Section 6.1 Area of a Region Between Two Curves

1. $A = \int_0^6 \left[0 - \left(x^2 - 6x\right)\right] dx = -\int_0^6 \left(x^2 - 6x\right) dx$

2. $A = \int_{-2}^{2} \left[(2x + 5) - \left(x^2 + 2x + 1\right)\right] dx$
$= \int_{-2}^{2} \left(-x^2 + 4\right) dx$

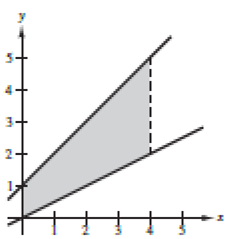
3. $A = \int_0^3 \left[\left(-x^2 + 2x + 3\right) - \left(x^2 - 4x + 3\right)\right] dx$
$= \int_0^3 \left(-2x^2 + 6x\right) dx$

4. $A = \int_0^1 \left(x^2 - x^3\right) dx$

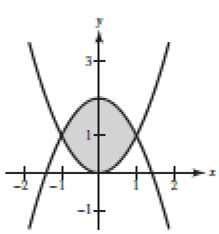
5. $A = 2\int_{-1}^{0} 3\left(x^3 - x\right) dx = 6\int_{-1}^{0} \left(x^3 - x\right) dx$
or $-6\int_0^1 \left(x^3 - x\right) dx$

6. $A = 2\int_0^1 \left[(x - 1)^3 - (x - 1)\right] dx$

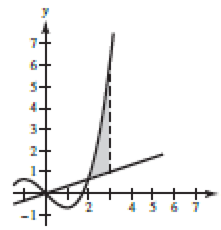
7. $\int_0^4 \left[(x + 1) - \frac{x}{2}\right] dx$



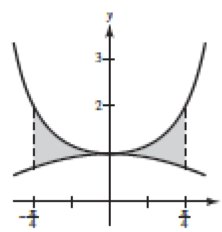
8. $\int_{-1}^{1} \left[\left(2 - x^2\right) - x^2\right] dx$



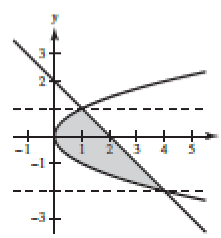
9. $\int_2^3 \left[\left(\frac{x^3}{3} - x\right) - \frac{x}{3}\right] dx$



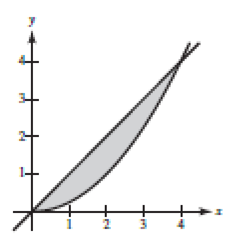
10. $\int_{-\pi/4}^{\pi/4} \left(\sec^2 x - \cos x\right) dx$



11. $\int_{-2}^{1} \left[(2 - y) - y^2\right] dy$



12. $\int_0^4 \left(2\sqrt{y} - y\right) dy$



13. The area is found by subtracting $g(x)$ from $f(x)$.

$$A = \int_{-2}^{0} \left[f(x) - g(x)\right] dx = \int_{-2}^{0} \left[\left(2x^3 + x^2 + 2\right) - \left(-x^2 + 4x + 2\right)\right] dx$$
$$= \int_{-2}^{0} \left(2x^3 + 2x^2 - 4x\right) dx$$

14. Because $g(x) \le f(x)$ on the interval $[-2, 0]$ and $f(x) \le g(x)$ on the interval $[0, 1]$, you need two integrals to find the sum of the areas of the regions R and S.

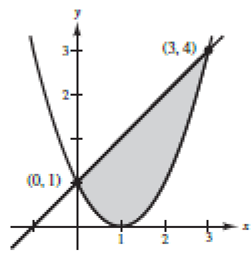
$$A = \int_{-2}^{0} \left[f(x) - g(x)\right] dx + \int_{0}^{1} \left[g(x) - f(x)\right] dx$$
$$= \int_{-2}^{0} \left(2x^3 + 2x^2 - 4x\right) dx + \int_{0}^{1} \left(-2x^3 - 2x^2 + 4x\right) dx$$

15. $f(x) = x + 1$

$g(x) = (x - 1)^2$

$A \simeq 4$

Matches (d)

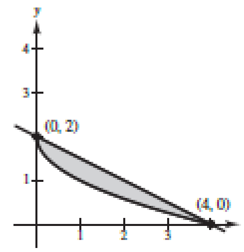


16. $f(x) = 2 - \frac{1}{2}x$

$g(x) = 2 - \sqrt{x}$

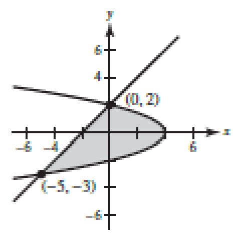
$A \approx 1$

Matches (a)



17. (a)
$$x = 4 - y^2$$
$$x = y - 2$$
$$4 - y^2 = y - 2$$
$$y^2 + y - 6 = 0$$
$$(y + 3)(y - 2) = 0$$

Intersection points: $(0, 2)$ and $(-5, -3)$



$$A = \int_{-5}^{0} \left[(x + 2) + \sqrt{4 - x}\right] dx + \int_{0}^{4} 2\sqrt{4 - x}\, dx = \frac{61}{6} + \frac{32}{3} = \frac{125}{6}$$

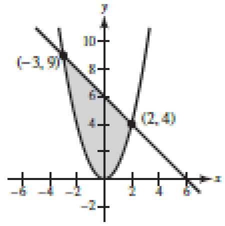
(b) $A = \int_{-3}^{2} \left[\left(4 - y^2\right) - (y - 2)\right] dy = \frac{125}{6}$

(c) The second method is simpler. Explanations will vary.

18. (a) $y = x^2$ and $y = 6 - x$

$x^2 = 6 - x \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x + 3)(x - 2) = 0$

Intersection points: $(2, 4)$ and $(-3, 9)$

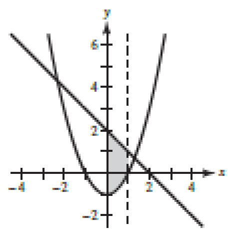


$$A = \int_{-3}^{2} \left[(6 - x) - x^2\right] dx = \frac{125}{6}$$

(b) $A = \int_{0}^{4} 2\sqrt{y}\, dy + \int_{4}^{9} \left[(6 - y) + \sqrt{y}\, dy\right. = \frac{32}{3} + \frac{61}{6} = \frac{125}{6}$

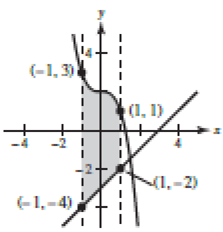
(c) The first method is simpler. Explanations will vary.

