2.3 Product and Quotient Rules and Higher Order Derivatives

Product Rule:

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)g'(x) + f'(x)g(x)$$

Extending the Product Rule:

$$\frac{d}{dx}[f(x)\cdot g(x)\cdot h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

Examples: Product Rule

Find the derivative of $h(x) = (3x+5)(x^2 - 2x^3)$

$$\begin{aligned} h'(x) &= (3x+5)(2x-6x^2) + (3)(x^2-2x^3) \\ &= 6x^2 + 10x - 18x^3 - 30x^2 + 3x^2 - 6x^3 \\ &= -24x^3 - 21x^2 + 10x \end{aligned}$$

Find the derivative of $y = x^2e^x$

$$\begin{aligned} y' &= x^2e^x + 2xe^x \\ &= xe^x(x+2) \end{aligned}$$

Find the derivative of $y = 4x^2 sinx - 7cosx$.

$$\begin{aligned} y' &= (4x^2)(\cos x) + 8x\sin x - 7(-\sin x) \\ &= 4x^2\cos x + 8x\sin x + 7\sin x \end{aligned}$$

Quotient Rule:

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

* Order Matters!

Examples: Quotient Rule

Find the derivative of $y = \frac{3x+1}{2x^2-5}$

$$y' = \frac{(2x^2-5)(3)-(3x+1)(4x)}{(2x^2-5)^2} = \frac{6x^2-15-12x^2-4x}{(2x^2-5)^2} =$$

$$\frac{-6x^2-4x-15}{(2x^2-5)^2}$$

Find the equation of the tangent line to the graph of $f(x) = \frac{2+\frac{1}{x}}{x-1}$ at $\left(2, \frac{5}{2}\right)$.

$$f(x) = \frac{2x+1}{x^2-x}$$

$$y - \frac{5}{2} = -\frac{11}{4}(x-2)$$

$$f'(x) = \frac{(x^2-x)(2)-(2x+1)(2x-1)}{(x^2-x)^2}$$

$$f'(2) = \frac{(2^2-2)(2)-(2(2)+1)(2(2)-1)}{(2^2-2)^2} = -\frac{11}{4}$$

Find $\frac{dy}{dx}$ if $y = \frac{2(x^3-x^2)}{5x}$

$$\frac{dy}{dx} = \frac{2}{5}(2x-1)$$

Proof: $\boldsymbol{\frac{d}{dx}[tanx] = sec^2x}$

$$\frac{d}{dx}\left[\frac{\sin x}{\cos x}\right] = \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Derivatives of Trigonometric Functions

$\frac{d}{dx}[tanx] =$ sec^2x

$\frac{d}{dx}[cotx] =$ $-csc^2x$

$\frac{d}{dx}[secx] =$ $secx\,tanx$

$\frac{d}{dx}[cscx] =$ $-cscx\,cotx$

Examples: Derivatives of Trigonometric Functions

Find the derivative of each function

$y = 3x^2 - cscx$

$y' = 6x - (-cscx\,cotx) = 6x + cscx\,cotx$

$y = x^3 cotx$

$y' = x^3(-csc^2x) + 3x^2 cotx$

$= x^2(-x\,csc^2x + 3cotx)$

Differentiate both forms of the trigonometric expression and show that the two derivatives are equal.

$$\frac{1 - cscx}{secx} = cosx - cotx$$

Higher Order Derivatives

First Derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$	$D_x[y]$
Second Derivative	y''	$f''(x)$	d^2y/dx^2	$\frac{d^2}{dx^2}[f(x)]$	$D_x^2[y]$
Third Derivative	y'''	$f'''(x)$	d^3y/dx^3		
Fourth Derivative	$y^{(4)}$	$f^{(4)}(x)$	d^4y/dx^4		
Nth Derivative	$y^{(n)}$	$f^{(n)}(x)$	d^ny/dx^n		

Position Function: $s(t)$

Velocity Function: $s'(t) = v(t)$

Acceleration Function: $s''(t) = v'(t) = a(t)$

Examples: Acceleration Due to Gravity

The position function of an object dropped on Mars is $s(t) = -1.85t^2 + 3$, where $s(t)$ is the height in meters and t is the time in seconds after the object is dropped. What is the ratio of the Earth's gravitational force to Mars?

$s(t) = -1.85t^2 + 3$

$s'(t) = -3.70t$

$s''(t) = -3.70$

$\frac{-9.8}{-3.70} \approx 2.65$

Finding the value of a derivative on the calculator: