Statistical and Machine Learning (01.113)
Homework 4

Adam Ilyas 1002010

April 16, 2019

Problem 1

Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be distributions over the set $\{1, 2\}$ and assume that p_1, p_2, q_1 and q_2 are all not zero. Find values for p and q such that $D_{KL}(p|q) \neq D_{KL}(q|p)$. Use base 2 for the logarithm and show all working.

Ans. To find values of p and q such that $D_{KL}(p|q) \neq D_{KL}(q|p)$, we can find values for $D_{KL}(p|q) - D_{KL}(q|p) \neq 0$

$$\begin{aligned}
D_{KL}(p|q) - D_{KL}(q|p) &= \sum_{i=1}^{2} p_i \log \frac{p_i}{q_i} - \sum_{i=1}^{2} q_i \log \frac{q_i}{p_i} \\
&= \sum_{i=1}^{2} \left(p_i \log \frac{p_i}{q_i} - q_i \log \frac{q_i}{p_i} \right) \\
&= \sum_{i=1}^{2} \left(p_i \log \frac{p_i}{q_i} + q_i \log \frac{p_i}{q_i} \right) \\
&= \sum_{i=1}^{2} (p_i + q_i) \log \frac{p_i}{q_i} \\
&= (p_1 + q_1) \log \frac{p_1}{q_1} + (p_2 + q_2) \log \frac{p_2}{q_2}
\end{aligned}$$

We observe that when $p_1 = q_1$, $D_{KL}(p|q) - D_{KL}(q|p) = 0$ because both $D_{KL}(p|q)$ and $D_{KL}(q|p)$ becomes 0.

We continue our approach. Since

$$p_1 + p_2 = 1, \text{ and } q_1 + q_2 = 1$$

$$
\begin{aligned}
D_{KL}(p|q) - D_{KL}(q|p) &= (p_1+q_1)\log\frac{p_1}{q_1} + (p_2+q_2)\log\frac{p_2}{q_2} \\
&= (p_1+q_1)\log\frac{p_1}{q_1} + (1-p_1+1-q_1)\log\frac{1-p_1}{1-q_1} \\
&= (p_1+q_1)\log\frac{p_1}{q_1} + 2\log\frac{1-p_1}{1-q_1} - (p_1+q_1)\log\frac{1-p_1}{1-q_1} \\
&= -(p_1+q_1)\log\frac{q_1}{p_1} + 2\log\frac{1-p_1}{1-q_1} - (p_1+q_1)\log\frac{1-p_1}{1-q_1}
\end{aligned}
$$

Here, we observe that when $q_1 + p_1 = 1$ (correspondingly $q_2 + p_2 = 1$), then $D_{KL}(p|q) - D_{KL}(q|p) = 0$ (symmetric). Thus, $D_{KL}(p|q) \neq D_{KL}(q|p)$ for 2 cases

1. $p_i \neq q_i \ \forall\ i$
2. $p_i + q_i \neq 1 \ \forall\ i$

For example: $p = (0.1, 0.9)$ and $q = (0.2, 0.8)$, then $D_{KL}(p|q) = 0.522$ but $D_{KL}(q|p) = 0.786$

Problem 2

Let $p(x, z)$ be a joint distribution on $\mathbb{R}^2$, with $p(x)$ and $p(z|x)$ denoting the corresponding marginal and conditional distributions respectively. Let $q(z)$ be any distribution on $\mathbb{R}$, and prove that

$$\log p(x) = \int_{\mathbb{R}} q(z) \log \frac{p(x,z)}{q(z)} \mathrm{d}z + D_{KL}[q(z)|p(z|x)]$$

We prove through multiple steps, first we introduce $q(z)$

$$\log p(x) = \log(\frac{p(x,z)}{p(z|x)}) = \log(\frac{p(x,z)}{q(z)}\frac{q(z)}{p(z|x)}) \tag{1}$$

Multiply both sides by $q(z)$

$$q(z)\log p(x) = q(z)\log(\frac{p(x,z)}{q(z)}) + q(z)\log(\frac{q(z)}{p(z|x)}) \tag{2}$$

Integrate both sides with respect to z for $z \in \mathbb{R}$

$$
\begin{aligned}
\int_{z\in\mathbb{R}} q(z)\log p(x) &= \int_{z\in\mathbb{R}} q(z)\log(\frac{p(x,z)}{q(z)})dz + \int_{z\in\mathbb{R}} q(z)\log(\frac{q(z)}{p(z|x)})dz \\
\log p(x) &= \int_{z\in\mathbb{R}} q(z)\log(\frac{p(x,z)}{q(z)})dz + D_{KL}[q(z)|p(z|x)] \\
RHS &= LHS
\end{aligned} \tag{3}
$$