19.



$$A = \int_0^1 \left[(-x + 2) - (x^2 - 1)\right] dx$$
$$= \int_0^1 (-x^2 - x + 3)\, dx$$
$$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 3x\right]_0^1$$
$$= \left(-\frac{1}{3} - \frac{1}{2} + 3\right) - 0 = \frac{13}{6}$$

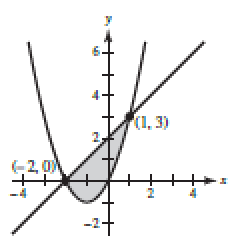
20.



$$A = \int_{-1}^1 \left[(-x^3 + 2) - (x - 3)\right] dx$$
$$= \int_{-1}^1 (-x^3 - x + 5)\, dx$$
$$= \left[\frac{-x^4}{4} - \frac{x^2}{2} + 5x\right]_{-1}^1$$
$$= \left(-\frac{1}{4} - \frac{1}{2} + 5\right) - \left(-\frac{1}{4} - \frac{1}{2} - 5\right) = 10$$

21. The points of intersection are given by:

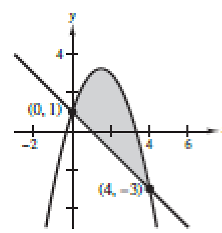
$$x^2 + 2x = x + 2$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0 \quad \text{when } x = -2, 1$$



$$A = \int_{-2}^1 \left[g(x) - f(x)\right] dx$$
$$= \int_{-2}^1 \left[(x + 2) - (x^2 + 2x)\right] dx$$
$$= \left[\frac{-x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^1$$
$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$$

22. The points of intersection are given by:

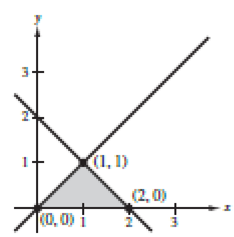
$$-x^2 + 3x + 1 = -x + 1$$
$$-x^2 + 4x = 0$$
$$x(4 - x) = 0 \quad \text{when } x = 0, 4$$



$$A = \int_0^4 \left[(-x^2 + 3x + 1) - (1 - x)\right] dx$$
$$= \int_0^4 (-x^2 + 4x)\, dx$$
$$= \left[\frac{-x^3}{3} + 2x^2\right]_0^4$$
$$= -\frac{64}{3} + 32 = \frac{32}{3}$$

23. The points of intersection are given by:

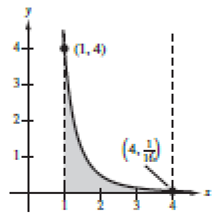
$$x = 2 - x \quad \text{and} \quad x = 0 \quad \text{and} \quad 2 - x = 0$$
$$x = 1 \qquad\qquad x = 0 \qquad\qquad x = 2$$



$$A = \int_0^1 \left[(2 - y) - (y)\right] dy = \left[2y - y^2\right]_0^1 = 1$$

Note that if you integrate with respect to x, you need two integrals. Also, note that the region is a triangle.

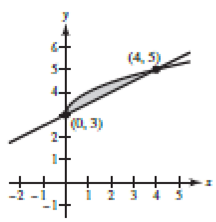
24.



$$A = \int_1^4 \frac{4}{x^3}\,dx = \int_1^4 4x^{-3}\,dx$$
$$= \left[-2x^{-2}\right]_1^4$$
$$= \left[\frac{-2}{x^2}\right]_1^4$$
$$= -\frac{2}{16} + 2 = \frac{15}{8}$$

25. The points of intersection are given by:

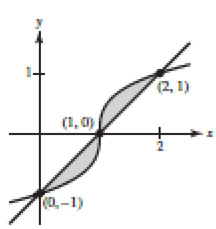
$$\sqrt{x} + 3 = \frac{1}{2}x + 3$$
$$\sqrt{x} = \frac{1}{2}x$$
$$x = \frac{x^2}{4} \quad \text{when } x = 0, 4$$



$$A = \int_0^4 \left[\left(\sqrt{x} + 3\right) - \left(\frac{1}{2}x + 3\right)\right] dx$$
$$= \left[\frac{2}{3}x^{3/2} - \frac{x^2}{4}\right]_0^4 = \frac{16}{3} - 4 = \frac{4}{3}$$

26. The points of intersection are given by:

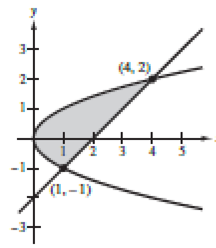
$$\sqrt[3]{x - 1} = x - 1$$
$$x - 1 = (x - 1)^3 = x^3 - 3x^2 + 3x - 1$$
$$x^3 - 3x^2 + 2x = 0$$
$$x\left(x^2 - 3x + 2\right) = 0$$
$$x(x - 2)(x - 1) = 0 \quad \text{when } x = 0, 1, 2$$



$$A = 2\int_0^1 \left[(x - 1) - \sqrt[3]{x - 1}\right] dx$$
$$= 2\left[\frac{x^2}{2} - x - \frac{3}{4}(x - 1)^{4/3}\right]_0^1$$
$$= 2\left[\left(\frac{1}{2} - 1 - 0\right) - \left(-\frac{3}{4}\right)\right] = \frac{1}{2}$$

27. The points of intersection are given by:

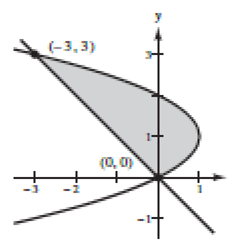
$$y^2 = y + 2$$
$$(y - 2)(y + 1) = 0 \quad \text{when } y = -1, 2$$



$$A = \int_{-1}^2 \left[g(y) - f(y)\right] dy$$
$$= \int_{-1}^2 \left[(y + 2) - y^2\right] dy$$
$$= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_{-1}^2 = \frac{9}{2}$$

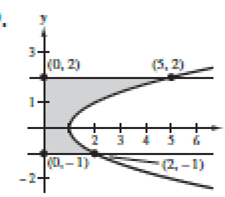
28. The points of intersection are given by:

$$2y - y^2 = -y$$
$$y(y - 3) = 0 \quad \text{when } y = 0, 3$$



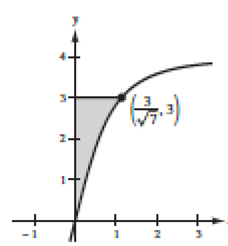
$$A = \int_0^3 \left[f(y) - g(y)\right] dy$$
$$= \int_0^3 \left[\left(2y - y^2\right) - (-y)\right] dy$$
$$= \int_0^3 \left(3y - y^2\right) dy$$
$$= \left[\tfrac{3}{2}y^2 - \tfrac{1}{3}y^3\right]_0^3 = \tfrac{9}{2}$$

29.



