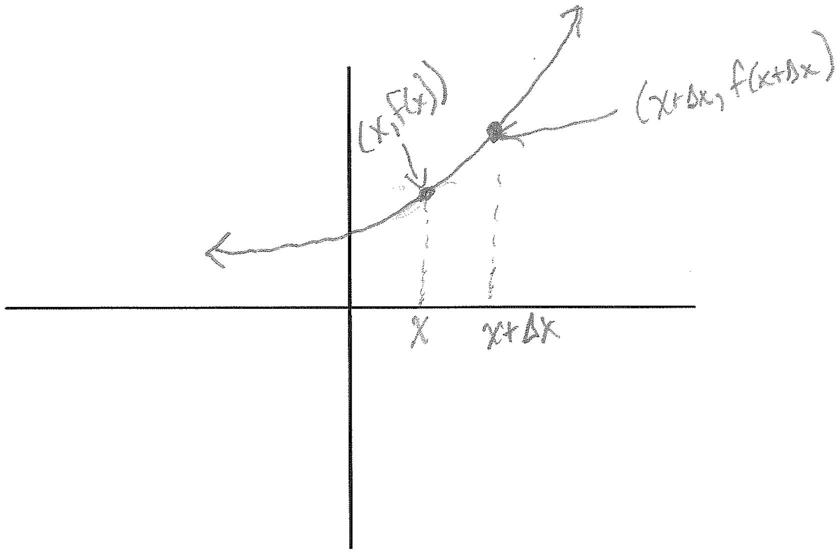
2.1 The Derivative and Tangent Line Problem



$$m_{sec} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Definition of Tangent Line with Slope m.
If f is defined on an open interval containing c and if the limit

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$$

Exists, then the line passing through $(c, f(c))$ with slope m is the tangent line to the graph of f at point $(c, f(c))$.

Definition of the Derivative of a Function

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(Instantaneous Rate of Change/Rate of Change)

Differentiation: Process of finding the derivative

Differentiable at a point: The derivative is defined at a point

Differentiable on an open interval: Differentiable at every point on the interval.

Notation: $f'(x)$, $\frac{dy}{dx}$, y', $\frac{d}{dx}[f(x)]$, $D_x[y]$

**Make sure you are comfortable with all of these!

Examples: Finding slopes of Tangent Lines

Use the definition of the slope of a tangent line to find the slope of the graph of $f(x) = 5x + 1$ when $c = 3$.

$$\lim_{\Delta x \to 0} \frac{5(3+\Delta x)+1 - (5(3)+1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{15} + 5\Delta x \cancel{+1} - \cancel{16}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 5 = 5$$

Find the slopes of the tangent lines to the graph of $f(x) = x^2 - 2$ at the points $(-3,7)$ and $(1,-1)$.

$$\lim_{\Delta x \to 0} \frac{[(x+\Delta x)^2 - 2] - (x^2-2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 2 - x^2 + 2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} 2x + \Delta x = 2x \qquad f'(x) = 2x$$

$2(-3) = -6 \therefore$ slope at $(-3,7)$ is -6

$2(1) = 2 \therefore$ slope at $(1,-1)$ is 2.

Examples: Finding and Using the Derivative

Find the derivative of $f(x) = 4x^2 - 5x$.

$$\lim_{\Delta x \to 0} \frac{4(x+\Delta x)^2 - 5(x+\Delta x) - (4x^2 - 5x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{4(x^2 + 2x\Delta x + (\Delta x)^2) - 5x - 5\Delta x - 4x^2 + 5x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{4x^2} + 8x\Delta x + 4(\Delta x)^2 - \cancel{5x} - 5\Delta x - \cancel{4x^2} + \cancel{5x}}{\Delta x} = \lim_{\Delta x \to 0} \frac{8x\Delta x + 4(\Delta x)^2 - 5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 8x + 4\Delta x - 5 = 8x - 5 \qquad \boxed{f'(x) = 8x - 5}$$

Find $f'(x)$ for $f(x) = \sqrt{x} + 1$. Then find the slopes of the graph at $(4,3)$ and $(9,4)$.

