CS-E4850 Computer Vision,
Answers to Exercise Round 2

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1 Pinhole Camera

The perspective projection equations for a pinhole camera are

$$x_p = f\frac{x_c}{z_c}, \quad y_p = f\frac{y_c}{z_c}, \tag{1}$$

where $\mathbf{x}_p = [x_p, y_p]^\intercal$ are the projected coordinates on the image plane, $\mathbf{x}_c = [x_c, y_c, z_c]^\intercal$ is the imaged point in the camera coordinate frame and f is the focal length. Give a geometric justification for the perspective projection equations.

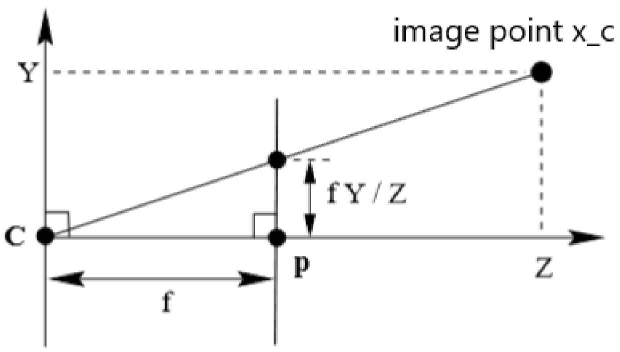


Figure 1: Similar triangles

Since $\triangle CZ\mathbf{x}_c$ and $\triangle Cp\mathbf{x}_p$ are similar triangles:

$$\frac{y_c}{z_c} = \frac{y_p}{f}, \quad y_p = f\frac{y_c}{z_c}$$

Since the image plane is parallel to the xy-plane (principal plane which is a plane that goes through the camera centre), similar triangles can be applied to the x-coordinates

$$\frac{x_c}{z_c} = \frac{x_p}{f}, \quad x_p = f\frac{x_c}{z_c}$$

2 Pixel Coordinate Frame

The image coordinates x_p and y_p given by the perspective projection equations (1) above are not in pixel units. The x_p and y_p coordinates have the same unit as distance f (typically millimetres) and the origin of the coordinate frame is the principal point (the point where the optical axis pierces the image plane). Now, give a formula which transforms the point x_p to its pixel coordinates $\mathbf{p} = [u, v]^\intercal$ when the number of pixels per unit distance in u and v directions are m_u and m_v, respectively, the pixel coordinates of the principal point are (u_0, v_0) and
a) u and v axis are parallel to x and y axis respectively.
b) u axis is parallel to x axis and the angle between u and v axis is θ.

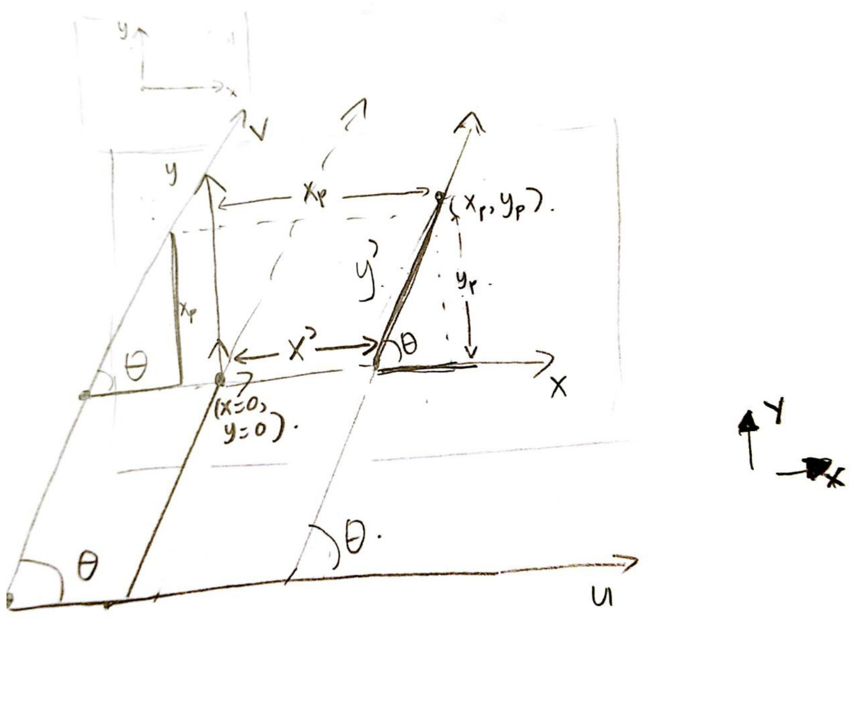
Solution
a) if u and v axis are parallel to x and y axis
We want to transform image coordinate to $(x_p, y_p)^\intercal \mapsto (u, v)^\intercal$ pixel coordinate by doing so

