Section 4.3 Riemann Sums and Definite Integrals

1. $f(x) = \sqrt{x},\ y = 0,\ x = 0,\ x = 3,\ c_i = \dfrac{3i^2}{n^2}$

$$\Delta x_i = \frac{3i^2}{n^2} - \frac{3(i-1)^2}{n^2} = \frac{3}{n^2}(2i-1)$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x_i &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{\frac{3i^2}{n^2}}\frac{3}{n^2}(2i-1)\\
&= \lim_{n\to\infty}\frac{3\sqrt{3}}{n^3}\sum_{i=1}^{n}\left(2i^2 - i\right)\\
&= \lim_{n\to\infty}\frac{3\sqrt{3}}{n^3}\left[2\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right]\\
&= \lim_{n\to\infty}3\sqrt{3}\left[\frac{(n+1)(2n+1)}{3n^2} - \frac{n+1}{2n^2}\right]\\
&= 3\sqrt{3}\left[\frac{2}{3} - 0\right] = 2\sqrt{3} \approx 3.464
\end{aligned}$$

2. $f(x) = \sqrt[3]{x},\ y = 0,\ x = 0,\ x = 1,\ c_i = \dfrac{i^3}{n^3}$

$$\Delta x_i = \frac{i^3}{n^3} - \frac{(i-1)^3}{n^3} = \frac{3i^2 - 3i + 1}{n^3}$$

$$\begin{aligned}
\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \lim_{n\to\infty}\sum_{i=1}^{n}\sqrt[3]{\frac{i^3}{n^3}}\left[\frac{3i^2 - 3i + 1}{n^3}\right]\\
&= \lim_{n\to\infty}\frac{1}{n^4}\sum_{i=1}^{n}\left(3i^3 - 3i^2 + i\right)\\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[3\left(\frac{n^2(n+1)^2}{4}\right) - 3\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}\right]\\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4 + 6n^3 + 3n^2}{4} - \frac{2n^3 + 3n^2 + n}{2} + \frac{n^2 + n}{2}\right]\\
&= \lim_{n\to\infty}\frac{1}{n^4}\left[\frac{3n^4}{4} + \frac{n^3}{2} - \frac{n^2}{4}\right] = \lim_{n\to\infty}\left[\frac{3}{4} + \frac{1}{2n} - \frac{1}{4n^2}\right] = \frac{3}{4}
\end{aligned}$$

3. $y = 8$ on $[2, 6]$. $\left(\textbf{Note: } \Delta x = \dfrac{6-2}{n} = \dfrac{4}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(2 + \frac{4i}{n}\right)\left(\frac{4}{n}\right) = \sum_{i=1}^{n} 8\left(\frac{4}{n}\right) = \sum_{i=1}^{n} \frac{32}{n} = \frac{1}{n}\sum_{i=1}^{n} 32 = \frac{1}{n}(32n) = 32$$

$$\int_2^6 8\,dx = \lim_{n\to\infty} 32 = 32$$

4. $y = x$ on $[-2, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-(-2)}{n} = \dfrac{5}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\,\Delta x_i = \sum_{i=1}^{n} f\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-2 + \frac{5i}{n}\right)\left(\frac{5}{n}\right) = -10 + \frac{25}{n^2}\sum_{i=1}^{n} i = -10 + \left(\frac{25}{n^2}\right)\frac{n(n+1)}{2} = -10 + \frac{25}{2}\left(1 + \frac{1}{n}\right) = \frac{5}{2} + \frac{25}{2n}$$

$$\int_{-2}^{3} x\,dx = \lim_{n\to\infty}\left(\frac{5}{2} + \frac{25}{2n}\right) = \frac{5}{2}$$

5. $y = x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-1 + \frac{2i}{n}\right)^3\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right)$$

$$= -2 + \frac{12}{n^2}\sum_{i=1}^{n} i - \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= -2 + 6\left(1 + \frac{1}{n}\right) - 4\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right) = \frac{2}{n}$$

$$\int_{-1}^{1} x^3\,dx = \lim_{n\to\infty}\frac{2}{n} = 0$$

6. $y = 4x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{4-1}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} 4\left(1 + \frac{3i}{n}\right)^2\left(\frac{3}{n}\right)$$

$$= \frac{12}{n}\sum_{i=1}^{n}\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right)$$

$$= \frac{12}{n}\left[n + \frac{6}{n}\frac{n(n+1)}{2} + \frac{9}{n^2}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 12 + 36\frac{n+1}{n} + 18\frac{(n+1)(2n+1)}{n^2}$$

$$\int_1^4 4x^2\,dx = \lim_{n\to\infty}\left[12 + \frac{36(n+1)}{n} + \frac{18(n+1)(2n+1)}{n^2}\right]$$

$$= 12 + 36 + 36 = 84$$

7. $y = x^2 + 1$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\,\Delta x_i &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1 + \frac{i}{n}\right)^2 + 1\right]\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[1 + \frac{2i}{n} + \frac{i^2}{n^2} + 1\right]\left(\frac{1}{n}\right) \\
&= 2 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2 = 2 + \left(1 + \frac{1}{n}\right) + \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) = \frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}
\end{aligned}$$

$$\int_1^2 \left(x^2 + 1\right) dx = \lim_{n\to\infty}\left(\frac{10}{3} + \frac{3}{2n} + \frac{1}{6n^2}\right) = \frac{10}{3}$$