$$\lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} + 1 - (\sqrt{x}+1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x} \frac{(\sqrt{x+\Delta x} + \sqrt{x})}{(\sqrt{x+\Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{x} + \cancel{\Delta x} - \cancel{x}}{(\cancel{\Delta x})(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \quad \boxed{f'(x) = \frac{1}{2\sqrt{x}}}$$

$$\boxed{\begin{array}{l} f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}, \text{ slope at } (4,3) \text{ is } \frac{1}{4} \\ f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}, \text{ Slope at } (9,4) \text{ is } \frac{1}{6} \end{array}}$$

Find the derivative with respect to t for the function $y = \frac{1}{2t^2}$. Then find the equation of the tangent line to the graph at the point $\left(1, \frac{1}{2}\right)$.

$$\lim_{\Delta t \to 0} \frac{\frac{1}{2(t+\Delta t)^2} - \frac{1}{2t^2}}{\Delta t} \frac{(2)(t+\Delta t)^2(t^2)}{(2)(t+\Delta t)^2(t^2)} = \lim_{\Delta t \to 0} \frac{t^2 - (t+\Delta t)^2}{(\Delta t)(2)(t+\Delta t)^2(t^2)}$$

$$\boxed{y' = -\frac{1}{t^3}}$$

$$y'(1) = -\frac{1}{1^3} = -1$$

$$\boxed{y - \frac{1}{2} = -1(t-1)}$$

★ Alternative Form of the Definition of the Derivative

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Example: Using the Alternative Form of the Definition of the Derivative

Use the alternative form of the definition of the derivative to find $f'(3)$ if $f(x) = 3x^2 - 2$.

$$\lim_{x \to 3} \frac{3x^2 - 2 - (3(3)^2 - 2)}{x - 3} = \lim_{x \to 3} \frac{3x^2 - 2 - 25}{x - 3}$$

$$= \lim_{x \to 3} \frac{3x^2 - 27}{x - 3} = \lim_{x \to 3} \frac{3\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim_{x \to 3} 3(x+3) = 18$$

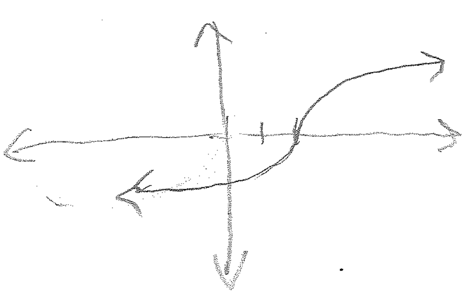
$$\boxed{f'(3) = 18}$$

Reasons that a function would not be differentiable at a point

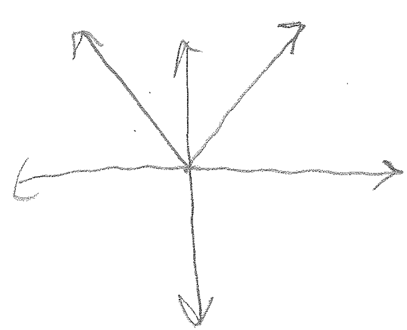
1. Not continuous (point doesn't exist)
2. Vertical Tangent Line (undefined slope)
3. Slope from the left does not equal the slope from the right.

Example: Determining Differentiability

Is $f(x) = (x-2)^{\frac{1}{5}}$ differentiable at $x = 2$.



$|x|$



Differentiability and Continuity

If f is differentiable at $x = c$, then f is continuous at c.

Is the converse true? Why?

Lesson Closer

1. Which of the following gives the slope of the tangent line to the graph of the function $f(x) = x^3$ at the point (2,8)?
   A. $\lim_{h \to 0} \frac{(x+2)^3 - x^3}{8}$
   B. $\lim_{h \to \infty} \frac{(2+h)^3 - 2^3}{8}$
   C. $\frac{(2+h)^3 - 2^3}{h}$
   D. $\lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h}$
   E. $\lim_{h \to 0} \frac{(8+h)^3 - 8^3}{h}$

2. The derivative of the function $f(x) = x^4$ may be expressed as a limit by which of the following?
   A. $\lim_{\Delta x \to 0} \frac{(x - \Delta x)^4 - x^4}{\Delta x}$
   B. $\lim_{\Delta x \to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x}$
   C. $\lim_{\Delta x \to 0} \frac{x^4 - (x - \Delta x)^4}{\Delta x}$
   D. Both A and B
   E. Both B and C