Section 4.6 The Natural Logarithmic Function: Integration

1. $\int \frac{5}{x}\,dx = 5\int \frac{1}{x}\,dx = 5\ln|x| + C$

2. $\int \frac{10}{x}\,dx = 10\int \frac{1}{x}\,dx = 10\ln|x| + C$

3. $u = x + 1, du = dx$

$$\int \frac{1}{x+1}\,dx = \ln|x+1| + C$$

4. $u = x - 5, du = dx$

$$\int \frac{1}{x-5}\,dx = \ln|x-5| + C$$

5. $u = 3x + 5, du = 3\,dx$

$$\int \frac{2}{3x+5}\,dx = \frac{2}{3}\int \frac{1}{3x+5}(3)\,dx = \frac{2}{3}\ln|3x+5| + C$$

6. $u = 5 - 4x, du = -4\,dx$

$$\int \frac{9}{5-4x}\,dx = -\frac{9}{4}\int \frac{1}{5-4x}(-4dx) = -\frac{9}{4}\ln|5-4x| + C$$

7. $u = x^2 - 3, du = 2x\,dx$

$$\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{1}{x^2-3}(2x)\,dx = \frac{1}{2}\ln|x^2-3| + C$$

8. $u = 5 - x^3, du = -3x^2\,dx$

$$\int \frac{x^2}{5-x^3}\,dx = -\frac{1}{3}\int \frac{1}{5-x^3}\left(-3x^2\right)dx = -\frac{1}{3}\ln|5-x^3| + C$$

9. $u = x^4 + 3x, du = \left(4x^3 + 3\right)dx$

$$\int \frac{4x^3+3}{x^4+3x}\,dx = \int \frac{1}{x^4+3x}\left(4x^3+3\right)dx = \ln|x^4+3x| + C$$

10. $u = x^3 - 3x^2, du = \left(3x^2 - 6x\right)dx = 3\left(x^2 - 2x\right)dx$

$$\int \frac{x^2-2x}{x^3-3x^2}\,dx = \frac{1}{3}\int \frac{1}{x^3-3x^2}\left(3x^2-6x\right)dx = \frac{1}{3}\ln|x^3-3x^2| + C$$

26. $\displaystyle\int \frac{x(x-2)}{(x-1)^3}\,dx = \int \frac{x^2-2x+1-1}{(x-1)^3}\,dx$

$$= \int \frac{(x-1)^2}{(x-1)^3}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \int \frac{1}{x-1}\,dx - \int \frac{1}{(x-1)^3}\,dx$$

$$= \ln|x-1| + \frac{1}{2(x-1)^2} + C$$

27. $u = 1 + \sqrt{2x},\ du = \dfrac{1}{\sqrt{2x}}\,dx \Rightarrow (u-1)\,du = dx$

$$\int \frac{1}{1+\sqrt{2x}}\,dx = \int \frac{(u-1)}{u}\,du = \int\left(1 - \frac{1}{u}\right)du$$

$$= u - \ln|u| + C_1$$

$$= \left(1+\sqrt{2x}\right) - \ln\left|1+\sqrt{2x}\right| + C_1$$

$$= \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

where $C = C_1 + 1$.

28. $u = 1 + \sqrt{3x},\ du = \dfrac{3}{2\sqrt{3x}}\,dx \Rightarrow dx = \dfrac{2}{3}(u-1)\,du$

$$\int \frac{1}{1+\sqrt{3x}}\,dx = \int \frac{1}{u}\frac{2}{3}(u-1)\,du$$

$$= \frac{2}{3}\int\left(1 - \frac{1}{u}\right)du$$

$$= \frac{2}{3}\left[u - \ln|u|\right] + C$$

$$= \frac{2}{3}\left[1 + \sqrt{3x} - \ln\left(1+\sqrt{3x}\right)\right] + C$$

$$= \frac{2}{3}\sqrt{3x} - \frac{2}{3}\ln\left(1+\sqrt{3x}\right) + C$$

29. $u = \sqrt{x} - 3,\ du = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow 2(u+3)du = dx$

$$\int \frac{\sqrt{x}}{\sqrt{x}-3}\,dx = 2\int \frac{(u+3)^2}{u}\,du$$

$$= 2\int \frac{u^2+6u+9}{u}\,du = 2\int\left(u + 6 + \frac{9}{u}\right)du$$

$$= 2\left[\frac{u^2}{2} + 6u + 9\ln|u|\right] + C_1$$

$$= u^2 + 12u + 18\ln|u| + C_1$$

$$= \left(\sqrt{x}-3\right)^2 + 12\left(\sqrt{x}-3\right) + 18\ln\left|\sqrt{x}-3\right| + C_1$$

$$= x + 6\sqrt{x} + 18\ln\left|\sqrt{x}-3\right| + C$$

where $C = C_1 - 27$.

