CS-E5740 Complex Networks, Answers to exercise set 2

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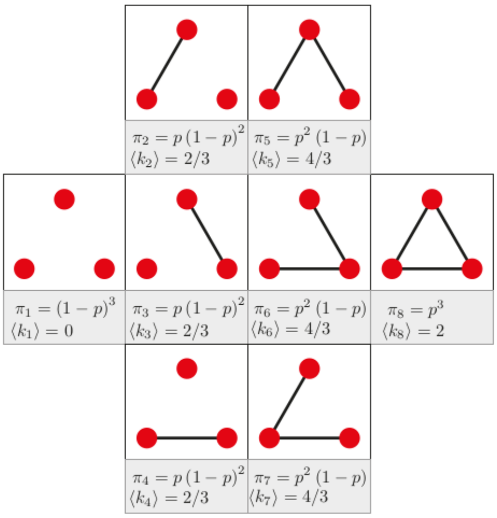
1 Ensemble averages by enumeration

Graph ensembles are distributions of graphs, where each graph G_i has a certain probability π_i. The ensemble average of a quantity X is defined as $\langle X \rangle = \sum_i \pi_i X(G_i)$. Note that this is the expected value of X across the ensemble. Let us define following quantities: $k(G)$ is the average degree of the graph G, $c(G)$ is the average clustering coefficient for graph G (assuming that the clustering coefficient gets value 0 for nodes of degree 0 and 1), and $d^*(G)$ is the diameter of the connected component of the graph G that has the largest diameter.



Ensemble i	Probability	Average Degree	Cluster Coefficient	Diameter
1	$(1-p)^3$	0	0	0
2	$p(1-p)^2$	2/3	0	1
3	$p(1-p)^2$	2/3	0	1
4	$p(1-p)^2$	2/3	0	1
5	$p^2(1-p)$	4/3	0	2
6	$p^2(1-p)$	4/3	0	2
7	$p^2(1-p)$	4/3	0	2
8	p^3	2	1	1

a) Calculate, using pen and paper, $\langle k \rangle$, $\langle c \rangle$, and $\langle d^* \rangle$ for $G(N = 3, p = 1/3)$ in which N is the number of nodes and p is the edge density

1. Average Degree $\langle k \rangle$

$$\begin{aligned}\langle k \rangle &= \sum_i^N \pi_i \langle k_i \rangle \\ &= (1-p)^3 \times 0 + 3p(1-p)^2 \times 2/3 + 3p^2(1-p) \times 4/3 + p^3 \times 2 \\ &= 8/27 + 8/27 + 2/27 = 18/27 \\ &= 2/3\end{aligned} \tag{1}$$

2. Average Coefficient $\langle c \rangle : 1/p^3 = 1/27$
3. $d * (G)$ Diameter (Furthest path):
$1 \times 4/27 \times 3 + 2 \times 2/27 \times 3 + 1 \times 1/27 = 25/27$

b) Calculate, using pen and paper, the formulas for $\langle k \rangle$, $\langle c \rangle$, and $\langle d^* \rangle$ for $G(N = 3, p)$ (that is, for N = 3 and all possible values of p). Remember to simplify the formulas you get as results.

1. Average number of edges $\langle m \rangle = \binom{N}{2} p$
Average degree

$$\begin{aligned}\langle k \rangle &= 2\langle m \rangle / N \\ &= p \times N(N-1)/N \\ &= p \times (N-1) \\ &= p \times 2\end{aligned} \tag{2}$$

2. Average clustering coefficient

$$\begin{aligned} \langle c \rangle &= \sum_{i=1}^{8} \pi_i \times \langle c_i \rangle \\ &= p^3 \end{aligned} \tag{3}$$

3. Average Diameter

$$\begin{aligned} \langle d* \rangle &= \sum_{i=1}^{8} \pi_i \times \langle d_i \rangle \\ &= 1 \times (1-p)^2 p \times 3 + 2 \times (1-p)p^2 \times 3 + 1 \times p^3 \\ &= 3p - 2p^3 \end{aligned} \tag{4}$$

2 Properties of Erds-Rnyi (ER) networks

Erds-Rnyi networks are random networks where N nodes are randomly connected such that the probability that a pair of nodes are linked is p. In network science, the ER random graphs are important because they provide the simplest reference to which one can compare real-world networks. In this exercise, we will analyze some of the properties of ER graphs.

a) The degree distribution $P(k)$ of ER networks is binomial:

$$P(k) = \binom{(N-1)}{k} p^k (1-p)^{(N-1)-k}$$

where N equals the number of nodes in the network. Explain in detail the origin of each of the three factors in the above formula. You dont need to derive the distribution, but it should be visible from your answer that you understand the meaning of each of the terms.

1. Each nodes number of links comes from N-1 independant trials where each trial has the probability p of success)
2. If there are k successes (k neighbours), there are N-1-k 'failures' (failed links), where each failure has the probability 1-p.
3. Since for N-1 trials, there can be $\binom{N-1}{k}$ combination of k success and N-1-k failures

b) In ER networks, the expected value of the average clustering coefficient $\langle c \rangle$ equals p. Explain, why this is the case. Hints:

- What is the expected value of the clustering coefficient for one node?

Clustering Coefficient for a node can be represented as

$$\frac{\#\text{ edges between its neighbours}}{\#\text{ possible edges between its neighbours}}$$

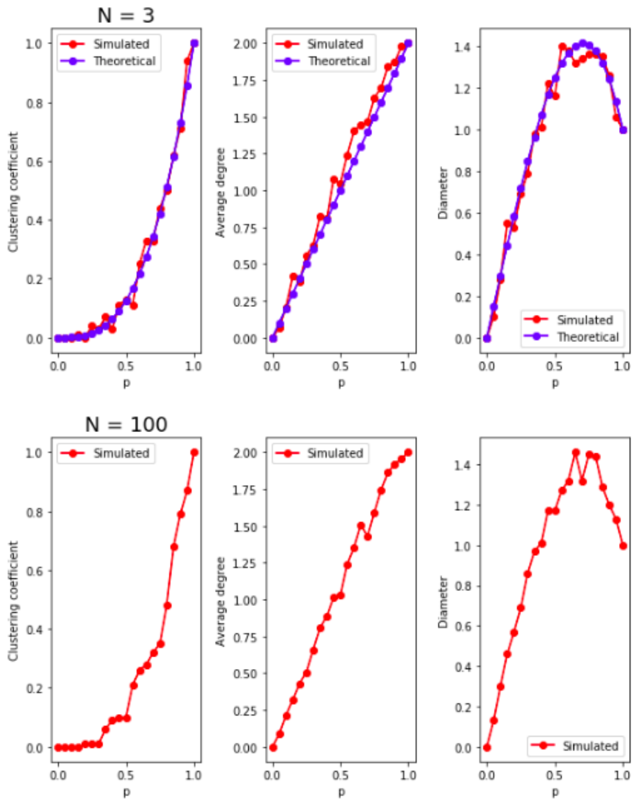
Which can also represent the probability of an edge between a node's neighbours. Hence, the average clustering coefficient, can be exactly this probability, since this is true for all nodes.

c) Explain, what happens to $\langle c \rangle$, if $N \to \infty$ with $\langle k \rangle$ bounded.

- Mathematically, if x is bounded then x is always smaller than some possibly big, but finite, number M .

Since clustering coefficient is independant of the total number of Nodes, then the average clustering coefficient $\langle k \rangle = p$

d) Use NetworkX to calculate estimates for the ensemble averages $\langle k \rangle$, $\langle c \rangle$, and $\langle d* \rangle$ defined in the Exercise 1. Do this by generating 100 ER networks for each value of p in range p = [0.00, 0.05, ..., 0.95, 1] and N = 3. An estimate for the ensemble average hXi can be calculated for each value of p by calculating the average value of X over the 100 realisations. Plot the quantities, such that p values are on the x-axis and $\langle X \rangle$ values are on the y-axis. If you solved the part b of Exercise 1, you can check the correctness of your results by including your analytical solution to these plots. Repeat the same exercise with N = 100.



