Section 1.6 Limits at Infinity

1. $f(x) = \dfrac{2x^2}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (f).

2. $f(x) = \dfrac{2x}{\sqrt{x^2 + 2}}$

No vertical asymptotes

Horizontal asymptotes: $y = \pm 2$

Matches (c).

3. $f(x) = \dfrac{x}{x^2 + 2}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) < 1$

Matches (d).

4. $f(x) = 2 + \dfrac{x^2}{x^4 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (a).

5. $f(x) = \dfrac{4 \sin x}{x^2 + 1}$

No vertical asymptotes

Horizontal asymptote: $y = 0$

$f(1) > 1$

Matches (b).

6. $f(x) = \dfrac{2x^2 - 3x + 5}{x^2 + 1}$

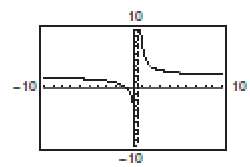
No vertical asymptotes

Horizontal asymptote: $y = 2$

Matches (e).

7. $f(x) = \dfrac{4x + 3}{2x - 1}$

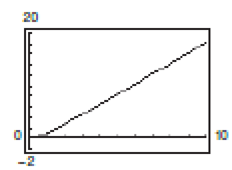
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	7	2.26	2.025	2.0025	2.0003	2	2



$\lim_{x \to \infty} f(x) = 2$

8. $f(x) = \dfrac{2x^2}{x + 1}$

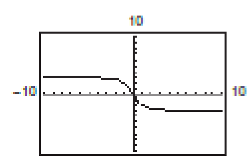
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	18.18	198.02	1998.02	19,998	199,998	1,999,998



$\lim_{x \to \infty} f(x) = \infty$ (Limit does not exist)

9. $f(x) = \dfrac{-6x}{\sqrt{4x^2 + 5}}$

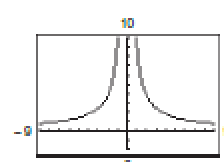
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	−2	−2.98	−2.9998	−3	−3	−3	−3



$\lim_{x \to \infty} f(x) = -3$

10. $f(x) = \dfrac{10}{\sqrt{2x^2 - 1}}$

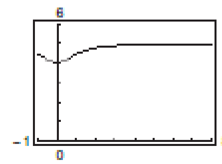
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	10.0	0.7089	0.0707	0.0071	0.0007	0.00007	0.000007



$\lim_{x \to \infty} f(x) = 0$

11. $f(x) = 5 - \dfrac{1}{x^2+1}$

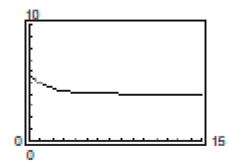
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	4.5	4.99	4.9999	4.999999	5	5	5



$\lim\limits_{x\to\infty} f(x) = 5$

12. $f(x) = 4 + \dfrac{3}{x^2+2}$

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	5	4.03	4.0003	4.0	4.0	4	4



$\lim\limits_{x\to\infty} f(x) = 4$

13. (a) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{5x^3-3}{x^2} = 5x - \dfrac{3}{x^2}$

$\lim\limits_{x\to\infty} h(x) = \infty$ (Limit does not exist.)

(b) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{5x^3-3}{x^3} = 5 - \dfrac{3}{x^3}$

$\lim\limits_{x\to\infty} h(x) = 5$

(c) $h(x) = \dfrac{f(x)}{x^4} = \dfrac{5x^3-3}{x^4} = \dfrac{5}{x} - \dfrac{3}{x^4}$

$\lim\limits_{x\to\infty} h(x) = 0$

14. (a) $h(x) = \dfrac{f(x)}{x} = \dfrac{-4x^2+2x-5}{x} = -4x + 2 - \dfrac{5}{x}$

$\lim\limits_{x\to\infty} h(x) = -\infty$ (Limit does not exist.)

(b) $h(x) = \dfrac{f(x)}{x^2} = \dfrac{-4x^2+2x-5}{x^2} = -4 + \dfrac{2}{x} - \dfrac{5}{x^2}$

$\lim\limits_{x\to\infty} h(x) = -4$

(c) $h(x) = \dfrac{f(x)}{x^3} = \dfrac{-4x^2+2x-5}{x^3} = -\dfrac{4}{x} + \dfrac{2}{x^2} - \dfrac{5}{x^3}$

$\lim\limits_{x\to\infty} h(x) = 0$

15. (a) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^3-1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x^2-1} = 1$

(c) $\lim\limits_{x\to\infty} \dfrac{x^2+2}{x-1} = \infty$ (Limit does not exist.)