8. $y = 2x^2 + 3$ on $[-2, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-2)}{n} = \dfrac{3}{n}, \|\Delta\| \to 0 \text{ as } n \to \infty\right)$

$$\begin{aligned}
\sum_{i=1}^{n} f(c_i)\Delta x_i &= \sum_{i=1}^{n} f\left(-2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) \\
&= \sum_{i=1}^{n}\left[2\left(-2 + \frac{3i}{n}\right)^2 + 3\right]\left(\frac{3}{n}\right) \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[2\left(4 - \frac{12i}{n} + \frac{9i^2}{n^2}\right) + 3\right] \\
&= \frac{3}{n}\sum_{i=1}^{n}\left[11 - \frac{24i}{n} + \frac{18i^2}{n^2}\right] \\
&= \frac{3}{n}\left[11n - \frac{24}{n}\frac{n(n+1)}{2} + \frac{18}{n^2}\frac{n(n+1)(2n+1)}{6}\right] = 33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}
\end{aligned}$$

$$\int_{-2}^{1}\left(2x^2 + 3\right) dx = \lim_{n\to\infty}\left[33 - 36\frac{n+1}{n} + 9\frac{(n+1)(2n+1)}{n^2}\right] = 33 - 36 + 18 = 15$$

9. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}(3c_i + 10)\,\Delta x_i = \int_{-1}^{5}(3x + 10)\,dx$

on the interval $[-1, 5]$.

10. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} 6c_i(4 - c_i)^2\,\Delta x_i = \int_0^4 6x(4 - x)^2\,dx$

on the interval $[0, 4]$.

11. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\sqrt{c_i^2 + 4}\,\Delta x_i = \int_0^3\sqrt{x^2 + 4}\,dx$

on the interval $[0, 3]$.

12. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left(\frac{3}{c_i^2}\right)\Delta x_i = \int_1^3 \frac{3}{x^2}\,dx$

on the interval $[1, 3]$.

13. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left(1 + \frac{3}{c_i}\right)\Delta x_i = \int_1^5\left(1 + \frac{3}{x}\right)dx$

on the interval $[1, 5]$.

14. $\displaystyle\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\left(2^{-c_i}\sin c_i\right)\Delta x_i = \int_0^{\pi} 2^{-x}\sin x\,dx$

on the interval $[0, \pi]$.

15. $\int_0^4 5\, dx$

16. $\int_0^2 (6 - 3x)\, dx$

17. $\int_{-4}^4 (4 - |x|)\, dx$

18. $\int_0^2 x^2\, dx$

19. $\int_{-5}^5 (25 - x^2)\, dx$

20. $\int_{-1}^1 \frac{4}{x^2 + 2}\, dx$

21. $\int_0^{\pi/2} \cos x\, dx$

22. $\int_0^{\pi/4} \tan x\, dx$

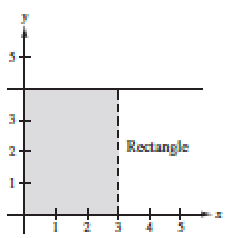
23. $\int_0^2 y^3\, dy$

24. $\int_0^2 (y - 2)^2\, dy$

25. $\int_1^4 \frac{2}{x}\, dx$

26. $\int_0^2 2e^{-x}\, dx$

27.

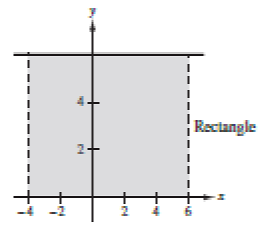


Rectangle

$A = bh = 3(4)$

$A = \int_0^3 4\, dx = 12$

28.

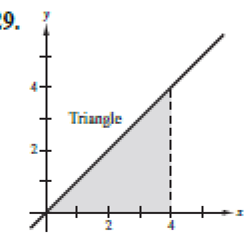


Rectangle

$A = bh = 10(6) = 60$

$A = \int_{-4}^6 6\, dx = 60$

29.

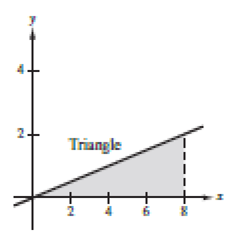


Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(4)(4) = 8$

$A = \int_0^4 x\, dx = 8$

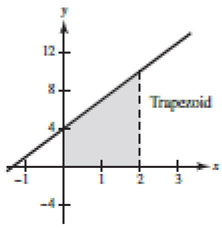
30.



Triangle

$A = \frac{1}{2}bh = \frac{1}{2}(8)(2) = 8$

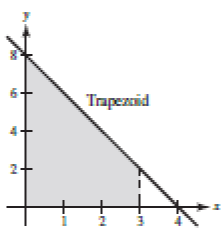
$A = \int_0^8 \frac{x}{4}\, dx = 8$

31. 

Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \left(\frac{4 + 10}{2}\right)2 = 14$$

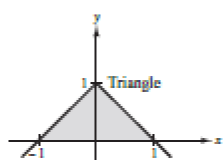
$$A = \int_0^2 (3x + 4)\,dx = 14$$

32. 

Trapezoid

$$A = \frac{b_1 + b_2}{2}h = \frac{8 + 2}{2}(3) = 15$$

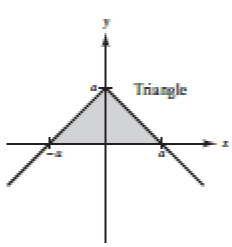
$$A = \int_0^3 (8 - 2x)\,dx = 15$$

33. 

Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2)(1) = 1$$

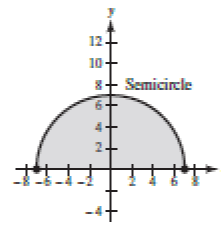
$$A = \int_{-1}^{1} (1 - |x|)\,dx = 1$$

34. 

Triangle

$$A = \tfrac{1}{2}bh = \tfrac{1}{2}(2a)\,a = a^2$$

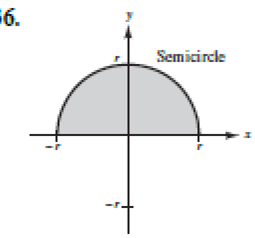
$$A = \int_{-a}^{a} (a - |x|)\,dx = a^2$$

35. 

Semicircle

$$A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi(7)^2 = \frac{49\pi}{2}$$

$$A = \int_{-7}^{7} \sqrt{49 - x^2}\,dx = \frac{49\pi}{2}$$

36. 

