

Computer Vision

CS-E4850, 5 study credits

Lecturer: Juho Kannala

Lecture 9: Two-view geometry & stereo vision

- **Two-view geometry** (a.k.a. epipolar geometry) describes the geometric constraints between two views
- **Stereo vision** is the principle of using two views to measure depths of scene points

Acknowledgement: many slides from Svetlana Lazebnik, Steve Seitz, Yuri Boykov, Noah Snavely, and others (detailed credits on individual slides)

Reading

- Szeliski's book, Section 7.2 and Chapter 11

and/or

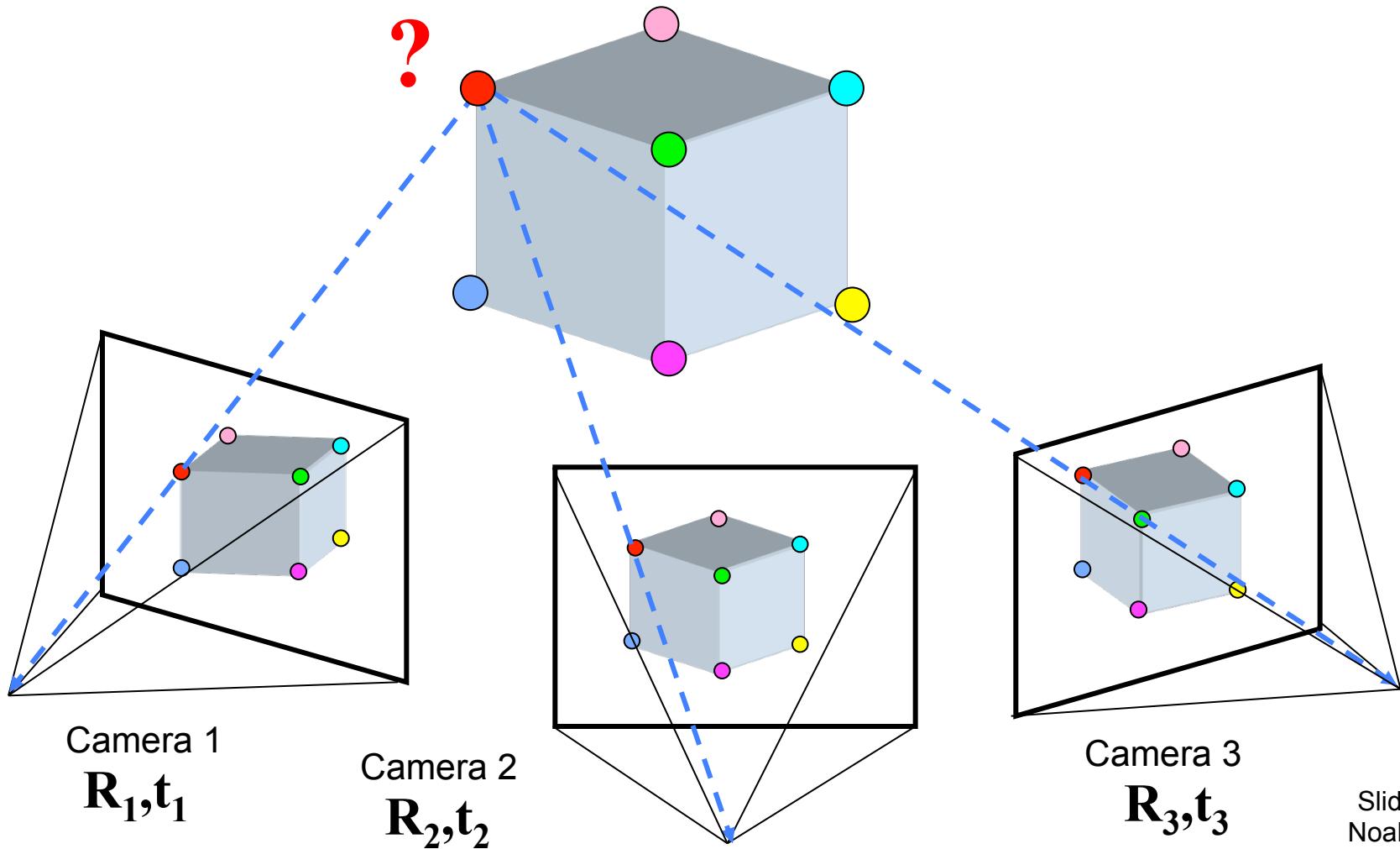
- Hartley & Zisserman book, Chapters 9-12

Multi-view geometry



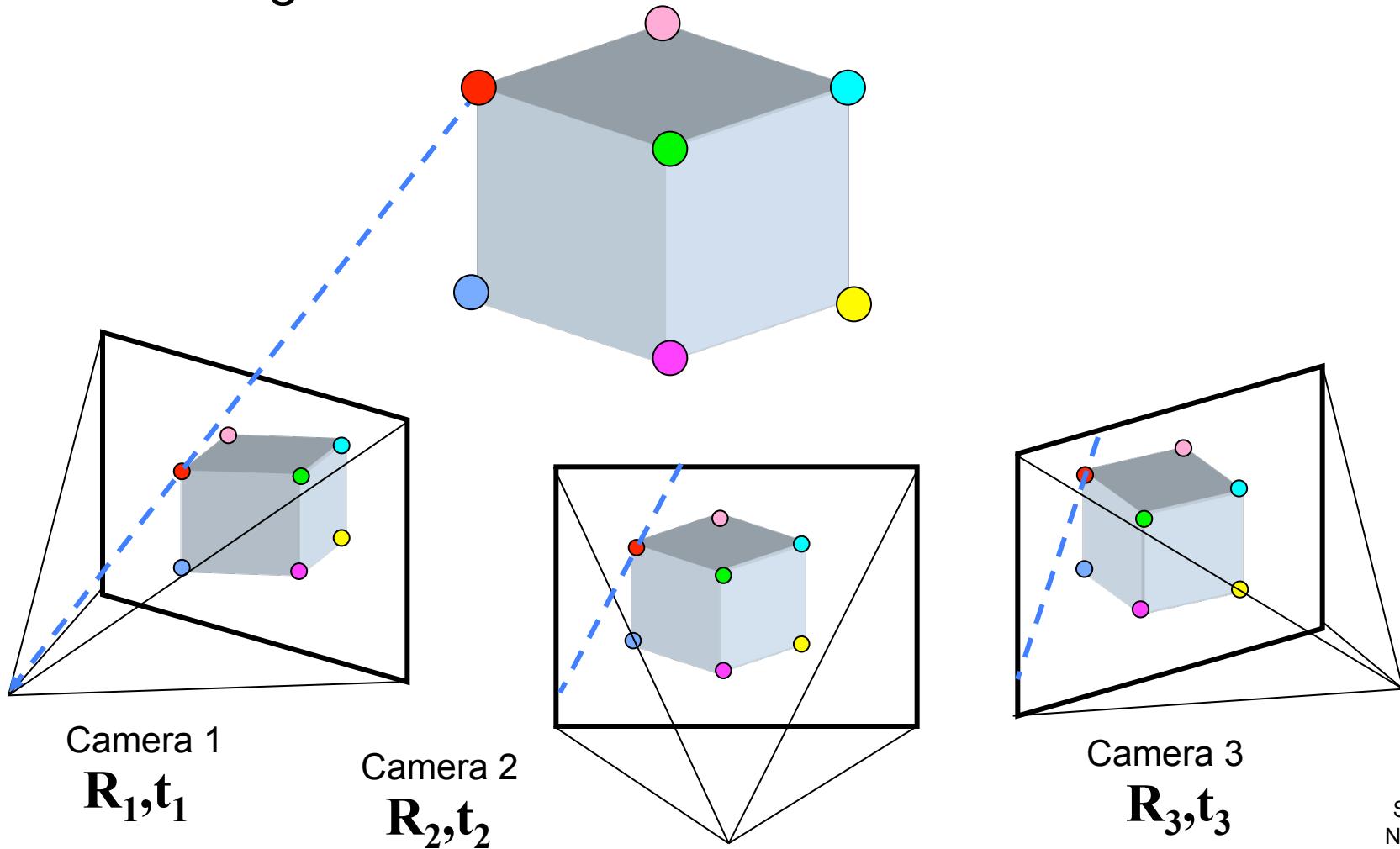
Multi-view geometry problems

- **Structure:** Given projections of the same 3D point in two or more images, compute the 3D coordinates of that point



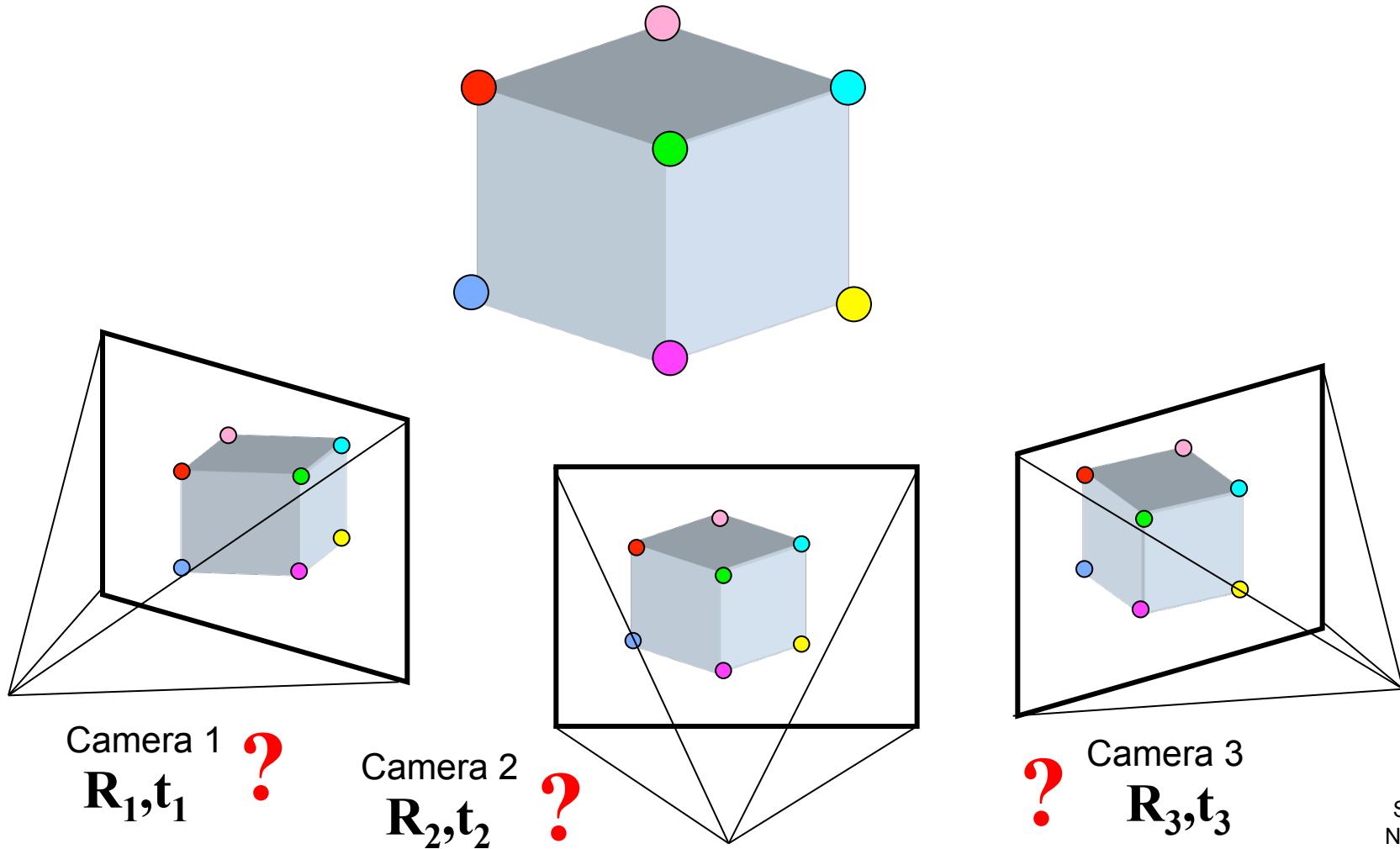
Multi-view geometry problems

- **Stereo correspondence:** Given a point in one of the images, where could its corresponding points be in the other images?

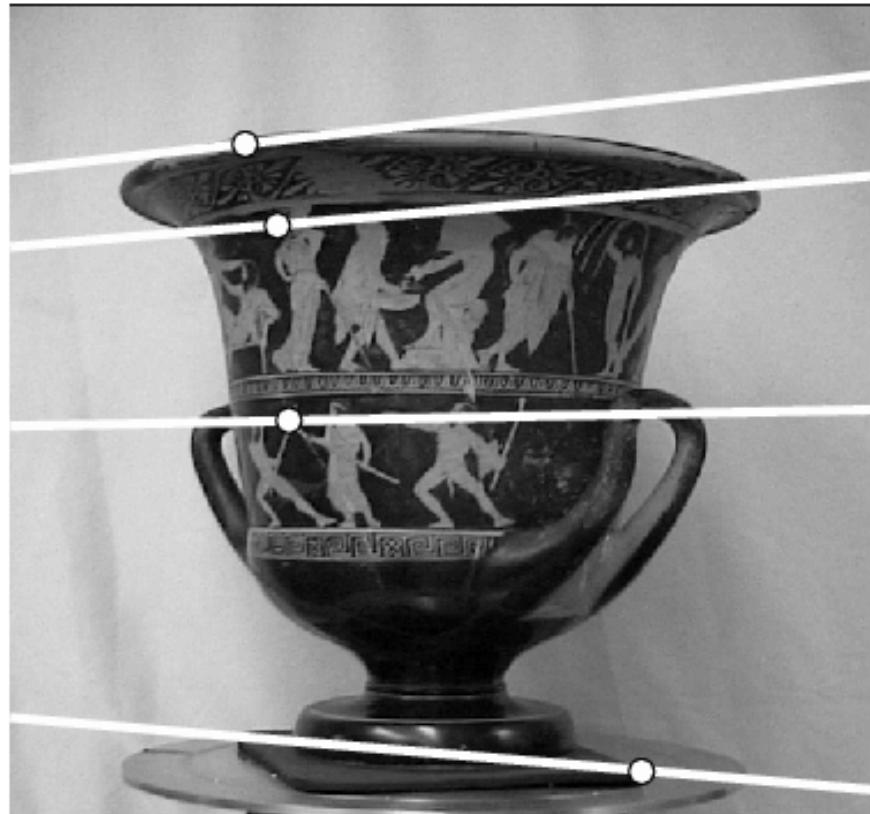


Multi-view geometry problems

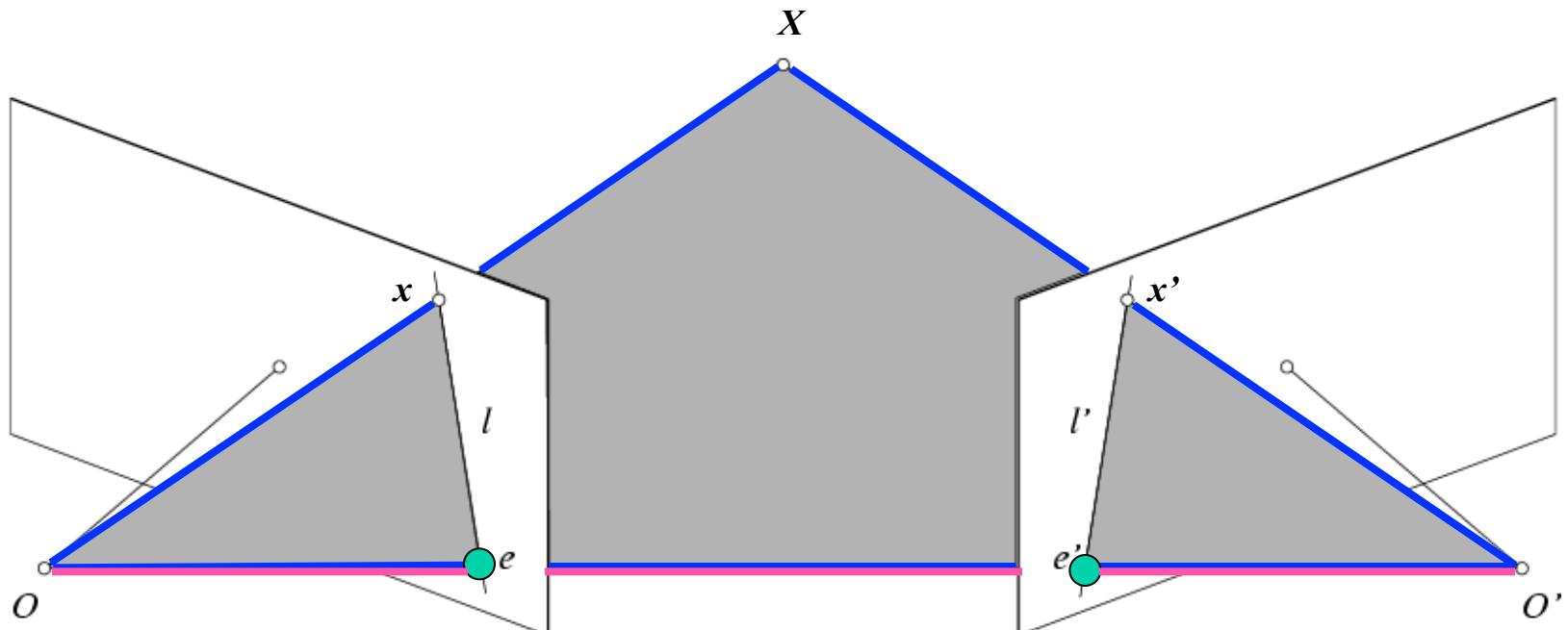
- **Motion:** Given a set of corresponding points in two or more images, compute the camera parameters



Two-view geometry

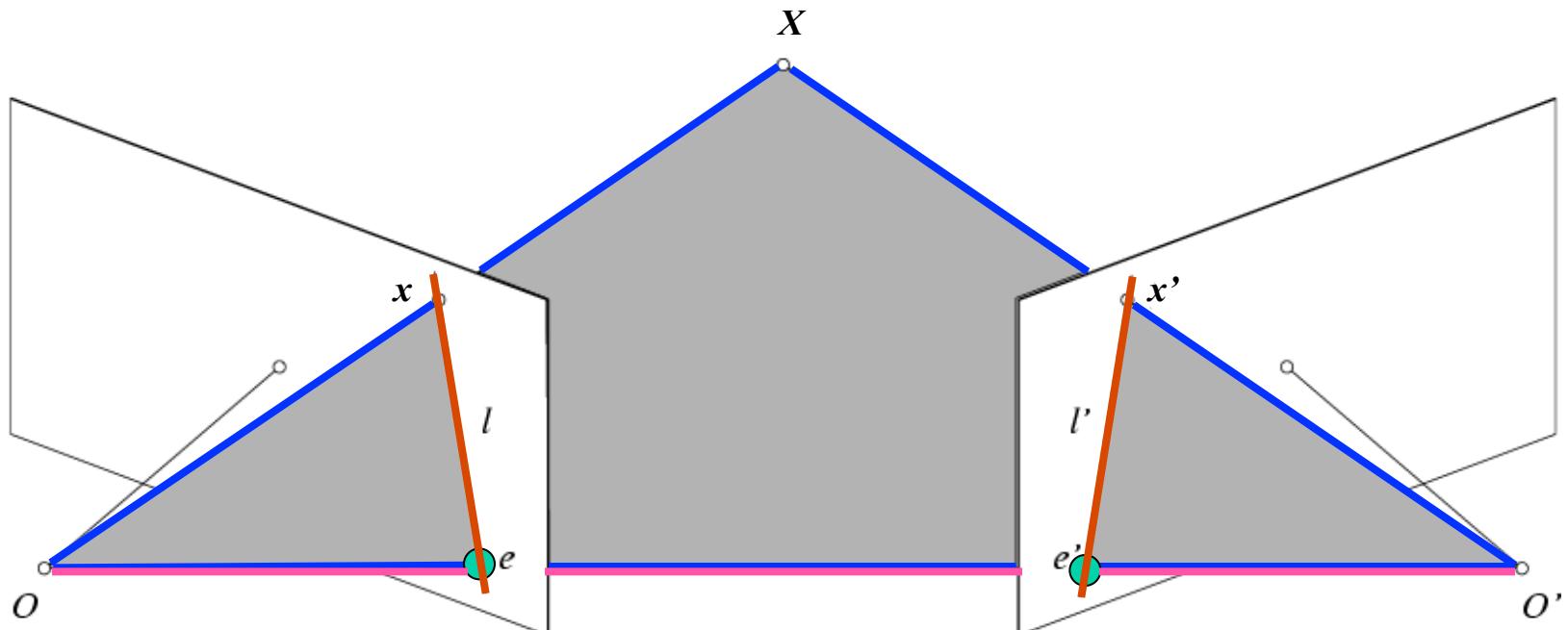


Epipolar geometry



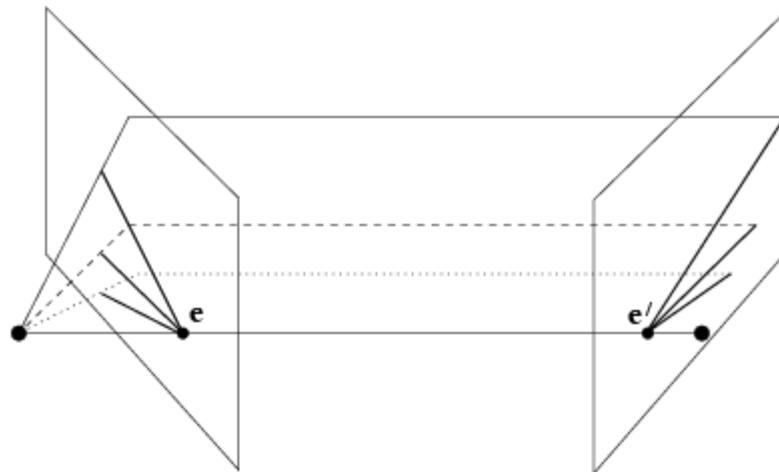
- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**
 - = intersections of baseline with image planes
 - = projections of the other camera center
 - = vanishing points of the motion direction

Epipolar geometry

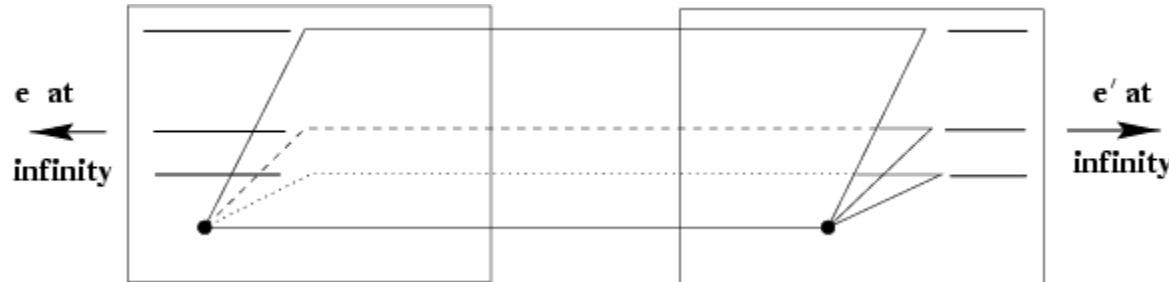


- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**
 - = intersections of baseline with image planes
 - = projections of the other camera center
 - = vanishing points of the motion direction
- **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)

Example: Converging cameras



Example: Motion parallel to image plane



Example: Motion perpendicular to image plane



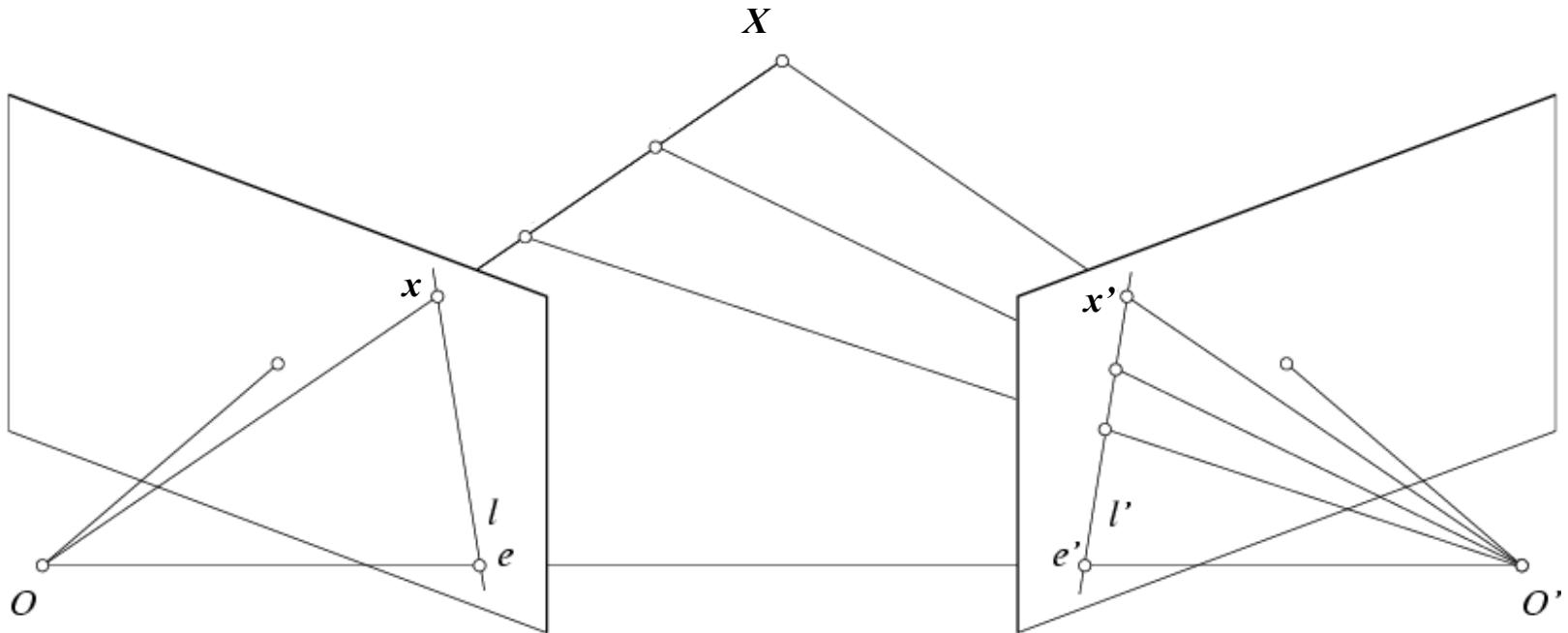
Source: S. Lazebnik

Example: Motion perpendicular to image plane



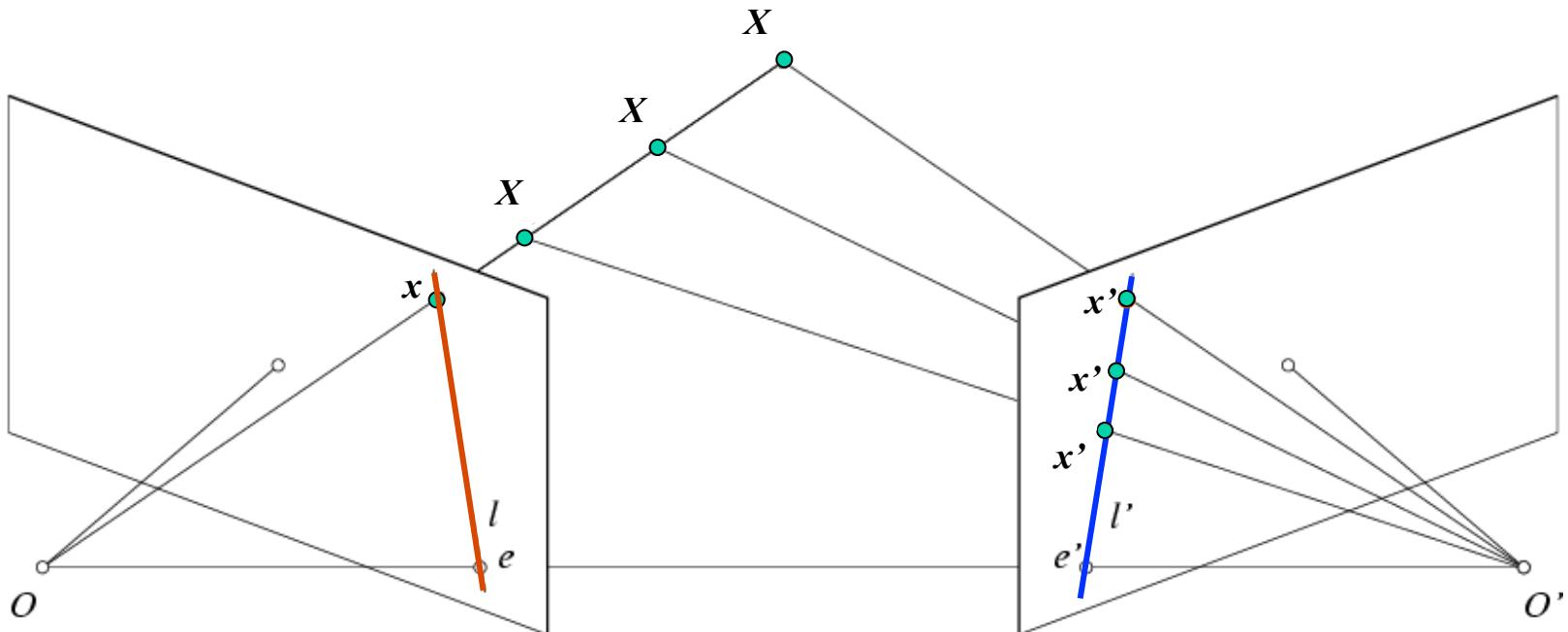
- Points move along lines radiating from the epipole: “focus of expansion”
- Epipole is the principal point

Epipolar constraint



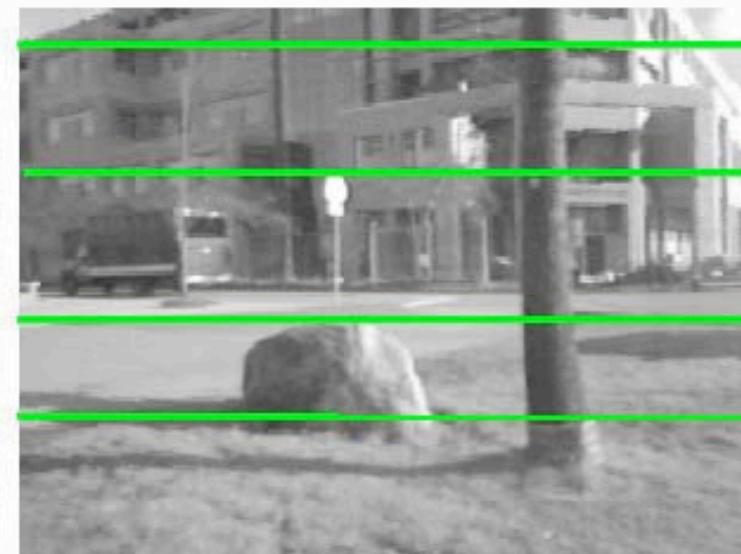
- If we observe a point x in one image, where can the corresponding point x' be in the other image?

Epipolar constraint

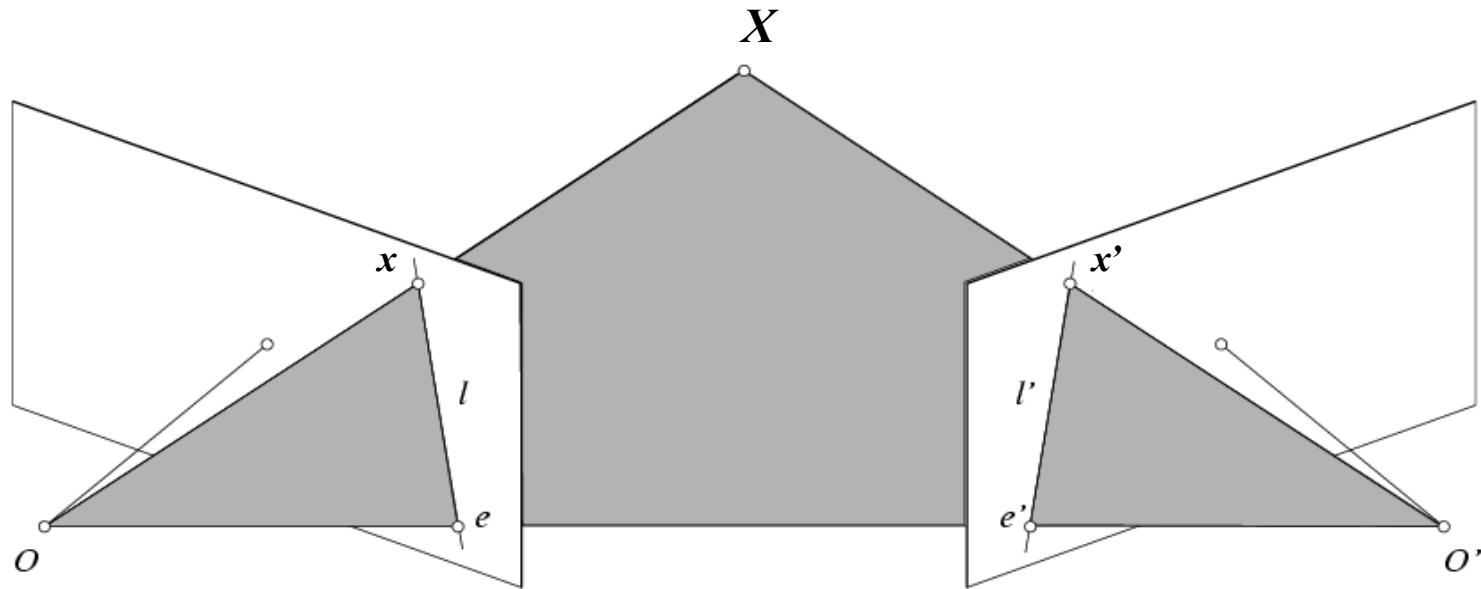


- Potential matches for x have to lie on the corresponding epipolar line l' .
- Potential matches for x' have to lie on the corresponding epipolar line l .

Epipolar constraint example



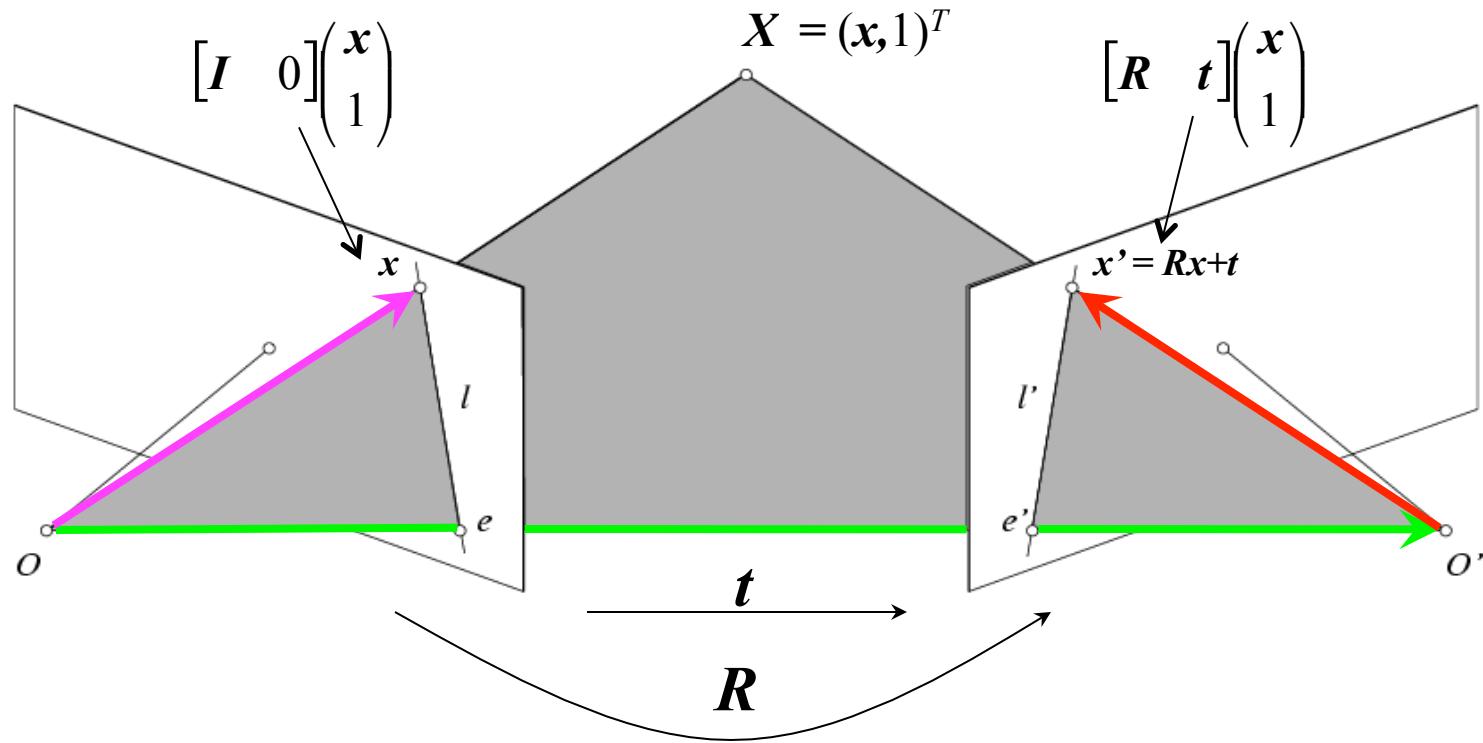
Epipolar constraint: Calibrated case



- Intrinsic and extrinsic parameters of the cameras are known, world coordinate system is set to that of the first camera
- Then the projection matrices are given by $K[I \mid \mathbf{0}]$ and $K'[R \mid t]$
- We can multiply the projection matrices (and the image points) by the inverse of the calibration matrices to get *normalized* image coordinates:

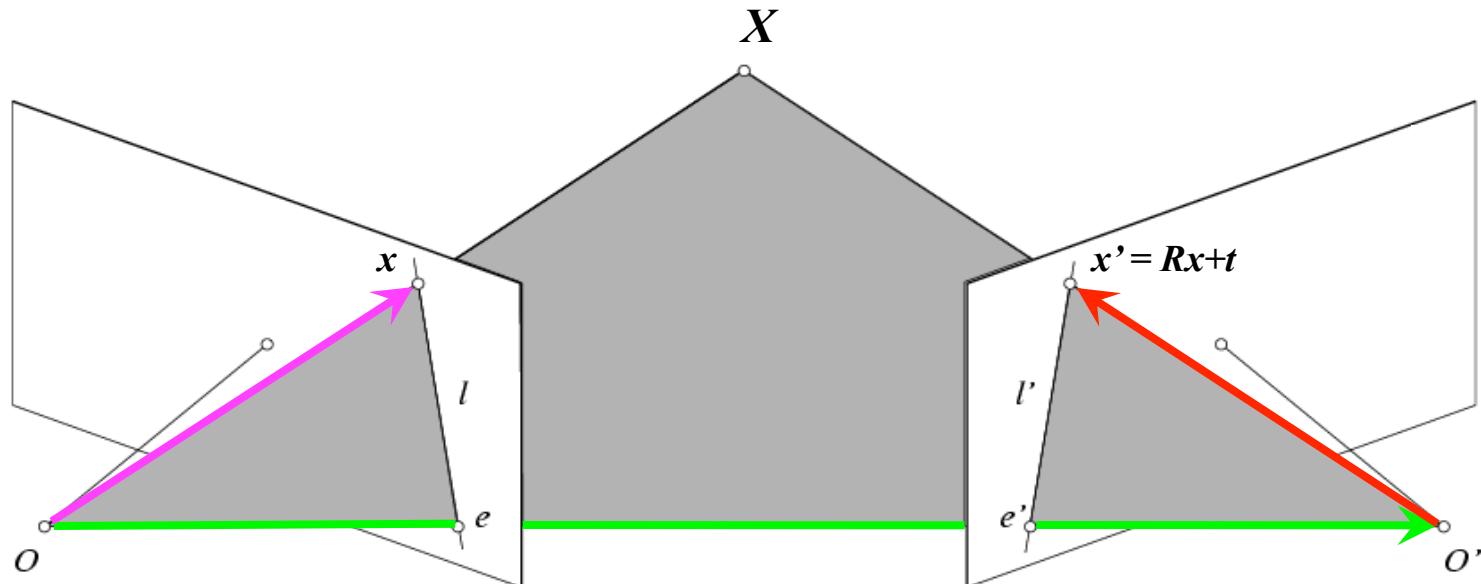
$$\mathbf{x}_{\text{norm}} = K^{-1} \mathbf{x}_{\text{pixel}} = [I \ 0] X, \quad \mathbf{x}'_{\text{norm}} = K'^{-1} \mathbf{x}'_{\text{pixel}} = [R \ t] X$$

Epipolar constraint: Calibrated case



The vectors Rx , t , and x' are coplanar

Epipolar constraint: Calibrated case

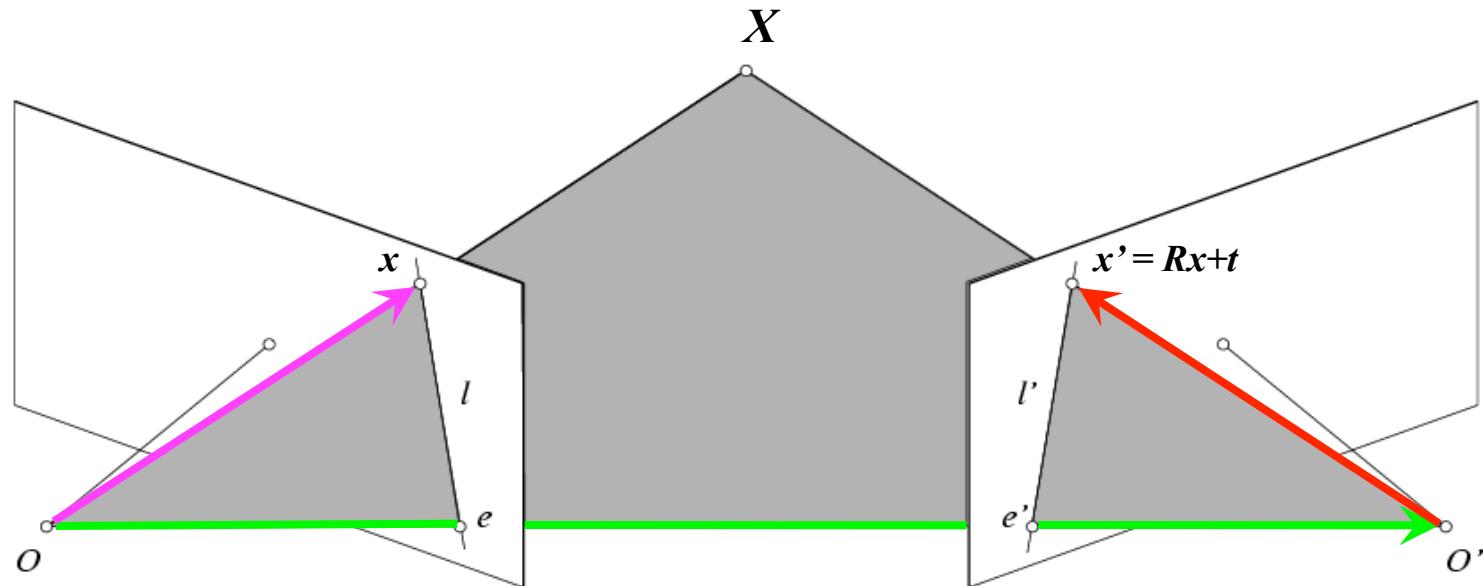


$$\mathbf{x}' \cdot [\mathbf{t} \times (\mathbf{R}\mathbf{x})] = 0 \quad \rightarrow \quad \mathbf{x}'^T [\mathbf{t}_x]^T \mathbf{R} \mathbf{x} = 0$$

Recall: $\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_x] \mathbf{b}$

The vectors \mathbf{Rx} , \mathbf{t} , and \mathbf{x}' are coplanar

Epipolar constraint: Calibrated case



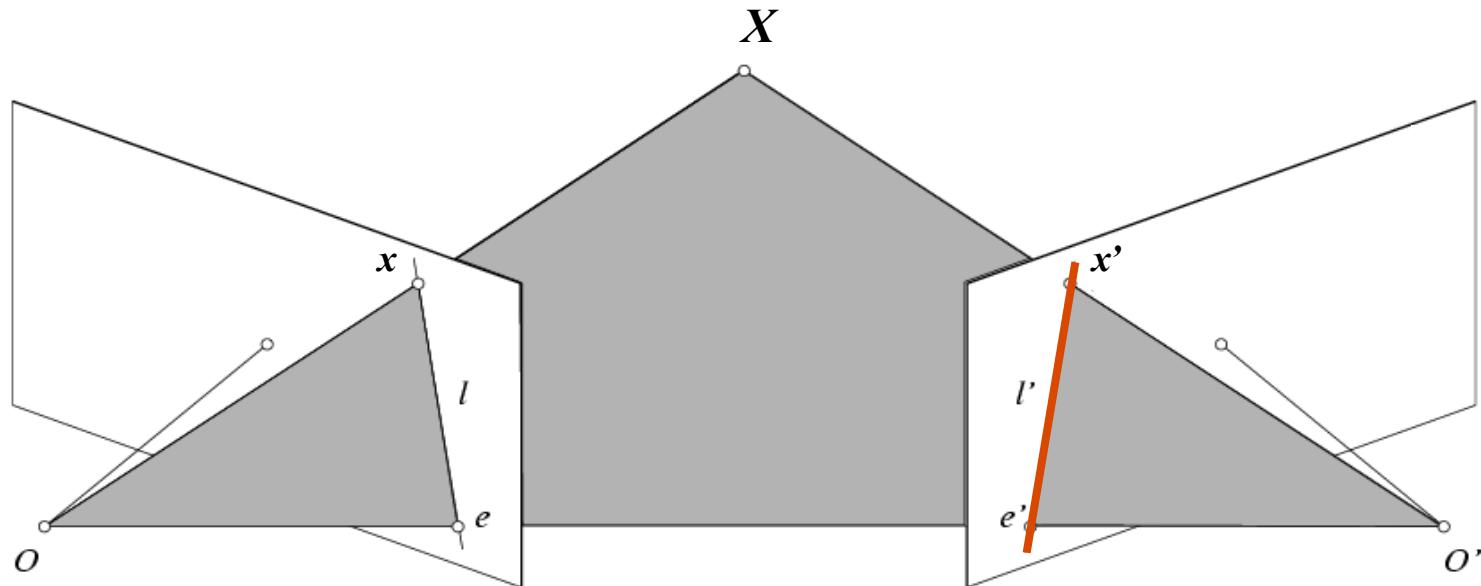
$$\mathbf{x}' \cdot [\mathbf{t} \times (\mathbf{R}\mathbf{x})] = 0 \quad \rightarrow \quad \mathbf{x}'^T [\mathbf{t}_x]^T \mathbf{R} \mathbf{x} = 0 \quad \rightarrow \quad \mathbf{x}'^T E \mathbf{x} = 0$$



Essential Matrix
(Longuet-Higgins, 1981)

The vectors $\mathbf{R}\mathbf{x}$, \mathbf{t} , and \mathbf{x}' are coplanar

Epipolar constraint: Calibrated case

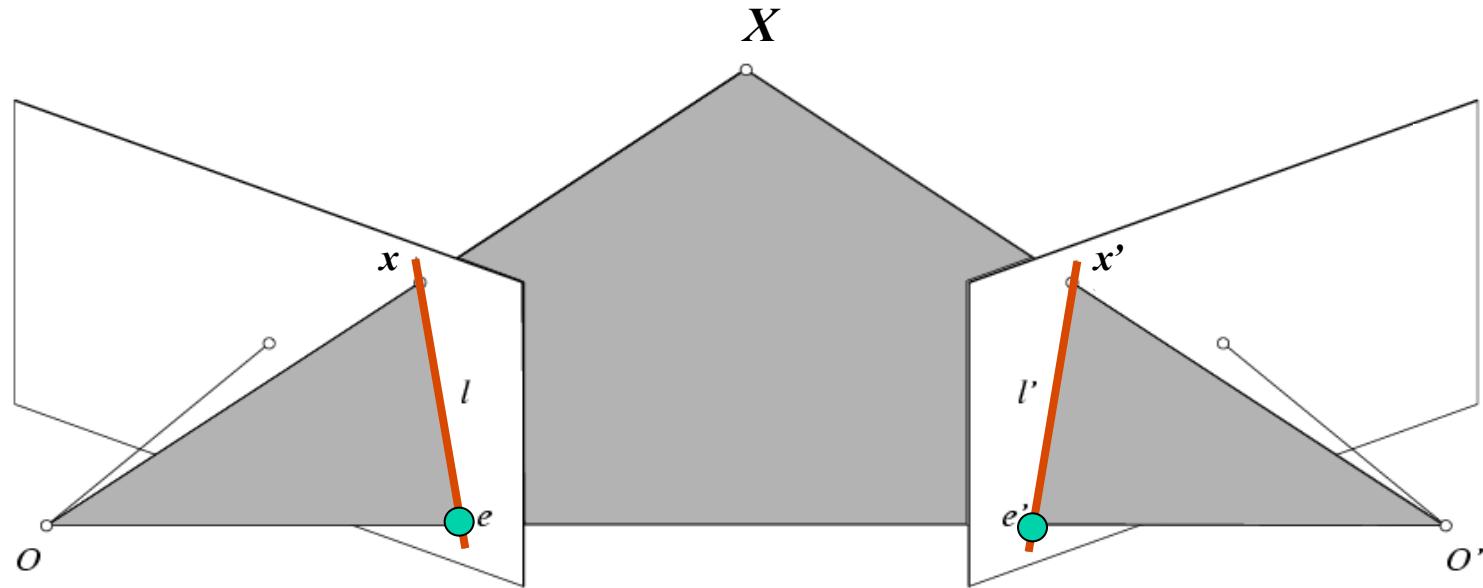


$$\mathbf{x}'^T \mathbf{E} \mathbf{x} = 0$$

- $\mathbf{E} \mathbf{x}$ is the epipolar line associated with \mathbf{x} ($\mathbf{l}' = \mathbf{E} \mathbf{x}$)
 - Recall: a line is given by $ax + by + c = 0$ or

$$\mathbf{l}^T \mathbf{x} = 0 \quad \text{where} \quad \mathbf{l} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

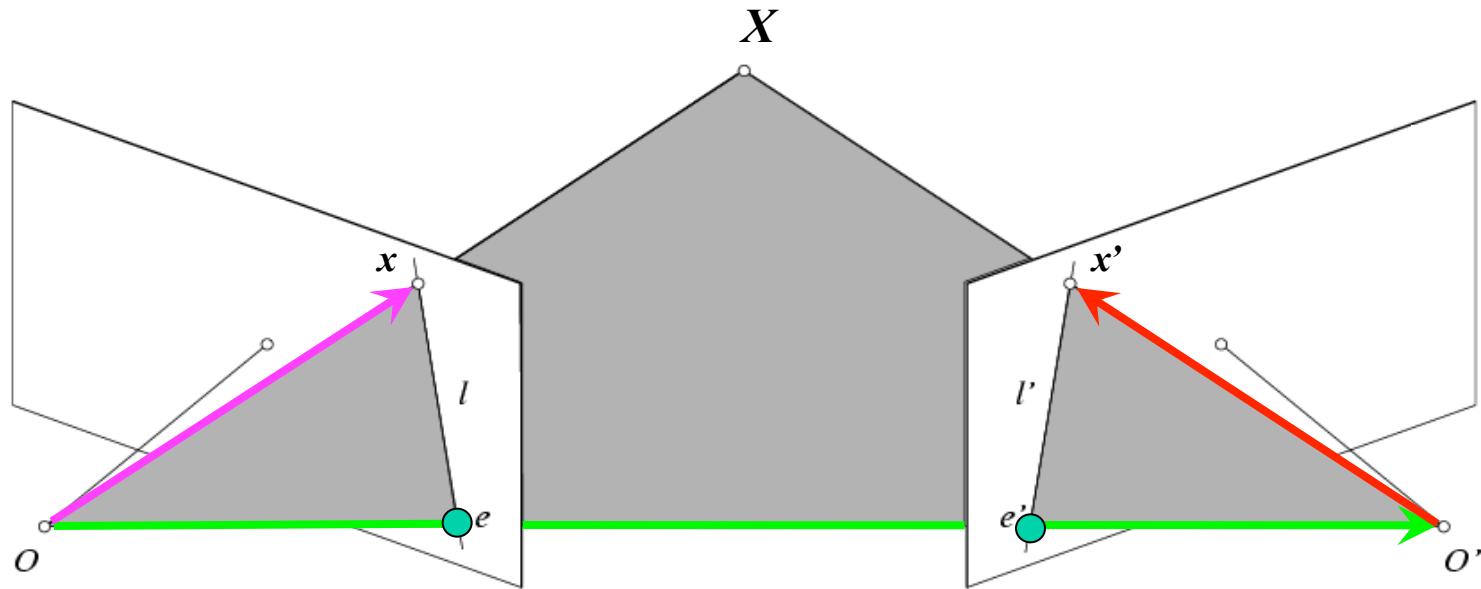
Epipolar constraint: Calibrated case



$$\mathbf{x}'^T \mathbf{E} \mathbf{x} = 0$$

- $\mathbf{E} \mathbf{x}$ is the epipolar line associated with \mathbf{x} ($\mathbf{l}' = \mathbf{E} \mathbf{x}$)
- $\mathbf{E}^T \mathbf{x}'$ is the epipolar line associated with \mathbf{x}' ($\mathbf{l} = \mathbf{E}^T \mathbf{x}'$)
- $\mathbf{E} \mathbf{e} = 0$ and $\mathbf{E}^T \mathbf{e}' = 0$
- \mathbf{E} is singular (rank two)
- \mathbf{E} has five degrees of freedom

Epipolar constraint: Uncalibrated case

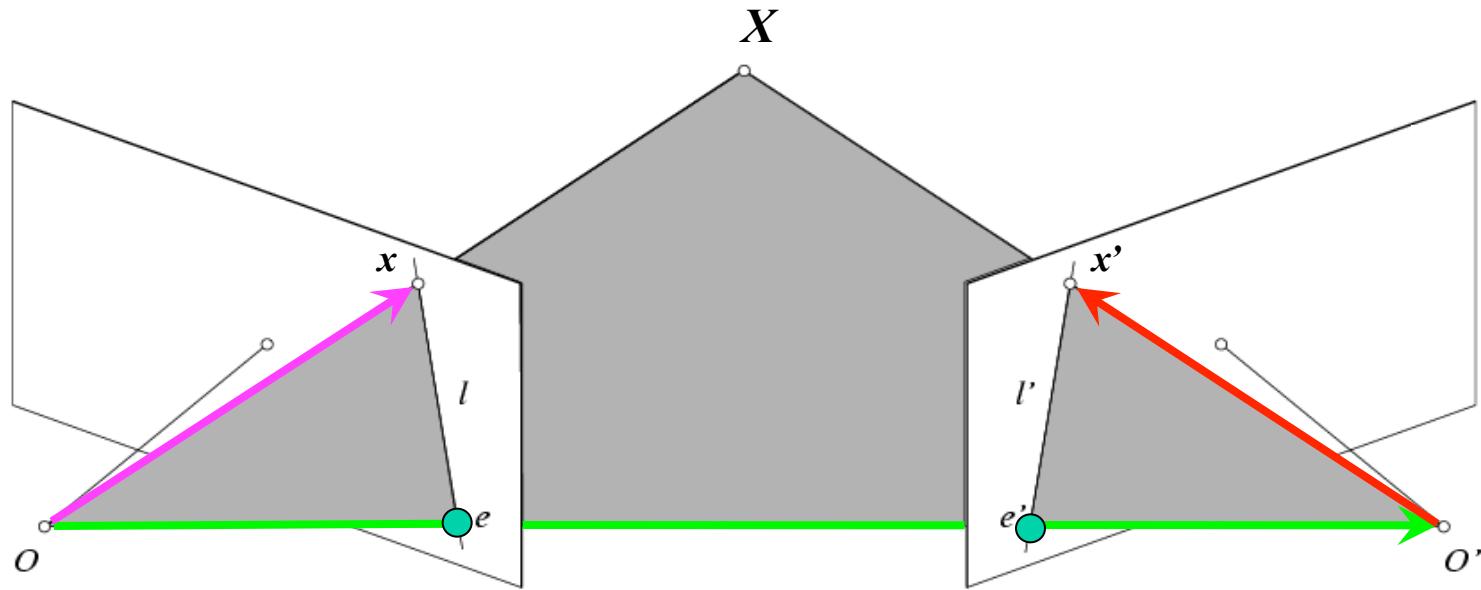


- The calibration matrices K and K' of the two cameras are unknown
- We can write the epipolar constraint in terms of *unknown* normalized coordinates:

$$\hat{x}'^T E \hat{x} = 0$$

$$\hat{x} = K^{-1}x, \quad \hat{x}' = K'^{-1}x'$$

Epipolar constraint: Uncalibrated case



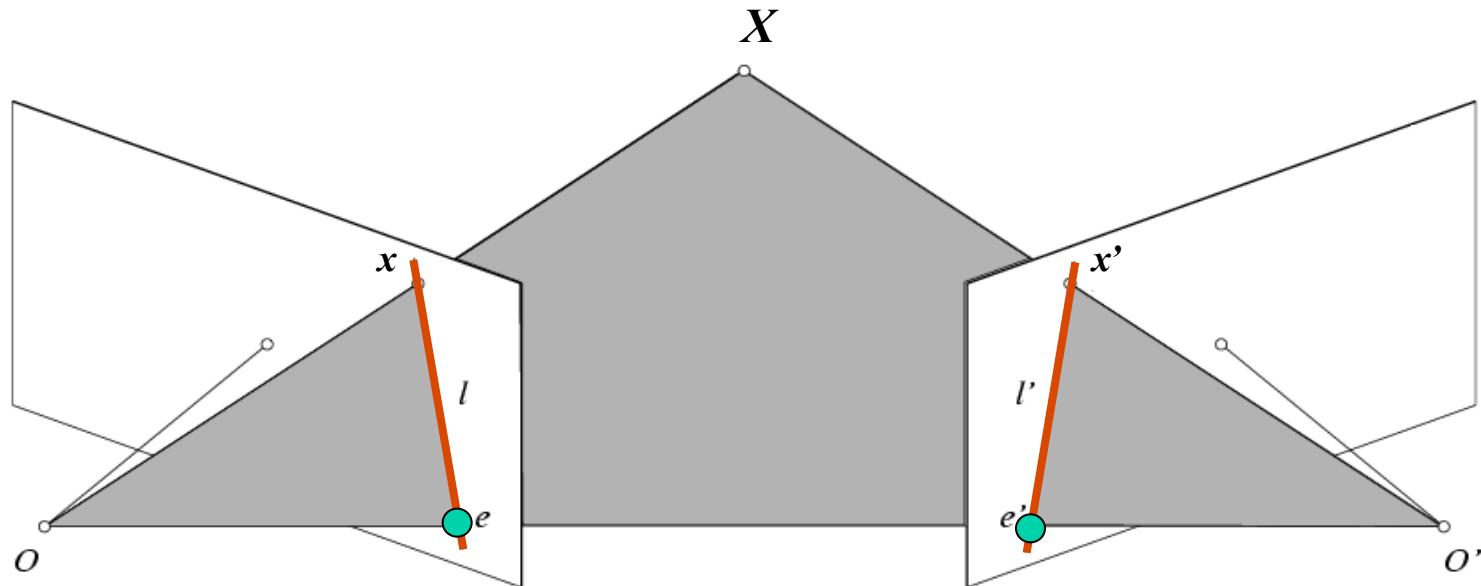
$$\hat{x}'^T E \hat{x} = 0 \quad \xrightarrow{\text{red arrow}} \quad x'^T F x = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1}$$

$$\hat{x} = K^{-1} x$$

$$\hat{x}' = K'^{-1} x'$$

Fundamental Matrix
(Faugeras and Luong, 1992)

Epipolar constraint: Uncalibrated case



$$\hat{x}'^T E \hat{x} = 0 \quad \rightarrow \quad x'^T F x = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1}$$

- $F x$ is the epipolar line associated with x ($l' = F x$)
- $F^T x'$ is the epipolar line associated with x' ($l = F^T x'$)
- $F e = 0$ and $F^T e' = 0$
- F is singular (rank two)
- F has seven degrees of freedom

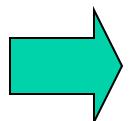
Estimating the fundamental matrix



The eight-point algorithm

$$\mathbf{x} = (u, v, 1)^T, \quad \mathbf{x}' = (u', v', 1)$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0$$

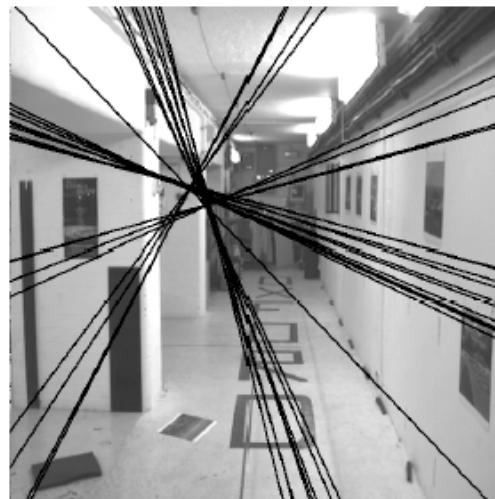


$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

Solve homogeneous
linear system using
eight or more matches



Enforce rank-2
constraint (take SVD
of \mathbf{F} and throw out the
smallest singular value)



Problem with eight-point algorithm

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Problem with eight-point algorithm

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48

$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Poor numerical conditioning

Can be fixed by rescaling the data

The normalized eight-point algorithm

(Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute \mathbf{F} from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of \mathbf{F} and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if \mathbf{T} and \mathbf{T}' are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is $\mathbf{T}'^T \mathbf{F} \mathbf{T}$

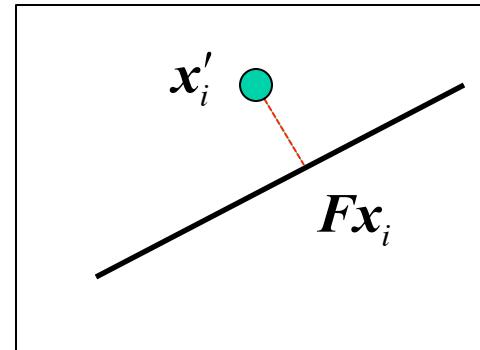
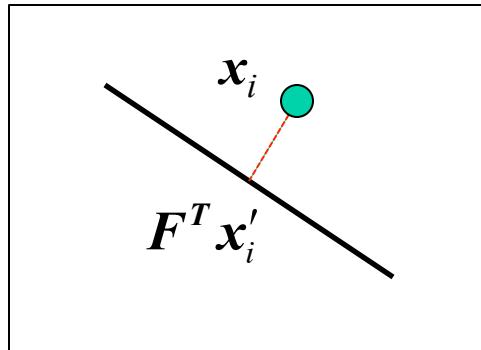
Nonlinear estimation

- Linear estimation minimizes the sum of squared *algebraic* distances between points \mathbf{x}'_i and epipolar lines $\mathbf{F} \mathbf{x}_i$ (or points \mathbf{x}_i and epipolar lines $\mathbf{F}^T \mathbf{x}'_i$):

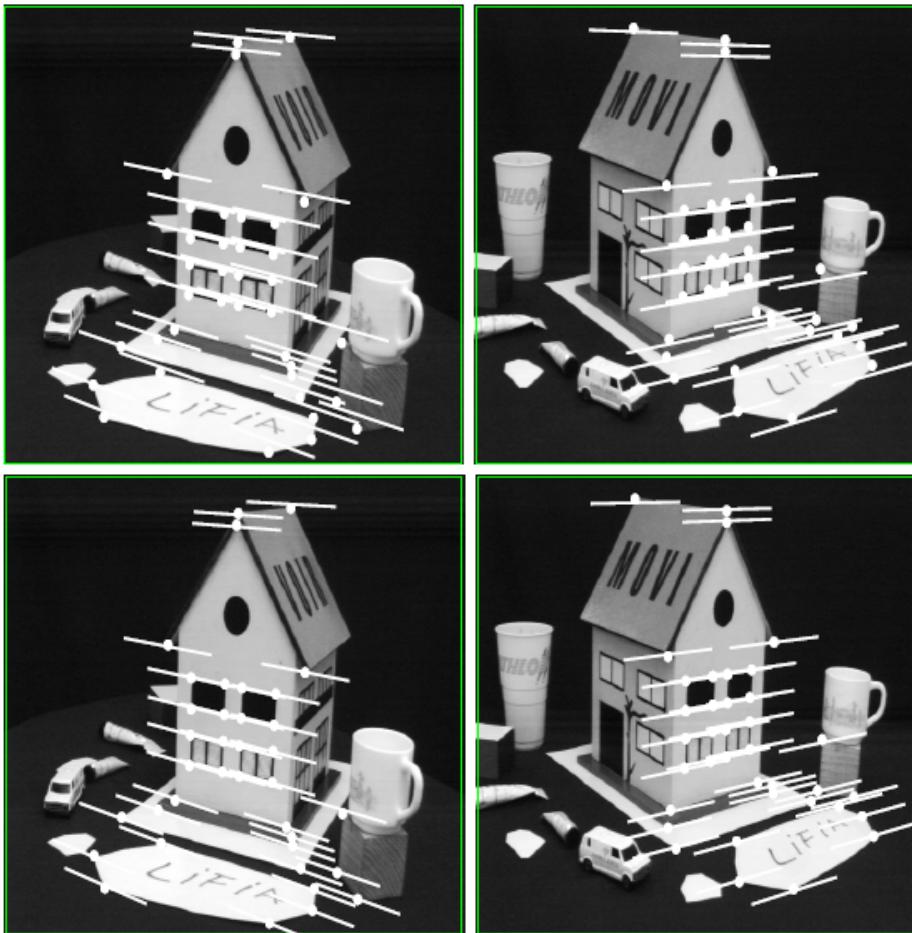
$$\sum_{i=1}^N (\mathbf{x}'_i^T \mathbf{F} \mathbf{x}_i)^2$$

- Nonlinear approach: minimize sum of squared *geometric* distances

$$\sum_{i=1}^N [d^2(\mathbf{x}'_i, \mathbf{F} \mathbf{x}_i) + d^2(\mathbf{x}_i, \mathbf{F}^T \mathbf{x}'_i)]$$

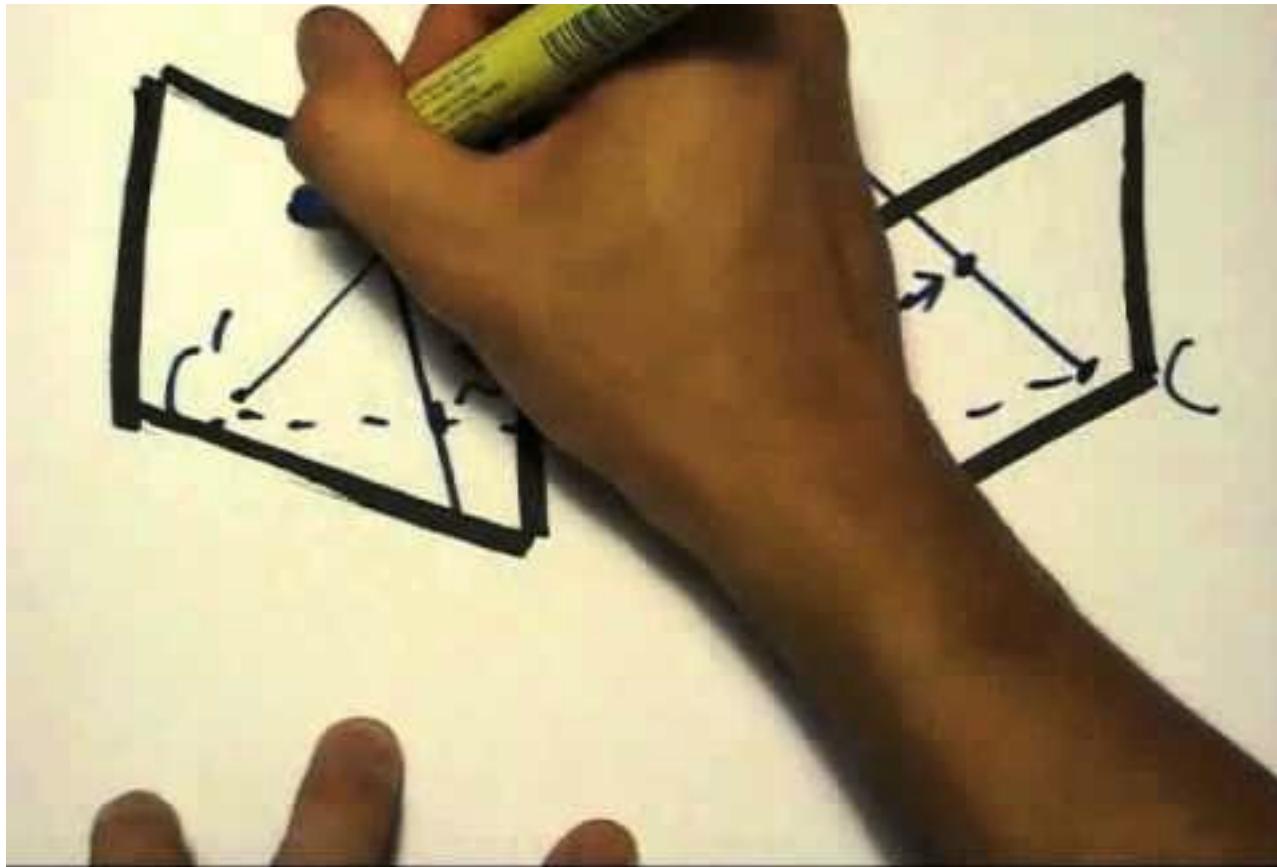


Comparison of estimation algorithms



	8-point	Normalized 8-point	Nonlinear least squares
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

The Fundamental Matrix Song



<http://danielwedge.com/fmatrix/>

From epipolar geometry to camera calibration

- Estimating the fundamental matrix is known as “weak calibration”
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $E = K'^T F K$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters

Stereo



Many slides adapted from Steve Seitz

Binocular stereo

- Given a calibrated binocular stereo pair, fuse it to produce a depth image

image 1



image 2



Dense depth map



Binocular stereo

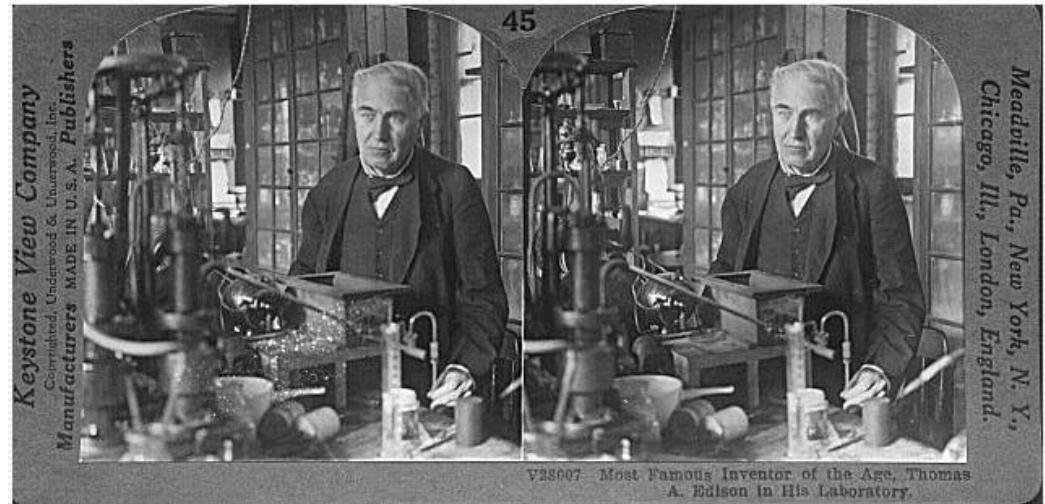
- Given a calibrated binocular stereo pair, fuse it to produce a depth image



Where does the depth information come from?

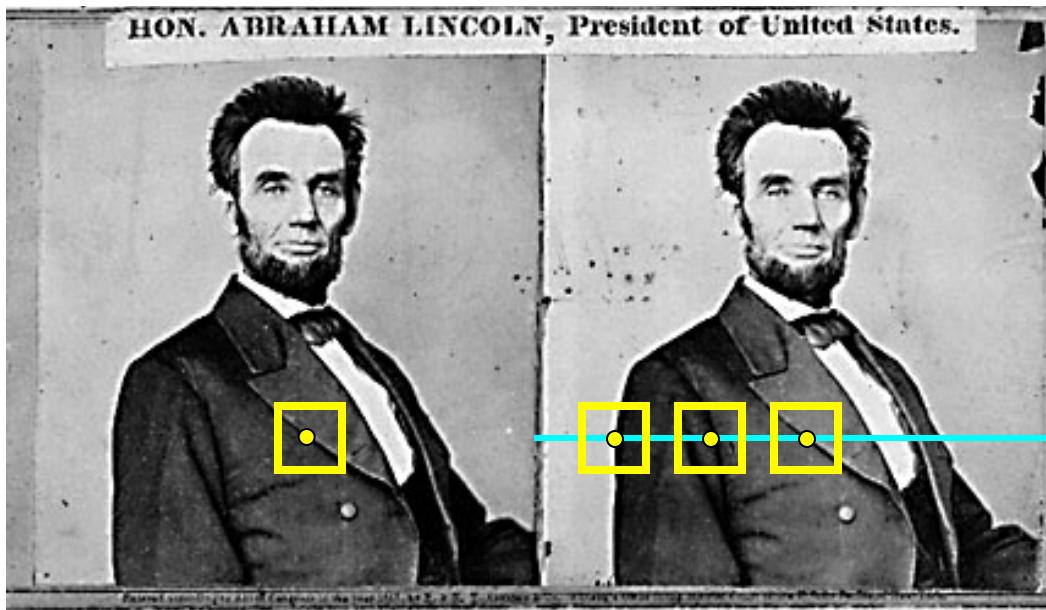
Binocular stereo

- Given a calibrated binocular stereo pair, fuse it to produce a depth image
 - Humans can do it



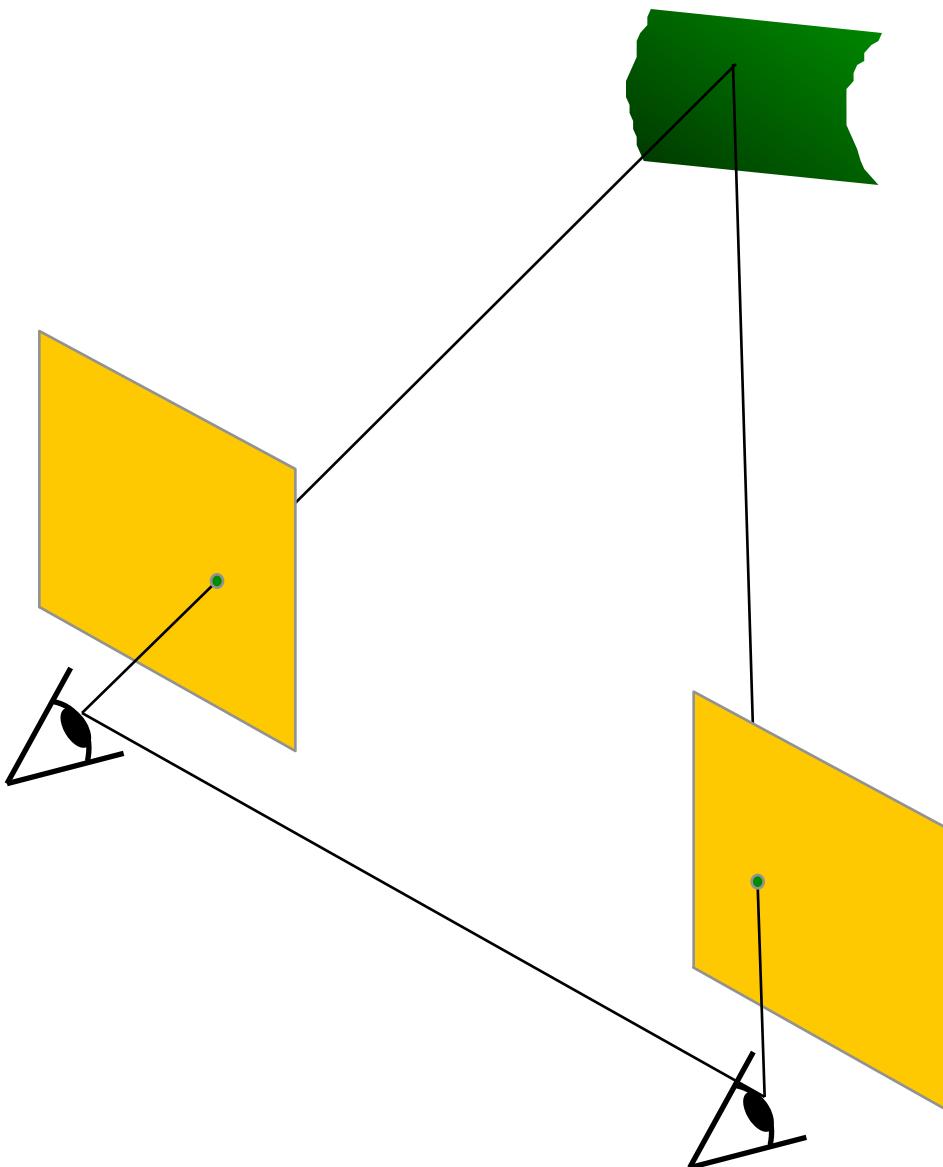
Stereograms: Invented by Sir Charles Wheatstone, 1838

Basic stereo matching algorithm



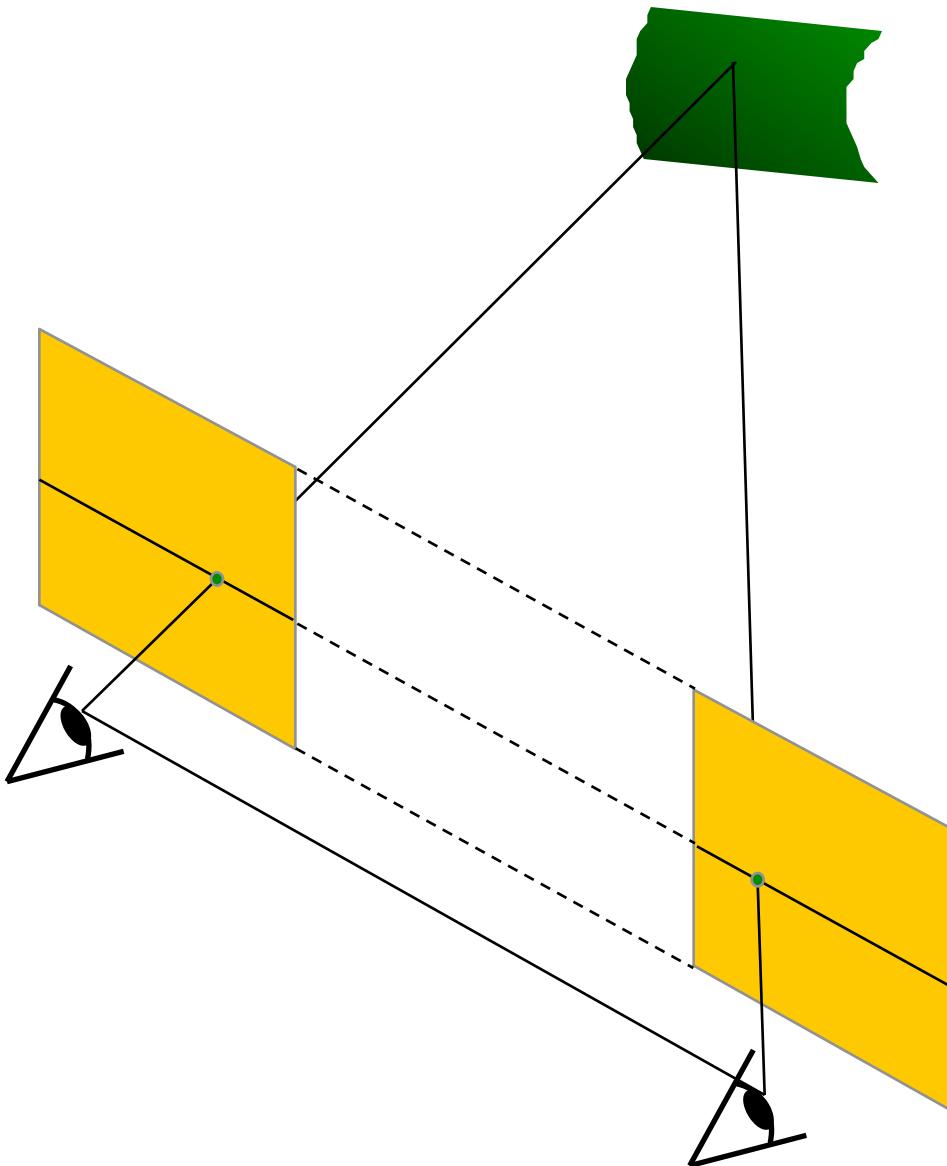
- For each pixel in the first image
 - Find corresponding epipolar line in the right image
 - Examine all pixels on the epipolar line and pick the best match
 - Triangulate the matches to get depth information
- Simplest case: epipolar lines are corresponding scanlines
 - When does this happen?