16. (a) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x^3-1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{3-2x}{3x-1} = -\dfrac{2}{3}$

(c) $\lim\limits_{x\to\infty} \dfrac{3-2x^2}{3x-1} = -\infty$ (Limit does not exist.)

17. (a) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^2-4} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x^{3/2}-4} = -\dfrac{2}{3}$

(c) $\lim\limits_{x\to\infty} \dfrac{5-2x^{3/2}}{3x-4} = -\infty$ (Limit does not exist.)

18. (a) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^2+1} = 0$

(b) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4x^{3/2}+1} = \dfrac{5}{4}$

(c) $\lim\limits_{x\to\infty} \dfrac{5x^{3/2}}{4\sqrt{x}+1} = \infty$ (Limit does not exist.)

19. $\lim\limits_{x\to\infty} \left(4 + \dfrac{3}{x}\right) = 4 + 0 = 4$

20. $\lim\limits_{x\to-\infty} \left(\dfrac{5}{x} - \dfrac{x}{3}\right) = \infty$ (Limit does not exist.)

21. $\lim\limits_{x\to\infty} \dfrac{2x-1}{3x+2} = \lim\limits_{x\to\infty} \dfrac{2-(1/x)}{3+(2/x)} = \dfrac{2-0}{3+0} = \dfrac{2}{3}$

22. $\lim\limits_{x\to-\infty} \dfrac{4x^2+5}{x^2+3} = \lim\limits_{x\to-\infty} \dfrac{4+(5/x^2)}{1+(3/x^2)} = 4$

23. $\lim_{x\to\infty} \dfrac{x}{x^2-1} = \lim_{x\to\infty} \dfrac{1/x}{1-\left(1/x^2\right)} = \dfrac{0}{1} = 0$

24. $\lim_{x\to\infty} \dfrac{5x^3+1}{10x^3-3x^2+7} = \lim_{x\to\infty} \dfrac{5+\left(1/x^3\right)}{10-(3/x)+\left(7/x^3\right)}$

$$= \frac{5+0}{10-0} = \frac{1}{2}$$

25. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2-x}}$

$$= \lim_{x\to-\infty} \frac{1}{\left(\dfrac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1-(1/x)}}$$

$$= -1, \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$$

26. $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}}$

$$= \lim_{x\to-\infty} \frac{1}{\left(\dfrac{\sqrt{x^2+1}}{-\sqrt{x^2}}\right)}$$

$$= \lim_{x\to-\infty} \frac{-1}{\sqrt{1+\left(1/x^2\right)}}$$

$$= -1, \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$$

27. $\lim_{x\to-\infty} \dfrac{2x+1}{\sqrt{x^2-x}}$

$$= \lim_{x\to-\infty} \frac{2+\dfrac{1}{x}}{\left(\dfrac{\sqrt{x^2-x}}{-\sqrt{x^2}}\right)}$$

$$= \lim_{x\to-\infty} \frac{-2-\left(\dfrac{1}{x}\right)}{\sqrt{1-\dfrac{1}{x}}}$$

$$= -2 \left(\text{for } x < 0, x = -\sqrt{x^2}\right)$$

28. $\lim_{x\to\infty} \dfrac{5x^2+2}{\sqrt{x^2+3}}$

$$= \lim_{x\to\infty} \frac{5x^2+2}{x\sqrt{1+\left(3/x^2\right)}}$$

$$= \lim_{x\to\infty} \frac{5x^2+(2/x)}{\sqrt{1+\left(3/x^2\right)}}$$

$$= \infty$$

Limit does not exist.