$$\begin{pmatrix} m_x & 0 & 0 \\ 0 & m_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f\frac{x_c}{z_c} + p_x \\ f\frac{y_c}{z_c} + p_y \\ 1 \end{pmatrix} = \begin{pmatrix} m_x f\frac{x_c}{z_c} + m_x p_x \\ m_y f\frac{y_c}{z_c} + m_y p_y \\ 1 \end{pmatrix} = \begin{pmatrix} m_x x_p + u_0 \\ m_y y_p + v_0 \\ 1 \end{pmatrix}$$

We represent the image coordinates of the principal point be $[p_x, p_y]^\intercal$. The formula to transform the point x_p to pixel coordinates is:

$$u = m_u x_p + u_0, \quad v = m_v y_p + v_0$$

b) u axis is parallel to x axis and the angle between u and v axis is θ



From the diagram above, in particular, we are interested in x' and y':

$$\frac{y_p}{y'} = sin(\theta), \quad y' = \frac{1}{sin(\theta)} y_p$$

$$\frac{y_p}{x_p - x'} = \tan\theta, \quad x' = x_p - \frac{1}{tan(\theta)} y_p$$

Once we considered the rotation of the new coordinates, we can modify the scale and shift of the image coordinate to the new pixel coordinates

$$\begin{aligned} u &= m_u x' + u_0, & v &= m_v y' + v_0 \\ u &= m_u x_p - \frac{m_u}{tan(\theta)} y_p + u_0, & v &= m_v \frac{1}{sin(\theta)} y_p + v_0 \end{aligned} \tag{2}$$

3 Intrinsic camera calibration matrix

Use homogeneous coordinates to represent case (2.b) above with a matrix $\mathbf{K}_{3\times3}$, also known as the camera's intrinsic calibration matrix, so that $\tilde{\mathbf{p}} = \mathbf{K}\mathbf{x_c}$. Where $\tilde{\mathbf{p}}$ is $\mathbf{p}$ in homogeneous coordinates.

Solution
We want to find $\mathbf{K}_{3\times3}$ such that:

$$\mathbf{K}_{3\times3} \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

From section 2.B:

$$\begin{aligned} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} &= \begin{pmatrix} m_u & -\frac{m_u}{\tan\theta} & u_0 \\ 0 & m_v(1/\sin\theta) & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_p \\ y_p \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} m_u & -\frac{m_u}{\tan\theta} & u_0 \\ 0 & m_v(1/\sin\theta) & v_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} fx_c \\ fy_c \\ fz_c \end{pmatrix} \end{aligned} \tag{3}$$

4 Camera projection matrix

Imaged points are often expressed in an arbitrary frame of reference called the world coordinate frame. The mapping from the world frame to the camera coordinate frame is a rigid transformation consisting of a 3D rotation R and translation t:

$$x_c = \mathbf{R} x_w + \mathbf{t}$$

Use homogeneous coordinates and the result of the exercise 3 above, to write down the 3×4 camera projection matrix P that projects a point form world coordinates x_w to pixel coordinates. That is, represent P as a function of the internal camera parameters K and the external camera parameters R,t.

The world $\mapsto$ pixel camera matrix :

$$\begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = \mathbf{K}\tilde{x}_c = \mathbf{K}(\mathbf{R}\tilde{x}_w + t) = \mathbf{K}[\mathbf{R} \quad t] \begin{pmatrix} \tilde{x}_w \\ 1 \end{pmatrix} = \mathbf{P} \begin{pmatrix} \tilde{x}_w \\ 1 \end{pmatrix}$$