Semicircle

$$A = \tfrac{1}{2}\pi r^2$$

$$A = \int_{-r}^{r} \sqrt{r^2 - x^2}\,dx = \tfrac{1}{2}\pi r^2$$

In Exercises 37 – 44, $\int_2^4 x^3\,dx = 60$, $\int_2^4 x\,dx = 6$, $\int_2^4 dx = 2$

37. $\int_4^2 x\,dx = -\int_2^4 x\,dx = -6$

38. $\int_2^2 x^3\,dx = 0$

39. $\int_2^4 8x\,dx = 8\int_2^4 x\,dx = 8(6) = 48$

40. $\int_2^4 25\,dx = 25\int_2^4 dx = 25(2) = 50$

41. $\int_2^4 (x - 9)\,dx = \int_2^4 x\,dx - 9\int_2^4 dx = 6 - 9(2) = -12$

42. $\int_2^4 (x^3 + 4)\,dx = \int_2^4 x^3\,dx + 4\int_2^4 dx = 60 + 4(2) = 68$

43. $\int_2^4 \left(\frac{1}{2}x^3 - 3x + 2\right) dx = \frac{1}{2}\int_2^4 x^3\, dx - 3\int_2^4 x\, dx + 2\int_2^4 dx$

$= \frac{1}{2}(60) - 3(6) + 2(2) = 16$

44. $\int_2^4 \left(10 + 4x - 3x^3\right) dx = 10\int_2^4 dx + 4\int_2^4 x\, dx - 3\, dx$

$= 10(2) + 4(6) - 3(60) = -136$

45. (a) $\int_0^7 f(x)\, dx = \int_0^5 f(x)\, dx + \int_5^7 f(x)\, dx = 10 + 3 = 13$

(b) $\int_5^0 f(x)\, dx = -\int_0^5 f(x)\, dx = -10$

(c) $\int_5^5 f(x)\, dx = 0$

(d) $\int_0^5 3f(x)\, dx = 3\int_0^5 f(x)\, dx = 3(10) = 30$

46. (a) $\int_2^6 \left[f(x) + g(x)\right] dx = \int_2^6 f(x)\, dx + \int_2^6 g(x)\, dx$

$= 10 + (-2) = 8$

(b) $\int_2^6 \left[g(x) - f(x)\right] dx = \int_2^6 g(x)\, dx - \int_2^6 f(x)\, dx$

$= -2 - 10 = -12$

(c) $\int_2^6 2g(x)\, dx = 2\int_2^6 g(x)\, dx = 2(-2) = -4$

(d) $\int_2^6 3f(x)\, dx = 3\int_2^6 f(x)\, dx = 3(10) = 30$

47. Lower estimate: $[24 + 12 - 4 - 20 - 36](2) = -48$

Upper estimate: $[32 + 24 + 12 - 4 - 20](2) = 88$

48. (a) Left endpoint estimate: $[-6 + 8 + 30](2) = 64$

(b) Right endpoint estimate: $[8 + 30 + 80](2) = 236$

(c) Midpoint estimate: $[0 + 18 + 50](2) = 136$

If f is increasing, then (a) is below the actual value and (b) is above.

49. (a) Quarter circle below x-axis:

$-\frac{1}{4}\pi r^2 = -\frac{1}{4}\pi(2)^2 = -\pi$

(b) Triangle: $\frac{1}{2}bh = \frac{1}{2}(4)(2) = 4$

(c) Triangle + Semicircle below x-axis:

$-\frac{1}{2}(2)(1) - \frac{1}{2}\pi(2)^2 = -(1 + 2\pi)$

(d) Sum of parts (b) and (c): $4 - (1 + 2\pi) = 3 - 2\pi$

(e) Sum of absolute values of (b) and (c):

$4 + (1 + 2\pi) = 5 + 2\pi$

(f) Answers to (d) plus

$2(10) = 20$: $(3 - 2\pi) + 20 = 23 - 2\pi$

50. (a) $\int_0^1 -f(x)\, dx = -\int_0^1 f(x)\, dx = \frac{1}{2}$

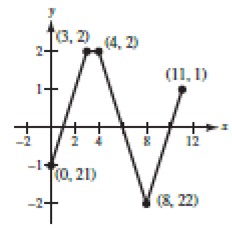
(b) $\int_3^4 3f(x)\, dx = 3(2) = 6$

(c) $\int_0^7 f(x)\, dx = -\frac{1}{2} + \frac{1}{2}(2)(2) + 2 + \frac{1}{2}(2)(2) - \frac{1}{2} = 5$

(d) $\int_5^{11} f(x)\, dx = \frac{1}{2} - \frac{1}{2}(4)(2) + \frac{1}{2} = -3$

(e) $\int_0^{11} f(x)\, dx = -\frac{1}{2} + 2 + 2 + 2 - 4 + \frac{1}{2} = 2$

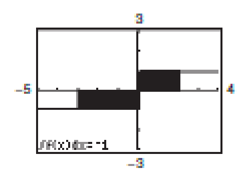
(f) $\int_4^{10} f(x)\, dx = 2 - 4 = -2$



51. (a) $\int_{-3}^{2} \frac{|x|}{x}\, dx$ is not continuous at $x = 0$.

(b) $\int_{-3}^{2} \frac{|x|}{x}\, dx = \int_{-3}^{0} (-1)\, dx + \int_0^2 (1)\, dx$

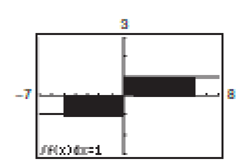
$= -3 + 2 = -1$



52. (a) $\int_{-5}^{6} \frac{x}{|x|}\, dx$ is not continuous at $x = 0$.

