4.7 Inverse Trigonometric Functions: Integration

Inverse Trigonometric Derivative Rules

$\frac{d}{dx}[arcsin u] = \frac{u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[arccos u] = \frac{-u'}{\sqrt{1-u^2}}$ $\frac{d}{dx}[arctan u] = \frac{u'}{1+u^2}$

$\frac{d}{dx}[arccsc u] = \frac{-u'}{|u|\sqrt{u^2-1}}$ $\frac{d}{dx}[arcsec u] = \frac{u'}{|u|\sqrt{u^2-1}}$ $\frac{d}{dx}[arccot u] = \frac{-u'}{1+u^2}$

Inverse Trigonometric Functions Integration Rules

$\int \frac{du}{\sqrt{a^2-u^2}} du = \arcsin\frac{u}{a} + C$ $\int \frac{du}{a^2+u^2} du = \frac{1}{a}\arctan\frac{u}{a} + C$ $\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$

Why are there only three? The derivative rules are the same, except for negatives

$\int \frac{dx}{\sqrt{16-3x^2}}$

$a^2 = 16$, $a = 4$

$u^2 = 3x^2$, $u = \sqrt{3}x$, $du = \sqrt{3}dx$, $dx = \frac{du}{\sqrt{3}}$

$\frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{4^2 - u^2}}$

$\frac{1}{\sqrt{3}} \arcsin \frac{\sqrt{3}x}{4} + C$

$\int \frac{dx}{\sqrt{e^{2x}-1}}$

$u = e^x$, $du = e^x dx$, $dx = \frac{du}{e^x}$, $dx = \frac{du}{u}$

$\int \frac{\frac{du}{u}}{\sqrt{u^2 - 1^2}}$

$= \int \frac{du}{u\sqrt{u^2-1^2}} = \frac{1}{1}\operatorname{arcsec}\frac{|u|}{1} + C$

$= 1\operatorname{arcsec} e^x + C$

Examples – Inverse Trigometry Integration

$\int \frac{dx}{2+9x^2}$

$a^2 = 2$, $a = \sqrt{2}$

$u^2 = 9x^2$, $u = 3x$, $du = 3dx$, $dx = \frac{du}{3}$

$\frac{1}{3}\int \frac{du}{(\sqrt{2})^2 + u^2}$

$\left(\frac{1}{3}\right)\left(\frac{1}{\sqrt{2}}\right)\arctan\frac{u}{\sqrt{2}} + C$

$= \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + C$

$\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

$a^2 = 4$, $a = 2$

$u^2 = x^2$, $u = x$, $du = dx$

$\arcsin\frac{x}{2}\Big|_0^1$

$= \arcsin\frac{1}{2} - \arcsin 0$

$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$

$\int \frac{dx}{x\sqrt{4x^2-9}}$ $\quad u^2=4x^2 \quad u=2x \quad du=2\,dx \quad dx=\frac{du}{2} \quad a=9 \quad a=3$

$\frac{1}{2}\int \frac{du}{\frac{u}{2}\sqrt{(2x)^2-3^2}}$

$\int \frac{du}{u\sqrt{u^2-3^2}} = \frac{1}{3}\operatorname{arcsec}\frac{|u|}{3}+C$

$= \frac{1}{3}\operatorname{arcsec}\frac{|2x|}{3}+C$

$\int \frac{t}{t^4+16}\,dt$ $\quad a^2=16 \quad a=4 \quad u^2=t^4 \quad u=t^2 \quad du=2t\,dt \quad dt=\frac{du}{2t}$

$\int \cancel{t}\cdot\frac{1}{u^2+4^2}\cdot\frac{du}{2\cancel{t}}$

$\frac{1}{2}\int\frac{1}{u^2+4^2}\,du = \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\arctan\frac{u}{4}+C$

$= \frac{1}{8}\arctan\frac{t^2}{4}+C$

$\int_{-\sqrt{3}}^{0}\frac{x}{1+x^2}\,dx$ $\quad u=1+x^2 \quad du=2x\,dx \quad dx=\frac{du}{2x} \quad u(0)=1 \quad u(-\sqrt{3})=4$