30. $u = x^{1/3} - 1,\ du = \dfrac{1}{3x^{2/3}}\,dx \Rightarrow dx = 3(u+1)^2\,du$

$$\int \frac{\sqrt[3]{x}}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u}3(u+1)^2\,du$$

$$= 3\int \frac{u+1}{u}\left(u^2+2u+1\right)du$$

$$= 3\int\left(u^2 + 3u + 3 + \frac{1}{u}\right)du$$

$$= 3\left[\frac{u^3}{3} + \frac{3u^2}{2} + 3u + \ln|u|\right] + C$$

$$= 3\left[\frac{\left(x^{1/3}-1\right)^3}{3} + \frac{3\left(x^{1/3}-1\right)^2}{2} + 3\left(x^{1/3}-1\right) + \ln\left|x^{1/3}-1\right|\right] + C$$

$$= 3\ln\left|x^{1/3}-1\right| + \frac{3x^{2/3}}{2} + 3x^{1/3} + x + C$$

31. $\int \cot\left(\frac{\theta}{3}\right) d\theta = 3\int \cot\left(\frac{\theta}{3}\right)\left(\frac{1}{3}\right) d\theta = 3 \ln\left|\sin \frac{\theta}{3}\right| + C$

32. $\int \tan 5\theta \, d\theta = \frac{1}{5}\int \frac{5 \sin 5\theta}{\cos 5\theta} d\theta = -\frac{1}{5} \ln\left|\cos 5\theta\right| + C$

33. $\int \csc 2x \, dx = \frac{1}{2}\int (\csc 2x)(2) \, dx$

$= -\frac{1}{2} \ln\left|\csc 2x + \cot 2x\right| + C$

34. $\int \sec \frac{x}{2} dx = 2\int \sec \frac{x}{2}\left(\frac{1}{2}\right) dx = 2 \ln\left|\sec \frac{x}{2} + \tan \frac{x}{2}\right| + C$

35. $\int (\cos 3\theta - 1) \, d\theta = \frac{1}{3}\int \cos 3\theta(3) \, d\theta - \int d\theta$

$= \frac{1}{3} \sin 3\theta - \theta + C$

36. $\int \left(2 - \tan \frac{\theta}{4}\right) d\theta = \int 2 d\theta - 4\int \tan \frac{\theta}{4}\left(\frac{1}{4}\right) d\theta$

$= 2\theta + 4 \ln\left|\cos \frac{\theta}{4}\right| + C$

37. $u = 1 + \sin t, du = \cos t \, dt$

$\int \frac{\cos t}{1 + \sin t} dt = \ln\left|1 + \sin t\right| + C$

38. $u = \cot t, du = -\csc^2 t \, dt$

$\int \frac{\csc^2 t}{\cot t} dt = -\ln\left|\cot t\right| + C$

39. $u = \sec x - 1, du = \sec x \tan x \, dx$

$\int \frac{\sec x \tan x}{\sec x - 1} dx = \ln\left|\sec x - 1\right| + C$

40. $\int (\sec 2x + \tan 2x) \, dx = \frac{1}{2}\int (\sec 2x + \tan 2x)(2) \, dx = \frac{1}{2} \ln\left|\sec 2x + \tan 2x\right| - \ln\left|\cos 2x\right| + C$

41. $\int e^{-x} \tan\left(e^{-x}\right) dx = -\int \tan\left(e^{-x}\right)\left(-e^{-x}\right) dx$

$= -\left(-\ln\left|\cos\left(e^{-x}\right)\right|\right) + C$

$= \ln\left|\cos\left(e^{-x}\right)\right| + C$

42. $\int \sec t(\sec t + \tan t) \, dt = \int \sec^2 t \, dt + \int \sec t \tan t \, dt$

$= \tan t + \sec t + C$

43. Because $\int \frac{1}{u} du = \ln|u| + C,$ the final answer should have absolute value symbols around $x + 3$.

$\int \frac{x^2 + 5x - 6}{x + 3} dx = \int \left(x - 8 + \frac{18}{x + 3}\right) dx$

$= \frac{x^2}{2} - 8x + 18 \ln\left|x + 3\right| + C$

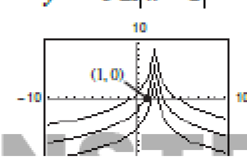
44. Because $f(x) = \frac{1}{x}$ has a nonremovable discontinuity at $x = 0, \frac{1}{x}$ is not differentiable on $[-1, 2]$.