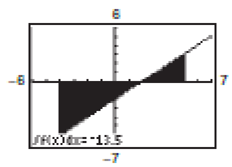
(b) $\int_{-5}^{6} \frac{x}{|x|}\, dx = \int_{-5}^{0} (-1)\, dx + \int_0^6 (1)\, dx$

$= -5 + 6 = 1$



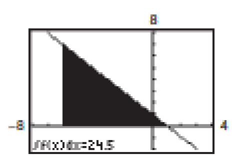
53. (a) $\int_{-4}^{5} \frac{x^2 - 4}{x + 2}\, dx$ is not continuous at $x = -2$.

(b) $$\int_{-4}^{5} \frac{x^2 - 4}{x + 2}\, dx = \int_{-4}^{5} (x - 2)\, dx$$
$$= \int_{-4}^{-2} (x - 2)\, dx + \int_{-2}^{5} (x - 2)\, dx$$
$$= \int_{-4}^{-2} x\, dx - 2\int_{-4}^{-2} dx + \int_{-2}^{5} x\, dx - 2\int_{-2}^{5} dx$$
$$= \frac{1}{2}(-12) - 2(2) + \frac{1}{2}(21) - 2(7)$$
$$= -\frac{27}{2}$$



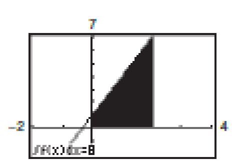
54. (a) $\int_{-6}^{1} \frac{1 - x^2}{x + 1}\, dx$ is not continuous at $x = -1$.

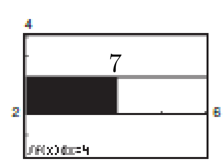
(b) $$\int_{-6}^{1} \frac{1 - x^2}{x + 1}\, dx = \int_{-6}^{1} (1 - x)\, dx$$
$$= \int_{-6}^{-1} (1 - x)\, dx + \int_{-1}^{1} (1 - x)\, dx$$
$$= \int_{-6}^{-1} dx - \int_{-6}^{-1} x\, dx + \int_{-1}^{1} dx - \int_{-1}^{1} x\, dx$$
$$= 5 - \frac{1}{2}(-35) + 2 - \frac{1}{2}(0)$$
$$= \frac{49}{2}$$



55. (a) $\int_{0}^{4} f(x)\, dx$ is not continuous at $x = 2$.

(b) $$\int_{0}^{4} f(x)\, dx = \int_{0}^{2} (3x + 1)\, dx + \int_{2}^{4} 2\, dx$$
$$= 3\int_{0}^{2} x\, dx + \int_{0}^{2} dx + 2\int_{2}^{4} dx$$
$$= 3\left(\tfrac{1}{2}\right)(4) + 2 + 2(2)$$
$$= 12$$



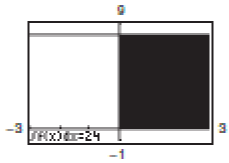


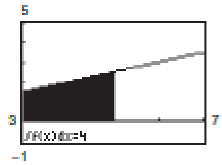
56. (a) $\int_0^5 g(x)\,dx$ is not continuous at $x = 3$.

(b) $\int_0^5 g(x)\,dx = 8\int_0^3 dx + \frac{1}{2}\int_3^5 x\,dx$

$= 8(3) + \frac{1}{2}\left(\frac{1}{2}\right)(16)$

$= 28$





57. Trapezoidal: $\int_0^2 x^2\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^2 + 2(1)^2 + 2\left(\frac{3}{2}\right)^2 + (2)^2\right] = \frac{11}{4} = 2.7500$

Exact: $\int_0^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} \approx 2.6667$

58. Trapezoidal: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx \approx \frac{1}{8}\left[\left(\frac{1^2}{4} + 1\right) + 2\left(\frac{(5/4)^2}{4} + 1\right) + 2\left(\frac{(3/2)^2}{4} + 1\right) + 2\left(\frac{(7/4)^2}{4} + 1\right) + \left(\frac{2^2}{4} + 1\right)\right] = \frac{203}{128} \approx 1.5859$

Exact: $\int_1^2 \left(\frac{x^2}{4} + 1\right) dx = \left[\frac{x^3}{12} + x\right]_1^2 = \frac{19}{12} \approx 1.5833$

59. Trapezoidal: $\int_0^2 x^3\,dx \approx \frac{1}{4}\left[0 + 2\left(\frac{1}{2}\right)^3 + 2(1)^3 + 2\left(\frac{3}{2}\right)^3 + (2)^3\right] = \frac{17}{4} = 4.2500$

Exact: $\int_0^2 x^3\,dx = \left[\frac{x^4}{4}\right]_0^2 = 4.0000$

60. Trapezoidal: $\int_2^3 \frac{2}{x^2}\,dx \approx \frac{1}{8}\left[\frac{2}{2^2} + 2\left(\frac{2}{(9/4)^2}\right) + 2\left(\frac{2}{(10/4)^2}\right) + 2\left(\frac{2}{(11/4)^2}\right) + \frac{2}{3^2}\right] \approx 0.3352$

Exact: $\int_2^3 \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_2^3 = -\frac{2}{3} + \frac{2}{2} = \frac{1}{3} \approx 0.3333$

61. Trapezoidal: $\int_1^3 x^3\,dx \approx \frac{1}{6}\left[1 + 2\left(\frac{4}{3}\right)^3 + 2\left(\frac{5}{3}\right)^3 + 2(2)^3 + 2\left(\frac{7}{3}\right)^3 + 2\left(\frac{8}{3}\right)^3 + 27\right] \approx 20.2222$

Exact: $\int_1^3 x^3\,dx = \left[\frac{x^4}{4}\right]_1^3 = \frac{81}{4} - \frac{1}{4} = 20.0000$

62. Trapezoidal: $\int_0^8 \sqrt[3]{x}\,dx \approx \frac{1}{2}\left[0 + 2 + 2\sqrt[3]{2} + 2\sqrt[3]{3} + 2\sqrt[3]{4} + 2\sqrt[3]{5} + 2\sqrt[3]{6} + 2\sqrt[3]{7} + 2\right] \approx 11.7296$