Simplest Case: Parallel images



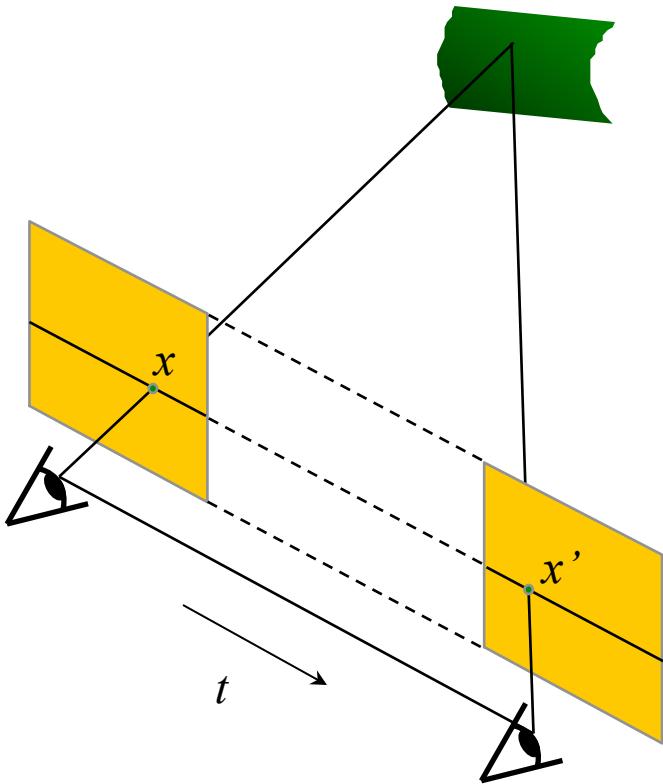
- Image planes of cameras are parallel to each other and to the baseline
- Camera centers are at same height
- Focal lengths are the same

Simplest Case: Parallel images



- Image planes of cameras are parallel to each other and to the baseline
- Camera centers are at same height
- Focal lengths are the same
- Then epipolar lines fall along the horizontal scan lines of the images

Essential matrix for parallel images



Epipolar constraint:

$$\mathbf{x}'^T \mathbf{E} \mathbf{x} = 0, \quad \mathbf{E} = [\mathbf{t}_x] \mathbf{R}$$

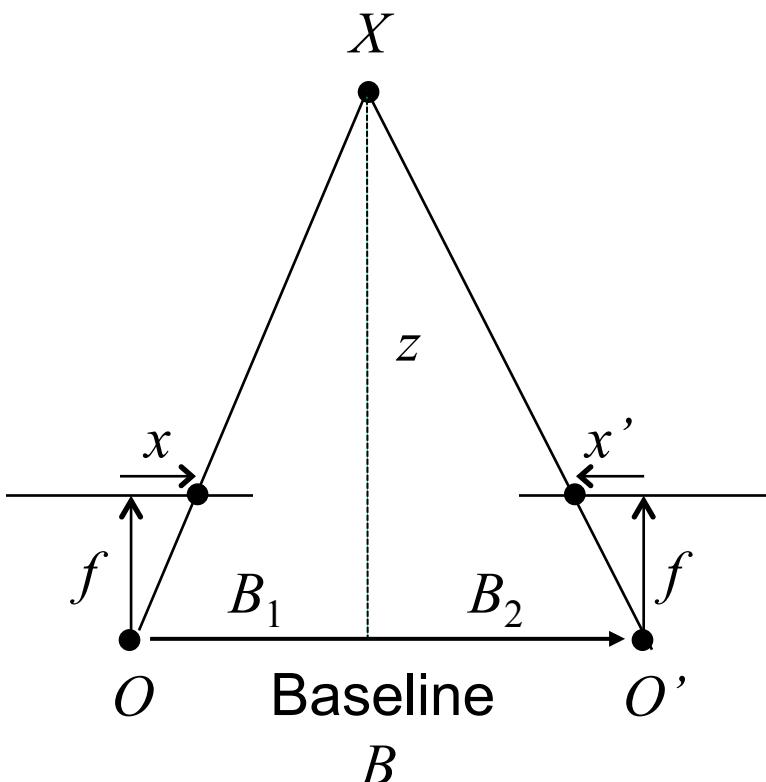
$$\mathbf{R} = \mathbf{I} \quad \mathbf{t} = (T, 0, 0)$$

$$\mathbf{E} = [\mathbf{t}_x] \mathbf{R} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix}$$

$$(u' \quad v' \quad 1) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -T \\ 0 & T & 0 \end{bmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = 0 \quad (u' \quad v' \quad 1) \begin{pmatrix} 0 \\ -T \\ Tv \end{pmatrix} = 0 \quad Tv' = Tv$$

The y-coordinates of corresponding points are the same!

Depth from disparity



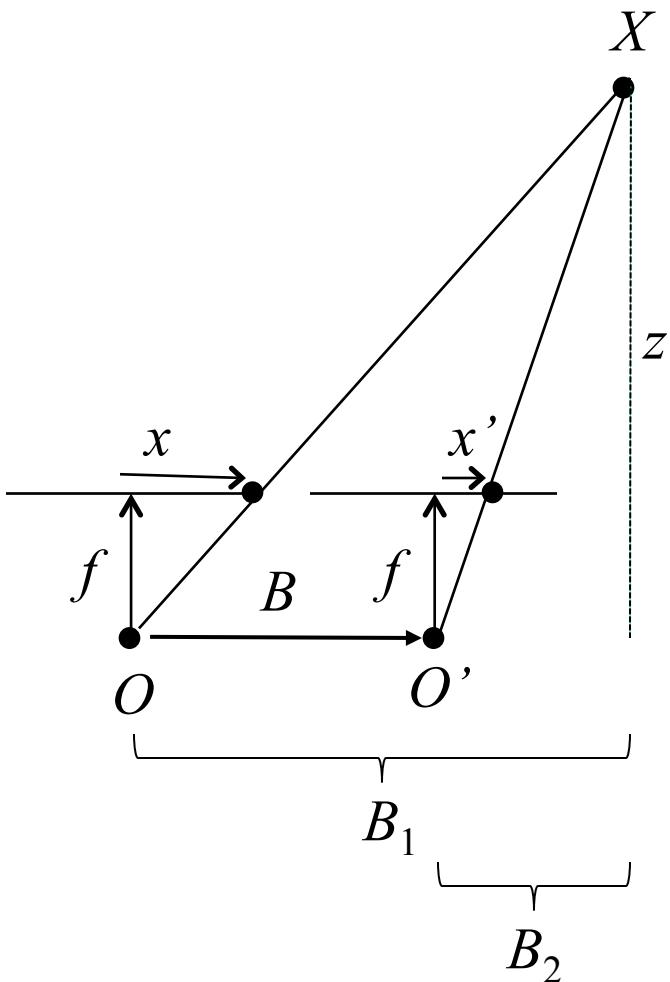
$$\frac{x}{f} = \frac{B_1}{z} \quad \frac{-x'}{f} = \frac{B_2}{z}$$

$$\frac{x - x'}{f} = \frac{B_1 + B_2}{z}$$

$$disparity = x - x' = \frac{B \cdot f}{z}$$

Disparity is inversely proportional to depth!

Depth from disparity

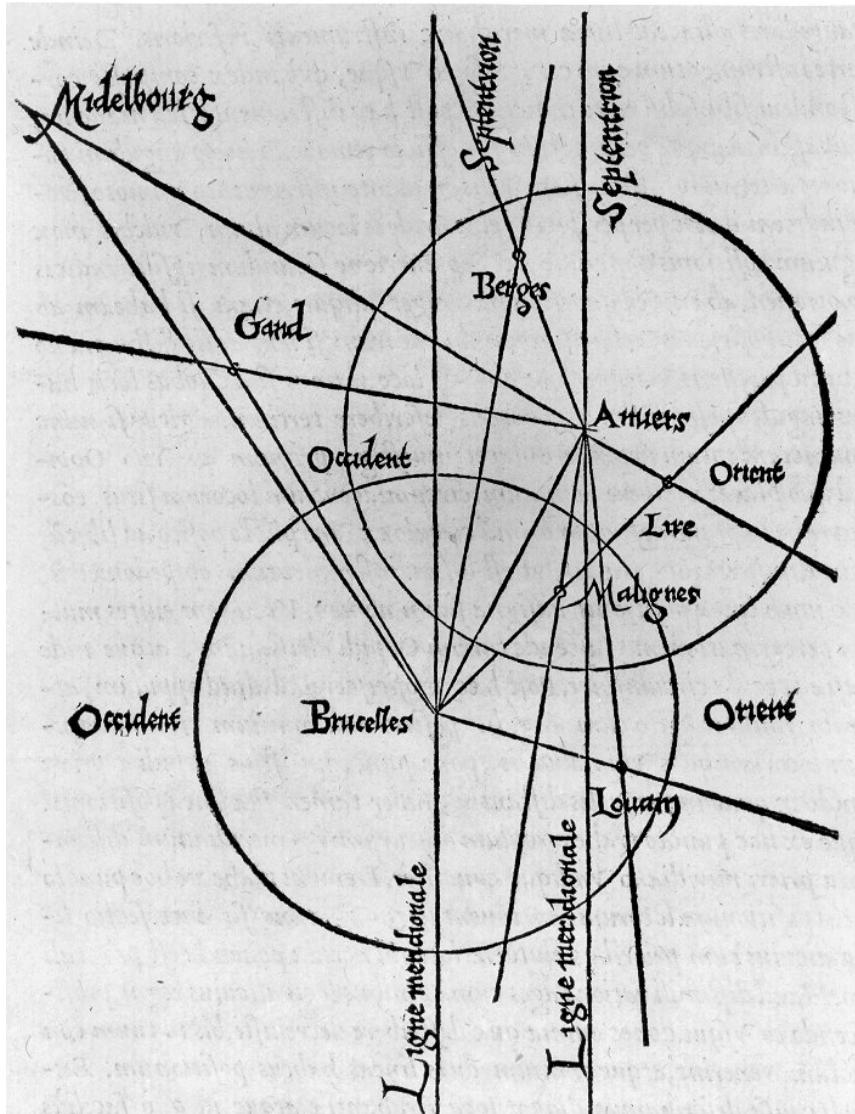


$$\frac{x}{f} = \frac{B_1}{z} \quad \frac{x'}{f} = \frac{B_2}{z}$$

$$\frac{x - x'}{f} = \frac{B_1 - B_2}{z}$$

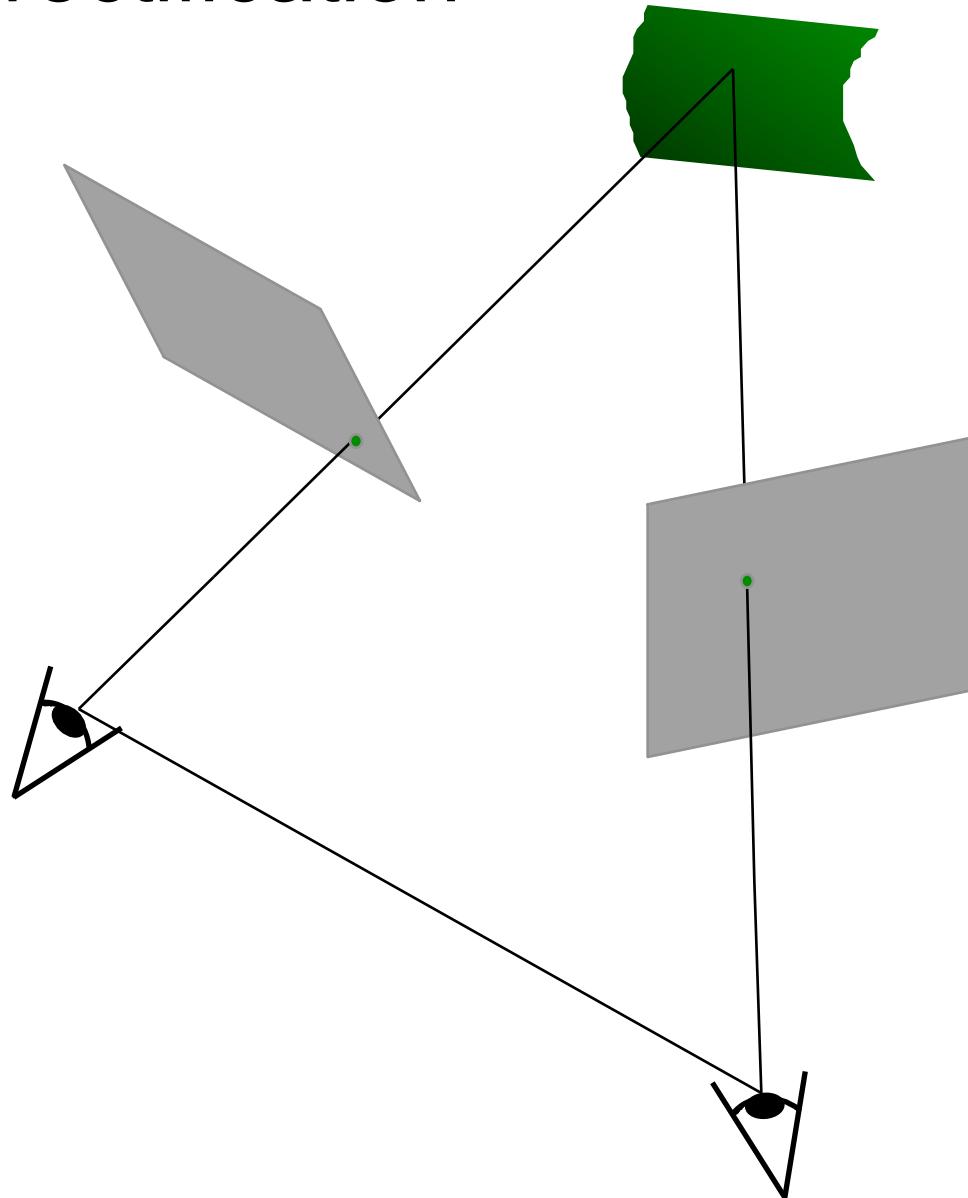
$$\boxed{\textit{disparity} = x - x' = \frac{B \cdot f}{z}}$$

Triangulation: History

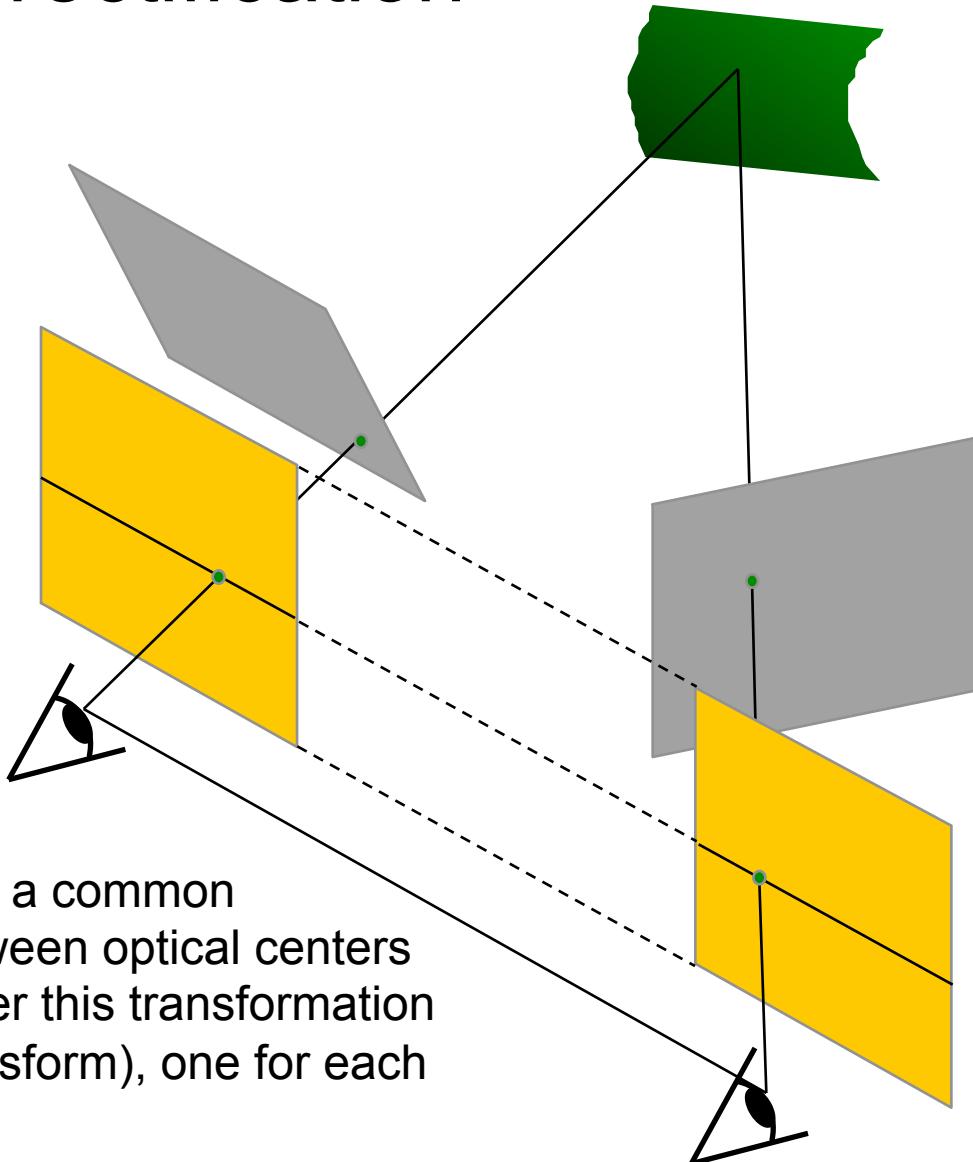


From [Wikipedia](#): Gemma Frisius's 1533 diagram introducing the idea of triangulation into the science of surveying. Having established a baseline, e.g. the cities of Brussels and Antwerp, the location of other cities, e.g. Middelburg, Ghent etc., can be found by taking a compass direction from each end of the baseline, and plotting where the two directions cross. This was only a theoretical presentation of the concept — due to topographical restrictions, it is impossible to see Middelburg from either Brussels or Antwerp. Nevertheless, the figure soon became well known all across Europe.

Stereo image rectification

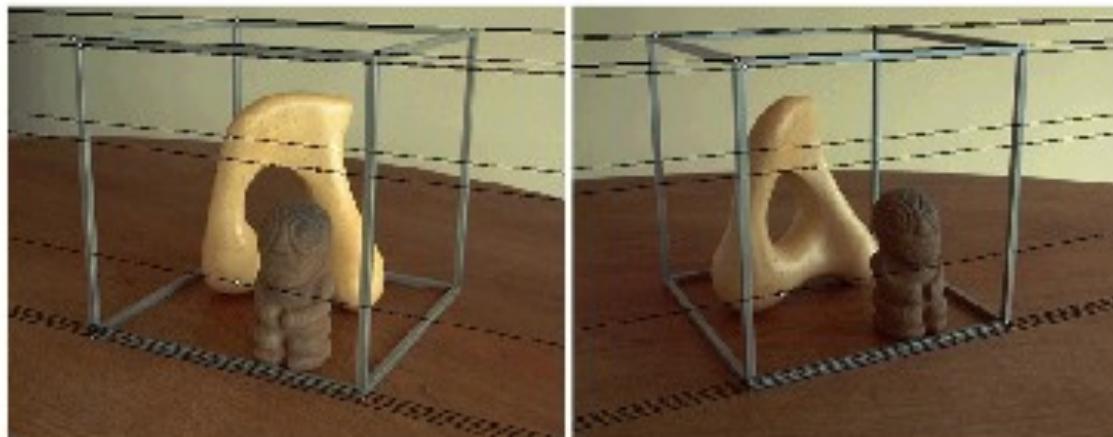


Stereo image rectification

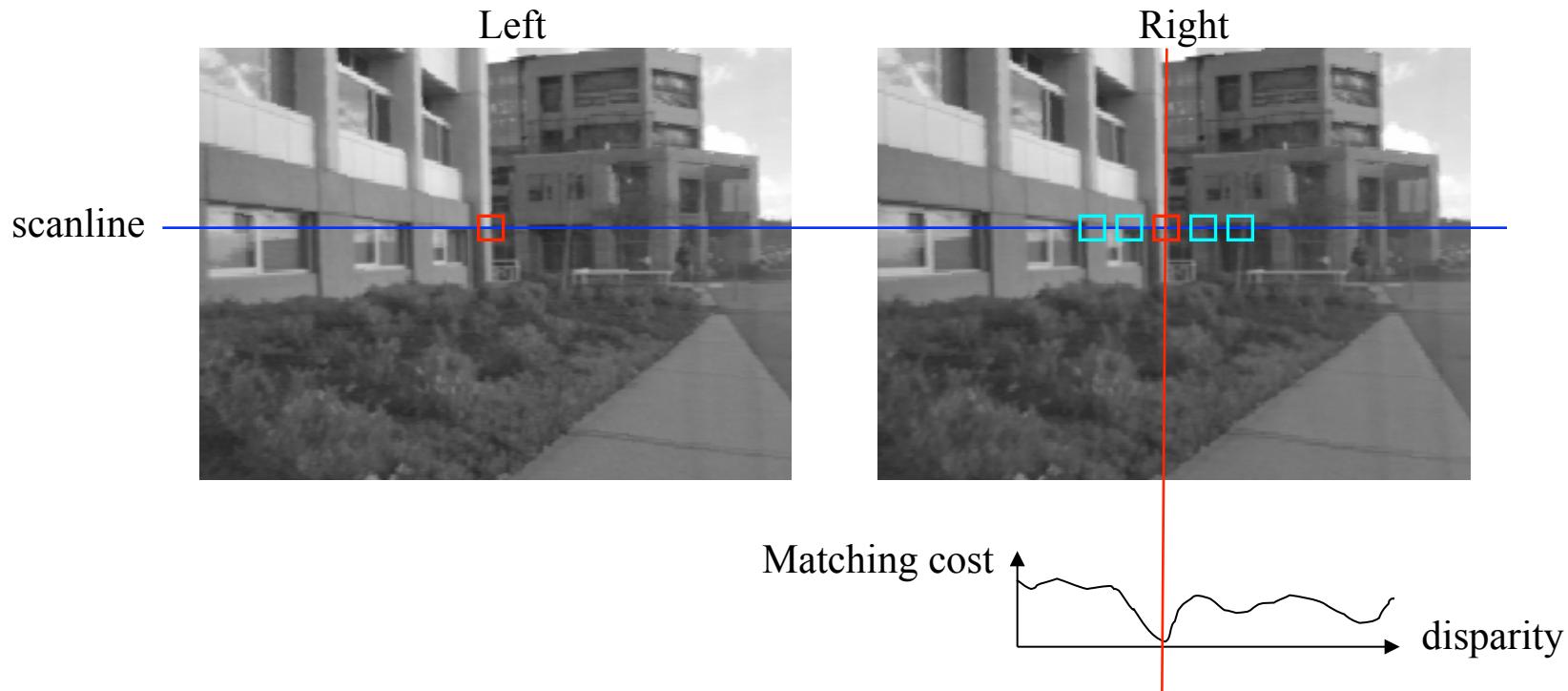


- Reproject image planes onto a common plane parallel to the line between optical centers
- Pixel motion is horizontal after this transformation
- Two homographies (3x3 transform), one for each input image reprojection

Rectification example

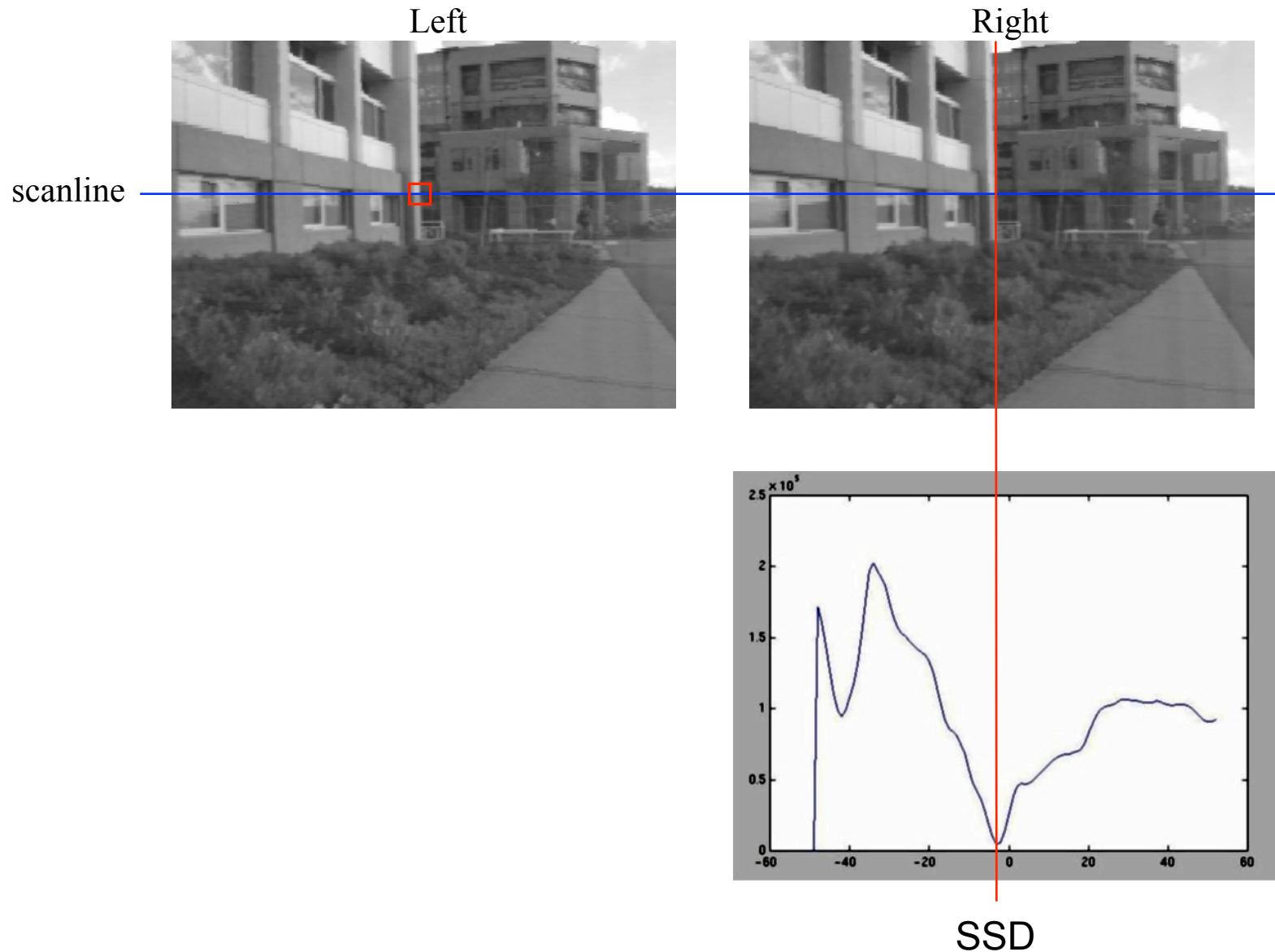


Correspondence search

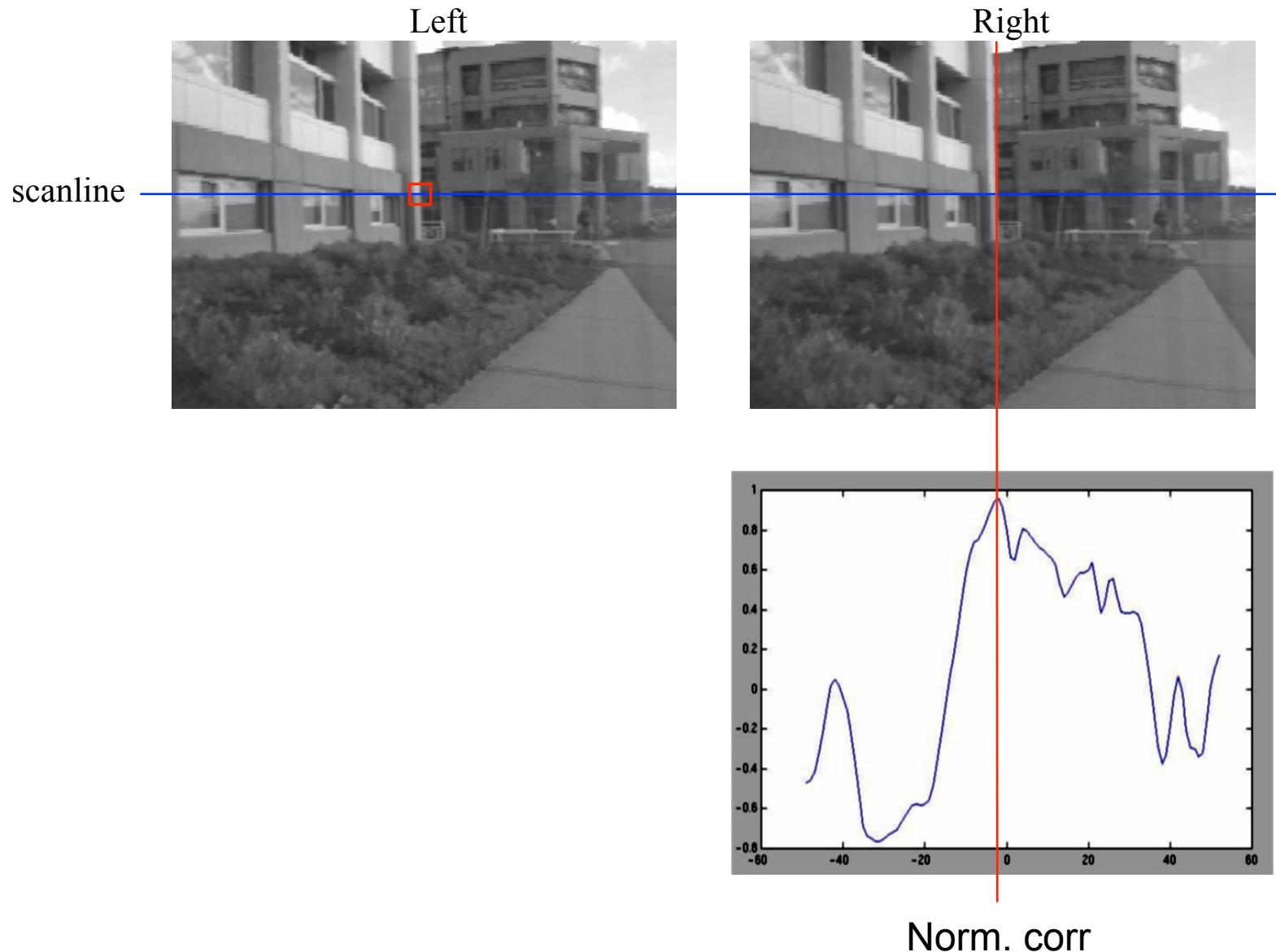


- Slide a window along the right scanline and compare contents of that window with the reference window in the left image
- Matching cost: SSD or normalized correlation

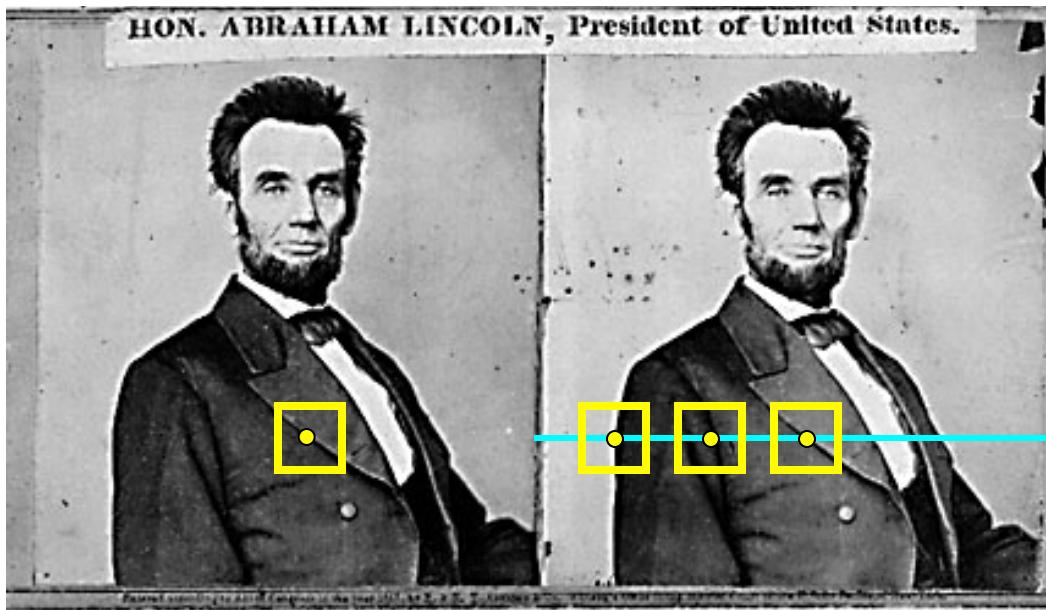
Correspondence search



Correspondence search

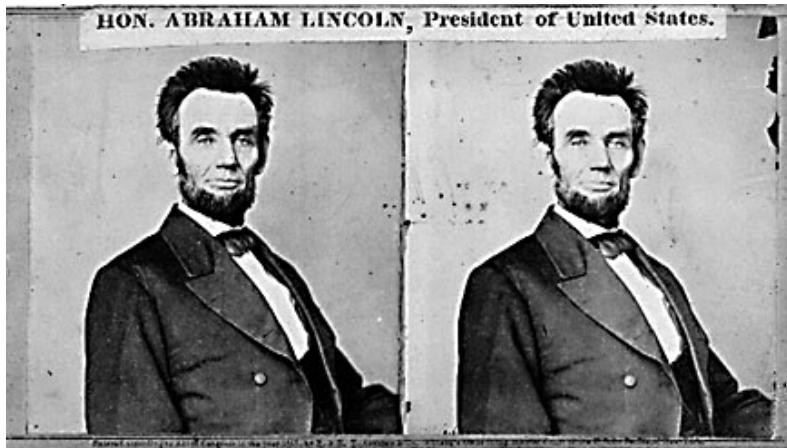


Basic stereo matching algorithm



- If necessary, rectify the two stereo images to transform epipolar lines into scanlines
- For each pixel x in the first image
 - Find corresponding epipolar scanline in the right image
 - Examine all pixels on the scanline and pick the best match x'
 - Compute disparity $x-x'$ and set $\text{depth}(x) = B*f/(x-x')$

Failures of correspondence search



Textureless surfaces



Occlusions, repetition

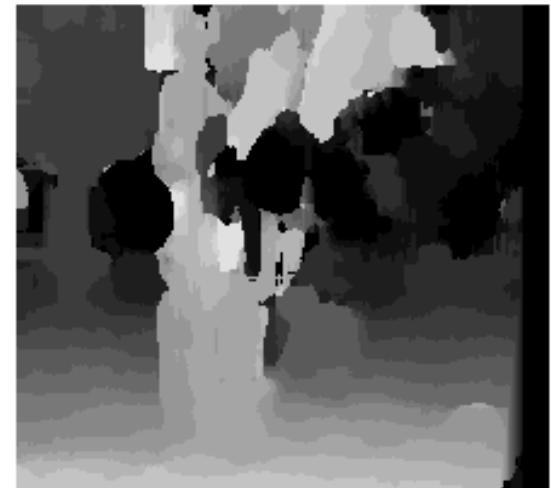


Non-Lambertian surfaces, specularities

Effect of window size



$$W = 3$$



$$W = 20$$

- Smaller window
 - + More detail
 - More noise

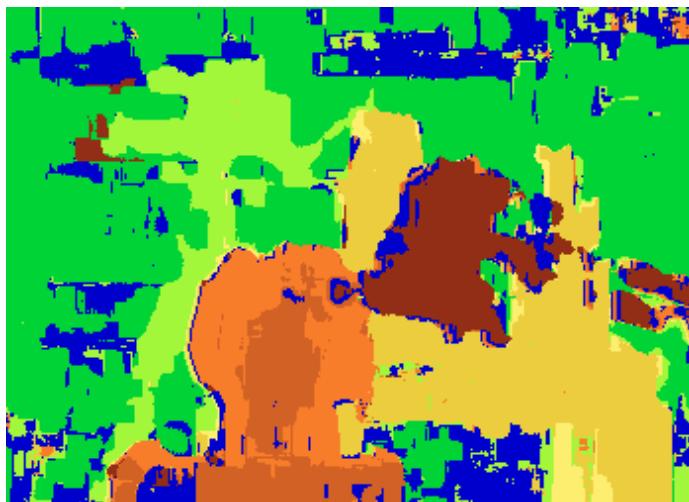
- Larger window
 - + Smoother disparity maps
 - Less detail

Results with window search

Data



Window-based matching



Ground truth



Better methods exist...



Graph cuts



Ground truth

Y. Boykov, O. Veksler, and R. Zabih,

Fast Approximate Energy Minimization via Graph Cuts, PAMI 2001

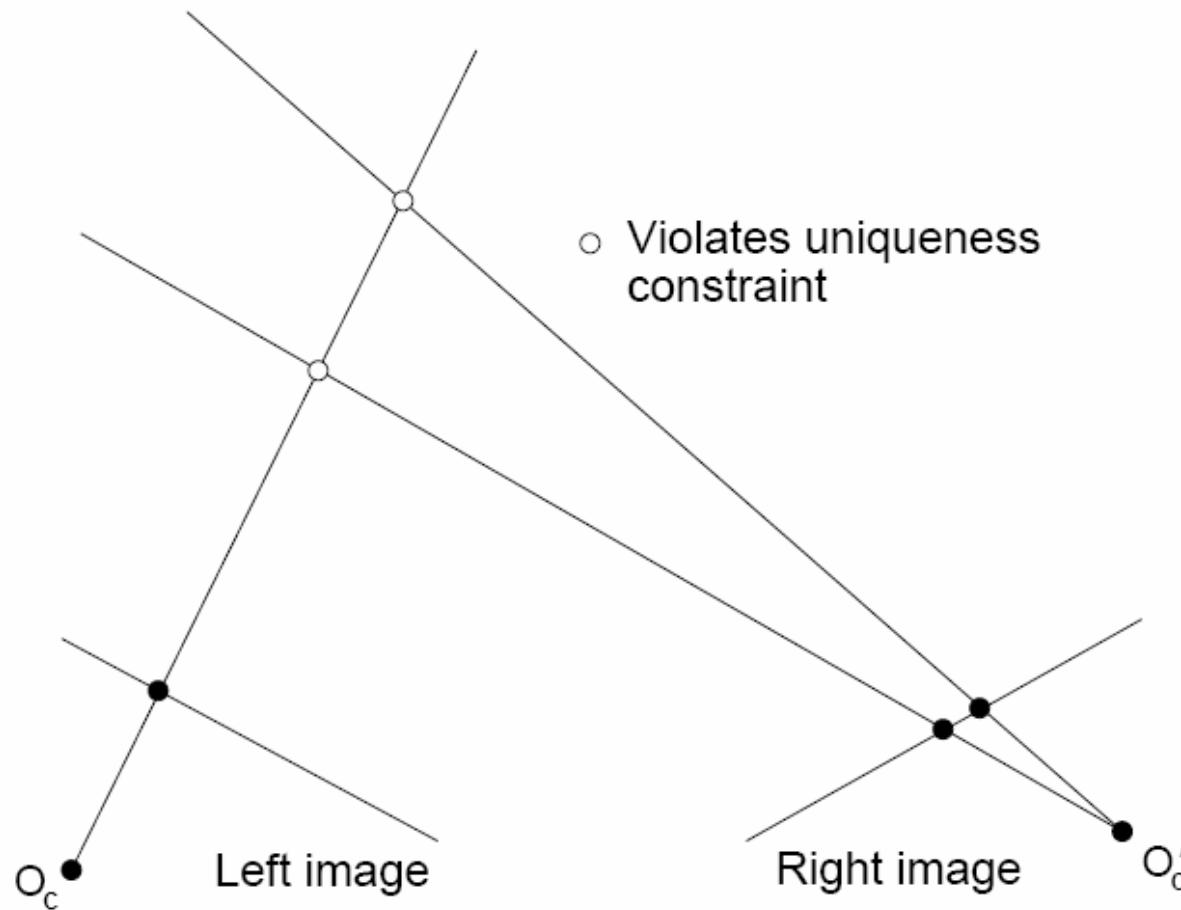
For the latest and greatest: <http://www.middlebury.edu/stereo/>

How can we improve window-based matching?

