## Section 5.1 Slope Fields and Euler's Method

**1.** Differential equation: $y' = 5y$

Solution: $y = Ce^{5x}$

Check: $y' = 5Ce^{5x} = 5y$

**2.** Differential equation: $3y' + 5y = -e^{-2x}$

Solution: $y = e^{-2x}$

$y' = -2e^{-2x}$

Check: $3\left(-2e^{-2x}\right) + 5\left(e^{-2x}\right) = -e^{-2x}$

**3.** Differential equation: $y' = \dfrac{2xy}{x^2 - y^2}$

Solution: $x^2 + y^2 = Cy$

Check: $2x + 2yy' = Cy'$

$$y' = \frac{-2x}{(2y - C)}$$

$$y' = \frac{-2xy}{2y^2 - Cy}$$

$$= \frac{-2xy}{2y^2 - \left(x^2 + y^2\right)}$$

$$= \frac{-2xy}{y^2 - x^2}$$

$$= \frac{2xy}{x^2 - y^2}$$

**4.** Differential equation: $\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$

Solution: $y^2 - 2\ln y = x^2$

Check: $2yy' - \dfrac{2}{y}y' = 2x$

$$\left(y - \frac{1}{y}\right)y' = x$$

$$y' = \frac{x}{y - \frac{1}{y}}$$

$$y' = \frac{xy}{y^2 - 1}$$

**5.** Differential equation: $y'' + y = 0$

Solution: $y = C_1 \sin x - C_2 \cos x$

$y' = C_1 \cos x + C_2 \sin x$

$y'' = -C_1 \sin x + C_2 \cos x$

Check: $y'' + y = \left(-C_1 \sin x + C_2 \cos x\right) + \left(C_1 \sin x - C_2 \cos x\right) = 0$

**6.** Differential equation: $y'' + 2y' + 2y = 0$

Solution: $y = C_1 e^{-x} \cos x + C_2 e^{-x} \sin x$

Check:

$$y' = -(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x$$

$$y'' = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x$$

$$y'' + 2y' + 2y = 2C_1 e^{-x} \sin x - 2C_2 e^{-x} \cos x +$$

$$2\left(-(C_1 + C_2)e^{-x} \sin x + (-C_1 + C_2)e^{-x} \cos x\right) + 2\left(C_1 e^{-x} \cos x + C_2 e^{-x} \sin x\right)$$

$$= (2C_1 - 2C_1 - 2C_2 + 2C_2)e^{-x} \sin x + (-2C_2 - 2C_1 + 2C_2 + 2C_1)e^{-x} \cos x = 0$$

7. Differential equation: $y'' + y = \tan x$

Solution: $y = -\cos x \ln|\sec x + \tan x|$

$$y' = (-\cos x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \sin x \ln|\sec x + \tan x|$$

$$= \frac{(-\cos x)}{\sec x + \tan x}(\sec x)(\tan x + \sec x) + \sin x \ln|\sec x + \tan x|$$

$$= -1 + \sin x \ln|\sec x + \tan x|$$

$$y'' = (\sin x)\frac{1}{\sec x + \tan x}(\sec x \cdot \tan x + \sec^2 x) + \cos x \ln|\sec x + \tan x|$$

$$= (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x|$$

Check: $y'' + y = (\sin x)(\sec x) + \cos x \ln|\sec x + \tan x| - \cos x \ln|\sec x + \tan x| = \tan x.$

8. Differential equation: $y'' + 4y' = 2e^x$

Solution: $y = \frac{2}{5}(e^{-4x} + e^x)$

$$y' = \frac{2}{5}(-4e^{-4x} + e^x) = -\frac{8}{5}e^{-4x} + \frac{2}{5}e^x$$

$$y'' = \frac{32}{5}e^{-4x} + \frac{2}{5}e^x$$

Check: $y'' + 4y' = \left(\frac{32}{5}e^{-4x} + \frac{2}{5}e^x\right) + 4\left(-\frac{8}{5}e^{-4x} + \frac{2}{5}e^x\right) = \left(\frac{2}{5} + \frac{8}{5}\right)e^x = 2e^x$

9. $y = \sin x \cos x - \cos^2 x$

$y' = -\sin^2 x + \cos^2 x + 2\cos x \sin x$

$= -1 + 2\cos^2 x + \sin 2x$

Differential equation:

$2y + y' = 2(\sin x \cos x - \cos^2 x) + (-1 + 2\cos^2 x + \sin 2x)$

$= 2\sin x \cos x - 1 + \sin 2x$

$= 2\sin 2x - 1$

Initial condition $\left(\frac{\pi}{4}, 0\right)$:

$$\sin\frac{\pi}{4}\cos\frac{\pi}{4} - \cos^2\frac{\pi}{4} = \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \left(\frac{\sqrt{2}}{2}\right)^2 = 0$$

10. $y = 6x - 4\sin x + 1$

$y' = 6 - 4\cos x$

Differential equation: $y' = 6 - 4\cos x$

Initial condition $(0, 1)$: $0 - 0 + 1 = 1$

11. $y = 4e^{-6x^2}$

$y' = 4e^{-6x^2}(-12x) = -48xe^{-6x^2}$

Differential equation:

$y' = -12xy = -12x\left(4e^{-6x^2}\right) = -48xe^{-6x^2}$

Initial condition $(0, 4)$: $4e^0 = 4$

12. $y = e^{-\cos x}$

$y' = e^{-\cos x}(\sin x) = \sin x \cdot e^{-\cos x}$

Differential equation:

$y' = \sin x \cdot e^{-\cos x} = \sin x(y) = y \sin x$

Initial condition $\left(\frac{\pi}{2}, 1\right)$: $e^{-\cos(\pi/2)} = e^0 = 1$

**In Exercises 13–18, the differential equation is $y^{(4)} - 16y = 0$.**

**13.** $y = 3\cos x$

$y^{(4)} = 3\cos x$

$y^{(4)} - 16y = -45\cos x \neq 0,$

No

**14.** $y = 3\sin 2x$

$y^{(4)} = 48\sin 2x$

$y^{(4)} - 16y = 48\sin 2x - 16(3\sin 2x) = 0$

Yes

**17.** $y = C_1e^{2x} + C_2e^{-2x} + C_3\sin 2x + C_4\cos 2x$

$y^{(4)} = 16C_1e^{2x} + 16C_2e^{-2x} + 16C_3\sin 2x + 16C_4\cos 2x$

$y^{(4)} - 16y = 0,$

Yes

**18.** $y = 3e^{2x} - 4\sin 2x$

$y^{(4)} = 48e^{2x} - 64\sin 2x$

$y^{(4)} - 16y = (48e^{2x} - 64\sin 2x) - 16(3e^{2x} - 4\sin 2x) = 0,$

Yes

**15.** $y = e^{-2x}$

$y^{(4)} = 16e^{-2x}$

$y^{(4)} - 16y = 16e^{-2x} - 16e^{-2x} = 0,$

Yes

**16.** $y = 5\ln x$

$y^{(4)} = -\dfrac{30}{x^4}$

$y^{(4)} - 16y = -\dfrac{30}{x^4} - 80\ln x \neq 0,$

No

**In Exercises 19–24, the differential equation is $xy' - 2y = x^3e^x$.**

**19.** $y = x^2, y' = 2x$

$xy' - 2y = x(2x) - 2(x^2) = 0 \neq x^3e^x,$

No

**20.** $y = \cos x, y' = -\sin x$

$xy' - 2y = x(-\sin x) - 2\cos x \neq x^3e^x$

No

**21.** $y = x^2e^x, y' = x^2e^x + 2xe^x = e^x(x^2 + 2x)$

$xy' - 2y = x(e^x(x^2 + 2x)) - 2(x^2e^x) = x^3e^x,$

Yes

**22.** $y = x^2(2 + e^x), y' = x^2(e^x) + 2x(2 + e^x)$

$xy' - 2y = x[x^2e^x + 2xe^x + 4x] - 2[x^2e^x + 2x^2]$

$= x^3e^x,$

Yes

**23.** $y = \ln x, y' = \dfrac{1}{x}$

$xy' - 2y = x\left(\dfrac{1}{x}\right) - 2\ln x \neq x^3e^x,$