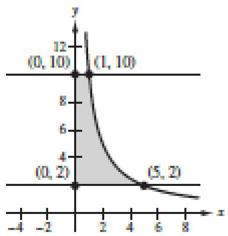
$$A = \int_{-1}^2 \left[f(y) - g(y)\right] dy$$
$$= \int_{-1}^2 \left[\left(y^2 + 1\right) - 0\right] dy$$
$$= \left[\frac{y^3}{3} + y\right]_{-1}^2 = 6$$

30.



$$A = \int_0^3 \left[f(y) - g(y)\right] dy$$
$$= \int_0^3 \left[\frac{y}{\sqrt{16 - y^2}} - 0\right] dy$$
$$= -\frac{1}{2}\int_0^3 \left(16 - y^2\right)^{-1/2}(-2y)\,dy$$
$$= \left[-\sqrt{16 - y^2}\right]_0^3 = 4 - \sqrt{7} \approx 1.354$$

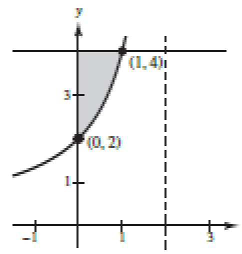
31. $y = \dfrac{10}{x} \Rightarrow x = \dfrac{10}{y}$



$$A = \int_2^{10} \frac{10}{y}\,dy$$
$$= \left[10 \ln y\right]_2^{10}$$
$$= 10(\ln 10 - \ln 2)$$
$$= 10 \ln 5 \approx 16.0944$$

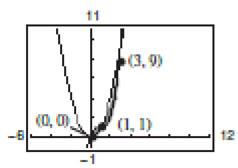
32. The point of intersection is given by:

$$\frac{4}{2-x} = 4$$
$$\frac{4}{2-x} - 4 = 0 \quad \text{when } x = 1$$



$$A = \int_0^1 \left(4 - \frac{4}{2-x}\right) dx$$
$$= \left[4x + 4 \ln|2 - x|\right]_0^1$$
$$= 4 - 4 \ln 2$$
$$\approx 1.227$$

33. (a)



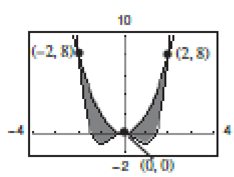
(b) The points of intersection are given by:

$$x^3 - 3x^2 + 3x = x^2$$
$$x(x-1)(x-3) = 0 \quad \text{when } x = 0, 1, 3$$

$$A = \int_0^1 \left[f(x) - g(x)\right] dx + \int_1^3 \left[g(x) - f(x)\right] dx$$
$$= \int_0^1 \left[\left(x^3 - 3x^2 + 3x\right) - x^2\right] dx + \int_1^3 \left[x^2 - \left(x^3 - 3x^2 + 3x\right)\right] dx$$
$$= \int_0^1 \left(x^3 - 4x^2 + 3x\right) dx + \int_1^3 \left(-x^3 + 4x^2 - 3x\right) dx = \left[\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3}{2}x^2\right]_0^1 + \left[\frac{-x^4}{4} + \frac{4}{3}x^3 - \frac{3}{2}x^2\right]_1^3 = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

(c) Numerical approximation: $0.417 + 2.667 \approx 3.083$

34. (a)



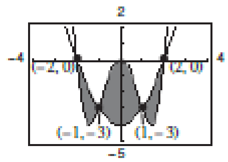
(b) The points of intersection are given by:

$$x^4 - 2x^2 = 2x^2$$
$$x^2\left(x^2 - 4\right) = 0 \quad \text{when } x = 0, \pm 2$$

$$A = 2\int_0^2 \left[2x^2 - \left(x^4 - 2x^2\right)\right] dx = 2\int_0^2 \left(4x^2 - x^4\right) dx = 2\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{128}{15}$$

(c) Numerical approximation: 8.533

35. (a) $f(x) = x^4 - 4x^2,\ \ g(x) = x^2 - 4$



(b) The points of intersection are given by:

$$x^4 - 4x^2 = x^2 - 4$$
$$x^4 - 5x^2 + 4 = 0$$
$$(x^2 - 4)(x^2 - 1) = 0 \quad \text{when } x = \pm 2, \pm 1$$

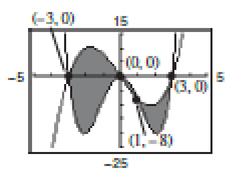
By symmetry:

$$A = 2\int_0^1 \left[(x^4 - 4x^2) - (x^2 - 4)\right] dx + 2\int_1^2 \left[(x^2 - 4) - (x^4 - 4x^2)\right] dx$$
$$= 2\int_0^1 (x^4 - 5x^2 + 4)\, dx + 2\int_1^2 (-x^4 + 5x^2 - 4)\, dx$$
$$= 2\left[\frac{x^5}{5} - \frac{5x^3}{3} + 4x\right]_0^1 + 2\left[-\frac{x^5}{5} + \frac{5x^3}{3} - 4x\right]_1^2$$
$$= 2\left[\frac{1}{5} - \frac{5}{3} + 4\right] + 2\left[\left(-\frac{32}{5} + \frac{40}{3} - 8\right) - \left(-\frac{1}{5} + \frac{5}{3} - 4\right)\right] = 8$$

(c) Numerical approximation:

$5.067 + 2.933 = 8.0$

36. (a)



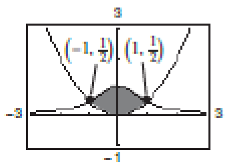
(b) The points of intersection are given by:

$$x^4 - 9x^2 = x^3 - 9x$$
$$x^4 - x^3 - 9x^2 + 9x = 0$$
$$x(x - 3)(x - 1)(x + 3) = 0 \quad \text{when } x = -3, 0, 1, 3$$

$$A = \int_{-3}^0 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx + \int_0^1 \left[(x^4 - 9x^2) - (x^3 - 9x)\right] dx + \int_1^3 \left[(x^3 - 9x) - (x^4 - 9x^2)\right] dx$$
$$= \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_{-3}^0 + \left[\frac{x^5}{5} - 3x^3 - \frac{x^4}{4} + \frac{9x^2}{2}\right]_0^1 + \left[\frac{x^4}{4} - \frac{9x^2}{2} - \frac{x^5}{5} + 3x^3\right]_1^3$$
$$= \frac{1053}{20} + \frac{29}{20} + \frac{68}{5} = \frac{677}{10}$$

(c) Numerical approximation: 67.7

37. (a)



(b) The points of intersection are given by:

$$\frac{1}{1+x^2} = \frac{x^2}{2}$$

$$x^4 + x^2 - 2 = 0$$

$$(x^2+2)(x^2-1) = 0 \quad \text{when } x = \pm 1$$

$$A = 2\int_0^1 \left[f(x) - g(x)\right] dx$$

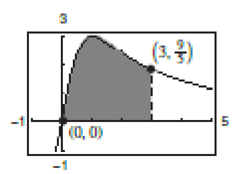
$$= 2\int_0^1 \left[\frac{1}{1+x^2} - \frac{x^2}{2}\right] dx$$

$$= 2\left[\arctan x - \frac{x^3}{6}\right]_0^1$$

$$= 2\left(\frac{\pi}{4} - \frac{1}{6}\right) = \frac{\pi}{2} - \frac{1}{3} \approx 1.237$$

(c) Numerical approximation: 1.237

38. (a)



(b) $A = \int_0^3 \left[\frac{6x}{x^2+1} - 0\right] dx$

$$= \left[3\ln(x^2+1)\right]_0^3$$

$$= 3\ln 10$$

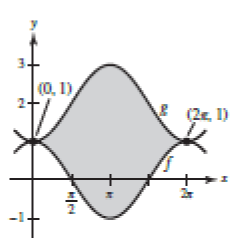
$$\approx 6.908$$

(c) Numerical approximation: 6.908

39. $A = \int_0^{2\pi} \left[(2 - \cos x) - \cos x\right] dx$

$$= 2\int_0^{2\pi} (1 - \cos x)\, dx$$

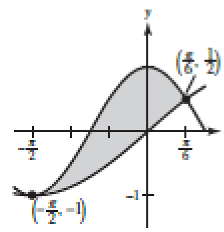
$$= 2\left[x - \sin x\right]_0^{2\pi} = 4\pi \approx 12.566$$



40. $A = \int_{-\pi/2}^{\pi/6} (\cos 2x - \sin x)\, dx$

$$= \left[\frac{1}{2}\sin 2x + \cos x\right]_{-\pi/2}^{\pi/6}$$

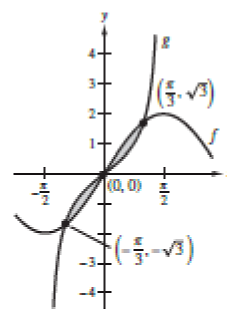
$$= \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) - (0) = \frac{3\sqrt{3}}{4} \approx 1.299$$



41. $A = 2\int_0^{\pi/3} \left[f(x) - g(x)\right] dx$

$$= 2\int_0^{\pi/3} (2\sin x - \tan x)\, dx$$

$$= 2\left[-2\cos x + \ln|\cos x|\right]_0^{\pi/3} = 2(1 - \ln 2) \approx 0.$$

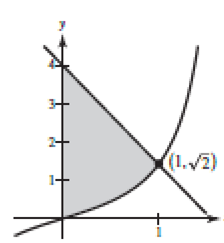


42. $A = \int_0^1 \left[\left(\sqrt{2} - 4\right)x + 4 - \sec\frac{\pi x}{4}\tan\frac{\pi x}{4}\right] dx$

$$= \left[\frac{\sqrt{2}-4}{2}x^2 + 4x - \frac{4}{\pi}\sec\frac{\pi x}{4}\right]_0^1$$

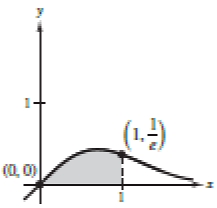
$$= \left(\frac{\sqrt{2}-4}{2} + 4 - \frac{4}{\pi}\sqrt{2}\right) - \left(-\frac{4}{\pi}\right)$$

$$= \frac{\sqrt{2}}{2} + 2 + \frac{4}{\pi}\left(1 - \sqrt{2}\right) \approx 2.1797$$



43. $A = \int_0^1 \left[xe^{-x^2} - 0\right] dx$

$= \left[-\frac{1}{2}e^{-x^2}\right]_0^1 = \frac{1}{2}\left(1 - \frac{1}{e}\right) \approx 0.316$



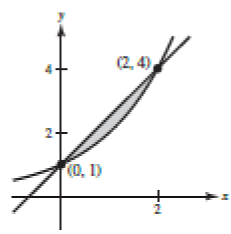
44. From the graph, f and g intersect at $x = 0$ and $x = 2$.

$A = \int_0^2 \left[\left(\frac{3}{2}x + 1\right) - 2^x\right] dx$

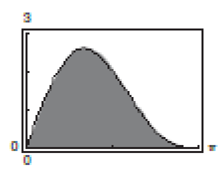
$= \left[\frac{3x^2}{4} + x - \frac{2^x}{\ln 2}\right]_0^2$

$= \left(3 + 2 - \frac{4}{\ln 2}\right) + \frac{1}{\ln 2}$

$= 5 - \frac{3}{\ln 2} \approx 0.672$



45. (a)



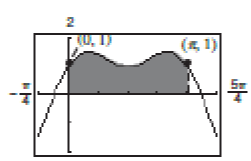
(b) $A = \int_0^\pi (2 \sin x + \sin 2x)\, dx$

$= \left[-2 \cos x - \frac{1}{2} \cos 2x\right]_0^\pi$

$= \left(2 - \frac{1}{2}\right) - \left(-2 - \frac{1}{2}\right) = 4$

(c) Numerical approximation: 4.0

46. (a)

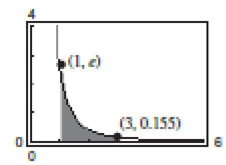


(b) $A = \int_0^\pi (2 \sin x + \cos 2x)\, dx$

$= \left[-2 \cos x + \frac{1}{2} \sin 2x\right]_0^\pi = 4$

(c) Numerical approximation: 4

47. (a)



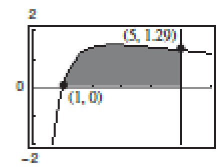
(b) $A = \int_1^3 \frac{1}{x^2} e^{1/x}\, dx$

$= \left[-e^{1/x}\right]_1^3$

$= e - e^{1/3}$

(c) Numerical approximation: 1.323

48. (a)



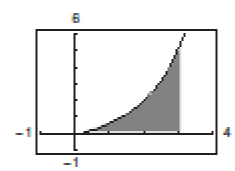
(b) $A = \int_1^5 \frac{4 \ln x}{x}\, dx$

$= \left[2(\ln x)^2\right]_1^5$

$= 2(\ln 5)^2$

(c) Numerical approximation: 5.181

49. (a)



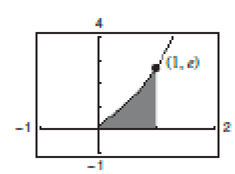
(b) The integral

$A = \int_0^3 \sqrt{\frac{x^3}{4 - x}}\, dx$

does not have an elementary antiderivative.

(c) $A \approx 4.7721$

50. (a)



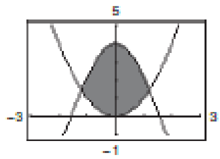
(b) The integral

$A = \int_0^1 \sqrt{x}e^x\, dx$

does not have an elementary antiderivative.

(c) 1.2556

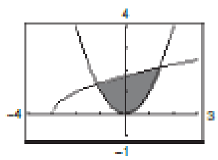
51. (a)



(b) The intersection points are difficult to determine by hand.

(c) Area $= \int_{-c}^{c} \left[4\cos x - x^2\right] dx \approx 6.3043$ where $c \approx 1.201538$.

52. (a)



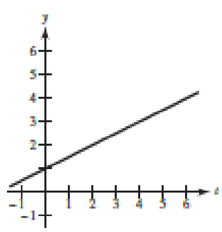
(b) The intersection points are difficult to determine.

(c) Intersection points: $(-1.164035, 1.3549778)$ and $(1.4526269, 2.1101248)$

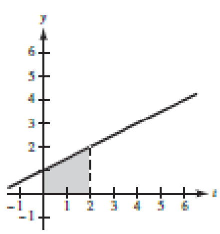
$$A = \int_{-1.164035}^{1.4526269} \left[\sqrt{3 + x} - x^2\right] dx \approx 3.0578$$

53. $F(x) = \int_0^x \left(\frac{1}{2}t + 1\right) dt = \left[\frac{t^2}{4} + t\right]_0^x = \frac{x^2}{4} + x$

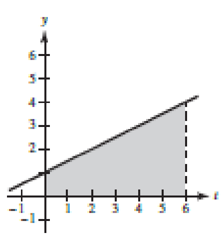
(a) $F(0) = 0$



(b) $F(2) = \frac{2^2}{4} + 2 = 3$

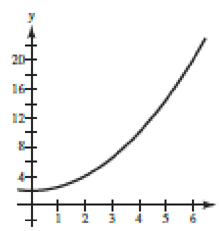


(c) $F(6) = \frac{6^2}{4} + 6 = 15$

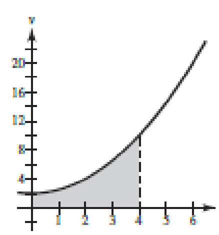


54. $F(x) = \int_0^x \left(\frac{1}{2}t^2 + 2\right) dt = \left[\frac{1}{6}t^3 + 2t\right]_0^x = \frac{x^3}{6} + 2x$

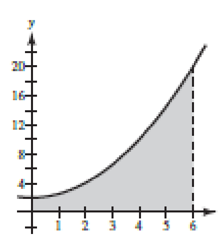
(a) $F(0) = 0$



(b) $F(4) = \frac{4^3}{6} + 2(4) = \frac{56}{3}$

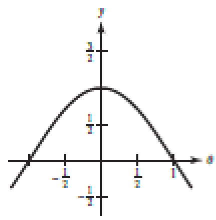


(c) $F(6) = 36 + 12 = 48$

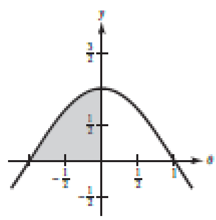


55. $F(\alpha) = \int_{-1}^{\alpha} \cos\frac{\pi\theta}{2}\,d\theta = \left[\frac{2}{\pi}\sin\frac{\pi\theta}{2}\right]_{-1}^{\alpha} = \frac{2}{\pi}\sin\frac{\pi\alpha}{2} + \frac{2}{\pi}$

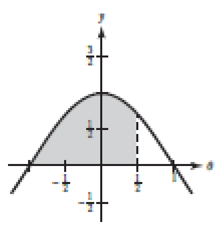
(a) $F(-1) = 0$



(b) $F(0) = \frac{2}{\pi} \approx 0.6366$

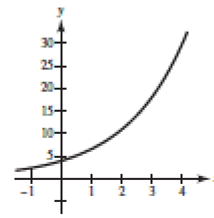


(c) $F\left(\frac{1}{2}\right) = \frac{2 + \sqrt{2}}{\pi} \approx 1.0868$

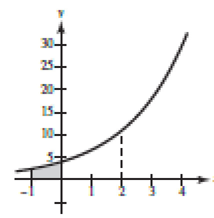


56. $F(y) = \int_{-1}^{y} 4e^{x/2}\,dx = \left[8e^{x/2}\right]_{-1}^{y} = 8e^{y/2} - 8e^{-1/2}$

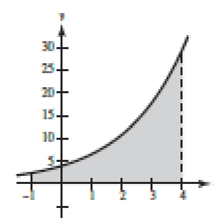
(a) $F(-1) = 0$



(b) $F(0) = 8 - 8e^{-1/2} \approx 3.1478$

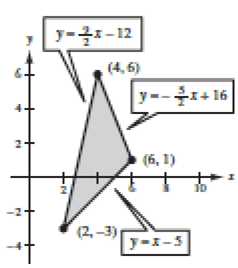


(c) $F(4) = 8e^2 - 8e^{-1/2} \approx 54.2602$



57. $A = \int_2^4 \left[\left(\frac{9}{2}x - 12\right) - (x - 5)\right] dx + \int_4^6 \left[\left(-\frac{5}{2}x + 16\right) - (x - 5)\right] dx$

$= \int_2^4 \left(\frac{7}{2}x - 7\right) dx + \int_4^6 \left(-\frac{7}{2}x + 21\right) dx = \left[\frac{7}{4}x^2 - 7x\right]_2^4 + \left[-\frac{7}{4}x^2 + 21x\right]_4^6 = 7 + 7 = 14$

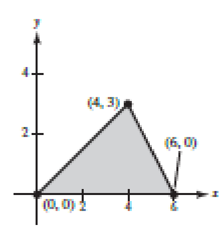


58. $A = \int_0^4 \frac{3}{4}x\,dx + \int_4^6 \left(9 - \frac{3}{2}x\right) dx$

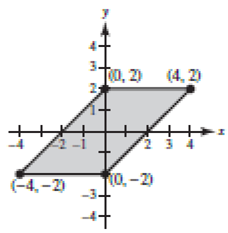
$= \left[\frac{3x^2}{8}\right]_0^4 + \left[9x - \frac{3x^2}{4}\right]_4^6$

$= 6 + (54 - 27) - (36 - 12)$

$= 6 + 3 = 9$



59.



Left boundary line: $y = x + 2 \Leftrightarrow x = y - 2$

Right boundary line: $y = x - 2 \Leftrightarrow x = y + 2$

$$A = \int_{-2}^{2} \left[(y + 2) - (y - 2)\right] dy$$

$$= \int_{-2}^{2} 4\, dy = \left[4y\right]_{-2}^{2} = 8 - (-8) = 16$$

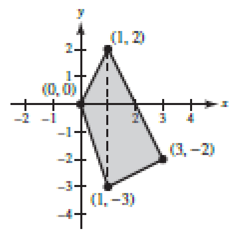
60. $A = \int_0^1 \left[2x - (-3x)\right] dx + \int_1^3 \left[(-2x + 4) - \left(\frac{1}{2}x - \frac{7}{2}\right)\right] dx$

$$= \int_0^1 5x\, dx + \int_1^3 \left(-\frac{5}{2}x + \frac{15}{2}\right) dx$$

$$= \left[\frac{5x^2}{2}\right]_0^1 + \left[-\frac{5x^2}{4} + \frac{15}{2}x\right]_1^3$$

$$= \frac{5}{2} + \left(-\frac{45}{4} + \frac{45}{2} + \frac{5}{4} - \frac{15}{2}\right)$$

$$= \frac{15}{2}$$



61. Answers will vary. *Sample answer*: If you let $\Delta x = 6$ and $n = 10$, $b - a = 10(6) = 60$.

(a) Area $\approx \frac{60}{2(10)}\left[0 + 2(14) + 2(14) + 2(12) + 2(12) + 2(15) + 2(20) + 2(23) + 2(25) + 2(26) + 0\right] = 3[322] = 966 \text{ ft}^2$

(b) Area $\approx \frac{60}{3(10)}\left[0 + 4(14) + 2(14) + 4(12) + 2(12) + 4(15) + 2(20) + 4(23) + 2(25) + 4(26) + 0\right] = 2[502] = 1004 \text{ ft}^2$

62. Answers will vary. *Sample answer*: $\Delta x = 4, n = 8, b - a = (8)(4) = 32$

(a) Area $\approx \frac{32}{2(8)}\left[0 + 2(11) + 2(13.5) + 2(14.2) + 2(14) + 2(14.2) + 2(15) + 2(13.5) + 0\right]$

$= 2[190.8]$

$= 381.6 \text{ mi}^2$

(b) Area $\approx \frac{32}{3(8)}\left[0 + 4(11) + 2(13.5) + 4(14.2) + 2(14) + 4(14.2) + 2(15) + 4(13.5) + 0\right]$

$= \frac{4}{3}[296.6]$

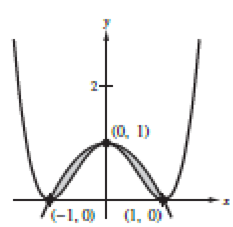
$= 395.5 \text{ mi}^2$

63. $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$

$$A = \int_{-1}^{1} \left[\left(1 - x^2\right) - \left(x^4 - 2x^2 + 1\right)\right] dx$$

$$= \int_{-1}^{1} \left(x^2 - x^4\right) dx$$

$$= \left[\frac{x^3}{3} - \frac{x^5}{5}\right]_{-1}^{1} = \frac{4}{15}$$



You can use a single integral because $x^4 - 2x^2 + 1 \le 1 - x^2$ on $[-1, 1]$.

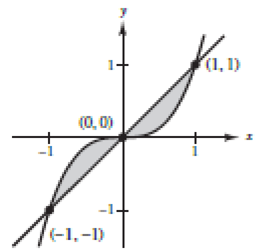
64. $x^3 \ge x$ on $[-1, 0]$, $x^3 \le x$ on $[0, 1]$

Both functions symmetric to origin.

$$\int_{-1}^{0} (x^3 - x)\,dx = -\int_{0}^{1} (x^3 - x)\,dx$$

Thus, $\int_{-1}^{1} (x^3 - x)\,dx = 0.$

$$A = 2\int_{0}^{1} (x - x^3)\,dx = 2\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{1}{2}$$



65. (a) $\int_0^5 [v_1(t) - v_2(t)]\,dt = 10$ means that Car 1 traveled 10 more meters than Car 2 on the interval $0 \le t \le 5$.

$\int_0^{10} [v_1(t) - v_2(t)]\,dt = 30$ means that Car 1 traveled 30 more meters than Car 2 on the interval $0 \le t \le 10$.

$\int_{20}^{30} [v_1(t) - v_2(t)]\,dt = -5$ means that Car 2 traveled 5 more meters than Car 1 on the interval $20 \le t \le 30$.

(b) No, it is not possible because you do not know the initial distance between the cars.

(c) At $t = 10$, Car 1 is ahead by 30 meters.

(d) At $t = 20$, Car 1 is ahead of Car 2 by 13 meters. From part (a), at $t = 30$, Car 1 is ahead by $13 - 5 = 8$ meters.

66. (a) The area between the two curves represents the difference between the accumulated deficit under the two plans.