- The similarity constraint is **local** (each reference window is matched independently)
- Need to enforce **non-local** correspondence constraints

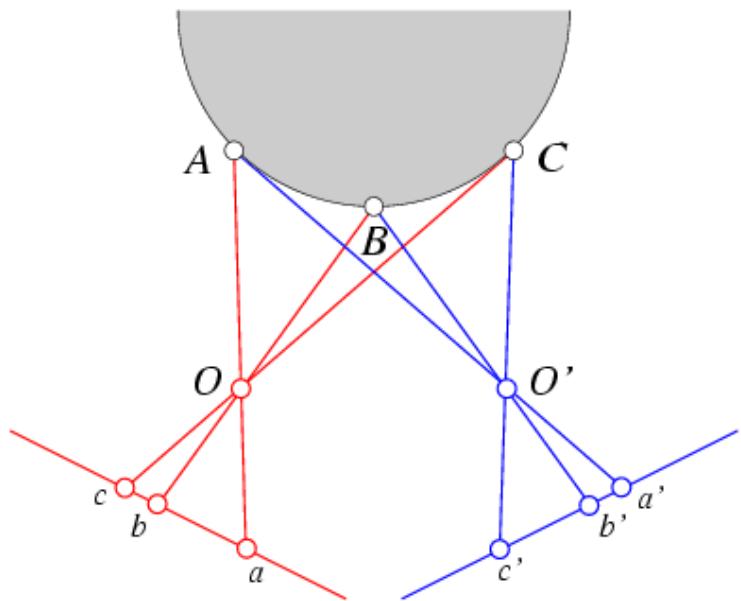
Non-local constraints

- Uniqueness
 - For any point in one image, there should be at most one matching point in the other image



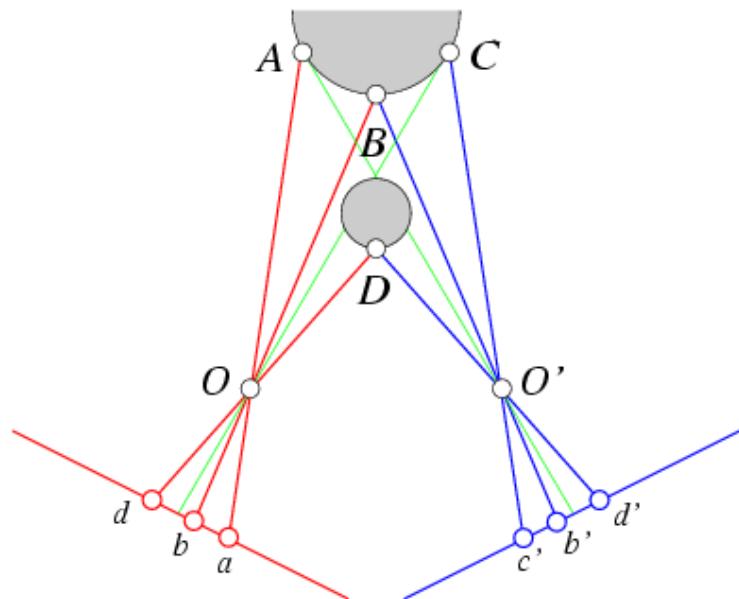
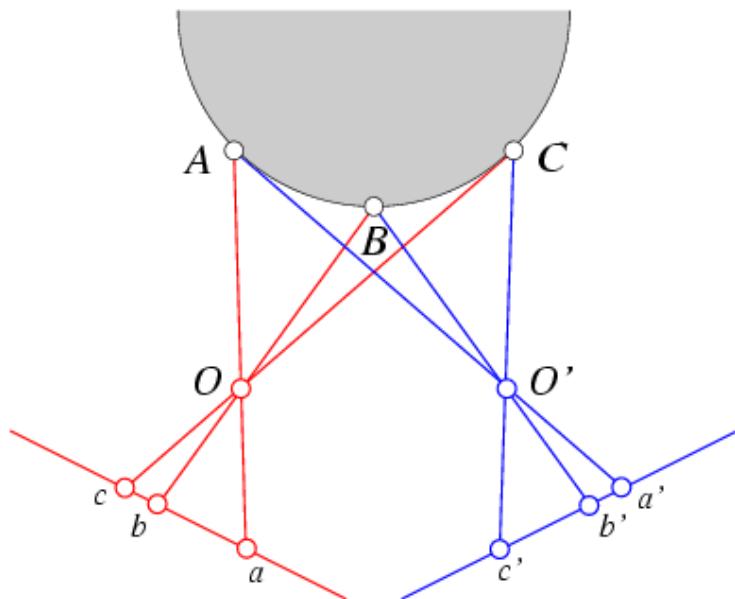
Non-local constraints

- Uniqueness
 - For any point in one image, there should be at most one matching point in the other image
- Ordering
 - Corresponding points should be in the same order in both views



Non-local constraints

- Uniqueness
 - For any point in one image, there should be at most one matching point in the other image
- Ordering
 - Corresponding points should be in the same order in both views



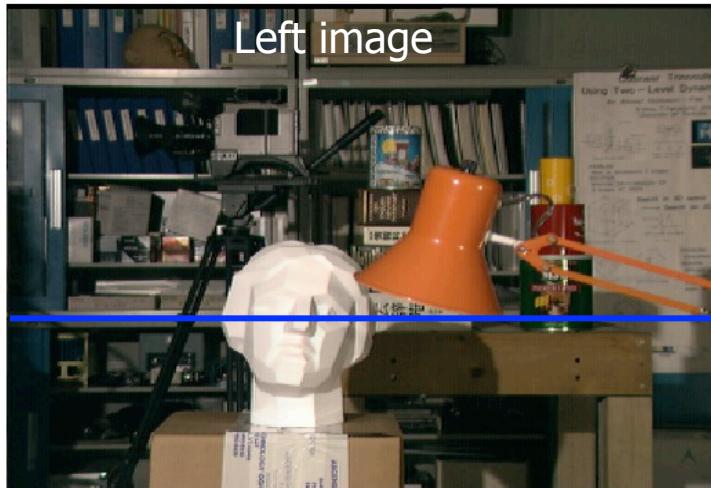
Ordering constraint doesn't hold

Non-local constraints

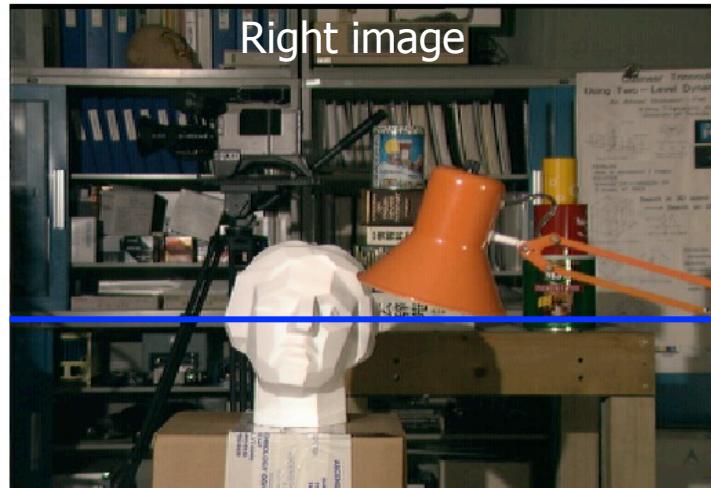
- Uniqueness
 - For any point in one image, there should be at most one matching point in the other image
- Ordering
 - Corresponding points should be in the same order in both views
- Smoothness
 - We expect disparity values to change slowly (for the most part)

Scanline stereo

- Try to coherently match pixels on the entire scanline
- Different scanlines are still optimized independently

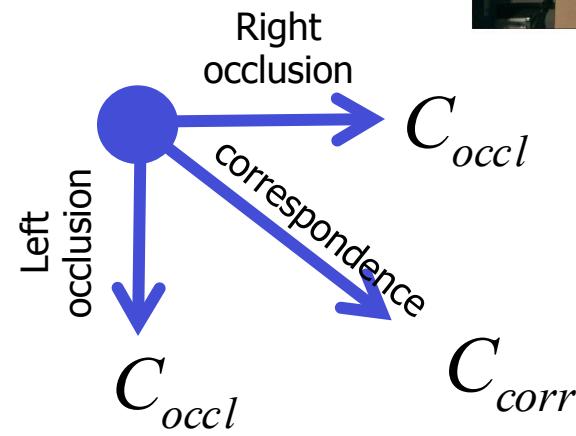
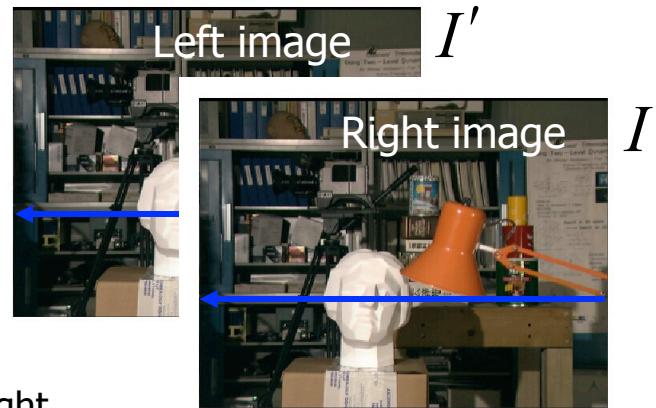
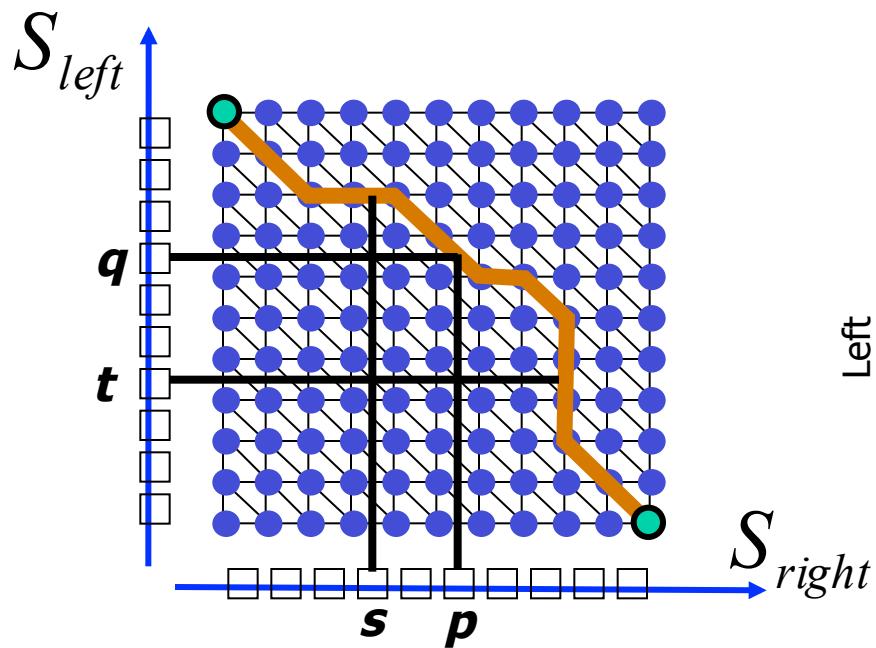


Left image



Right image

“Shortest paths” for scan-line stereo

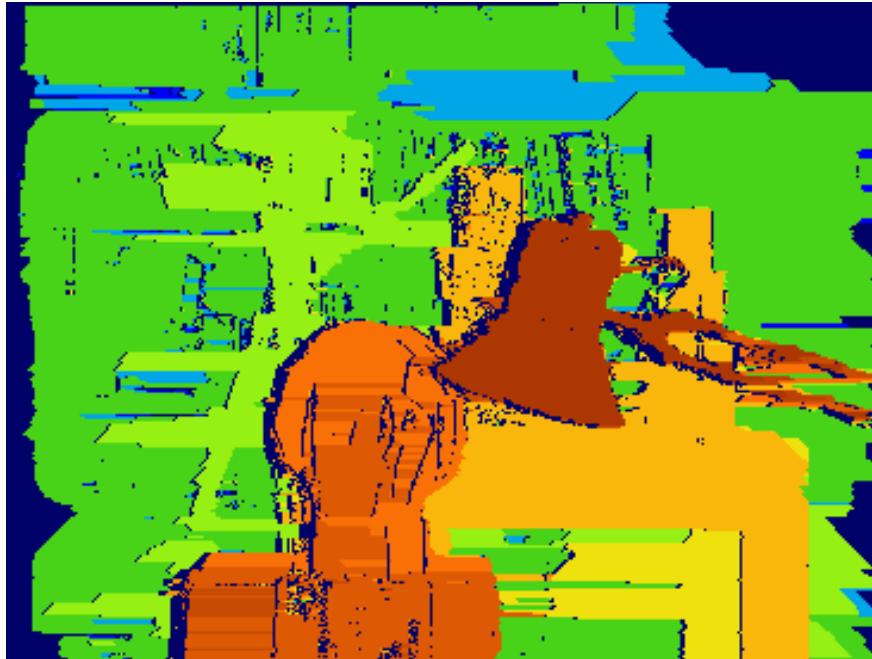


Can be implemented with dynamic programming

Ohta & Kanade '85, Cox et al. '96

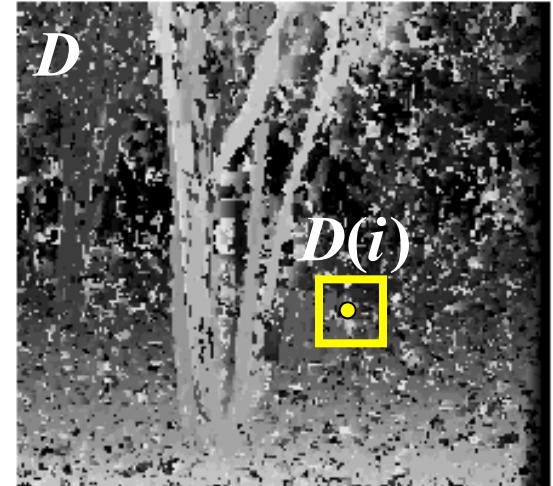
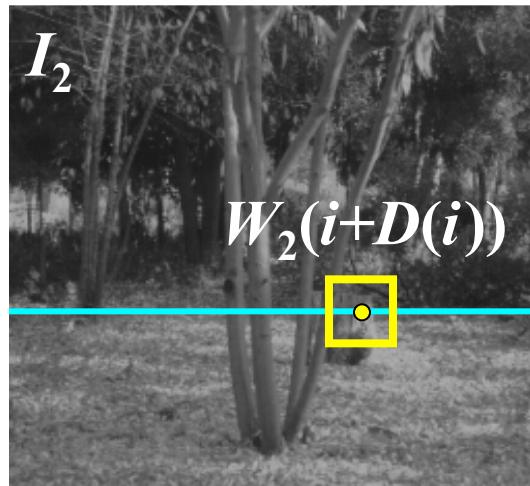
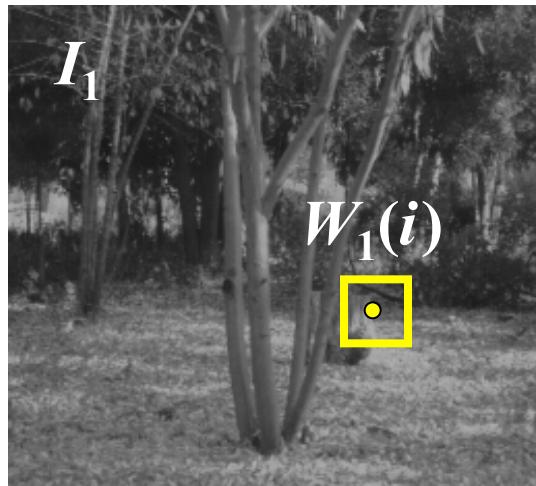
Coherent stereo on 2D grid

- Scanline stereo generates streaking artifacts



- Can't use dynamic programming to find spatially coherent disparities/ correspondences on a 2D grid

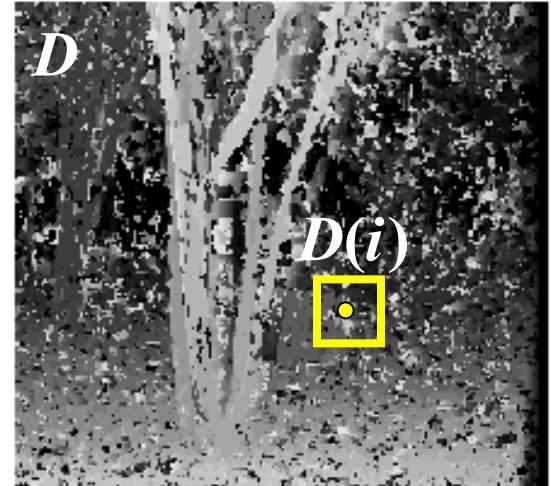
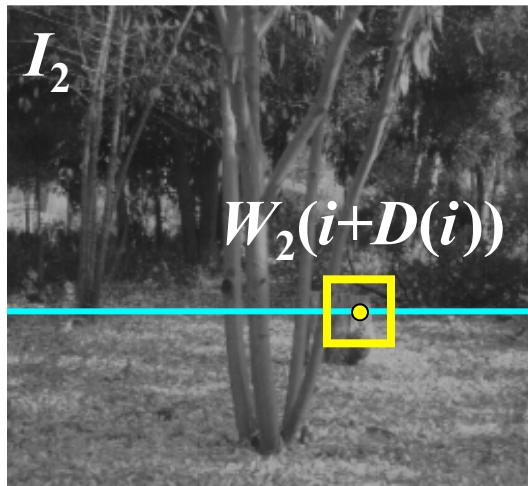
Stereo matching as energy minimization



$$E(D) = \underbrace{\sum_i (W_1(i) - W_2(i + D(i)))^2}_{\text{data term}} + \lambda \underbrace{\sum_{\text{neighbors } i,j} \rho(D(i) - D(j))}_{\text{smoothness term}}$$

- Energy functions of this form can be minimized using *graph cuts*

Stereo matching as energy minimization



- Probabilistic interpretation: we want to find a Maximum A Posteriori (MAP) estimate of disparity image D :

$$P(D | I_1, I_2) \propto P(I_1, I_2 | D)P(D)$$

$$-\log P(D | I_1, I_2) \propto -\log P(I_1, I_2 | D) - \log P(D)$$

$$E = E_{\text{data}}(I_1, I_2, D) + \lambda E_{\text{smooth}}(D)$$

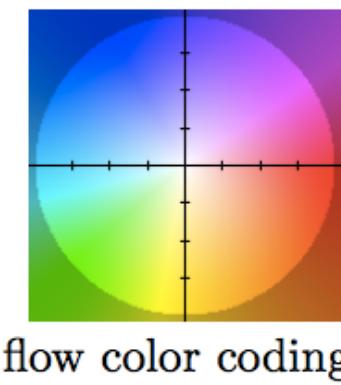
Stereo matching as energy minimization

- Note: the above formulation does not treat the two images symmetrically, does not enforce uniqueness, and does not take occlusions into account
- It is possible to come up with an energy that does all these things, but it's a bit more complex
 - Defined over all possible sets of matches, not over all disparity maps with respect to the first image
 - Includes an *occlusion term*
 - The smoothness term looks different and more complicated

Optical flow estimation for stereo



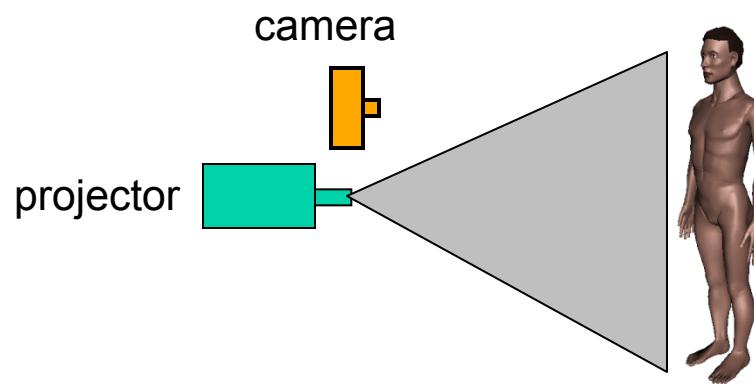
Source: <http://people.csail.mit.edu/celiu/OpticalFlow/>



