 1)

4. A random number generator of a computer draws samples from a $U(0, 1)$ distribution. Assume that the generator has generated a sample $u = 0.77306$. What is the corresponding value of a random variable X, when X is the number of trials before the first six appears when rolling a dice?

   We can describe the random variable X which is number of trials (k) before the first six (success) appears as a GEOMETRIC distribution which follows:

   $$\begin{aligned} P(X = k) &= (1 - p)^{k-1} p \\ &= (\frac{5}{6})^k \frac{1}{6} \text{ where } p = \frac{1}{6} \end{aligned} \tag{1}$$

Hence, by drawing U from a uniform distribution $U(0,1)$, we can generate X using the following:

$$X = \left\lfloor \frac{\log U}{\log(1-p)} \right\rfloor$$

As such, if we draw $u = 0.77306$, we would generate $X = int(1.41) = 1$

5. Apply the inverse transformation method to generate rv:s from the Weibull distribution with the cumulative distribution function

$$F(x) = 1 - e^{-(\lambda x)^\beta}$$

Also, give the algorithm to generate the samples.

First we find the inverse function F^{-1}:

$$\begin{aligned} y &= 1 - e^{-(\lambda x)^\beta} \\ \log(1-y) &= -(\lambda x)^\beta \\ \log(1-y)^{1/\beta} &= -\lambda x \\ F^{-1}(y) &= -\frac{1}{\lambda}\log(1-y)^{1/\beta} \end{aligned} \tag{2}$$

Algorithm:
If we draw y from a uniform distribution $U \sim U(0,1)$, $(1-y)$ will also follow a unifrom distribution. The inverse transformation

$$X = F^{-1}(1-U) = -(\frac{1}{\lambda})\log(U)^{\frac{1}{\beta}}$$