(b) Proposal 2 is better because the cumulative deficit (the area under the curve) is less.

67.

$$A = \int_{-3}^{3} (9 - x^2)\,dx = 36$$

$$\int_{-\sqrt{9-b}}^{\sqrt{9-b}} \left[(9 - x^2) - b\right] dx = 18$$

$$\int_0^{\sqrt{9-b}} \left[(9 - b) - x^2\right] dx = 9$$

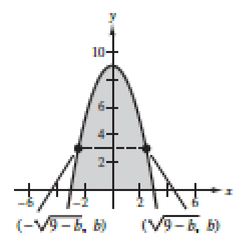
$$\left[(9 - b)x - \frac{x^3}{3}\right]_0^{\sqrt{9-b}} = 9$$

$$\frac{2}{3}(9 - b)^{3/2} = 9$$

$$(9 - b)^{3/2} = \frac{27}{2}$$

$$9 - b = \frac{9}{\sqrt[3]{4}}$$

$$b = 9 - \frac{9}{\sqrt[3]{4}} \approx 3.330$$



68.

$$A = 2\int_0^9 (9 - x)\,dx = 2\left[9x - \frac{x^2}{2}\right]_0^9 = 81$$

$$2\int_0^{9-b} \left[(9 - x) - b\right] dx = \frac{81}{2}$$

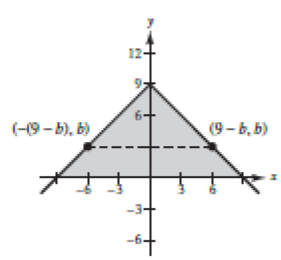
$$2\int_0^{9-b} \left[(9 - b) - x\right] dx = \frac{81}{2}$$

$$2\left[(9 - b)x - \frac{x^2}{2}\right]_0^{9-b} = \frac{81}{2}$$

$$(9 - b)(9 - b) = \frac{81}{2}$$

$$9 - b = \frac{9}{\sqrt{2}}$$

$$b = 9 - \frac{9}{\sqrt{2}} \approx 2.636$$



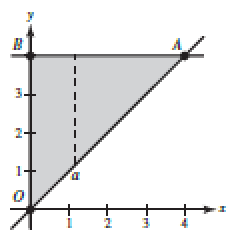
69. Area of triangle OAB is $\frac{1}{2}(4)(4) = 8$.

$$4 = \int_0^a (4 - x)\,dx = \left[4x - \frac{x^2}{2}\right]_0^a = 4a - \frac{a^2}{2}$$

$$a^2 - 8a + 8 = 0$$

$$a = 4 \pm 2\sqrt{2}$$

Because $0 < a < 4$, select $a = 4 - 2\sqrt{2} \approx 1.172$.



70. Total area $= \int_{-2}^{2} (4 - y^2)\,dy = 2\int_0^2 (4 - y^2)\,dy$

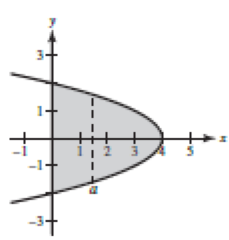
$$= 2\left[4y - \frac{y^3}{3}\right]_0^2 = 2\left[8 - \frac{8}{3}\right] = \frac{32}{3}$$

$$\frac{16}{3} = 2\int_a^4 \sqrt{4 - x}\,dx = -\frac{4}{3}(4 - x)^{3/2}\Big]_a^4 = \frac{4}{3}(4 - a)^{3/2}$$

$$4 = (4 - a)^{3/2}$$

$$4^{2/3} = 4 - a$$

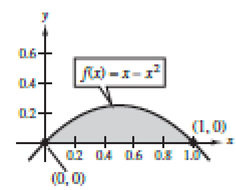
$$a = 4 - 4^{2/3} \approx 1.48$$



71. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(x_i - x_i^2\right) \Delta x$

where $x_i = \frac{i}{n}$ and $\Delta x = \frac{1}{n}$ is the same as

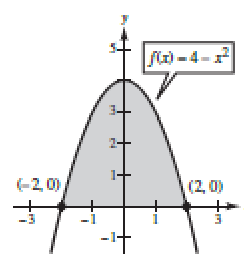
$$\int_0^1 \left(x - x^2\right) dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}.$$



72. $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} \left(4 - x_i^2\right) \Delta x$

where $x_i = -2 + \frac{4i}{n}$ and $\Delta x = \frac{4}{n}$ is the same as

$$\int_{-2}^{2} \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2} = \frac{32}{3}.$$



73. (a) $A = 2\left[\int_0^5 \left(1 - \frac{1}{3}\sqrt{5 - x}\right) dx + \int_5^{5.5} (1 - 0)\,dx\right]$

$$= 2\left(\left[x + \frac{2}{9}(5 - x)^{3/2}\right]_0^5 + \left[x\right]_5^{5.5}\right)$$

$$= 2\left(5 - \frac{10\sqrt{5}}{9} + 5.5 - 5\right) \approx 6.031 \text{ m}^2$$

(b) $V = 2A \approx 2(6.031) \approx 12.062 \text{ m}^3$

(c) $5000\,V \approx 5000(12.062) = 60{,}310$ pounds

74. 5%: $P_1 = 15.9e^{0.05t}$ (in millions)

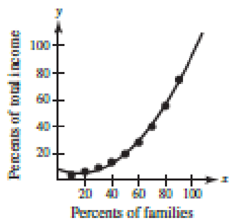
3.5%: $P_2 = 15.9e^{0.035t}$ (in millions)

Difference in profits over 5 years:

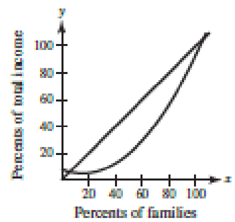
$$\int_0^5 (P_1 - P_2)\,dt = \int_0^5 15.9\left(e^{0.05t} - e^{0.035t}\right) dt = 15.9\left[\frac{e^{0.05t}}{0.05} - \frac{e^{0.035t}}{0.035}\right]_0^5 \approx \$3.44 \text{ million}$$

75. (a) $y_1 = 0.0124x^2 - 0.385x + 7.85$

(b)



(c)



(d) Income inequality $= \int_0^{100} [x - y_1]\, dx \approx 2006.7$

76. The curves intersect at the point where the slope of y_2 equals that of y_1, 1.

$$y_2 = 0.08x^2 + k \Rightarrow y_2' = 0.16x = 1 \Rightarrow x = \frac{1}{0.16} = 6.25$$