29. $\lim_{x\to\infty} \dfrac{\sqrt{x^2-1}}{2x-1} = \lim_{x\to\infty} \dfrac{\sqrt{x^2-1}/\sqrt{x^2}}{2-(1/x)}$

$$= \lim_{x\to\infty} \frac{\sqrt{1-\left(1/x^2\right)}}{2-(1/x)} = \frac{1}{2}$$

30. $\lim_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1} = \lim_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x^3-1}\left(\dfrac{1/\left(-\sqrt{x^6}\right)}{1/x^3}\right)$

$$= \lim_{x\to-\infty} \frac{\sqrt{\left(1/x^2\right)-\left(1/x^6\right)}}{-1+\left(1/x^3\right)} = 0$$

$\left(\text{for } x < 0, -\sqrt{x^6} = x^3\right)$

31. $\lim_{x\to\infty} \dfrac{x+1}{\left(x^2+1\right)^{1/3}} = \lim_{x\to\infty} \dfrac{x+1}{\left(x^2+1\right)^{1/3}}\left(\dfrac{1/x^{2/3}}{1/\left(x^2\right)^{1/3}}\right)$

$$= \lim_{x\to\infty} \frac{x^{1/3}+\left(1/x^{2/3}\right)}{\left[1+\left(1/x^2\right)\right]^{1/3}} = \infty$$

Limit does not exist.

32. $\lim_{x\to-\infty} \dfrac{2x}{\left(x^6-1\right)^{1/3}} = \lim_{x\to-\infty} \dfrac{2x}{\left(x^6-1\right)^{1/3}}\left(\dfrac{1/x^2}{1/\left(x^6\right)^{1/3}}\right)$

$$= \lim_{x\to-\infty} \frac{2/x}{\left[1-\left(1/x^6\right)\right]^{1/3}} = 0$$

33. $\lim_{x\to\infty} \dfrac{1}{2x+\sin x} = 0$

34. $\lim_{x\to\infty} \cos\left(\dfrac{1}{x}\right) = \cos 0 = 1$

35. Because $(-1/x) \le (\sin 2x)/x \le (1/x)$ for all $x \ne 0$,

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin 2x}{x} \le 0$$

by the Squeeze Theorem.

Therefore, $\lim_{x\to\infty} \dfrac{\sin 2x}{x} = 0.$

36. $\lim_{x\to\infty} \dfrac{x-\cos x}{x} = \lim_{x\to\infty} \left(1-\dfrac{\cos x}{x}\right)$

$$= 1 - 0 = 1$$

Note:

Because $-\dfrac{1}{x} \le \dfrac{\cos x}{x} \le \dfrac{1}{x}$, $\lim_{x\to\infty} \dfrac{\cos x}{x} = 0$ by the Squeeze Theorem.

37. $\lim_{x\to\infty}\left(2 - 5e^{-x}\right) = 2$

38. $\lim_{x\to\infty}\dfrac{8}{4 - 10^{-x/2}} = 2$

39. $\lim_{x\to\infty}\log_{10}\left(1 + 10^{-x}\right) = 0$

40. $\lim_{x\to\infty}\left(\dfrac{5}{2} + \ln\dfrac{x^2+1}{x^2}\right) = \dfrac{5}{2}$

41. $\lim_{t\to\infty}\left(8t^{-1} - \arctan t\right) = \lim_{t\to\infty}\left(\dfrac{8}{t}\right) - \lim_{t\to\infty}\arctan t$

$$= 0 - \frac{\pi}{2} = -\frac{\pi}{2}$$

42. $\lim_{u\to\infty}\operatorname{arcsec}(u+1) = \dfrac{\pi}{2}$

43. $\lim_{x\to-\infty}\dfrac{5}{x^3}$ should be 0.

$$\lim_{x\to-\infty}\frac{5x^3}{6x^3-5} = \lim_{x\to-\infty}\frac{5}{6-\left(5/x^3\right)} = \frac{5}{6-0} = \frac{5}{6}$$

44. $\lim_{x\to\infty}\dfrac{8}{x^2}$ should be 0.

$$\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+8}} = \lim_{x\to\infty}\frac{4x/x}{\sqrt{x^2+8}/\sqrt{x^2}}$$

$$= \lim_{x\to\infty}\frac{4}{\sqrt{1+\left(8/x^2\right)}}$$

$$= \frac{4}{\sqrt{1}}$$

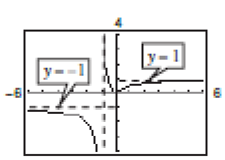
$$= 4$$

45. $f(x) = \dfrac{|x|}{x+1}$

$\lim_{x\to\infty}\dfrac{|x|}{x+1} = 1$

$\lim_{x\to-\infty}\dfrac{|x|}{x+1} = -1$

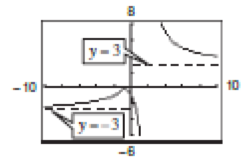
Therefore, $y = 1$ and $y = -1$ are both horizontal asymptotes.