$\int_{-1}^{2} \frac{1}{x} dx$ does not exist.

45. $y = \int \frac{3}{2 - x} dx = -3\int \frac{1}{x - 2} dx = -3 \ln\left|x - 2\right| + C$

$(1, 0)$: $0 = -3 \ln\left|1 - 2\right| + C \Rightarrow C = 0$

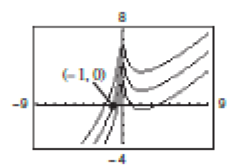
$y = -3 \ln\left|x - 2\right|$



46. $y = \int \frac{x - 2}{x} dx = \int \left(1 - \frac{2}{x}\right) dx = x - 2 \ln\left|x\right| + C$

$(-1, 0)$: $0 = -1 - 2 \ln\left|-1\right| + C = -1 + C \Rightarrow C = 1$

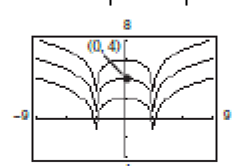
$y = x - 2 \ln\left|x\right| + 1$



47. $y = \int \frac{2x}{x^2 - 9} dx = \ln\left|x^2 - 9\right| + C$

$(0, 4)$: $4 = \ln\left|0 - 9\right| + C \Rightarrow C = 4 - \ln 9$

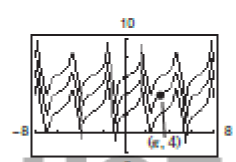
$y = \ln\left|x^2 - 9\right| + 4 - \ln 9$



48. $r = \int \frac{\sec^2 t}{\tan t + 1} dt = \ln\left|\tan t + 1\right| + C$

$(\pi, 4)$: $4 = \ln\left|0 + 1\right| + C \Rightarrow C = 4$

$r = \ln\left|\tan t + 1\right| + 4$



49. $f''(x) = \dfrac{2}{x^2} = 2x^{-2}, \quad x > 0$

$f'(x) = \dfrac{-2}{x} + C$

$f'(1) = 1 = -2 + C \Rightarrow C = 3$

$f'(x) = \dfrac{-2}{x} + 3$

$f(x) = -2\ln x + 3x + C_1$

$f(1) = 1 = -2(0) + 3 + C_1 \Rightarrow C_1 = -2$

$f(x) = -2\ln x + 3x - 2$

50. $f''(x) = \dfrac{-4}{(x-1)^2} - 2 = -4(x-1)^{-2} - 2, \quad x > 1$

$f'(x) = \dfrac{4}{(x-1)} - 2x + C$

$f'(2) = 0 = 4 - 4 + C \Rightarrow C = 0$

$f'(x) = \dfrac{4}{x-1} - 2x$

$f(x) = 4\ln(x-1) - x^2 + C_1$

$f(2) = 3 = 4(0) - 4 + C_1 \Rightarrow C_1 = 7$

$f(x) = 4\ln(x-1) - x^2 + 7$

51. $\displaystyle\int_0^4 \frac{5}{3x+1}\,dx = \left[\frac{5}{3}\ln|3x+1|\right]_0^4 = \frac{5}{3}\ln 13 \approx 4.275$

52. $\displaystyle\int_{-1}^{1} \frac{1}{2x+3}\,dx = \frac{1}{2}\Big[\ln|2x+3|\Big]_{-1}^{1}$

$= \dfrac{1}{2}[\ln 5 - \ln 1] = \dfrac{1}{2}\ln 5 \approx 0.805$

53. $u = 1 + \ln x,\ du = \dfrac{1}{x}\,dx$

$\displaystyle\int_1^e \frac{(1+\ln x)^2}{x}\,dx = \left[\frac{1}{3}(1+\ln x)^3\right]_1^e = \frac{7}{3}$

54. $u = \ln x,\ du = \dfrac{1}{x}dx$

$\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx = \int_e^{e^2}\left(\frac{1}{\ln x}\right)\frac{1}{x}\,dx = \Big[\ln|\ln|x|\Big]_e^{e^2} = \ln 2$

≈ 0.693

55. $\displaystyle\int_0^2 \frac{x^2-2}{x+1}\,dx = \int_0^2\left(x - 1 - \frac{1}{x+1}\right)dx$

$= \left[\dfrac{1}{2}x^2 - x - \ln|x+1|\right]_0^2 = -\ln 3$

≈ -1.099

56. $\displaystyle\int_0^1 \frac{x-1}{x+1}\,dx = \int_0^1 1\,dx + \int_0^1 \frac{-2}{x+1}\,dx$

$= \Big[x - 2\ln|x+1|\Big]_0^1 = 1 - 2\ln 2$

≈ -0.386

57. $\displaystyle\int_1^2 \frac{1-\cos\theta}{\theta - \sin\theta}\,d\theta = \Big[\ln|\theta - \sin\theta|\Big]_1^2$

$= \ln\left|\dfrac{2 - \sin 2}{1 - \sin 1}\right| \approx 1.929$

58. $u = 2\theta,\ du = 2\,d\theta,\ \theta = \dfrac{\pi}{8} \Rightarrow u = \dfrac{\pi}{4},\ \theta = \dfrac{\pi}{4} \Rightarrow u = \dfrac{\pi}{2}$

$\displaystyle\int_{\pi/8}^{\pi/4}(\csc 2\theta - \cot 2\theta)\,d\theta = \frac{1}{2}\int_{\pi/4}^{\pi/2}(\csc u - \cot u)\,du$

$= \dfrac{1}{2}\Big[-\ln|\csc u + \cot u| - \ln|\sin u|\Big]_{\pi/4}^{\pi/2}$

$= \dfrac{1}{2}\left[-\ln(1+0) - \ln(1) + \ln\left(\sqrt{2}+1\right) + \ln\dfrac{\sqrt{2}}{2}\right]$

$= \dfrac{1}{2}\left[\ln\left(\sqrt{2}+1\right) + \ln\dfrac{\sqrt{2}}{2}\right]$

$= \dfrac{1}{2}\ln\left(1 + \dfrac{\sqrt{2}}{2}\right)$

≈ 0.267

59. $\displaystyle\int \frac{1}{1+\sqrt{x}}\,dx = 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

60. $\displaystyle\int \frac{x^2}{x-1}\,dx = \ln|x-1| + \frac{x^2}{2} + x + C$

61. $\int_{\pi/4}^{\pi/2}(\csc x - \sin x)\,dx = \ln\left(\sqrt{2}+1\right) - \dfrac{\sqrt{2}}{2} \approx 0.174$

62. $\int_{-\pi/4}^{\pi/4} \dfrac{\sin^2 x - \cos^2 x}{\cos x}\,dx = \ln\left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right) - 2\sqrt{2}$

≈ -1.066

Note: In Exercises 63–66, you can use the Second Fundamental Theorem of Calculus or integrate the function.

63. $F(x) = \int_1^x \dfrac{1}{t}\,dt$

$F'(x) = \dfrac{1}{x}$

64. $F(x) = \int_0^x \tan t\,dt$

$F'(x) = \tan x$

65. $F(x) = \int_1^{3x} \dfrac{1}{t}\,dt$

$F'(x) = \dfrac{1}{3x}(3) = \dfrac{1}{x}$

(by Second Fundamental Theorem of Calculus)

Alternate Solution:

$F(x) = \int_1^{3x} \dfrac{1}{t}\,dt = \Big[\ln|t|\Big]_1^{3x} = \ln|3x|$

$F'(x) = \dfrac{1}{3x}(3) = \dfrac{1}{x}$

66. $F(x) = \int_1^{x^2} \dfrac{1}{t}\,dt$

$F'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

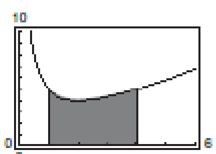
67. $A = \int_1^3 \dfrac{6}{x}\,dx = \Big[6\ln|x|\Big]_1^3 = 6\ln 3$

68. $A = \int_2^4 \dfrac{2}{x\ln x}\,dx = 2\int_2^4 \dfrac{1}{\ln x}\dfrac{1}{x}\,dx = 2\ln|\ln x|\Big]_2^4 = 2\big[\ln(\ln 4) - \ln(\ln 2)\big] = 2\ln\left(\dfrac{2\ln 2}{\ln 2}\right) = 2\ln 2$

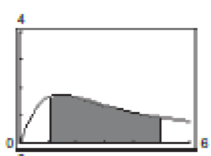
69. $A = \int_0^{\pi/4} \tan x\,dx = -\ln|\cos x|\Big]_0^{\pi/4} = -\ln\dfrac{\sqrt{2}}{2} + 0 = \ln\sqrt{2} = \dfrac{\ln 2}{2}$

70. $A = \int_{\pi/4}^{3\pi/4} \dfrac{\sin x}{1+\cos x}\,dx = -\ln|1+\cos x|\Big]_{\pi/4}^{3\pi/4} = -\ln\left(1 - \dfrac{\sqrt{2}}{2}\right) + \ln\left(1 + \dfrac{\sqrt{2}}{2}\right) = \ln\left(\dfrac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \ln\left(3 + 2\sqrt{2}\right)$

71. $A = \int_1^4 \dfrac{x^2+4}{x}\,dx = \int_1^4\left(x + \dfrac{4}{x}\right)dx = \left[\dfrac{x^2}{2} + 4\ln x\right]_1^4 = (8 + 4\ln 4) - \dfrac{1}{2} = \dfrac{15}{2} + 8\ln 2 \approx 13.045$

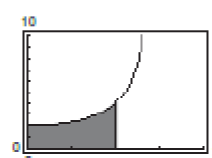


72. $A = \int_1^5 \dfrac{5x}{x^2+2}\,dx = \dfrac{5}{2}\int_1^5 \dfrac{1}{x^2+2}(2x\,dx) = \left[\dfrac{5}{2}\ln\left|x^2+2\right|\right]_1^5 = \dfrac{5}{2}(\ln 27 - \ln 3) = \dfrac{5}{2}\ln 9 = 5\ln 3 \approx 5.4931$

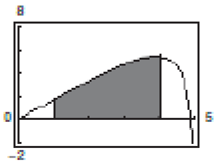


73. $\int_0^2 2\sec\dfrac{\pi x}{6}\,dx = \dfrac{12}{\pi}\int_0^2 \sec\left(\dfrac{\pi x}{6}\right)\dfrac{\pi}{6}\,dx = \dfrac{12}{\pi}\left[\ln\left|\sec\dfrac{\pi x}{6} + \tan\dfrac{\pi x}{6}\right|\right]_0^2$

$= \dfrac{12}{\pi}\left(\ln\left|\sec\dfrac{\pi}{3} + \tan\dfrac{\pi}{3}\right| - \ln|1+0|\right) = \dfrac{12}{\pi}\ln\left(2 + \sqrt{3}\right) \approx 5.0304$



74. $\int_1^4 (2x - \tan(0.3x))\,dx = \left[x^2 + \frac{10}{3}\ln\left|\cos(0.3x)\right|\right]_1^4 = \left[16 + \frac{10}{3}\ln\cos(1.2)\right] - \left[1 + \frac{10}{3}\ln\cos(0.3)\right] \approx 11.7686$



75. $f(x) = \frac{12}{x}, b - a = 5 - 1 = 4, n = 4$

Trapezoid: $\frac{4}{2(4)}\left[f(1) + 2f(2) + 2f(3) + 2f(4) + f(5)\right] = \frac{1}{2}[12 + 12 + 8 + 6 + 2.4] = 20.2$

Calculator: $\int_1^5 \frac{12}{x}\,dx \approx 19.3133$

Exact: $12 \ln 5$

76. $f(x) = \frac{8x}{x^2 + 4}, b - a = 4 - 0 = 4, n = 4$

Trapezoid: $\frac{4}{2(4)}\left[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)\right] = \frac{1}{2}[0 + 3.2 + 4 + 3.6923 + 1.6] \approx 6.2462$

Calculator: $\int_0^4 \frac{8x}{x^2 + 4}\,dx \approx 6.438$

Exact: $4 \ln 5$

77. $f(x) = \ln x, b - a = 6 - 2 = 4, n = 4$

Trapezoid: $\frac{4}{2(4)}\left[f(2) + 2f(3) + 2f(4) + 2f(5) + f(6)\right] = \frac{1}{2}[0.6931 + 2.1972 + 2.7726 + 3.2189 + 1.7918] \approx 5.3368$

Calculator: $\int_2^6 \ln x\,dx \approx 5.3643$

78. $f(x) = \sec x, b - a = \frac{\pi}{3} - \left(-\frac{\pi}{3}\right) = \frac{2\pi}{3}, n = 4$

Trapezoid: $\frac{2\pi/3}{2(4)}\left[f\left(-\frac{\pi}{3}\right) + 2f\left(-\frac{\pi}{6}\right) + 2f(0) + 2f\left(\frac{\pi}{6}\right) + f\left(\frac{\pi}{3}\right)\right] \approx \frac{\pi}{12}[2 + 2.3094 + 2 + 2.3094 + 2] \approx 2.7800$

Calculator: $\int_{-\pi/3}^{\pi/3} \sec x\,dx \approx 2.6339$

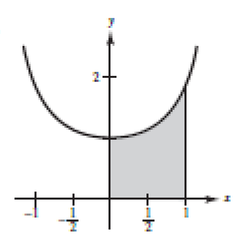
79. Power Rule

80. Substitution: $(u = x^2 + 4)$ and Power Rule

81. Substitution: $(u = x^2 + 4)$ and Log Rule

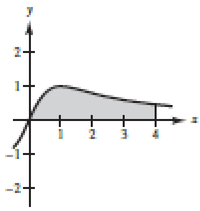
82. Substitution: $(u = \tan x)$ and Log Rule

83.



$A \approx 1.25$; Matches (d)

84.



$A \approx 3$; Matches (a)

85. $\displaystyle\int_1^x \frac{3}{t}\,dt = \int_{1/4}^x \frac{1}{t}\,dt$

$$\Big[3\ln|t|\Big]_1^x = \Big[\ln|t|\Big]_{1/4}^x$$

$$3\ln x = \ln x - \ln\left(\frac{1}{4}\right)$$

$$2\ln x = -\ln\left(\frac{1}{4}\right) = \ln 4$$

$$\ln x = \frac{1}{2}\ln 4 = \ln 2$$

$$x = 2$$

86. $\displaystyle\int_1^x \frac{1}{t}\,dt = \Big[\ln|t|\Big]_1^x = \ln x \qquad (\text{assume } x > 0)$

(a) $\ln x = \ln 5 \Rightarrow x = 5$

(b) $\ln x = 1 \Rightarrow x = e$

87. $\displaystyle\int \cot u\,du = \int \frac{\cos u}{\sin u}\,du = \ln|\sin u| + C$

Alternate solution:

$$\frac{d}{du}\Big[\ln|\sin u| + C\Big] = \frac{1}{\sin u}\cos u + C = \cot u + C$$

88. $\displaystyle\int \csc u\,du = \int \csc u\left(\frac{\csc u + \cot u}{\csc u + \cot u}\right)du = -\int \frac{1}{\csc u + \cot u}\left(-\csc u\cot u - \csc^2 u\right)du = -\ln|\csc u + \cot u| + C$

Alternate solution:

$$\frac{d}{du}\Big[-\ln|\csc u + \cot u| + C\Big] = -\frac{1}{\csc u + \cot u}\left(-\csc u\cot u - \csc^2 u\right) = \frac{\csc u(\cot u + \csc u)}{\csc u + \cot u} = \csc u$$

89. $\displaystyle -\ln|\cos x| + C = \ln\left|\frac{1}{\cos x}\right| + C = \ln|\sec x| + C$

90. $\displaystyle \ln|\sin x| + C = \ln\left|\frac{1}{\csc x}\right| + C = -\ln|\csc x| + C$

91.
$$\begin{aligned}\ln|\sec x + \tan x| + C &= \ln\left|\frac{(\sec x + \tan x)(\sec x - \tan x)}{(\sec x - \tan x)}\right| + C\\ &= \ln\left|\frac{\sec^2 x - \tan^2 x}{\sec x - \tan x}\right| + C\\ &= \ln\left|\frac{1}{\sec x - \tan x}\right| + C = -\ln|\sec x - \tan x| + C\end{aligned}$$

92.
$$\begin{aligned}-\ln|\csc x + \cot x| + C &= -\ln\left|\frac{(\csc x + \cot x)(\csc x - \cot x)}{(\csc x - \cot x)}\right| + C\\ &= -\ln\left|\frac{\csc^2 x - \cot^2 x}{\csc x - \cot x}\right| + C\\ &= -\ln\left|\frac{1}{\csc x - \cot x}\right| + C = \ln|\csc x - \cot x| + C\end{aligned}$$

93. Average value $= \frac{1}{4-2}\int_2^4 \frac{8}{x^2}\,dx$

$= 4\int_2^4 x^{-2}\,dx$

$= \left[-4\frac{1}{x}\right]_2^4$

$= -4\left(\frac{1}{4} - \frac{1}{2}\right) = 1$

94. Average value $= \frac{1}{4-2}\int_2^4 \frac{4(x+1)}{x^2}\,dx$

$= 2\int_2^4\left(\frac{1}{x} + \frac{1}{x^2}\right)dx$

$= 2\left[\ln x - \frac{1}{x}\right]_2^4$

$= 2\left[\ln 4 - \frac{1}{4} - \ln 2 + \frac{1}{2}\right]$

$= 2\left[\ln 2 + \frac{1}{4}\right] = \ln 4 + \frac{1}{2} \approx 1.8863$

95. Average value $= \frac{1}{e-1}\int_1^e \frac{2\ln x}{x}\,dx$

$= \frac{2}{e-1}\left[\frac{(\ln x)^2}{2}\right]_1^e$

$= \frac{1}{e-1}(1-0)$

$= \frac{1}{e-1} \approx 0.582$

96. Average value $= \frac{1}{2-0}\int_0^2 \sec\frac{\pi x}{6}\,dx$

$= \left[\frac{1}{2}\left(\frac{6}{\pi}\right)\ln\left|\sec\frac{\pi x}{6} + \tan\frac{\pi x}{6}\right|\right]_0^2$

