ELE-E4760
Modeling and Simulation
Exercise 3

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1. Apply the inverse transform method and give an algorithm to generate samples of X with the pdf

   Solution:

$$f(x) = \begin{cases} \frac{1}{4}, & 0 \leq x \leq 2 \\ 1 - \frac{x}{4}, & 2 < x \leq 4 \\ 0, & \text{otherwise} \end{cases} \qquad F(x) = \begin{cases} \frac{x}{4} + c_1, & 0 \leq x \leq 2 \\ x - \frac{x^2}{8} + c_2, & 2 < x \leq 4 \\ 0, & \text{otherwise} \end{cases}$$

   We have to choose c_1, c_2 such that $F(x)$ is continuous and monotonically increasing from $[0, 1]$, via the $F(4) = 1$ and

   When $x = 2$:

$$\frac{x}{4} + c_1 = x - \frac{x^2}{8} + c_2$$

   Thus, $c_1 = 0, c_2 = -1$

   Inverse transform method: We can generate samples of X. We let $y = F(x)$

   For $0 \leq x \leq 2$,

$$y = \frac{x}{4}$$

$$F^{-1}(x) = 4x$$

   For $2 < x \leq 4$,

$$y = x - \frac{x^2}{8} - 1$$
$$8y = 8x - x^2 - 1$$

$$F^{-1}(x) = 4 - 2\sqrt{2 - 2x}$$

$$X = \begin{cases} 4U, & 0 \leq U \leq \frac{1}{4} \\ 4 - 2\sqrt{2 - 2U}, & \frac{1}{4} < U \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

where $U \sim U(0, 1)$. We can generate X from U

2. Consider a queueing system during a time period (0, 10). Initially the system is empty, $L_0 = 0$. New customers enter the queue at times a_i and depart at times d_i.

a_i	1.5	2.0	3.5	4.3	4.7	6.5	8.5
d_i	4.0	5.0	5.5	7.0	7.5	-	-

Compute the average number of customers in the system during the time period (0, 10). Apply, the ideas on slide 13 in Statistical analysis, part 1.

Solution:

If we are interested in the average queue length (in this case the average number of customers) , over the time interval (0, 10), then $t = 10$ and we can use:

$$\begin{aligned} X &= \frac{1}{10} \int_0^{10} L(t)dt \\ &= \frac{1}{10} \sum_i (d_i - a_i) \\ &= \frac{18}{10} = \mathbf{1.8} \end{aligned}$$

3. In seven independent simulation replications of a system it has been observed that the average waiting time of the customers arriving during a 2 hour interval has been (3.62, 4.55, 3.95, 3.71, 4.12, 4.61, 3.24) min. The expectation of this average waiting time has to be estimated such that the error is 0.4 min at 95 % confidence level. Assume that all the seven samples obey the Normal distribution. Estimate how many independent replications are needed to achieve the desired accuracy? Hint: use the Student-t distribution (see, e.g., Wikipedia for fractile values).

   2 hour interval:

Normally distributed

Waiting time X: 3.62, 4.55, 3.95, 3.71, 4.12, 4.61, 3.24

Sample mean $\bar{x} = 3.971$

Sample Variance $s^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x}) = 0.249$

Sample standard deviation $s = 0.499$

Degrees of freedom = N - 1 = 6

$p = \frac{1-0.95}{2} = 0.025$,

$$t_{6,0.025} = error \times \sqrt{\frac{n}{s^2}},$$
$$2.446 = 0.4 \times \sqrt{\frac{n}{0.249}},$$
$$n = 9.298 \approx 10$$

4. Write a simulation script with Mathematica to generate samples of $X \in (0, 1)$ obeying the beta distribution $\beta(2, 4)$ with the pdf

$$f(x) = 20x(1-x)^3, \quad 0 \leq x \leq 1$$

To estimate the mean value of X, E[X], generate 10000 samples and give the 95% confidence interval. Hint: for sample generation see slide 4 in Generation of random numbers, part 2. Also, remember that all samples of X are independent.

```
Needs["HypothesisTesting`"];

pdf = PDF[BetaDistribution[2, 4], x];
dist = ProbabilityDistribution[pdf, {x, 0, 1}];

values = RandomVariate[dist, 100000];

mean = Mean[values]
interval = MeanCI[values]
```

Estimated Mean: 0.33329395% Confidence Interval: [0.33219, 0.334398]

5. We can estimate the value of the constant e by the following algorithm. Generate samples of N according to

$$N = \min\{n : \sum_{i=1}^{n} U > 1\}$$

where U_i are independent and $U_i \sim U(0,1)$, i.e., one generates random numbers until their sum exceed 1 and one records the number of trials required for that. Then it can be shown that $E[N] = e$. Implement the above algorithm in Mathematica. Generate 10000 samples to estimate e and give its 95% confidence interval. Hint: remember that all samples of N generated with above algorithm are independent.