46. $f(x) = \dfrac{|3x+2|}{x-2}$

$y = 3$ is a horizontal asymptote (to the right).

$y = -3$ is a horizontal asymptote (to the left).

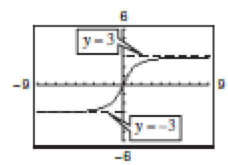


47. $f(x) = \dfrac{3x}{\sqrt{x^2+2}}$

$\lim_{x\to\infty} f(x) = 3$

$\lim_{x\to-\infty} f(x) = -3$

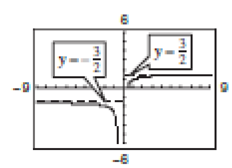
Therefore, $y = 3$ and $y = -3$ are both horizontal asymptotes.



48. $f(x) = \dfrac{\sqrt{9x^2-2}}{2x+1}$

$y = \dfrac{3}{2}$ is a horizontal asymptote (to the right).

$y = -\dfrac{3}{2}$ is a horizontal asymptote (to the left).



49. $\lim_{x\to\infty} x\sin\dfrac{1}{x} = \lim_{t\to0^+}\dfrac{\sin t}{t} = 1$

(Let $x = 1/t$.)

50. $\lim_{x\to\infty} x\tan\dfrac{1}{x} = \lim_{x\to0^+}\dfrac{\tan t}{t} = \lim_{x\to0^+}\left[\dfrac{\sin t}{t}\cdot\dfrac{1}{\cos t}\right]$

$$= (1)(1) = 1$$

(Let $x = 1/t$.)

51. $\lim_{x\to-\infty}\left(x + \sqrt{x^2+3}\right) = \lim_{x\to-\infty}\left[\left(x+\sqrt{x^2+3}\right)\cdot\dfrac{x-\sqrt{x^2+3}}{x-\sqrt{x^2+3}}\right] = \lim_{x\to-\infty}\dfrac{-3}{x-\sqrt{x^2+3}} = 0$

52. $\lim_{x\to\infty}\left(x-\sqrt{x^2+x}\right) = \lim_{x\to\infty}\left[\left(x-\sqrt{x^2+x}\right)\cdot\frac{x+\sqrt{x^2+x}}{x+\sqrt{x^2+x}}\right] = \lim_{x\to\infty}\frac{-x}{x+\sqrt{x^2+x}} = \lim_{x\to\infty}\frac{-1}{1+\sqrt{1+(1/x)}} = -\frac{1}{2}$

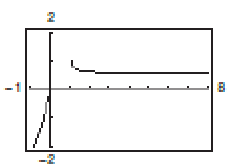
53. $$\lim_{x\to-\infty}\left(3x+\sqrt{9x^2-x}\right) = \lim_{x\to-\infty}\left[\left(3x+\sqrt{9x^2-x}\right)\cdot\frac{3x-\sqrt{9x^2-x}}{3x-\sqrt{9x^2-x}}\right]$$
$$= \lim_{x\to-\infty}\frac{x}{3x-\sqrt{9x^2-x}}$$
$$= \lim_{x\to-\infty}\frac{1}{3-\dfrac{\sqrt{9x^2-x}}{-\sqrt{x^2}}}\quad\left(\text{for } x<0,\ x=-\sqrt{x^2}\right)$$
$$= \lim_{x\to-\infty}\frac{1}{3+\sqrt{9-(1/x)}} = \frac{1}{6}$$

54. $$\lim_{x\to\infty}\left(4x-\sqrt{16x^2-x}\right)\frac{4x+\sqrt{16x^2-x}}{4x+\sqrt{16x^2-x}} = \lim_{x\to\infty}\frac{16x^2-\left(16x^2-x\right)}{4x+\sqrt{\left(16x^2-x\right)}}$$
$$= \lim_{x\to\infty}\frac{x}{4x+\sqrt{16x^2-x}}$$
$$= \lim_{x\to\infty}\frac{1}{4+\sqrt{16-1/x}}$$
$$= \frac{1}{4+4} = \frac{1}{8}$$

55.

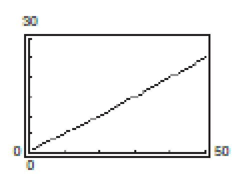
x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1	0.513	0.501	0.500	0.500	0.500	0.500

$$\lim_{x\to\infty}\left[x-\sqrt{x(x-1)}\right] = \lim_{x\to\infty}\frac{x-\sqrt{x^2-x}}{1}\cdot\frac{x+\sqrt{x^2-x}}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x}{x+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{1}{1+\sqrt{1-(1/x)}} = \frac{1}{2}$$



56.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	1.0	5.1	50.1	500.1	5000.1	50,000.1	500,000.1

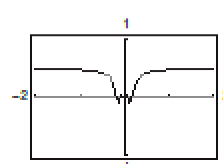


$$\lim_{x\to\infty}\frac{x^2-x\sqrt{x^2-x}}{1}\cdot\frac{x^2+x\sqrt{x^2-x}}{x^2+x\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{x^3}{x^2+x\sqrt{x^2-x}} = \infty$$

Limit does not exist.

57.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	0.479	0.500	0.500	0.500	0.500	0.500	0.500



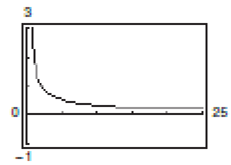
Let $x = 1/t$.

$$\lim_{x\to\infty} x\sin\left(\frac{1}{2x}\right) = \lim_{t\to0^+}\frac{\sin(t/2)}{t} = \lim_{t\to0^+}\frac{1}{2}\,\frac{\sin(t/2)}{t/2} = \frac{1}{2}$$

58.

x	10^0	10^1	10^2	10^3	10^4	10^5	10^6
$f(x)$	2.000	0.348	0.101	0.032	0.010	0.003	0.001

$\lim_{x\to\infty} \frac{x+1}{x\sqrt{x}} = 0$

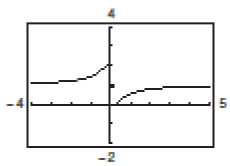


59. (a) $\lim_{x\to\infty} f(x) = 4$ means that $f(x)$ approaches 4 as x becomes large.

(b) $\lim_{x\to-\infty} f(x) = 2$ means that $f(x)$ approaches 2 as x becomes very large (in absolute value) and negative.

60. Answers will vary.

61. (a)



(b) Answers will vary. Sample answer: When x increases without bound, $1/x$ approaches zero and $e^{1/x}$ approaches 1. Therefore, $f(x)$ approaches $2/(1+1) = 1$. So, $f(x)$ has a horizontal asymptote at $y = 1$. As x approaches zero from the right, $1/x$ approaches ∞, $e^{1/x}$ approaches ∞, and $f(x)$ approaches zero. As x approaches zero from the left, $1/x$ approaches $-\infty$, $e^{1/x}$ approaches zero, and $f(x)$ approaches 2. The limit does not exist because the left limit does not equal the right limit. Therefore, $x = 0$ is a nonremovable discontinuity.

62. (a) $\lim_{t\to 0^+} T = 1700°$

This is the temperature of the kiln.

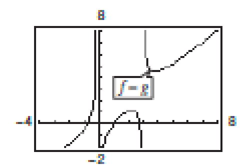
(b) $\lim_{t\to\infty} T = 72°$

This is the temperature of the room.

(c) Because $y = 72$ is the horizontal asymptote, the temperature of the glass will never actually reach room temperature.

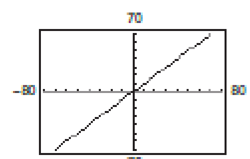
63. $f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)}$, $g(x) = x + \frac{2}{x(x-3)}$

(a)



(b) $$f(x) = \frac{x^3 - 3x^2 + 2}{x(x-3)} = \frac{x^2(x-3)}{x(x-3)} + \frac{2}{x(x-3)} = x + \frac{2}{x(x-3)} = g(x)$$

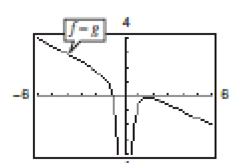
(c)



The graph appears as the slant asymptote $y = x$.

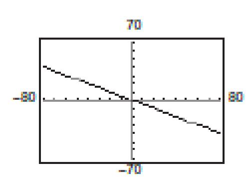
64. $f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2}$, $g(x) = -\frac{1}{2}x + 1 - \frac{1}{x^2}$

(a)



(b) $$f(x) = -\frac{x^3 - 2x^2 + 2}{2x^2} = -\left[\frac{x^3}{2x^2} - \frac{2x^2}{2x^2} + \frac{2}{2x^2}\right] = -\frac{1}{2}x + 1 - \frac{1}{x^2} = g(x)$$

(c)



The graph appears as the slant asymptote $y = -\frac{1}{2}x + 1$.

65. $\lim_{v_1/v_2 \to \infty} 100\left[1 - \frac{1}{(v_1/v_2)^c}\right] = 100[1 - 0] = 100\%$

66. $C = 0.5x + 500$

$\overline{C} = \frac{C}{x}$

$\overline{C} = 0.5 + \frac{500}{x}$

$\lim_{x \to \infty}\left(0.5 + \frac{500}{x}\right) = 0.5$

67. $f(x) = \frac{2x^2}{x^2 + 2}$

(a) $\lim_{x \to \infty} f(x) = 2 = L$

(b) $f(x_1) + \varepsilon = \frac{2x_1^2}{x_1^2 + 2} + \varepsilon = 2$

$$2x_1^2 + \varepsilon x_1^2 + 2\varepsilon = 2x_1^2 + 4$$
$$x_1^2\varepsilon = 4 - 2\varepsilon$$
$$x_1 = \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) Let $M = \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}} > 0$. For $x > M$:

$$x > \sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$$
$$x^2\varepsilon > 4 - 2\varepsilon$$
$$2x^2 + x^2\varepsilon + 2\varepsilon > 2x^2 + 4$$
$$\frac{2x^2}{x^2 + 2} + \varepsilon > 2$$
$$\left|\frac{2x^2}{x^2 + 2} - 2\right| > |-\varepsilon| = \varepsilon$$
$$|f(x) - L| > \varepsilon$$

(d) Similarly, $N = -\sqrt{\frac{4 - 2\varepsilon}{\varepsilon}}$.

68. $f(x) = \frac{6x}{\sqrt{x^2 + 2}}$

(a) $\lim_{x \to \infty} f(x) = 6 = L$

$\lim_{x \to -\infty} f(x) = -6 = K$

(b) $f(x_1) + \varepsilon = \frac{6x_1}{\sqrt{x_1^2 + 2}} + \varepsilon = 6$

$$6x_1 = (6 - \varepsilon)\sqrt{x_1^2 + 2}$$
$$36x_1^2 = (x_1^2 + 2)(6 - \varepsilon)^2$$
$$36x_1^2 - (6 - \varepsilon)^2 x_1^2 = 2(6 - \varepsilon)^2$$
$$x_1^2\left[36 - 36 + 12\varepsilon - \varepsilon^2\right] = 2(6 - \varepsilon)^2$$
$$x_1^2 = \frac{2(6 - \varepsilon)^2}{12\varepsilon - \varepsilon^2}$$
$$x_1 = (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$$
$$x_2 = -x_1 \text{ by symmetry}$$

(c) $M = x_1 = (6 - \varepsilon)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$

(d) $N = x_2 = (\varepsilon - 6)\sqrt{\frac{2}{12\varepsilon - \varepsilon^2}}$

69. $\lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 3}} = 3$

$f(x_1) + \varepsilon = \frac{3x_1}{\sqrt{x_1^2 + 3}} + \varepsilon = 3$

$$3x_1 = (3 - \varepsilon)\sqrt{x_1^2 + 3}$$
$$9x_1^2 = (3 - \varepsilon)^2(x_1^2 + 3)$$
$$9x_1^2 - (3 - \varepsilon)^2 x_1^2 = 3(3 - \varepsilon)^2$$
$$x_1^2(9 - 9 + 6\varepsilon - \varepsilon^2) = 3(3 - \varepsilon)^2$$
$$x_1^2 = \frac{3(3 - \varepsilon)^2}{6\varepsilon - \varepsilon^2}$$
$$x_1 = (3 - \varepsilon)\sqrt{\frac{3}{6\varepsilon - \varepsilon^2}}$$

Let $M = x_1 = (3 - \varepsilon)\sqrt{\frac{3}{6\varepsilon - \varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$M = (3 - 0.5)\sqrt{\frac{3}{6(0.5) - (0.5)^2}} = \frac{5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$M = (3 - 0.1)\sqrt{\frac{3}{6(0.1) - (0.1)^2}} = \frac{29\sqrt{177}}{59}$$