Exact: $\int_0^8 \sqrt[3]{x}\,dx = \left[\frac{3}{4}x^{4/3}\right]_0^8 = 12.0000$

63. Trapezoidal: $\int_4^9 \sqrt{x}\,dx \approx \frac{5}{16}\left[2 + 2\sqrt{\frac{37}{8}} + 2\sqrt{\frac{21}{4}} + 2\sqrt{\frac{47}{8}} + 2\sqrt{\frac{26}{4}} + 2\sqrt{\frac{57}{8}} + 2\sqrt{\frac{31}{4}} + 2\sqrt{\frac{67}{8}} + 3\right] \approx 12.6640$

Exact: $\int_4^9 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_4^9 = 18 - \frac{16}{3} = \frac{38}{3} \approx 12.6667$

64. Trapezoidal: $\int_1^4 \left(4 - x^2\right) dx \approx \frac{1}{4}\left\{3 + 2\left[4 - \left(\frac{3}{2}\right)^2\right] + 2(0) + 2\left[4 - \left(\frac{5}{2}\right)^2\right] + 2(-5) + 2\left[4 - \left(\frac{7}{2}\right)^2\right] - 12\right\} \approx -9.1250$

Exact: $\int_1^4 \left(4 - x^2\right) dx = \left[4x - \frac{x^3}{3}\right]_1^4 = -\frac{16}{3} - \frac{11}{3} = -9.0000$

65. Trapezoidal: $\int_0^2 \frac{1}{\sqrt{1 + x^3}}\,dx \approx \frac{1}{4}\left[1 + 2\left(\frac{1}{\sqrt{1 + (1/2)^3}}\right) + 2\left(\frac{1}{\sqrt{1 + 1^3}}\right) + 2\left(\frac{1}{\sqrt{1 + (3/2)^3}}\right) + \frac{1}{3}\right] \approx 1.397$

Graphing utility: 1.402

66. Trapezoidal: $\int_0^2 \sqrt{1 + x^3}\,dx \approx \frac{1}{4}\left[1 + 2\sqrt{1 + \left(\frac{1}{8}\right)} + 2\sqrt{2} + 2\sqrt{1 + \left(\frac{27}{8}\right)} + 3\right] \approx 3.283$

Graphing utility: 3.241

67. $\int_0^1 \sqrt{x}\sqrt{1 - x}\,dx = \int_0^1 \sqrt{x(1 - x)}\,dx$

Trapezoidal: $\int_0^1 \sqrt{x(1 - x)}\,dx \approx \frac{1}{8}\left[0 + 2\sqrt{\frac{1}{4}\left(1 - \frac{1}{4}\right)} + 2\sqrt{\frac{1}{2}\left(1 - \frac{1}{2}\right)} + 2\sqrt{\frac{3}{4}\left(1 - \frac{3}{4}\right)}\right] \approx 0.342$

Graphing utility: 0.393

68. Trapezoidal: $\int_0^4 \sqrt{x}e^x\,dx \approx \frac{1}{2}\left[0 + 2e^1 + 2\sqrt{2}e^2 + 2\sqrt{3}e^3 + 2e^4\right] \approx 102.555$

Graphing utility: 92.744

69. Trapezoidal: $\int_0^{\sqrt{\pi/2}} \sin\left(x^2\right) dx \approx \frac{\sqrt{\pi/2}}{8}\left[\sin 0 + 2\sin\left(\frac{\sqrt{\pi/2}}{4}\right)^2 + 2\sin\left(\frac{\sqrt{\pi/2}}{2}\right)^2 + 2\sin\left(\frac{3\sqrt{\pi/2}}{4}\right)^2 + \sin\left(\sqrt{\frac{\pi}{2}}\right)^2\right] \approx 0.550$

Graphing utility: 0.549

70. Trapezoidal: $\int_0^{\sqrt{\pi/4}} \tan\left(x^2\right) dx \approx \frac{\sqrt{\pi/4}}{8}\left[\tan 0 + 2\tan\left(\frac{\sqrt{\pi/4}}{4}\right)^2 + 2\tan\left(\frac{\sqrt{\pi/4}}{2}\right)^2 + 2\tan\left(\frac{3\sqrt{\pi/4}}{4}\right)^2 + \tan\left(\sqrt{\frac{\pi}{4}}\right)^2\right] \approx 0.271$

Graphing utility: 0.256

71. Trapezoidal: $\int_3^{3.1} \cos x^2\,dx \approx \frac{0.1}{8}\left[\cos(3)^2 + 2\cos(3.025)^2 + 2\cos(3.05)^2 + 2\cos(3.075)^2 + \cos(3.1)^2\right] \approx -0.098$

Graphing utility: -0.098

72. Trapezoidal: $\int_0^{\pi/2} \sqrt{1+\sin^2 x}\,dx \approx \frac{\pi}{16}\left[1 + 2\sqrt{1+\sin^2\left(\frac{\pi}{8}\right)} + 2\sqrt{1+\sin^2\left(\frac{\pi}{4}\right)} + 2\sqrt{1+\sin^2\left(\frac{3\pi}{8}\right)} + \sqrt{2}\right] \approx 1.910$

Graphing utility: 1.910

73. Trapezoidal: $\int_0^2 x\ln(x+1)\,dx \approx \frac{1}{4}\left[0 + 2(0.5)\ln(1.5) + 2\ln(2) + 2(1.5)\ln(2.5) + 2\ln(3)\right] \approx 1.684$

Graphing utility: 1.648

74. Trapezoidal: $\int_1^3 \ln x\,dx \approx \frac{1}{4}\left[0 + 2\ln(1.5) + 2\ln 2 + 2\ln(2.5) + \ln 3\right] \approx \frac{5.1284}{4} \approx 1.282$