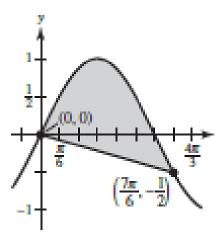
(a) The value of k is given by

$$\begin{aligned} y_1 &= y_2 \\ 6.25 &= (0.08)(6.25)^2 + k \\ k &= 3.125. \end{aligned}$$

(b)
$$\begin{aligned} \text{Area} &= 2\int_0^{6.25} (y_2 - y_1)\, dx \\ &= 2\int_0^{6.25} \left(0.08x^2 + 3.125 - x\right) dx \\ &= 2\left[\frac{0.08x^3}{3} + 3.125x - \frac{x^2}{2}\right]_0^{6.25} \\ &= 2(6.510417) \approx 13.02083 \end{aligned}$$

77. Line: $y = \dfrac{-3}{7\pi}x$

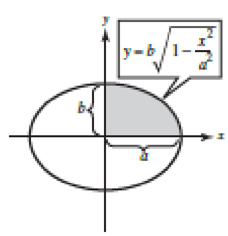
$$\begin{aligned} A &= \int_0^{7\pi/6} \left[\sin x + \frac{3x}{7\pi}\right] dx \\ &= \left[-\cos x + \frac{3x^2}{14\pi}\right]_0^{7\pi/6} \\ &= \frac{\sqrt{3}}{2} + \frac{7\pi}{24} + 1 \\ &\approx 2.7823 \end{aligned}$$



78. $A = 4\int_0^a b\sqrt{1 - \dfrac{x^2}{a^2}}\, dx = \dfrac{4b}{a}\int_0^a \sqrt{a^2 - x^2}\, dx$

$\int_0^a \sqrt{a^2 - x^2}\, dx$ is the area of $\dfrac{1}{4}$ of a circle $= \dfrac{\pi a^2}{4}$.

So, $A = \dfrac{4b}{a}\left(\dfrac{\pi a^2}{4}\right) = \pi ab.$



79. False. Let $f(x) = x$ and $g(x) = 2x - x^2$, f and g intersect at $(1, 1)$, the midpoint of $[0, 2]$, but

$$\int_a^b \left[f(x) - g(x)\right] dx = \int_0^2 \left[x - \left(2x - x^2\right)\right] dx = \tfrac{2}{3} \neq 0.$$

80. True. The area under $f(x)$ between 0 and 1 is $\frac{1}{6}$. The curves intersect at $x = \frac{1}{2}^{1/3}$, and the area between $y = \left(1 - \frac{1}{2}^{1/3}\right)x$ and f on the interval $\left[0, \frac{1}{2}^{1/3}\right]$ is $\frac{1}{12}$.

81. Find the points of intersection of the graphs.

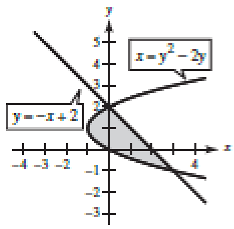
$$\begin{aligned} 2 - 4x - x^2 &= -2x - 1 \\ 0 &= x^2 + 2x - 3 \\ 0 &= (x + 3)(x - 1) \end{aligned}$$

So, $a = -3$ and $b = 1$. Because $y = -2x - 1 \le y = 2 - 4x - x^2$ for all x in the interval $[-3, -1]$, the area of the region is shown below

$$\begin{aligned} A &= \int_{-3}^{1} \left[\left(2 - 4x - x^2\right) - (-2x - 1)\right] dx \\ &= \int_{-3}^{1} \left(-x^2 - 2x + 3\right) dx \\ &= \left[-\tfrac{1}{3}x^3 - x^2 + 3x\right]_{-3}^{1} \\ &= \left(-\tfrac{1}{3} - 1 + 3\right) - (9 - 9 - 9) \\ &= \tfrac{32}{3} \end{aligned}$$

So, the answer is C.

82.



To find the area of the region, integrate with respect to y. Find the points of intersection of the graphs $y = -x + 2 \Rightarrow x = -y + 2$ and $x = y^2 - 2y$.

$$\begin{aligned} -y + 2 &= y^2 - 2y \\ 0 &= y^2 - y - 2 \\ 0 &= (y - 2)(y + 1) \end{aligned}$$

So, $a = -1$ and $b = 2$.

Because $x = y^2 - 2y \le x = -y + 2$ for all y in the interval $[-1, 2]$, the area of the region is shown below.

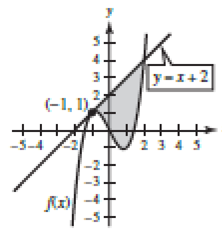
$$\begin{aligned} A &= \int_{-1}^{2} \left[(-y + 2) - (y^2 - 2y)\right] dy \\ &= \int_{-1}^{2} \left[(-y^2 + y + 2)\right] dy \\ &= \left[-\tfrac{1}{3}y^3 + \tfrac{1}{2}y^2 + 2y\right]_{-1}^{2} \\ &= \left(-\tfrac{8}{3} + 2 + 4\right) - \left(\tfrac{1}{3} + \tfrac{1}{2} - 2\right) \\ &= \tfrac{9}{2} \end{aligned}$$

So, the answer is C.

83. First, find the tangent line of f at $(-1, 1)$.

$$\begin{aligned} f(x) &= x^3 - 2x \\ f'(x) &= 3x^2 - 2 \\ f'(-1) &= 3(-1)^2 - 2 = 1 \end{aligned}$$

So, the tangent line is $y = 1[x - (-1)] + 1 = x + 2$.



Find the points of intersection.

$$\begin{aligned} x^3 - 2x &= x + 2 \\ x^3 - 3x - 2 &= 0 \\ (x + 1)(x^2 - x - 2) &= 0 \\ (x + 1)^2(x - 2) &= 0 \end{aligned}$$

So, $a = -1$ and $b = 2$. Because $f(x) = x^3 - 2x \le y = x + 2$ for all x in the interval $[-1, 2]$, the area of the region is given by

$$\begin{aligned} A &= \int_{-1}^{2} \left[(x + 2) - (x^3 - 2x)\right] dx \\ &= \int_{-1}^{2} \left(-x^3 + 3x + 2\right) dx \end{aligned}$$

So, the answer is D.