70. $\lim_{x\to-\infty} \dfrac{3x}{\sqrt{x^2+3}} = -3$

$$f(x_1) - \varepsilon = \frac{3x_1}{\sqrt{x_1^2+3}} - \varepsilon = -3$$

$$\begin{aligned} 3x_1 &= (\varepsilon - 3)\sqrt{x_1^2+3} \\ 9x_1^2 &= (\varepsilon-3)^2(x_1^2+3) \\ 9x_1^2 - (\varepsilon-3)^2 x_1^2 &= 3(\varepsilon-3)^2 \\ x_1^2(9 - \varepsilon^2 + 6\varepsilon - 9) &= 3(\varepsilon-3)^2 \\ x_1^2 &= \frac{3(\varepsilon-3)^2}{6\varepsilon - \varepsilon^2} \\ x_1 &= (\varepsilon-3)\sqrt{\frac{3}{6\varepsilon-\varepsilon^2}} \end{aligned}$$

Let $x_1 = N = (\varepsilon - 3)\sqrt{\dfrac{3}{6\varepsilon - \varepsilon^2}}$

(a) When $\varepsilon = 0.5$:

$$N = (0.5-3)\sqrt{\frac{3}{6(0.5)-(0.5)^2}} = \frac{-5\sqrt{33}}{11}$$

(b) When $\varepsilon = 0.1$:

$$N = (0.1-3)\sqrt{\frac{3}{6(0.1)-(0.1)^2}} = \frac{-29\sqrt{177}}{59}$$

71. $\lim_{x\to\infty} \dfrac{1}{x^2} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$|f(x) - L| = \left|\frac{1}{x^2} - 0\right| = \frac{1}{x^2} < \varepsilon \text{ whenever } x > M.$$

$$x^2 > \frac{1}{\varepsilon} \Rightarrow x > \frac{1}{\sqrt{\varepsilon}}$$

Let $M = \dfrac{1}{\sqrt{\varepsilon}}$.

For $x > M$, you have

$$x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

72. $\lim_{x\to\infty} \dfrac{2}{\sqrt{x}} = 0$. Let $\varepsilon > 0$ be given. You need $M > 0$ such that

$$|f(x) - L| = \left|\frac{2}{\sqrt{x}} - 0\right| = \frac{2}{\sqrt{x}} < \varepsilon \text{ whenever } x > M.$$

$$\frac{2}{\sqrt{x}} < \varepsilon \Rightarrow \frac{\sqrt{x}}{2} > \frac{1}{\varepsilon} \Rightarrow x > \frac{4}{\varepsilon^2}$$

Let $M = 4/\varepsilon^2$.

For $x > M = 4/\varepsilon^2$, you have

$$\sqrt{x} > 2/\varepsilon \Rightarrow \frac{2}{\sqrt{x}} < \varepsilon \Rightarrow |f(x) - L| < \varepsilon.$$

73. $\lim_{x\to-\infty} \dfrac{1}{x^3} = 0$. Let $\varepsilon > 0$. You need $N < 0$ such that

$$|f(x) - L| = \left|\frac{1}{x^3} - 0\right| = \frac{-1}{x^3} < \varepsilon \text{ whenever } x < N.$$

$$\frac{-1}{x^3} < \varepsilon \Rightarrow -x^3 > \frac{1}{\varepsilon} \Rightarrow x < \frac{-1}{\varepsilon^{1/3}}$$

Let $N = \dfrac{-1}{\sqrt[3]{\varepsilon}}$.

For $x < N = \dfrac{-1}{\sqrt[3]{\varepsilon}}$,

$$\frac{1}{x} > -\sqrt[3]{\varepsilon}$$

$$-\frac{1}{x} < \sqrt[3]{\varepsilon}$$

$$-\frac{1}{x^3} < \varepsilon$$

$$\Rightarrow |f(x) - L| < \varepsilon.$$

74. $\lim_{x\to-\infty} \dfrac{1}{x-2} = 0$. Let $\varepsilon > 0$ be given.

