2.6 Derivatives of Inverse Functions

Continuity and Differentiability of Inverse Functions

Let f be a function whose domain is an interval I. If f has an inverse function, then the following statements are true:

1. If f is continuous on its domain, then f^{-1} is continuous on its domain.
2. If f is differentiable on an interval containing c and $f'(c) \neq 0$ then f^{-1} is differentiable at $f(c)$.

The Derivative of an Inverse Function

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$ and:

$$g'(x) = \frac{1}{f'(g(x))} \qquad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Examples: Using the Theorem to Find the Derivative of an Inverse Function

Let $f(x) = 3x^3 - x^2 - 8$

What is the value of $f^{-1}(x)$ when $x = 12$?

$12 = 3x^3 - x^2 - 8$

$f^{-1}(12) = 2$

$\therefore f(2) = 12$

What is the value of $(f^{-1})'(x)$ when $x = 12$?

$f'(x) = 9x^2 - 2x$

$f'(2) = 9(2)^2 - 2(2) = 9(4) - 4 = 32$

$\therefore (f^{-1})'(12) = \frac{1}{32}$

Examples: Slope Relationships in Inverse Functions

Let $f(x) = 2x^3$

Find $f^{-1}(x)$ $= \sqrt[3]{\frac{x}{2}}$

$$(f^{-1})'(x) = \left(\frac{1}{6}\right)\left(\frac{x}{2}\right)^{-2/3}$$

$f'(x) = 6x^2$

Find $f'(x)$ at $(1,2)$ $= 6$

Find $[f^{-1}(x)]'$ at $(2,1)$

$$[f^{-1}(2)]' = \left(\frac{1}{6}\right)\left(\frac{2}{2}\right)^{-2/3} = \frac{1}{6}$$

Find $f'(x)$ at $(-2,-16)$ $= 24$

Find $[f^{-1}(x)]'$ at $(-16,-2)$ $= \frac{1}{24}$

$$\left(\frac{1}{6}\right)\left(\frac{-16}{2}\right)^{-2/3} = \left(\frac{1}{6}\right)(-8)^{-2/3}$$

Example: Finding the Derivative of an Inverse Function

$$\frac{d}{dx}[arccosx] = \frac{1}{f'(arccos x)} = \frac{1}{-\sin(arccos x)}$$

$$= \frac{1}{-\sqrt{1-x^2}}$$

Derivatives of Inverse Trigonometric Functions * u is a function of x

$\frac{d}{dx}[arcsinu] = \frac{u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[arccosu] = \frac{-u'}{\sqrt{1-u^2}}$

$\frac{d}{dx}[arctanu] = \frac{u'}{1+u^2}$

$\frac{d}{dx}[arccotu] = \frac{-u'}{1+u^2}$

$\frac{d}{dx}[arcsecu] = \frac{u'}{|u|\sqrt{u^2-1}}$

$\frac{d}{dx}[arccscu] = \frac{-u'}{|u|\sqrt{u^2-1}}$

Examples: Finding the Derivatives of Inverse Functions

Find $f'(x)$ if $f(x) = arccscx^2$

Find $f'(x)$ if $f(x) = arccos4x$

$\frac{d}{dx}[arccote^{\frac{x}{3}}]$

Find the derivative of $y = arccosx - x\sqrt{1-x^2}$

$$y' = \frac{-1}{\sqrt{1-x^2}} - \left[x \cdot \frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x) + (1)(1-x^2)^{\frac{1}{2}}\right]$$

$$= \frac{-1}{\sqrt{1-x^2}} - \left[(1-x^2)^{-\frac{1}{2}}(-x^2 + 1 - x^2)\right]$$

$$= \frac{-1}{\sqrt{1-x^2}} - \left[(1-x^2)^{-\frac{1}{2}}(1-2x^2)\right]$$

$$= \frac{-1}{\sqrt{1-x^2}} - \frac{1-2x^2}{\sqrt{1-x^2}} = \frac{-2+2x^2}{\sqrt{1-x^2}}$$

$$= \frac{-2(1-x^2)}{\sqrt{1-x^2}} = -2\sqrt{1-x^2}$$