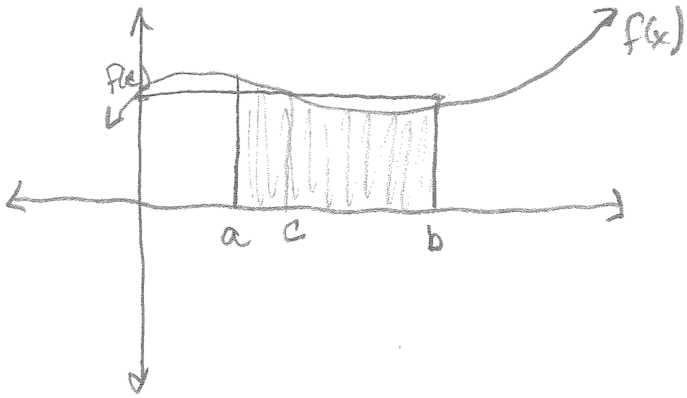
4.4 The Fundamental Theorem of Calculus, Day 2

Mean Value Theorem for Integrals

If f is continuous on the closed interval $[a, b]$, then there exists a number c, in the closed interval $[a, b]$ such that

$$\int_a^b f(x)dx = f(c)(b-a)$$

Let's think through this....



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$$f(c)\,(b-a) = \int_a^b f(x)dx$$

Average Value of a Function

If f is integrable on the closed interval $[a, b]$, then the average value of f on the interval is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Let's think through this.......

$\int_a^b f(x)dx \rightarrow$ accumulation (adding the values)

$\frac{1}{b-a} \rightarrow$ divides how many there were

$$\int_a^b f(x)\,dx = f(c)\,(b-a)$$

$$\therefore f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Find the average value of the function $f(x) = \frac{x^2+1}{x^2}$ on the interval $\left[\frac{1}{2}, 2\right]$.

$$\frac{1}{2-\frac{1}{2}} \int_{\frac{1}{2}}^{2} \frac{x^2+1}{x^2}\, dx = \frac{1}{\frac{3}{2}} \int_{\frac{1}{2}}^{2} 1 + x^{-2}\, dx = \frac{2}{3}\left(x - x^{-1}\right)\Big|_{\frac{1}{2}}^{2}$$

$$\frac{2}{3}\left[\left(2-\frac{1}{2}\right) - \left(\frac{1}{2} - 2\right)\right] = \frac{2}{3}\left[\frac{3}{2} - \left(-\frac{3}{2}\right)\right] = 2$$

Find all values of x in the interval for which the function equals its average value.

$$\frac{x^2+1}{x^2} = 2 \qquad x = \pm 1$$

$$x^2+1 = 2x^2$$

$$1 = x^2$$

$c = 1$

Examples – Average Value

Find the average value of $f(x) = 2x^3 + x$ on the interval $[1,2]$

$$\frac{1}{2-1}\int_1^2 (2x^3 + x)\,dx = \frac{1}{1}\left(2\left(\frac{x^4}{4} + \frac{1}{2}x^2\right)\right)\Big|_1^2 = \frac{1}{2}x^4 + \frac{1}{2}x^2\Big|_1^2$$

$$= \left[\frac{1}{2}(16) + \frac{1}{2}(4)\right] - \left[\frac{1}{2} + \frac{1}{2}\right] = 8 + 2 - 1 = 9$$

At different altitudes in Earth's atmosphere, sound travels at different speeds. The speed of sound $s(x)$ (in meters per second) can be modeled by the following function

$$s(x) = \begin{cases} -4x + 341, & 0 \le x < 11.5 \\ 295, & 11.5 \le x < 22 \\ \frac{3}{4}x + 278.5, & 22 \le x < 32 \\ \frac{3}{2}x + 254.5, & 32 \le x < 50 \\ -\frac{3}{2}x + 404.5, & 50 \le x \le 80 \end{cases}$$

$$\left[\frac{3}{8}x^2 + 278.5x\right]\Big|_{22}^{32} + \left[\frac{3}{4}x^2 + 254.5x\right]\Big|_{32}^{50}$$

$$[9296 - 6308.5] + [14{,}600 - 8912]$$

Determine the average speed of sound over the interval [22 ,50].

$$\frac{1}{50-22}\left[\int_{22}^{32} \frac{3}{4}x + 278.5\, dx + \int_{32}^{50} \frac{3}{2}x + 254.5\, dx\right]$$

$$\frac{1}{50-22}\left[\left[\left(\frac{3}{4}\right)\left(\frac{x^2}{2}\right) + 278.5x\right]\Big|_{22}^{32} + \left[\frac{3}{2}\left(\frac{x^2}{2}\right) + 254.5x\right]\Big|_{32}^{50}\right]$$

$$2987.5 + 5688$$

$$= 8675.5\left(\frac{1}{28}\right) \approx 309.839 \text{ m/sec}$$

The Second Fundamental Theorem of Calculus

If f is continuous on an open interval I containing a, then, for every x in the interval

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Let's think through this....

The derivative of a function that is defined by an integral is the function with the upper limit plugged in.

Can you show that it works using the Fundamental Theorem of Calculus?

$$\frac{d}{dx}\left[F(t)\Big|_a^x\right] = \frac{d}{dx}\left[F(x) - F(a)\right]$$

$$= f(x) - 0$$

$$= f(x)$$

Why does the lower limit need to be a constant?

- So when we differentiate it goes to 0.

What if the upper limit is not simply a function of x?

It's rate of change will need to be considered.

$$\frac{d}{dx}\int_a^{g(x)} f(t)\,dt = \frac{d}{dx}\left[F(t)\Big|_a^{g(x)}\right] = \frac{d}{dx}\left[F(g(x)) - F(a)\right]$$

$$= f(g(x)) \cdot g'(x)$$

What could you do if both the lower an upper limits were functions?

Split it up.

$$\frac{d}{dx}\left[\int_{h(x)}^{g(x)} f(t)\,dt\right] = \frac{d}{dx}\left[\int_{h(x)}^{a} f(t)\,dt + \int_a^{g(x)} f(t)\,dt\right]$$

Examples – Second Fundamental Theorem of Calculus

Find $F'(x)$ if $F(x) = \int_1^{4} \sqrt[4]{t}\; dt$

$$F'(x) = \sqrt[4]{x}$$

$$\frac{d}{dx}\int_2^{x^2} \left[\frac{1}{t^3}\right] dt = \left(\frac{1}{(x^2)^3}\right)(2x) = \frac{2x}{x^6} = \frac{2}{x^5}$$

Find the derivative of $F(x) = \int_0^{x^2} \sin\theta^2 \, d\theta$

$$F'(x) = \sin(x^2)^2 \, (2x)$$

$$= 2x \sin x^4$$

Differentiate $F(x) = \int_{x^2}^{\sin x} \sqrt{t^2+1} \, dt$

$$= \int_{x^2}^{a} \sqrt{t^2+1} \, dt + \int_{a}^{\sin x} \sqrt{t^2+1} \, dt$$

$$= -\int_{a}^{x^2} \sqrt{t^2+1} \, dt + \int_{a}^{\sin x} \sqrt{t^2+1} \, dt$$

$$= -\left(\sqrt{x^4+1}\right)(2x) + \sqrt{\sin^2 x + 1} \, (\cos x)$$

$$= \cos x \sqrt{\sin^2 x + 1} - 2x\sqrt{x^4+1}$$