You need $N < 0$ such that

$$|f(x) - L| = \left|\frac{1}{x-2} - 0\right| = \frac{-1}{x-2} < \varepsilon$$

whenever $x < N$.

$$\frac{-1}{x-2} < \varepsilon \Rightarrow x - 2 < \frac{-1}{\varepsilon} \Rightarrow x < 2 - \frac{1}{\varepsilon}$$

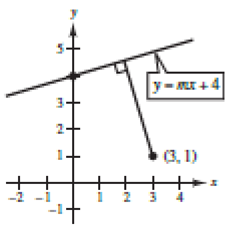
Let $N = 2 - \dfrac{1}{\varepsilon}$. For $x < N = 2 - \dfrac{1}{\varepsilon}$,

$$x - 2 < \frac{-1}{\varepsilon}$$

$$\frac{-1}{x-2} < \varepsilon$$

$$\Rightarrow |f(x) - L| < \varepsilon.$$

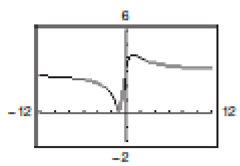
75. Line: $y = mx + 4$



(a) $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \dfrac{|m(3) - 1(1) + 4|}{\sqrt{m^2 + 1}}$

$= \dfrac{|3m + 3|}{\sqrt{m^2 + 1}}$

(b)



(c) $\lim_{m \to \infty} d(m) = 3 = \lim_{m \to -\infty} d(m)$

The line approaches the vertical line $x = 0$. So, the distance from $(3, 1)$ approaches 3.

76. $\lim_{x \to \infty} x^3 = \infty$. Let $M > 0$ be given. You need $N > 0$ such that $f(x) = x^3 > M$ whenever $x > N$.

$x^3 > M \Rightarrow x > M^{1/3}$. Let $N = M^{1/3}$. For $x > N = M^{1/3}$, $x > M^{1/3} \Rightarrow x^3 > M \Rightarrow f(x) > M$.

77. Evaluate each statement.

A: $f(10)$ may or may not be undefined based on the function f.
The statement may or may not be true.

B: $\lim_{x \to 10} f(x)$ may or may not exist based on the function f.
The statement may or may not be true.

C: Because $y = 10$ is a horizontal asymptote, $\lim_{x \to \infty} f(x) = 10$.
The statement must be true.

D: Even though $y = 10$ is a horizontal asymptote, there may be at least one value of x for which $f(x) = 10$.
The statement may or may not be true.

So, the answer is C.

78. $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^3 + x - 5}{4x^2 + 8 - 5x^3} = \lim_{x \to \infty} \frac{\frac{x^3}{x^3} + \frac{x}{x^3} - \frac{5}{x^3}}{\frac{4x^2}{x^3} + \frac{8}{x^3} - \frac{5x^3}{x^3}} = \lim_{x \to \infty} \frac{1 + \frac{1}{x^2} - \frac{5}{x^3}}{\frac{4}{x} + \frac{8}{x^3} - 5} = \frac{1 + 0 - 0}{0 + 0 - 5} = -\frac{1}{5}$$

Because $\lim_{x \to \infty} f(x) = -\frac{1}{5}$, the graph has a horizontal asymptote at $y = -\frac{1}{5}$. So, the answer is B.

79. $$\lim_{x \to \infty} \frac{4x - 3}{\sqrt{x^2 + 6}} = \lim_{x \to \infty} \frac{\frac{4x}{x} - \frac{3}{x}}{\sqrt{\frac{x^2}{x^2} + \frac{6}{x^2}}} = \lim_{x \to \infty} \frac{4 - \frac{3}{x}}{\sqrt{1 + \frac{6}{x^2}}} = \frac{4 - 0}{\sqrt{1 + 0}} = 4$$

So, the answer is C.

80. Evaluate each statement.

I. Because $\lim_{x \to 4} f(x) = 2$ and $f(4) = 8$, f is not continuous at $x = 4$.

The statement is true.

II. $$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2 + 4x - 32}{x^2 - 2x - 8} = \lim_{x \to \infty} \frac{1 + \frac{4}{x} - \frac{32}{x^2}}{1 - \frac{2}{x} - \frac{8}{x^2}} = \frac{1}{1} = 1 \neq 4$$

The statement is false.

III. $$f(x) = \frac{x^2 + 4x - 32}{x^2 - 2x - 8} = \frac{(x + 8)(x - 4)}{(x - 4)(x + 2)} = \frac{x + 8}{x + 2}, x \neq 4$$

f has a removable discontinuity at $x = 4$, not a vertical asymptote.

The statement is false.

Because I is the only statement that is true, the answer is A.