$\int_4^1 \cancel{x}\cdot\frac{1}{u}\cdot\frac{du}{2x}$

$= \frac{1}{2}\ln u\Big|_4^1 = \frac{1}{2}\ln 1-\frac{1}{2}\ln 4$

$= 0-\frac{1}{2}\ln 4 = \ln\frac{1}{2}$

$\int\frac{2x-5}{x^2+2x+2}\,dx$

$\int\frac{2x-5}{x^2+2x+1+2-1}\,dx = \int\frac{2x-5}{(x+1)^2+1}\,dx$ $\quad u=x+1 \quad du=dx \quad x=u-1 \quad a^2=1 \quad a=1$

$= \int\frac{2(u-1)-5}{u^2+1}\,du = \int\frac{2u-2-5}{u^2+1}\,du$

$= \int\frac{2u-7}{u^2+1}\,du = \int\frac{2u}{u^2+1}\,du - \int\frac{7}{u^2+1}\,du$

$w=u^2+1 \quad dw=2u\,du \quad du=\frac{dw}{2u}$

$= \int\cancel{2u}\cdot\frac{1}{w}\,\frac{dw}{2u} - 7\left(\frac{1}{1}\right)\arctan\frac{u}{1}+C = \ln(u^2+1)-7\arctan(x+1)+C$

$= \ln\left[(x+1)^2+1\right]-7\arctan(x+1)+C$

$\int\frac{1}{4+(x-1)^2}\,dx$ $\quad a^2=4 \quad a=2 \quad u^2=(x-1)^2 \quad u=x-1 \quad du=dx$

$\int\frac{1}{2^2+u^2}\,du = \frac{1}{2}\arctan\frac{u}{2}+C$

$= \frac{1}{2}\arctan\frac{x-1}{2}+C$

$\int_0^{\frac{1}{\sqrt{2}}}\frac{\arccos x}{\sqrt{1-x^2}}\,dx$ $\quad u=\arccos x \quad du=\frac{-1}{\sqrt{1-x^2}}\,dx \quad u\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi}{4} \quad u(0)=\frac{\pi}{2}$

$dx = du\cdot\sqrt{1-x^2}$

$-1\int_{\pi/2}^{\pi/4} u\cdot\frac{1}{\cancel{\sqrt{1-x^2}}}\cdot du\,\cancel{\sqrt{1-x^2}}$

$= (-1)\frac{u^2}{2}\Big|_{\pi/2}^{\pi/4} = \left(\frac{-1}{2}\right)\left(\frac{\pi}{4}\right)^2-\left(\frac{-1}{2}\right)\left(\frac{\pi}{2}\right)^2$

$= -\frac{\pi^2}{32}+\frac{\pi^2}{8} = \frac{3\pi^2}{32}$

$\int_{-3}^{-1}\frac{1}{x^2+6x+13}\,dx$

$\int_{-3}^{-1}\frac{1}{x^2+6x+9+13-9}\,dx = \int_{-3}^{-1}\frac{1}{(x+3)^2+4}\,dx$ $\quad u^2=(x+3)^2 \quad u=x+3 \quad du=dx \quad a^2=4 \quad a=2$

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$\int_0^2\frac{1}{u^2+a^2}\,du = \frac{1}{2}\arctan\frac{u}{2}\Big|_0^2$ $\quad u(-1)=2 \quad u(-3)=0$

$\frac{1}{2}\arctan 1-\frac{1}{2}\arctan 0$

$\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right)-\left(\frac{1}{2}\right)(0) = \frac{\pi}{8}$

$\int\frac{x^5+5x^2}{x^6+1}\,dx = \int\frac{x^5}{x^6+1}\,dx + \int\frac{5x^2}{x^6+1}\,dx$

$u=x^6+1 \quad du=6x^5\,dx \quad dx=\frac{du}{6x^5}$

$u^2=x^6 \quad u=x^3 \quad du=3x^2\,dx \quad dx=\frac{du}{3x^2}$

$\int x^5\cdot\frac{1}{u}\,\frac{du}{6x^5} + \int 5x^2\cdot\frac{1}{u^2+1}\,\frac{du}{3x^2}$

$\frac{1}{6}\ln u + \frac{5}{3}\left(\frac{1}{1}\right)\arctan\frac{x^3}{1}+C$

$\frac{1}{6}\ln|x^6+1| + \frac{5}{3}\arctan|x^3|+C$