No

**24.** $y = x^2e^x - 5x^2, y' = x^2e^x + 2xe^x - 10x$

$xy' - 2y = x[x^2e^x + 2xe^x - 10x] - 2[x^2e^x - 5x^2]$

$= x^3e^x,$

Yes

**25.** $y = Ce^{-x/2}$ passes through $(0, 3)$.

$3 = Ce^0 = C \Rightarrow C = 3$

Particular solution: $y = 3e^{-x/2}$

**26.** $2x^2 - y^2 = C$ passes through $(3, 4)$.

$2(9) - 16 = C \Rightarrow C = 2$

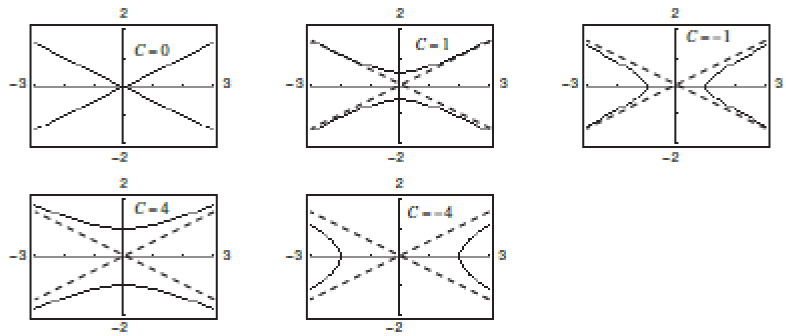
Particular solution: $2x^2 - y^2 = 2$

**27.** Differential equation: $4yy' - x = 0$

General solution: $4y^2 - x^2 = C$

Particular solutions: $C = 0$, Two intersecting lines

$C = \pm 1, C = \pm 4$, Hyperbolas



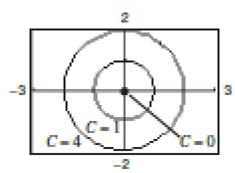

**28.** Differential equation: $yy' + x = 0$

General solution: $x^2 + y^2 = C$

Particular solutions:

$C = 0$, Point

$C = 1, C = 4$, Circles




**29.** Differential equation: $y' + 2y = 0$

General solution: $y = Ce^{-2x}$

$y' + 2y = C(-2)e^{-2x} + 2(Ce^{-2x}) = 0$

Initial condition $(0, 3)$: $3 = Ce^0 = C$

Particular solution: $y = 3e^{-2x}$

**30.** Differential equation: $3x + 2yy' = 0$

General solution: $3x^2 + 2y^2 = C$

$$6x + 4yy' = 0$$
$$2(3x + 2yy') = 0$$
$$3x + 2yy' = 0$$

Initial condition $(1, 3)$:

$3(1)^2 + 2(3)^2 = 3 + 18 = 21 = C$

Particular solution: $3x^2 + 2y^2 = 21$

**31.** Differential equation: $y'' + 9y = 0$

General solution: $y = C_1 \sin 3x + C_2 \cos 3x$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x,$$
$$y'' = -9C_1 \sin 3x - 9C_2 \cos 3x$$
$$y'' + 9y = (-9C_1 \sin 3x - 9C_2 \cos 3x) + 9(C_1 \sin 3x + C_2 \cos 3x) = 0$$

Initial conditions $\left(\frac{\pi}{6}, 2\right)$ and $y' = 1$ when $x = \frac{\pi}{6}$:

$$2 = C_1 \sin\left(\frac{\pi}{2}\right) + C_2 \cos\left(\frac{\pi}{2}\right) \Rightarrow C_1 = 2$$

$$y' = 3C_1 \cos 3x - 3C_2 \sin 3x$$

$$1 = 3C_1 \cos\left(\frac{\pi}{2}\right) - 3C_2 \sin\left(\frac{\pi}{2}\right) = -3C_2 \Rightarrow C_2 = -\frac{1}{3}$$

Particular solution: $y = 2 \sin 3x - \frac{1}{3} \cos 3x$

**32.** Differential equation: $xy'' + y' = 0$

General solution: $y = C_1 + C_2 \ln x$

$y' = C_2\left(\frac{1}{x}\right), y'' = -C_2\left(\frac{1}{x^2}\right)$

$xy'' + y' = x\left(-C_2\frac{1}{x^2}\right) + C_2\frac{1}{x} = 0$

Initial conditions $(2, 0)$ and $y' = \frac{1}{2}$ when $x = 2$:

$0 = C_1 + C_2 \ln 2$

$y' = \frac{C_2}{x}$

$\frac{1}{2} = \frac{C_2}{2} \Rightarrow C_2 = 1, C_1 = -\ln 2$

Particular solution: $y = -\ln 2 + \ln x = \ln\frac{x}{2}$

**33.** Differential equation: $x^2y'' - 3xy' + 3y = 0$

General solution: $y = C_1x + C_2x^3$

$y' = C_1 + 3C_2x^2, y'' = 6C_2x$

$x^2y'' - 3xy' + 3y = x^2(6C_2x) - 3x\left(C_1 + 3C_2x^2\right) + 3\left(C_1x + C_2x^3\right) = 0$

Initial conditions $(2, 0)$ and $y' = 4$ when $x = 2$:

$0 = 2C_1 + 8C_2$

$y' = C_1 + 3C_2x^2$

$4 = C_1 + 12C_2$

$\left.\begin{aligned} C_1 + 4C_2 &= 0 \\ C_1 + 12C_2 &= 4 \end{aligned}\right\} C_2 = \frac{1}{2}, C_1 = -2$

Particular solution: $y = -2x + \frac{1}{2}x^3$

**34.** Differential equation: $9y'' - 12y' + 4y = 0$

General solution: $y = e^{2x/3}(C_1 + C_2x)$

$y' = \frac{2}{3}e^{2x/3}(C_1 + C_2x) + C_2e^{2x/3} = e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right)$

$y'' = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + e^{2x/3}\frac{2}{3}C_2 = \frac{2}{3}e^{2x/3}\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right)$

$9y'' - 12y' + 4y = 9\left(\frac{2}{3}e^{2x/3}\right)\left(\frac{2}{3}C_1 + 2C_2 + \frac{2}{3}C_2x\right) - 12\left(e^{2x/3}\right)\left(\frac{2}{3}C_1 + C_2 + \frac{2}{3}C_2x\right) + 4\left(e^{2x/3}\right)(C_1 + C_2x) = 0$

Initial conditions $(0, 4)$ and $(3, 0)$:

$0 = e^2(C_1 + 3C_2)$

$4 = (1)(C_1 + 0) \Rightarrow C_1 = 4$

$0 = e^2(4 + 3C_2) \Rightarrow C_2 = -\frac{4}{3}$



Particular solution: $y = e^{2x/3}\left(4 - \frac{4}{3}x\right)$

35. $\dfrac{dy}{dx} = 12x^2$

$$y = \int 12x^2\,dx = 4x^3 + C$$

36. $\dfrac{dy}{dx} = 10x^4 - 2x^3$

$$y = \int \left(10x^4 - 2x^3\right) dx = 2x^5 - \frac{x^4}{2} + C$$

37. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$

$$y = \int \frac{x}{1 + x^2}\,dx = \frac{1}{2}\ln\left(1 + x^2\right) + C$$

$\left(u = 1 + x^2, du = 2x\,dx\right)$

38. $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

$$y = \int \frac{e^x}{4 + e^x}\,dx = \ln\left(4 + e^x\right) + C$$

39. $\dfrac{dy}{dx} = \dfrac{x - 2}{x} = 1 - \dfrac{2}{x}$

$$y = \int\left(1 - \frac{2}{x}\right)dx$$

$$= x - 2\ln|x| + C = x - \ln x^2 + C$$

40. $\dfrac{dy}{dx} = x\cos x^2$

$$y = \int x\cos\left(x^2\right)dx = \frac{1}{2}\sin\left(x^2\right) + C$$

$\left(u = x^2, du = 2x\,dx\right)$

41. $\dfrac{dy}{dx} = \sin 2x$

$$y = \int \sin 2x\,dx = -\frac{1}{2}\cos 2x + C$$

$\left(u = 2x, du = 2\,dx\right)$

42. $\dfrac{dy}{dx} = \tan^2 x = \sec^2 x - 1$

$$y = \int\left(\sec^2 x - 1\right)dx = \tan x - x + C$$

43. $\dfrac{dy}{dx} = x\sqrt{x - 6}$

Let $u = \sqrt{x - 6}$, then $x = u^2 + 6$ and $dx = 2u\,du$.

$$\begin{aligned} y = \int x\sqrt{x - 6}\,dx &= \int\left(u^2 + 6\right)(u)(2u)\,du \\ &= 2\int\left(u^4 + 6u^2\right)du \\ &= 2\left(\frac{u^5}{5} + 2u^3\right) + C \\ &= \frac{2}{5}(x - 6)^{5/2} + 4(x - 6)^{3/2} + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x - 6 + 10) + C \\ &= \frac{2}{5}(x - 6)^{3/2}(x + 4) + C \end{aligned}$$

44. $\dfrac{dy}{dx} = 2x\sqrt{4x^2 + 1}$

$$\begin{aligned} y &= \int 2x\sqrt{4x^2 + 1}\,dx \\ &= \frac{1}{4}\int\sqrt{4x^2 + 1}\,(8x)\,dx \\ &= \frac{1}{4}\,\frac{\left(4x^2 + 1\right)^{3/2}}{(3/2)} + C \\ &= \frac{1}{6}\left(4x^2 + 1\right)^{3/2} + C \end{aligned}$$

45. $\dfrac{dy}{dx} = xe^{x^2}$

$$y = \int xe^{x^2}dx = \frac{1}{2}e^{x^2} + C$$

$\left(u = x^2, du = 2x\,dx\right)$

46. $\dfrac{dy}{dx} = 5e^{-x/2}$

$$y = \int 5e^{-x/2}\,dx = 5(-2)\int e^{-x/2}\left(-\frac{1}{2}\right)dx = -10e^{-x/2} + C$$

47.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $8$ |
| $y$ | $2$ | $0$ | $4$ | $4$ | $6$ | $8$ |
| $dy/dx$ | $-4$ | Undef. | $0$ | $1$ | $\frac{4}{3}$ | $2$ |

48.

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $8$ |
| $y$ | $2$ | $0$ | $4$ | $4$ | $6$ | $8$ |
| $dy/dx$ | $6$ | $2$ | $4$ | $2$ | $2$ | $0$ |

**49.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $8$ |
| $y$ | $2$ | $0$ | $4$ | $4$ | $6$ | $8$ |
| $dy/dx$ | $-2\sqrt{2}$ | $-2$ | $0$ | $0$ | $-2\sqrt{2}$ | $-8$ |

**50.**

| | | | | | | |
|---|---|---|---|---|---|---|
| $x$ | $-4$ | $-2$ | $0$ | $2$ | $4$ | $8$ |
| $y$ | $2$ | $0$ | $4$ | $4$ | $6$ | $8$ |
| $dy/dx$ | $\sqrt{3}$ | $0$ | $-\sqrt{3}$ | $-\sqrt{3}$ | $0$ | $\sqrt{3}$ |

**51.** $\dfrac{dy}{dx} = \sin 2x$

For $x = 0, \dfrac{dy}{dx} = 0$. Matches (b).

**52.** $\dfrac{dy}{dx} = \dfrac{1}{2}\cos x$

For $x = 0, \dfrac{dy}{dx} = \dfrac{1}{2}$. Matches (c).

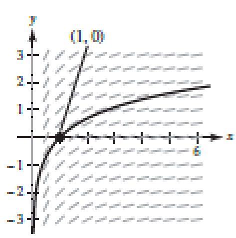
**53.** $\dfrac{dy}{dx} = e^{-2x}$

As $x \to \infty, \dfrac{dy}{dx} \to 0$. Matches (d).

**54.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

For $x = 0, \dfrac{dy}{dx}$ is undefined (vertical tangent). Matches (a).

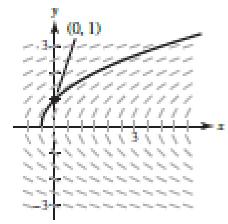
**55.** $y' = \dfrac{1}{x}, (1, 0)$




As $x \to \infty, y \to \infty$

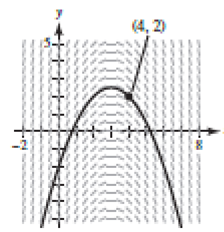
[Note: The solution is $y = \ln x$.]

**56.** $y' = \dfrac{1}{y}, (0, 1)$




As $x \to \infty, y \to \infty$

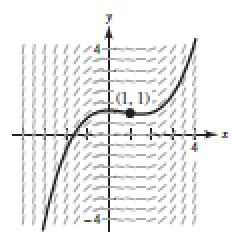
**57.** (a), (b)




(c) As $x \to \infty, y \to -\infty$

As $x \to -\infty, y \to -\infty$

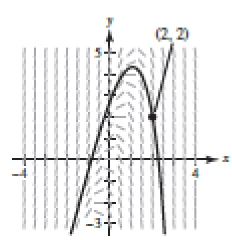
**58.** (a), (b)