3 Implementing the Watts-Strogatz small-world model

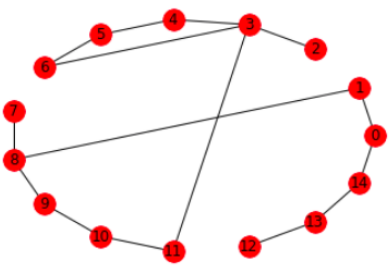
In this exercise, you will implement the Watts-Strogatz small-world model [1], which is a very simple network model that yields small diameter as well as high level of clustering. In practice, the Watts-Strogatz model is a ring lattice where some of the links have been randomly rewired. The model has three parameters: network size N , m (each node on the ring connects to m nearest neighbors both to the left and to the right), and p, the probability of rewiring one end of each link to a random endpoint node.

a) Implement the Watts-Strogatz small world model and visualize the network using N = 15, m = 2 p = 0.1, and N = 100, m = 2 p = 0.05 using a circular layout algorithm nx.draw_circular , and check that the networks look right. For each network Re- port the total number of links and also the number of rewired links.

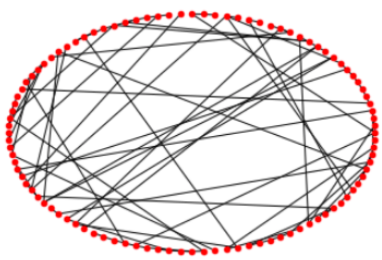
```
def ws(n, m, p):
```

```
network = ring(n, m)
    all_edges = list(network.edges()) # same as copy

    rewired_num = 0 # tracks the number of rewired links
    total_num = len(all_edges)

    for edges in all_edges:
        if np.random.rand() < p:
            rewired_num += 1
            # rewire
            u, v = edges
            network.remove_edge(u, v)
            other_node = random.choice(
                list(nx.non_neighbors(network, u))
            )
            network.add_edge(u, other_node)

    print("total number of links:")
    print(total_num)
    print("number of rewired links:")
    print(rewired_num)
    return network
```

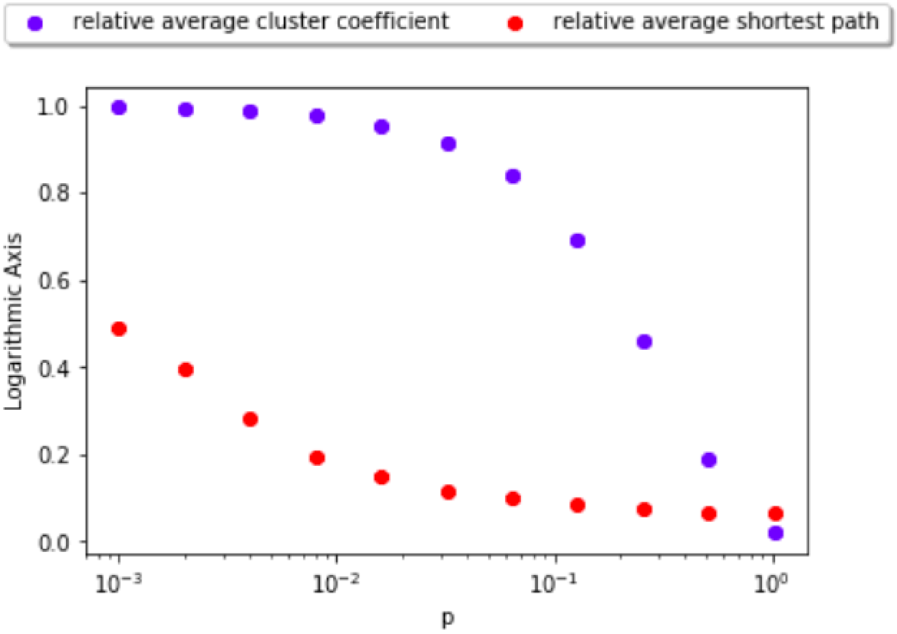
```
total number of links:
15
number of rewired links:
3
```




```
total number of links:
100
number of rewired links:
51
```



b) Plot the relative average clustering coefficient c(p)/c(p = 0) and average shortest path length l(p)/l(p = 0) vs. p in your network, for p = 0.001, . . . , 1.0 (see template for the log-spaced values). Here, relative=average value for given p divided by the same value for p = 0. Use N = 500 and m = 2. Use a logarithmic x-axis in your plot (ax.semilogx). Check that your results are in line with the plots in the lecture slides.


relative average cluster coefficient
relative average shortest path
Logarithmic Axis
1.0
0.8
0.6
0.4
0.2
0.0
10^{-3}
10^{-2}
10^{-1}
10^{0}
p