Graphing utility: 1.296

75. Trapezoidal: $\int_0^2 xe^{-x}\,dx \approx \frac{1}{4}\left[0 + e^{-1/2} + 2e^{-1} + 3e^{-3/2} + 2e^{-2}\right] \approx \frac{2.2824}{4} \approx 0.571$

Graphing utility: 0.594

76. Trapezoidal: $\int_0^{\pi} \frac{\sin x}{x}\,dx \approx \frac{\pi}{8}\left[1 + \frac{2\sin(\pi/4)}{\pi/4} + \frac{2\sin(\pi/2)}{\pi/2} + \frac{2\sin(3\pi/4)}{3\pi/4} + 0\right] \approx 1.836$

Graphing utility: 1.852

77.
$$f(x) = 2x^3$$
$$f'(x) = 6x^2$$
$$f''(x) = 12x$$
$$f'''(x) = 12$$
$$f^{(4)}(x) = 0$$

$|E| \le \frac{(3-1)^3}{12(4^2)}(36) = 1.5$ because $|f''(x)|$ is maximum in [1, 3] when $x = 3$.

78.
$$f(x) = 5x + 2$$
$$f'(x) = 5$$
$$f''(x) = 0$$

The error is 0.

79.
$$f(x) = (x-1)^{-2}$$
$$f'(x) = -2(x-1)^{-3}$$
$$f''(x) = 6(x-1)^{-4}$$
$$f'''(x) = -24(x-1)^{-5}$$
$$f^{(4)}(x) = 120(x-1)^{-6}$$

$|E| \le \frac{(4-2)^3}{12(4^2)}(6) = \frac{1}{4}$ because $|f''(x)|$ is a maximum of 6 at $x = 2$.

80.
$$f(x) = \cos x$$
$$f'(x) = -\sin x$$
$$f''(x) = -\cos x$$
$$f'''(x) = \sin x$$
$$f^{(4)}(x) = \cos x$$

$|E| \le \frac{(\pi-0)^3}{12(4^2)}(1) = \frac{\pi^3}{192} \approx 0.1615$ because $|f''(x)|$ is at most 1 on $[0, \pi]$.

81.
$$f(x) = x^{-1}, \quad 1 \le x \le 3$$
$$f'(x) = -x^{-2}$$
$$f''(x) = 2x^{-3}$$
$$f'''(x) = -6x^{-4}$$
$$f^{(4)}(x) = 24x^{-5}$$

Maximum of $|f''(x)| = |2x^{-3}|$ is 2.

$|E| \le \frac{2^3}{12n^2}(2) \le 0.00001$, $n^2 \ge 133{,}333.33$, $n \ge 365.15$ Let $n = 366$.

82. $f(x) = (1+x)^{-1}, \quad 0 \le x \le 1$

$f'(x) = -(1+x)^{-2}$

$f''(x) = 2(1+x)^{-3}$

$f'''(x) = -6(1+x)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5}$

Maximum of $|f''(x)| = \left|2(1+x)^{-3}\right|$ is 2.

$$|E| \le \frac{1}{12n^2}(2) \le 0.00001$$

$$n^2 \ge 16{,}666.67$$

$n \ge 129.10$. Let $n = 130$.

83. $f(x) = (x+2)^{1/2}, \quad 0 \le x \le 2$

$f'(x) = \frac{1}{2}(x+2)^{-1/2}$

$f''(x) = -\frac{1}{4}(x+2)^{-3/2}$

$f'''(x) = \frac{3}{8}(x+2)^{-5/2}$

$f^{(4)}(x) = \frac{-15}{16}(x+2)^{-7/2}$

Maximum of $|f''(x)| = \left|\frac{-1}{4(x+2)^{3/2}}\right|$ is $\frac{\sqrt{2}}{16} \approx 0.0884$.

$$|E| \le \frac{(2-0)^3}{12n^2}\left(\frac{\sqrt{2}}{16}\right) \le 0.00001$$

$$n^2 \ge \frac{8\sqrt{2}}{12(16)}10^5 = \frac{\sqrt{2}}{24}10^5$$

$n \ge 76.8$. Let $n = 77$.

84. $f(x) = \sin x, \quad 0 \le x \le \frac{\pi}{2}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

All derivatives are bounded by 1.

$$|E| \le \frac{(\pi/2)^3}{12n^2}(1) \le 0.00001$$

$$n^2 \ge \frac{\pi^3}{96}10^5$$

$n \ge 179.7$. Let $n = 180$.

85. (a) $\int_0^5 [f(x) + 2]\,dx = \int_0^5 f(x)\,dx + \int_0^5 2\,dx = 4 + 10 = 14$

(b) $\int_{-2}^{3} f(x+2)\,dx = \int_0^5 f(x)\,dx = 4$ (Let $u = x + 2$.)

(c) $\int_{-5}^{5} f(x)\,dx = 2\int_0^5 f(x)\,dx = 2(4) = 8$ (f even)

(d) $\int_{-5}^{5} f(x)\,dx = 0$ (f odd)

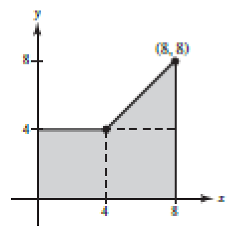
86. (a) The left endpoint approximation will be greater than the actual area, so

$$\sum_{i=1}^{n} f(x_i)\Delta x > \int_1^5 f(x)\,dx.$$

(b) The right endpoint approximation will be less than the actual area so,

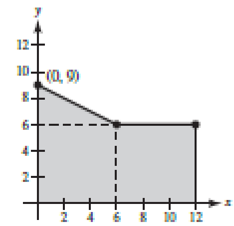
$$\sum_{i=1}^{n} f(x_i)\Delta x < \int_1^5 f(x)\,dx.$$

87. $f(x) = \begin{cases} 4, & x < 4 \\ x, & x \geq 4 \end{cases}$



$$\int_0^8 f(x)\,dx = 4(4) + 4(4) + \tfrac{1}{2}(4)(4) = 40$$

88. $f(x) = \begin{cases} 6, & x > 6 \\ -\frac{1}{2}x + 9, & x \leq 6 \end{cases}$



$$\int_0^{12} f(x)\,dx = 6(6) + \tfrac{1}{2}6(3) + 6(6) = 36 + 9 + 36 = 81$$

89. (a) No; The intervals are not of equal width.