(c) As $x \to \infty, y \to \infty$

As $x \to -\infty, y \to -\infty$

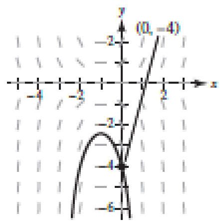
**59.** (a), (b)




(c) As $x \to \infty, y \to -\infty$

As $x \to -\infty, y \to -\infty$

**60.** (a), (b)

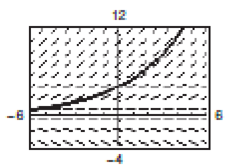



(c) As $x \to \infty$, $y \to -\infty$

As $x \to -\infty$, $y \to -\infty$

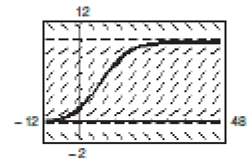
**61.** $\dfrac{dy}{dx} = 0.25y,\ y(0) = 4$

(a), (b)



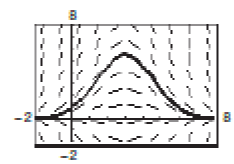

**62.** $\dfrac{dy}{dx} = 0.02y(10 - y),\ y(0) = 2$

(a), (b)



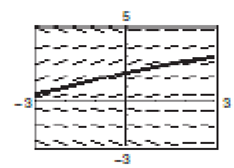

**63.** $\dfrac{dy}{dx} = 0.4y(3 - x),\ y(0) = 1$

(a), (b)




**64.** $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8}\sin\dfrac{\pi y}{4},\ y(0) = 2$

(a), (b)




**65.** $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.1)(0 + 2) = 2.2$

$y_2 = y_1 + hF(x_1, y_1) = 2.2 + (0.1)(0.1 + 2.2) = 2.43$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 2 | 2.2 | 2.43 | 2.693 | 2.992 | 3.332 | 3.715 | 4.146 | 4.631 | 5.174 | 5.781 |

**66.** $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 2 + (0.05)(0 + 2) = 2.1$

$y_2 = y_1 + hF(x_1, y_1) = 2.1 + (0.05)(0.05 + 2.1) = 2.2075$, etc.

The table shows the values for $n = 0, 2, 4, \ldots, 20$.

| $n$ | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 9 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 2 | 2.208 | 2.447 | 2.720 | 3.032 | 3.387 | 3.788 | 4.240 | 4.749 | 5.320 | 5.960 |

**67.** $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

$y_1 = y_0 + hF(x_0, y_0) = 3 + (0.05)(3(0) - 2(3)) = 2.7$

$y_2 = y_1 + hF(x_1, y_1) = 2.7 + (0.05)(3(0.05) + 2(2.7)) = 2.4375$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 | 0.35 | 0.4 | 0.45 | 0.5 |
| $y_n$ | 3 | 2.7 | 2.438 | 2.209 | 2.010 | 1.839 | 1.693 | 1.569 | 1.464 | 1.378 | 1.308 |

**68.** $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.4)(0.5(0)(3 - 1)) = 1$

$y_2 = y_1 + hF(x_1, y_1) = 1 + (0.4)(0.5(0.4)(3 - 1)) = 1.16$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.4 | 0.8 | 1.2 | 1.6 | 2.0 |
| $y_n$ | 1 | 1 | 1.16 | 1.454 | 1.825 | 2.201 |

**69.** $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 1 + (0.1)e^{0(1)} = 1.1$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + (0.1)e^{(0.1)(1.1)} \approx 1.2116$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 1 | 1.1 | 1.212 | 1.339 | 1.488 | 1.670 | 1.900 | 2.213 | 2.684 | 3.540 | 5.958 |

**70.** $y' = \cos x + \sin y, y(0) = 5, n = 10, h = 0.1$

$y_1 = y_0 + hF(x_0, y_0) = 5 + (0.1)(\cos 0 + \sin 5) \approx 5.0041$

$y_2 = y_1 + hF(x_1, y_1) = 5.0041 + (0.1)(\cos(0.1) + \sin(5.0041)) \approx 5.0078$, etc.

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $x_n$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 | 1.0 |
| $y_n$ | 5 | 5.004 | 5.008 | 5.010 | 5.010 | 5.007 | 4.999 | 4.985 | 4.965 | 4.938 | 4.903 |

**71.** $\frac{dy}{dx} = y, y = 3e^x, (0, 3)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 3 | 3.6642 | 4.4755 | 5.4664 | 6.6766 | 8.1548 |
| $y(x)$ $(h = 0.2)$ | 3 | 3.6000 | 4.3200 | 5.1840 | 6.2208 | 7.4650 |
| $y(x)$ $(h = 0.1)$ | 3 | 3.6300 | 4.3923 | 5.3147 | 6.4308 | 7.7812 |

**72.** $\dfrac{dy}{dx} = \dfrac{2x}{y}, y = \sqrt{2x^2 + 4}, \quad (0, 2)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 2 | 2.0199 | 2.0785 | 2.1726 | 2.2978 | 2.4495 |
| $y(x)$ $(h = 0.2)$ | 2 | 2.000 | 2.0400 | 2.1184 | 2.2317 | 2.3751 |
| $y(x)$ $(h = 0.1)$ | 2 | 2.0100 | 2.0595 | 2.1460 | 2.2655 | 2.4131 |

**73.** $\dfrac{dy}{dx} = y + \cos x, y = \dfrac{1}{2}(\sin x - \cos x + e^x), \quad (0, 0)$

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y(x)$ (exact) | 0 | 0.2200 | 0.4801 | 0.7807 | 1.1231 | 1.5097 |
| $y(x)$ $(h = 0.2)$ | 0 | 0.2000 | 0.4360 | 0.7074 | 1.0140 | 1.3561 |
| $y(x)$ $(h = 0.1)$ | 0 | 0.2095 | 0.4568 | 0.7418 | 1.0649 | 1.4273 |

**74.** As $h$ increases (from 0.1 to 0.2), the error increases.

**75.** $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.1$

(a)

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Euler | 140 | 112.7 | 96.4 | 86.6 |

(b) $y = 72 + 68e^{-t/2}$ exact

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Exact | 140 | 113.24 | 97.016 | 87.173 |

(c) $\dfrac{dy}{dt} = -\dfrac{1}{2}(y - 72), \quad (0, 140), h = 0.05$

| $t$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| Euler | 140 | 112.98 | 96.7 | 86.9 |

The approximations are better using $h = 0.05$.

**76.** When $x = 0, y' = 0$. Therefore, (d) is not possible.

When $x > 0$ and $y > 0, y' < 0$ (decreasing function). Therefore, (c) is the equation.

**77.** The general solution is a family of curves that satisfies the differential equation. A particular solution is one member of the family that satisfies given conditions.

**78.** A slope field for the differential equation $y' = F(x, y)$ consists of small line segments at various points $(x, y)$ in the plane. The line segment equals the slope $y' = F(x, y)$ of the solution $y$ at the point $(x, y)$.

**79.** Consider $y' = F(x, y), y(x_0) = y_0$. Begin with a point $(x_0, y_0)$ that satisfies the initial condition, $y(x_0) = y_0$. Then, using a step size of $h$, find the point $(x_1, y_1) = (x_0 + h, y_0 + hF(x_0, y_0))$. Continue generating the sequence of points $(x_{n+1}, y_{n+1}) = (x_n + h, y_n + hF(x_n, y_n))$.

**80.** $y = Ce^{kx}$

$\dfrac{dy}{dx} = Cke^{kx}$

Because $dy/dx = 0.07y$, you have $Cke^{kx} = 0.07Ce^{kx}$.

So, $k = 0.07$.

$C$ cannot be determined.

**81.** Because $y = e^{-4x}$ and $y' = -4e^{-4x}$, $y''$ should be $16e^{-4x}$.

$$\begin{aligned} y'' - 3y' + 4y &= 16e^{-4x} - 3(-4e^{-4x}) + 4(e^{-4x}) \\ &= 16e^{-4x} + 12e^{-4x} + 4e^{-4x} \\ &= 32e^{-4x} \\ &\neq 0 \end{aligned}$$

So, $y = e^{-4x}$ is *not* a solution.

82. Because $y = x^3 + 4x + \frac{2}{x}$, $y'$ should be

$3x^2 + 4 - \frac{2}{x^2}$.

$$\begin{aligned} xy' + y &= x\left(3x^2 + 4 - \frac{2}{x^2}\right) + x^3 + 4x + \frac{2}{x} \\ &= 3x^3 + 4x - \frac{2}{x} + x^3 + 4x + \frac{2}{x} \\ &= 4x^3 + 8x \\ &= 4x\left(x^2 + 2\right) \\ &\neq 0 \end{aligned}$$

So, $y = x^3 + 4x + \frac{2}{x}$ is *not* a solution.