$= \frac{3}{\pi}\left[\ln\left(2+\sqrt{3}\right) - \ln(1+0)\right]$

$= \frac{3}{\pi}\ln\left(2+\sqrt{3}\right)$

≈ 1.2576

97. $P(t) = \int \frac{3000}{1+0.25t}\,dt = (3000)(4)\int \frac{0.25}{1+0.25t}\,dt$

$= 12{,}000\ln|1+0.25t| + C$

$P(0) = 12{,}000\ln|1+0.25(0)| + C = 1000$

$C = 1000$

$P(t) = 12{,}000\ln|1+0.25t| + 1000$

$= 1000\left[12\ln|1+0.25t| + 1\right]$

$P(3) = 1000\left[12(\ln 1.75) + 1\right] \approx 7715$

98. $\frac{dS}{dt} = \frac{k}{t}$

$S(t) = \int \frac{k}{t}\,dt = k\ln|t| + C = k\ln t + C$ because $t > 1$.

$S(2) = k\ln 2 + C = 200$

$S(4) = k\ln 4 + C = 300$

Solving this system yields $k = 100/\ln 2$ and $C = 100$. So,

$S(t) = \frac{100\ln t}{\ln 2} + 100 = 100\left(\frac{\ln t}{\ln 2} + 1\right)$.

99. $t = \frac{10}{\ln 2}\int_{250}^{300}\frac{1}{T-100}\,dT$

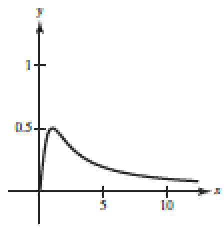
$= \frac{10}{\ln 2}\left[\ln(T-100)\right]_{250}^{300} = \frac{10}{\ln 2}\left[\ln 200 - \ln 150\right]$

$= \frac{10}{\ln 2}\left[\ln\left(\frac{4}{3}\right)\right] \approx 4.1504$ min

100. $\frac{1}{50-40}\int_{40}^{50}\frac{90{,}000}{400+3x}\,dx = \left[3000\ln|400+3x|\right]_{40}^{50}$

$\approx \$168.27$

101. $f(x) = \dfrac{x}{1 + x^2}$



(a) $y = \dfrac{1}{2}x$ intersects $f(x) = \dfrac{x}{1 + x^2}$:

$$\frac{1}{2}x = \frac{x}{1 + x^2}$$

$$1 + x^2 = 2$$

$$x = 1$$

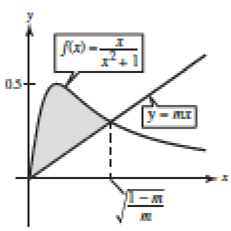
$$A = \int_0^1 \left(\left[\frac{x}{1 + x^2}\right] - \frac{1}{2}x\right) dx = \left[\frac{1}{2}\ln\left(x^2 + 1\right) - \frac{x^2}{4}\right]_0^1 = \frac{1}{2}\ln 2 - \frac{1}{4}$$

(b) $f'(x) = \dfrac{\left(1 + x^2\right) - x(2x)}{\left(1 + x^2\right)^2} = \dfrac{1 - x^2}{\left(1 + x^2\right)^2}$

$f'(0) = 1$

So, for $0 < m < 1$, the graphs of f and $y = mx$ enclose a finite region.

(c)



$f(x) = \dfrac{x}{x^2 + 1}$ intersects $y = mx$:

$$\frac{x}{1 + x^2} = mx$$

$$1 = m + mx^2$$

$$x^2 = \frac{1 - m}{m}$$

$$x = \sqrt{\frac{1 - m}{m}}$$

$$A = \int_0^{\sqrt{(1-m)/m}} \left(\frac{x}{1 + x^2} - mx\right) dx, \quad 0 < m < 1$$

$$= \left[\frac{1}{2}\ln\left(1 + x^2\right) - \frac{mx^2}{2}\right]_0^{\sqrt{(1-m)/m}}$$

$$= \frac{1}{2}\ln\left(1 + \frac{1 - m}{m}\right) - \frac{1}{2}m\left(\frac{1 - m}{m}\right)$$

$$= \frac{1}{2}\ln\left(\frac{1}{m}\right) - \frac{1}{2}(1 - m)$$

$$= \frac{1}{2}\left[m - \ln(m) - 1\right]$$

102. (a) At $x = -1$, $f'(-1) \approx \frac{1}{2}$.