Active stereo with structured light



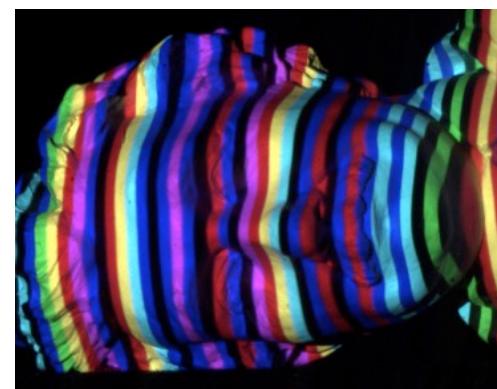
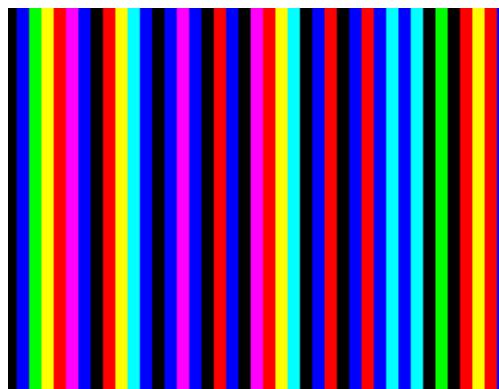
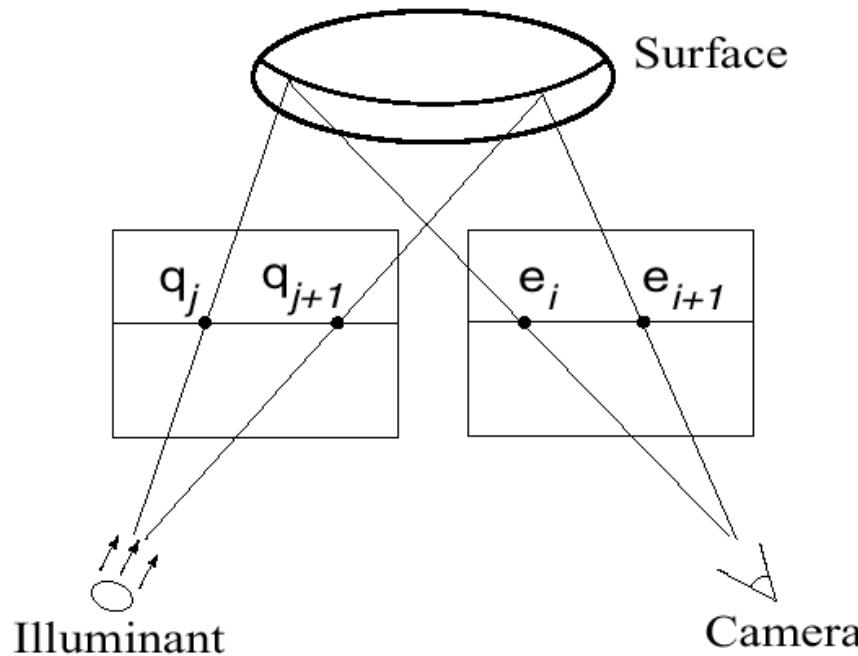
- Project “structured” light patterns onto the object
 - Simplifies the correspondence problem
 - Allows us to use only one camera



L. Zhang, B. Curless, and S. M. Seitz.

[Rapid Shape Acquisition Using Color Structured Light and Multi-pass Dynamic Programming. 3DPVT 2002](#)

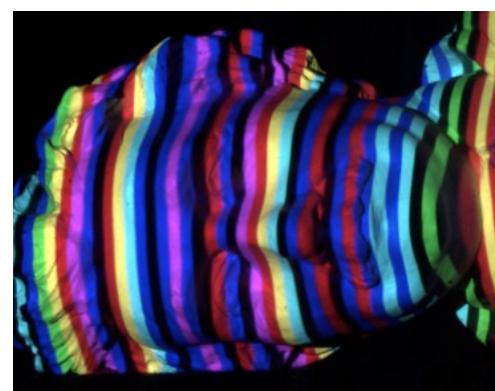
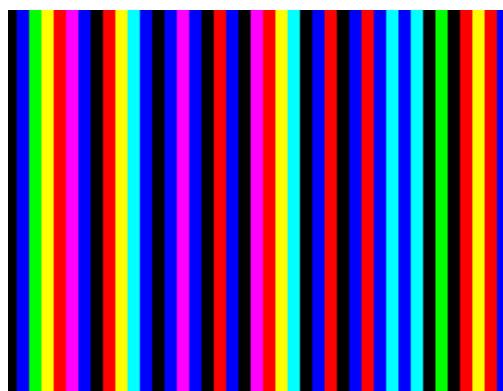
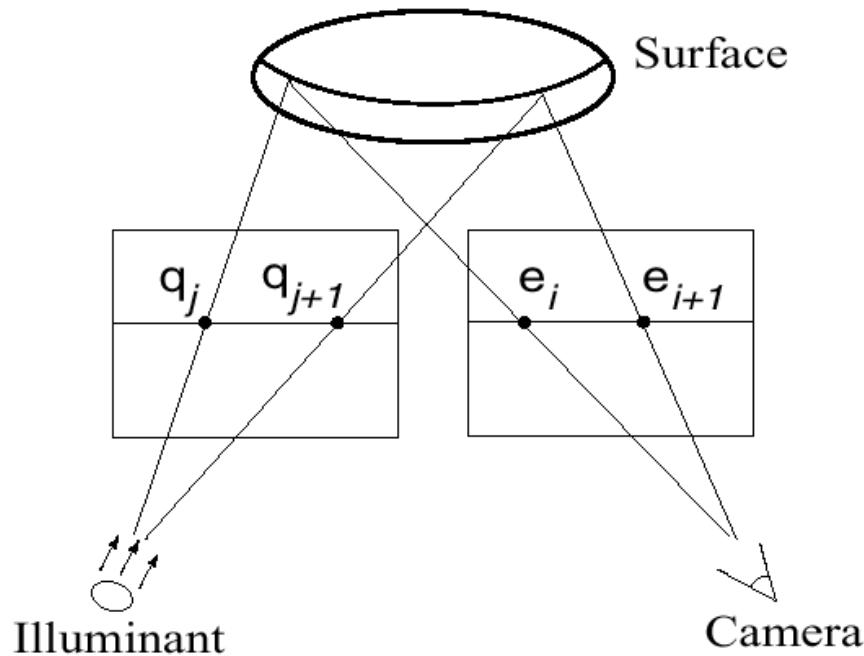
Active stereo with structured light



L. Zhang, B. Curless, and S. M. Seitz.

[Rapid Shape Acquisition Using Color Structured Light and Multi-pass Dynamic Programming.](#) 3DPVT 2002

Active stereo with structured light



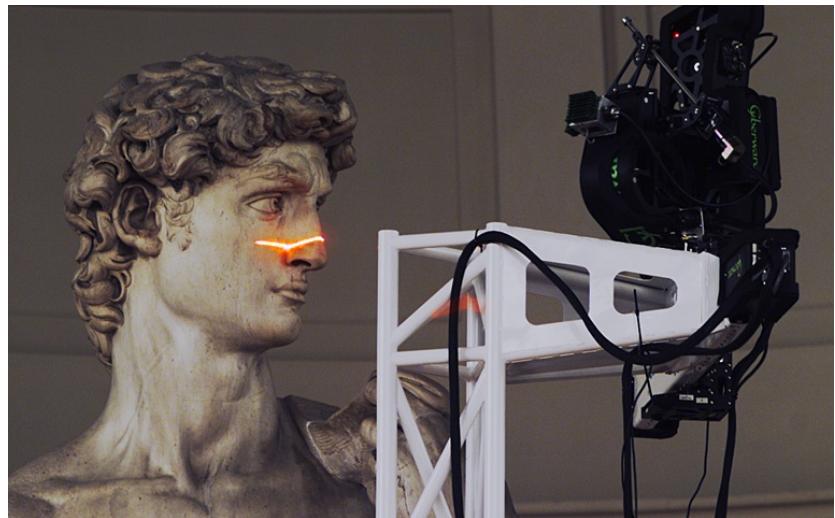
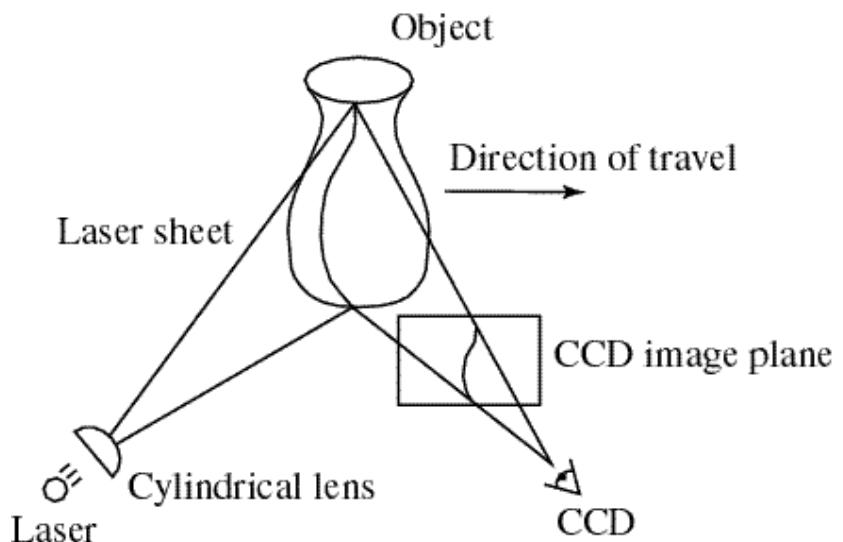
http://en.wikipedia.org/wiki/Structured-light_3D_scanner

Kinect: Structured infrared light



<http://bbzippo.wordpress.com/2010/11/28/kinect-in-infrared/>

Laser scanning



Digital Michelangelo Project
Levoy et al.

<http://graphics.stanford.edu/projects/mich/>

Optical triangulation

- Project a single stripe of laser light
- Scan it across the surface of the object
- This is a very precise version of structured light scanning

Laser scanned models



The Digital Michelangelo Project, Levoy et al.

Source: S. Seitz

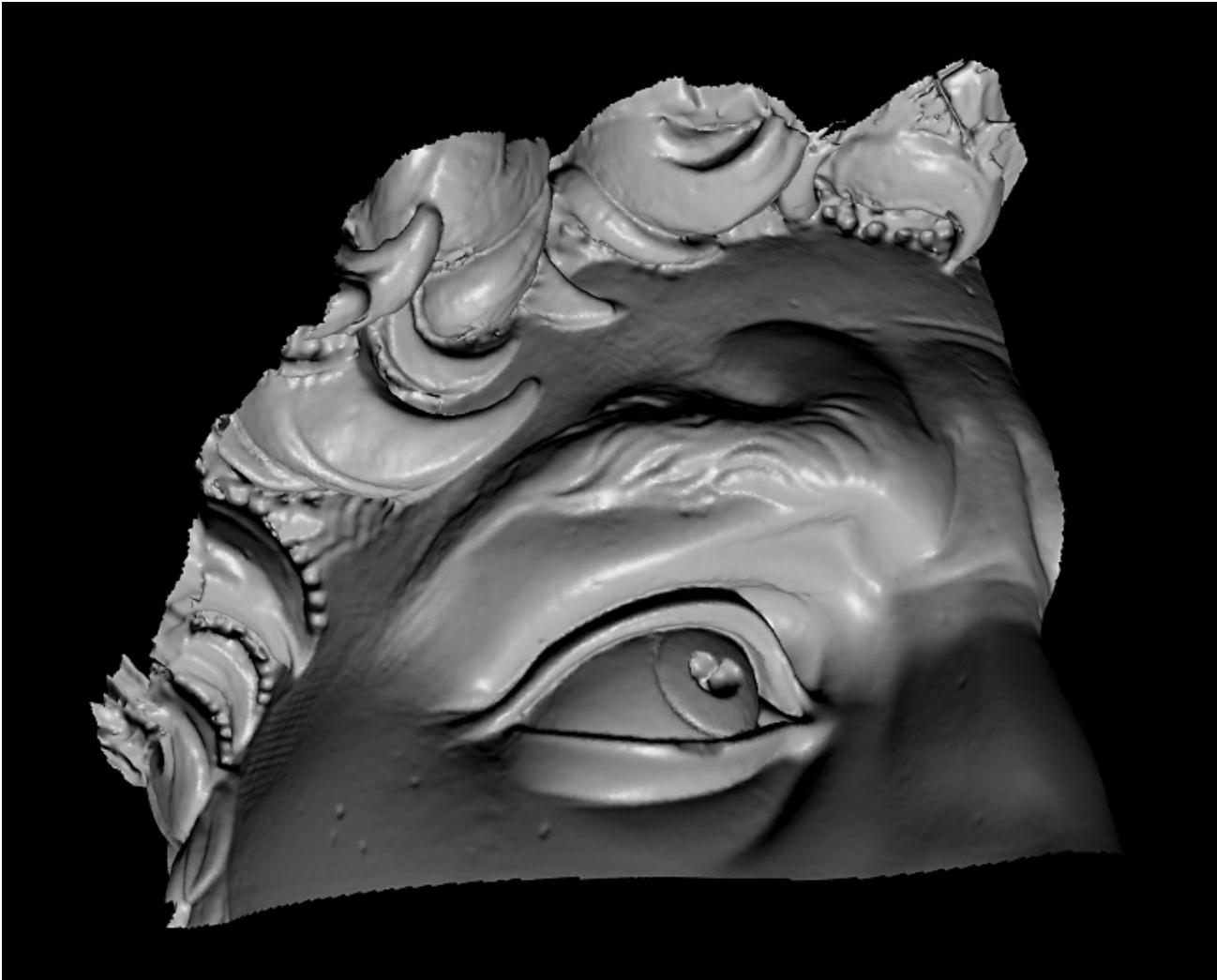
Laser scanned models



The Digital Michelangelo Project, Levoy et al.

Source: S. Seitz

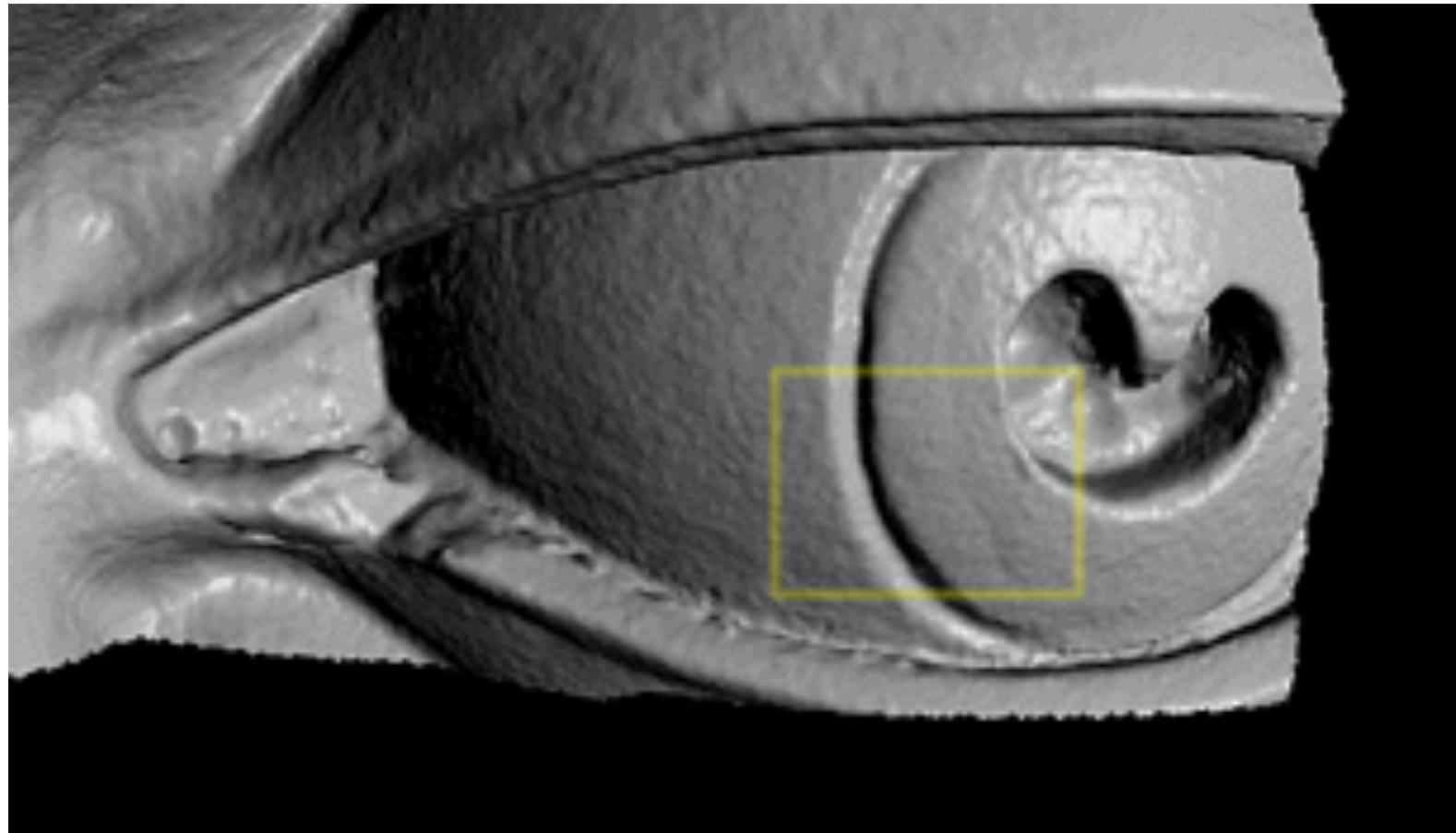
Laser scanned models



The Digital Michelangelo Project, Levoy et al.

Source: S. Seitz

Laser scanned models

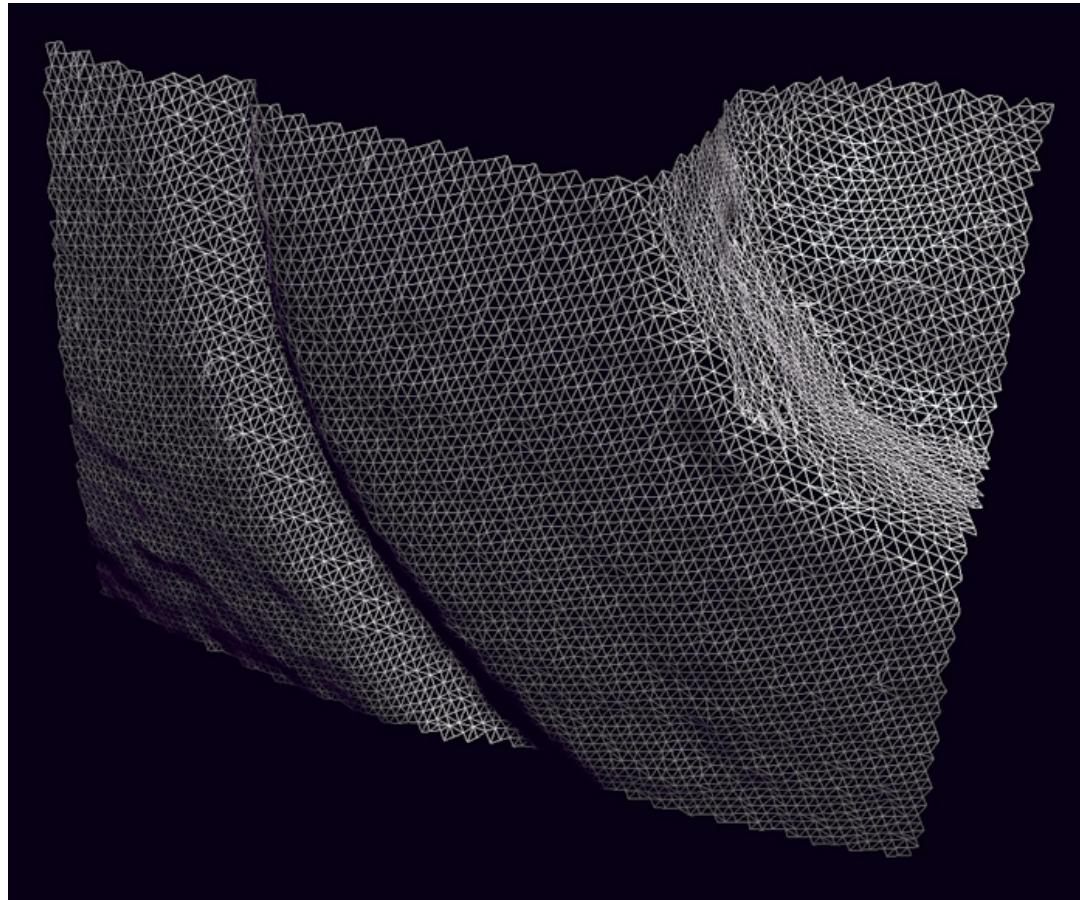


The Digital Michelangelo Project, Levoy et al.

Source: S. Seitz

Laser scanned models

1.0 mm resolution (56 million triangles)



The Digital Michelangelo Project, Levoy et al.

Source: S. Seitz

Aligning range images

- A single range scan is not sufficient to describe a complex surface
- Need techniques to register multiple range images



B. Curless and M. Levoy,
[A Volumetric Method for Building Complex Models from Range Images](#), SIGGRAPH
1996

Aligning range images

- A single range scan is not sufficient to describe a complex surface
- Need techniques to register multiple range images
 - ... which brings us to *multi-view stereo*