83. False. Consider Example 2. $y = x^3$ is a solution to $xy' - 3y = 0$, but $y = x^3 + 1$ is not a solution.

84. True

85. True

86. False. The slope field could represent many different differential equations, such as $y' = 2x + 4y$.

87. $\frac{dy}{dx} = -2y$, $y(0) = 4$, $y = 4e^{-2x}$

(a)

| $x$ | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1 |
|---|---|---|---|---|---|---|
| $y$ | 4 | 2.6813 | 1.7973 | 1.2048 | 0.8076 | 0.5413 |
| $y_1$ | 4 | 2.5600 | 1.6384 | 1.0486 | 0.6711 | 0.4295 |
| $y_2$ | 4 | 2.4000 | 1.4400 | 0.8640 | 0.5184 | 0.3110 |
| $e_1$ | 0 | 0.1213 | 0.1589 | 0.1562 | 0.1365 | 0.1118 |
| $e_2$ | 0 | 0.2813 | 0.3573 | 0.3408 | 0.2892 | 0.2303 |
| $r$ | | 0.4312 | 0.4447 | 0.4583 | 0.4720 | 0.4855 |

(b) If $h$ is halved, then the error is approximately halved $(r \approx 0.5)$.

(c) When $h = 0.05$, the errors will again be approximately halved.

88.
$$\begin{aligned} y &= e^{kt} \\ y' &= ke^{kt} \\ y'' &= k^2e^{kt} \\ y'' - 16y &= 0 \\ k^2e^{kt} - 16e^{kt} &= 0 \\ k^2 - 16 &= 0 \qquad \left(\text{because } e^{kt} \neq 0\right) \\ k &= \pm 4 \end{aligned}$$

89. $y = A \sin \omega t$

$y' = A\omega \cos \omega t$

$y'' = -A\omega^2 \sin \omega t$

$y'' + 16y = 0$

$-A\omega^2 \sin \omega t + 16A \sin \omega t = 0$

$A \sin \omega t\left[16 - \omega^2\right] = 0$

If $A \neq 0$, then $\omega = \pm 4$

90. You can see that the slope when $x = 1$ is 0 and there is no slope when $x = 0$.

Evaluate each equation.

A: When $x = 1$, $\frac{dy}{dx} = \frac{1}{0} \neq 0$.

B: When $x = 0$, $\frac{dy}{dx} = -1$, which is not undefined.

C: When $x = 0$, $\frac{dy}{dx}$ is undefined. When $x = 1$, $\frac{dy}{dx} = 0$. When $x = 5$, $\frac{dy}{dx} \approx 1.609$, which is steeper than the slope at $x = 5$.

D: When $x = 0$, $\frac{dy}{dx}$ is undefined. When $x = 1$, $\frac{dy}{dx} = 0$. When $1.5 \leq x \leq 5$, $\frac{dy}{dx}$ matches the slope field of the graph.

So, the answer is D.

**91.** Using $h = 0.2, x_0 = 0, y_0 = -1$, and $F(x, y) = y - 6x$,

$x_0 = 0, x_1 = 0.2, x_2 = 0.4$, and $x_3 = 0.6$.

The first three approximations are shown.

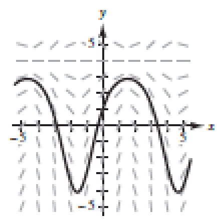
$y_1 = y_0 + hF(x_0, y_0) = -1 + 0.2[-1 - 6(0)] = -1.2$

$y_2 = y_1 + hF(x_1, y_1) = -1.2 + 0.2[-1.2 - 6(0.2)] = -1.68$

$y_3 = y_2 + hF(x_2, y_2) = -1.68 + 0.2[-1.68 - 6(0.4)] = -2.496$

So, the answer is B.

**92.** (a)




(b) At $(\pi, 1)$, $\dfrac{dy}{dx} = (4 - 1)\cos\pi$

$= 3(-1) = -3.$

Using $m = -3$, the equation of the tangent line through $(\pi, 1)$ is

$y - 1 = -3(x - \pi)$

$= -3x + 3\pi + 1.$

So, $f(3.2) = -3(3.2) + 3\pi + 1 \approx 0.825.$