The slope of f at $x = -1$ is approximately $\frac{1}{2}$.

(b) Because the slope is positive for $x > -2$, f is increasing on $(-2, \infty)$. Similarly, f is decreasing on $(-\infty, -2)$.

103. True

$$\int \frac{1}{x}\,dx = \ln|x| + C_1 = \ln|x| + \ln|C| = \ln|Cx|, C \neq 0$$

104. False

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

105. $\frac{d}{dx}\ln|x| = \frac{1}{x}$ implies that

$$\int \frac{1}{x}\,dx = \ln|x| + C.$$

The second formula follows by the Chain Rule.

106. $u = 7 - 8x$, $du = -8dx$

$$\int \frac{12}{7-8x}\,dx = \frac{12}{-8}\int \frac{1}{7-8x}(-8\,dx)$$

$$= -\frac{3}{2}\ln|7 - 8x| + C$$

So, the answer is A.

107. $F(x) = \int f(x)\,dx$

$$= \int \frac{2(\ln x)^4}{x}\,dx \left(u = \ln x,\ du = \frac{1}{x}\,dx\right)$$

$$= 2\int (\ln x)^4\left(\frac{1}{x}\,dx\right)$$

$$= \frac{2}{5}(\ln x)^5 + C$$

Find C when $F(1) = 0$.

$$F(1) = \frac{2}{5}[\ln(1)]^5 + C = 0$$

$$C = 0$$

$$F(6) = \frac{2}{5}(\ln 6)^5 \approx 7.387$$

So, the answer is C.

108. (a)

$$f(x) = -\tan x + 1$$

$$f'(x) = -\sec^2 x\,dx$$

$$f'\left(\frac{\pi}{6}\right) = -\sec^2\left(\frac{\pi}{6}\right) = -\left(\frac{2}{\sqrt{3}}\right)^2 = -\frac{4}{3}$$

$$f\left(\frac{\pi}{6}\right) = -\tan\frac{\pi}{6} + 1 = -\frac{1}{\sqrt{3}} + 1$$

Tangent line:

$$y - \left(-\frac{1}{\sqrt{3}} + 1\right) = -\frac{4}{3}\left(x - \frac{\pi}{6}\right)$$

$$y = -\frac{4}{3}x + \frac{2}{9}\pi - \frac{1}{\sqrt{3}} + 1$$

$$y = -\frac{4}{3}x + \frac{2\pi + 9 - 3\sqrt{3}}{9}$$

(b) Because $f(x) = 0$ when $x = \frac{\pi}{4}$, the region is bounded by the x-axis and y-axis on the interval $\left[0, \frac{\pi}{4}\right]$.

So, an expression is $\int_0^{\pi/4}(-\tan x + 1)\,dx$.

(c) $\dfrac{1}{\pi/6 - \pi/4}\displaystyle\int_{\pi/4}^{\pi/6}(-\tan x + 1)\,dx = \dfrac{12}{\pi}\int_{\pi/4}^{\pi/6}(\tan x - 1)\,dx$

$$= \frac{12}{\pi}\Big[-\ln|\cos x| - x\Big]_{\pi/4}^{\pi/6}$$

$$= \frac{12}{\pi}\left[\left(-\ln\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) - \left(-\ln\frac{\sqrt{2}}{2} - \frac{\pi}{4}\right)\right]$$

$$= \frac{12}{\pi}\left[\frac{1}{2}\ln 6 - \ln 3 + \frac{\pi}{12}\right]$$

$$\approx 0.2256$$

Note: $\ln\dfrac{\sqrt{2}}{2} - \ln\dfrac{\sqrt{3}}{2} = \ln\dfrac{\sqrt{2}/2}{\sqrt{3}/2}$

$$= \ln\frac{\sqrt{6}}{3}$$

$$= \frac{1}{2}\ln 6 - \ln 3$$

So, the average value is about 0.2256.

109. (a) $f'(x) = -\dfrac{3}{x-2} + x$

$$f(x) = \int\left(-\frac{3}{x-2} + x\right)dx$$

$$= -3\int\frac{1}{x-2}\,dx + \int x\,dx$$

$$= -3\ln|x-2| + \frac{1}{2}x^2 + C$$

$$f(3) = -3\ln|3-2| + \frac{1}{2}(3)^2 + C = 4$$

$$C = -\frac{1}{2}$$

So, $f(x) = -3\ln|x-2| + \dfrac{1}{2}x^2 - \dfrac{1}{2}$.

(b) As x increases without bound, so does $f(x)$. So, $\displaystyle\lim_{x\to\infty} f(x) = \infty$.