40.004 Statistics 2018: **Problem set 2**

`due Monday, 12 March, 2018 at 11:59 pm. Submit on` ***e-dimension***.

1. (A twist on the German Tank Problem). Suppose that the enemy has tanks numbered $0, 1, 2, \ldots, N$. You observe n of the tanks *with* replacement at random and note down their numbers. Using the sample mean of these numbers, find an unbiased estimator for the total number of tanks (with justification for your claim).

2. Let S^2 denote the sample variance computed from a random sample of size n from a $\mathcal{N}(\mu, \sigma^2)$ distribution. Find the probability that the sample variance S^2 exceeds the true variance σ^2 by a factor of two, i.e., $\Pr(S^2 > 2\sigma^2)$ when $n = 8, 17, 21$. Comment on your results. You may use R or Excel or a standard table in the book to find the probabilities.

3. A random sample of size 100, drawn from a normal distribution, has sample mean $\overline{x} = 16.3$.
   (a) Calculate the 95% two-sided confidence interval for μ, if $\sigma = 6$.
   (b) Calculate the 95% two-sided confidence interval for μ, if $s = 6$ and σ is unknown.
   (c) Calculate the upper and lower 95% one-sided confidence intervals for μ, if $s = 6$ and σ is unknown.
   (d) Why is the confidence interval in (b) wider than the CI in (a)?

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution with mean μ and variance σ^2. Show that $\mathbb{E}(\overline{X}^2)$ is a biased estimator of μ^2.

5. In each of the following cases, state the two competing hypotheses that should be tested and specify which would you set up as the null hypothesis and which one as the alternative hypothesis. Explain your choice briefly.
   (a) A consumer watchdog group suspects that a yogurt advertised to be 98% fat free has actually a higher fat content. The group plans to measure the fat contents of 25 yogurt cups (each containing 170 grams) to verify its suspicion.
   (b) It is claimed that cloud seeding is an effective technique technique to increase precipitation.

6. Consider testing $\mathbf{H}_0 : \mu = 0$ vs $\mathbf{H}_\mathrm{A} : \mu \neq 0$ based on a random sample of size n from a $\mathcal{N}(\mu, 1)$ distribution.
   (a) Calculate the p-values for the following three cases:
   (i) $\overline{x} = 0.1$, $n = 100$; (ii) $\overline{x} = 0.1$, $n = 400$; (iii) $\overline{x} = 0.1$, $n = 900$.
   (b) Given the significance level $\alpha = 0.01$, conduct the hypothesis tests for the three cases in (a).

7. A tire company has developed a new tread design. To determine the newly designed tire has a mean of 60,000 miles or more, a random sample of 16 prototype tires are tested. The mean life for this sample is 60,758 miles. Assume that the tire life is normally distributed with unknown μ and standard deviation $\sigma = 1500$ miles. Test the hypothesis $\mathbf{H}_0 : \mu = 60,000$ vs. $\mathbf{H}_\mathrm{A} : \mu > 60,000$.
   (a) Compute the test statistic and the p-value. Based on the p-value, state whether $\mathbf{H}_0$ can be rejected at $\alpha = 0.01$.
   (b) What is the power of the 0.01-level test in (a) if the true mean life for the new tread design is $61,000$ miles?
   (c) Suppose that at least 90% power is needed to identify a tread design that has the mean life of $61,000$ miles. How many tires should be tested?

8. Two methods of measuring the atomic weight of carbon (the nominal atomic weight is 12) yielded the following results.

Method 1	12.0129	12.0072	12.0064	12.0054	12.0016
	11.9853	11.9949	11.9985	12.0077	12.0061
Method 2	12.0318	12.0246	12.0069	12.0006	12.0075

   (a) Test $\mathbf{H}_0 : \mu_1 = \mu_2$ vs. $\mathbf{H}_\mathrm{A} : \mu_1 \neq \mu_2$ at $\alpha = 0.05$, assuming $\sigma_1^2 = \sigma_2^2$. What is your conclusion?

   (b) Repeat (a) without assuming $\sigma_1^2 = \sigma_2^2$. Compare the results.

9. A restaurant purchased a new oven, which is hoped to have more even heating than the old oven. By testing 9 locations inside each oven on the same temperature setting, it is found that the sample standard deviation for the temperature in the old oven is $s_1 = 2.3$, while that for the new oven is $s_2 = 1.1$. Set up a hypothesis test with $\alpha = 0.05$ to check whether the new oven indeed provides more even heating.

10. A person claims to be able to taste whether tea or milk was added first to a cup of English tea. To test her claim, 12 cups of visually indistinguishable tea are prepared, of which 6 of the cups are prepared tea-first, the other 6 milk-first. Being aware of this experimental setup, she would always try to pick 6 of the cups as tea-first, and the other 6 as milk-first. After tasting each cup of tea, she correctly identifies 5 of the tea-first cups (making 1 mistake), and 5 of the milk-first cups (also making 1 mistake).

    Compute the p-value, that is, the probability that one can do at least as well as her by guessing, and hence perform a hypothesis test at the $\alpha = 0.05$ level.