(b) Using a trapezoidal sum,

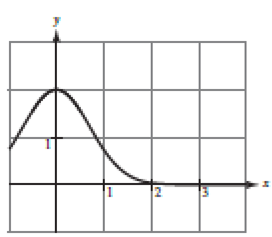
$$\int_0^2 f(x)\,dx \approx \left(\frac{4.32 + 4.58}{2}\right)(0.50 - 0) + \left(\frac{4.58 + 5.79}{2}\right)(0.75 - 0.50) + \cdots + \left(\frac{8.08 + 8.14}{2}\right)(2 - 1.75)$$
$$= 12.450.$$

(c) Using a graphing utility, $y = -1.25603x^3 + 3.7287x^2 - 0.513x + 4.29.$

$$\int_0^2 y\,dx \approx 12.473$$

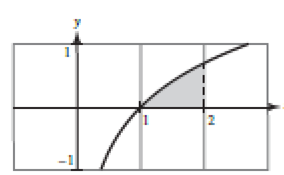
90. Area $\approx \dfrac{1000}{2(10)}\big[125 + 2(125) + 2(120) + 2(112) + 2(90) + 2(90) + 2(95) + 2(88) + 2(75) + 2(35)\big] = 89{,}250\text{ m}^2$

91. $\int_0^2 2e^{-x^2}\,dx$



(c) $A \approx 2$ square units

92. $\int_1^2 \ln x\,dx$

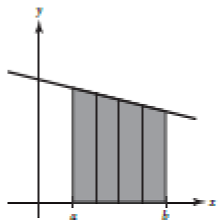


(a) $A \approx \dfrac{1}{3}$ square units

93. $f(x) = \dfrac{1}{x - 4}$

is not integrable on the interval [3, 5] because f has a discontinuity at $x = 4$.

94. For a linear function, the Trapezoidal Rule is exact. The error formula says that $E \le \frac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$ and $f''(x) = 0$ for a linear function. Geometrically, a linear function is approximated exactly by trapezoids.



95. $\int_{-2}^{1} f(x)\,dx + \int_{1}^{5} f(x)\,dx = \int_{-2}^{5} f(x)\,dx$

$a = -2, b = 5$

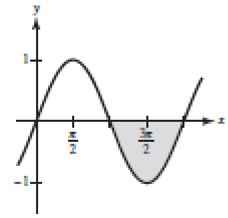
96. $\int_{-3}^{3} f(x)\,dx + \int_{3}^{6} f(x)\,dx - \int_{a}^{b} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

$\int_{-3}^{6} f(x)\,dx + \int_{b}^{a} f(x)\,dx = \int_{-1}^{6} f(x)\,dx$

$a = -3, b = -1$

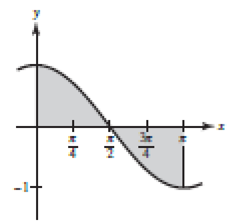
97. Answers will vary. *Sample answer:* $a = \pi$, $b = 2\pi$

$\int_{\pi}^{2\pi} \sin x\,dx < 0$



98. Answers will vary. *Sample answer:* $a = 0$, $b = \pi$

$\int_{0}^{\pi} \cos x\,dx = 0$



99. False

$\int_0^1 x\sqrt{x}\,dx \ne \left(\int_0^1 x\,dx\right)\left(\int_0^1 \sqrt{x}\,dx\right)$

100. True

101. $f(x) = x^2 + 3x, [0, 8]$

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 7,\ x_4 = 8$

$\Delta x_1 = 1,\ \Delta x_2 = 2,\ \Delta x_3 = 4,\ \Delta x_4 = 1$

$c_1 = 1, c_2 = 2,\ c_3 = 5,\ c_4 = 8$

$$\sum_{i=1}^{4} f(c_i)\Delta x = f(1)\Delta x_1 + f(2)\Delta x_2 + f(5)\Delta x_3 + f(8)\Delta x_4$$

$$= (4)(1) + (10)(2) + (40)(4) + (88)(1) = 272$$

102. $f(x) = \sin x, [0, 2\pi]$

$x_0 = 0,\ x_1 = \frac{\pi}{4},\ x_2 = \frac{\pi}{3},\ x_3 = \pi,\ x_4 = 2\pi$

$\Delta x_1 = \frac{\pi}{4},\ \Delta x_2 = \frac{\pi}{12},\ \Delta x_3 = \frac{2\pi}{3},\ \Delta x_4 = \pi$

$c_1 = \frac{\pi}{6},\ c_2 = \frac{\pi}{3},\ c_3 = \frac{2\pi}{3},\ c_4 = \frac{3\pi}{2}$

$$\sum_{i=1}^{4} f(c_i)\,\Delta x_i = f\left(\frac{\pi}{6}\right)\Delta x_1 + f\left(\frac{\pi}{3}\right)\Delta x_2 + f\left(\frac{2\pi}{3}\right)\Delta x_3 + f\left(\frac{3\pi}{2}\right)\Delta x_4$$

$$= \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{12}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{2\pi}{3}\right) + (-1)(\pi) \approx -0.708$$

103. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_a^b x\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} a + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i\right] \\
&= \lim_{n\to\infty} \left[\frac{b-a}{n}(an) + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2}\right] \\
&= \lim_{n\to\infty} \left[a(b-a) + \frac{(b-a)^2}{n}\frac{n+1}{2}\right] \\
&= a(b-a) + \frac{(b-a)^2}{2} \\
&= (b-a)\left[a + \frac{b-a}{2}\right] \\
&= \frac{(b-a)(a+b)}{2} = \frac{b^2-a^2}{2}
\end{aligned}$$

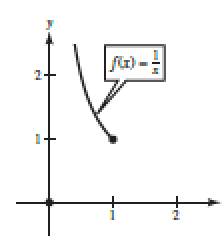
104. $\Delta x = \dfrac{b-a}{n},\ c_i = a + i(\Delta x) = a + i\left(\dfrac{b-a}{n}\right)$

$$\begin{aligned}
\int_a^b x^2\,dx &= \lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[a + i\left(\frac{b-a}{n}\right)\right]^2\left(\frac{b-a}{n}\right) \\
&= \lim_{n\to\infty} \left[\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} \left(a^2 + \frac{2ai(b-a)}{n} + i^2\left(\frac{b-a}{n}\right)^2\right)\right] \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\left[na^2 + \frac{2a(b-a)}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)(2n+1)}{6}\right] \\
&= \lim_{n\to\infty} \left[a^2(b-a) + \frac{a(b-a)^2(n+1)}{n} + \frac{(b-a)^3}{6}\frac{(n+1)(2n+1)}{n^2}\right] \\
&= a^2(b-a) + a(b-a)^2 + \frac{1}{3}(b-a)^3 = \frac{1}{3}\left(b^3 - a^3\right)
\end{aligned}$$

105. $f(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

is not integrable on the interval $[0, 1]$. As $\|\Delta\| \to 0$, $f(c_i) = 1$ or $f(c_i) = 0$ in each subinterval because there are an infinite number of both rational and irrational numbers in any interval, no matter how small.

106. $f(x) = \begin{cases} 0, & x = 0 \\ \dfrac{1}{x}, & 0 < x \le 1 \end{cases}$



The limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

does not exist.

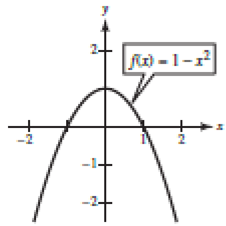
This does not contradict Theorem 4.4 because f is not continuous on $[0, 1]$.

107. The function f is nonnegative between $x = -1$ and $x = 1$.

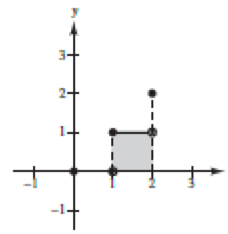
So, $\int_a^b (1 - x^2)\, dx$

is a maximum for

$a = -1$ and $b = 1$.



108. To find $\int_0^2 [\![x]\!]\, dx$, use a geometric approach.



So, $\int_0^2 [\![x]\!]\, dx = 1(2 - 1) = 1$.

109. Let $f(x) = x^2$, $0 \le x \le 1$, and $\Delta x_i = 1/n$. The appropriate Riemann Sum is

$$\sum_{i=1}^{n} f(c_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 \frac{1}{n} = \frac{1}{n^3}\sum_{i=1}^{n} i^2.$$

$$\lim_{n\to\infty} \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right] = \lim_{n\to\infty} \frac{1}{n^3}\cdot\frac{n(2n+1)(n+1)}{6} = \lim_{n\to\infty}\frac{2n^2 + 3n + 1}{6n^2} = \lim_{n\to\infty}\left(\frac{1}{3} + \frac{1}{2n} + \frac{1}{6n^2}\right) = \frac{1}{3}$$

110. $$\int_{-3}^{6}\left(-\tfrac{2}{3}x + 5\right) dx = -\tfrac{2}{3}\int_{-3}^{6} x\, dx + 5\int_{-3}^{6} dx$$
$$= -\tfrac{2}{3}\left(\tfrac{1}{2}\right)(27) + 5(9)$$
$$= 36$$

So, the answer is C.

111. Because $\int_0^1 x^2\, dx$ can be partitioned into ten rectangles where the width of each rectangle is $\frac{1}{10}, \frac{2}{10}, \cdots, \frac{9}{10}, \frac{10}{10}$, the Reimann sum approximation is

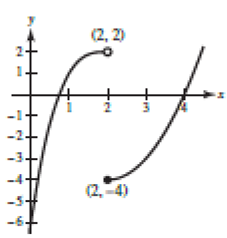
$$\tfrac{1}{10}\left[\left(\tfrac{1}{10}\right)^2 + \left(\tfrac{2}{10}\right)^2 + \left(\tfrac{3}{10}\right)^2 + \cdots + \left(\tfrac{10}{10}\right)^2\right].$$

So, the answer is A.

112. $$\int_2^8 f(x)\, dx = \int_2^3 f(x)\, dx + \int_3^5 f(x)\, dx + \int_5^8 f(x)\, dx$$
$$\approx \frac{3-2}{2}\left[f(2) + f(3)\right] + \frac{5-3}{2}\left[f(3) + f(5)\right] + \frac{8-5}{2}\left[f(5) + f(8)\right]$$
$$= \frac{1}{2}(8 + 22) + 1(22 + 72) + \frac{3}{2}(72 + 142)$$
$$= 430$$

So, the answer is C.

113. (a)



(b) No; f has a nonremovable discontinuity at $x = 2$, so f is not differentiable at $x = 2$.

(c) Yes; f can be integrated on [0, 4] by integrating on the intervals [0, 0.75], [0.75, 2], and [2, 4].

$$\text{So, } \int_0^4 f(x)\, dx = \int_0^{0.75} f(x)\, dx + \int_{0.75}^{2} f(x)\, dx + \int_2^4 f(x)\, dx$$
$$\approx -\tfrac{1}{2}(0.75)(6) + \tfrac{1}{2}(1.25)(2) + \tfrac{1}{2}(2)(4)$$
$$= -5.$$