4.1 Antiderivatives and Indefinite Integrals, Day 1

Definition of Antiderivative

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

- If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$, for all x in I where C is a constant.
- $G(x)$ is called the "general solution" of the differential equation (equation that involves the derivatives of a function

Notation for Antiderivatives

Finding the general solution to a differential equation is called antidifferentiation OR indefinite integration

$$y = \int f(x)dx = F(x) + C$$

**Integration "undoes" differentiation

Basic Integration Rules

Differentiation Rules	Integration Formula
$\frac{d}{dx}[C] = 0$	$\int 0\,dx = C$
$\frac{d}{dx}[kx] = k$	$\int k\,dx = kx + C$
$\frac{d}{dx}[kf(x)] = kf'(x)$	$\int kf'(x)\,dx = kf(x) + C$
$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	$\int f'(x) \pm g'(x)\,dx = f(x) + g(x) + C$
$\frac{d}{dx}[x^n] = nx^{n-1}$	$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$
$\frac{d}{dx}[sinx] = cosx$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}[cosx] = -sinx$	$\int \sin x\,dx = -\cos x + C$

$$\frac{d}{dx}[tanx] = sec^2x$$

$\int sec^2x\, dx = tanx + C$

$$\frac{d}{dx}[secx] = secxtanx$$

$\int secx\, tanx\, dx = secx + C$

$$\frac{d}{dx}[cotx] = -csc^2x$$

$\int csc^2x\, dx = -cotx + C$

$$\frac{d}{dx}[cscx] = -cscxcotx$$

$\int cscx\, cotx\, dx = -cscx + C$

$$\frac{d}{dx}[e^x] = e^x$$

$\int e^x\, dx = e^x + C$

$$\frac{d}{dx}[a^x] = (lna)a^x$$

$\int a^x = \left(\frac{1}{lna}\right) a^x + C$

$$\frac{d}{dx}[lnx] = \frac{1}{x}, \qquad x > 0$$

$\int \frac{1}{x}\, dx = ln|x| + C$

$$\int \frac{x-2}{2x^2}\, dx = \int \frac{1}{2}x^{-1} - x^{-2}\, dx$$

$\frac{1}{2}(ln|x|) - \frac{x^{-1}}{-1} + C$

$\frac{1}{2}ln|x| + \frac{1}{x} + C$

$$\int \frac{sinx+1}{cos^2x}\, dx = \int \frac{sinx}{cos^2x}\, dx + \int \frac{1}{cos^2x}\, dx$$

$\int tanx\, secx\, dx + \int sec^2x\, dx$

$secx + tanx + C$

Examples – Finding Antiderivatives

$\int (x^4+3)dx$

$\frac{x^5}{5} + 3x + c = \boxed{\frac{1}{5}x^5 + 3x + C}$

$\int \sqrt[5]{x^2}\, dx = \int x^{2/5} dx$

$\frac{x^{7/5}}{7/5} + C = \boxed{\frac{5}{7}x^{7/5} + C}$

$\int 5\sec^2 x\; dx$

$\boxed{5\tan x + C}$

$\int \frac{1}{2x}\, dx = \int \frac{1}{2}x^{-1} dx$

$= \boxed{\frac{1}{2}\ln|x| + C}$

$\int (x-2)(x+2)dx = \int x^2 - 4\, dx$

$\boxed{\frac{1}{3}x^3 - 4x + C}$

$\int \left(\sqrt{x} + \frac{1}{2\sqrt{x}}\right) dx = \int x^{1/2} + \frac{1}{2}x^{-1/2} dx$

$\frac{x^{3/2}}{3/2} + \left(\frac{1}{2}\right)\left(\frac{x^{1/2}}{1/2}\right) + C = \boxed{\frac{2}{3}x^{3/2} + x^{1/2} + C}$

$\int (t^2 - \sin t)\, dt$

$\frac{1}{3}t^3 - (-\cos t) + C$

$\boxed{\frac{1}{3}t^3 + \cos t + C}$

$\int \frac{\sin x}{1-\sin^2 x}\, dx = \int \frac{\sin x}{\cos^2 x} dx$

$= \int \tan x \sec x\, dx$

$= \boxed{\sec x + C}$

$\int 4\, dx = \boxed{4x + C}$

$\int (2x-1)dx$

$= 2\left(\frac{x^2}{2}\right) - x + C = \boxed{x^2 - x + C}$

$\int (x^3 - 4x - 7)dx$

$\frac{1}{4}x^4 - 4\left(\frac{x^2}{2}\right) - 7x + C = \boxed{\frac{1}{4}x^4 - 2x^2 - 7x + C}$

$\int \frac{3}{\sqrt[3]{x^2}}\, dx = \int 3x^{-2/3} dx$

$3\left(\frac{x^{1/3}}{1/3}\right) + C = \boxed{9x^{1/3} + C}$

$\int x(x^2+3)dx = \int x^3 + 3x\, dx$

$\frac{1}{4}x^4 + 3\left(\frac{x^2}{2}\right) + C = \boxed{\frac{1}{4}x^4 + \frac{3}{2}x^2 + C}$

$\int \frac{x^2+1}{x^2}\, dx \quad \int 1 + x^{-2} dx$

$x + \frac{x^{-1}}{-1} + C = \boxed{x - \frac{1}{x} + C}$

$\int \sec y(\tan y - \sec y)dy = \int \sec y \tan y - \sec^2 y\; dy$

$= \boxed{\sec y - \tan y + C}$

$\int \sqrt[3]{x}(x-4)dx = \int x^{4/3} - 4x^{1/3} dx$

$\frac{x^{7/3}}{7/3} - 4\left(\frac{x^{4/3}}{4/3}\right) + C$

$\boxed{\frac{3}{7}x^{7/3} - 3x^{4/3} + C}$

Solving Differential Equations

$\frac{dy}{dx} = f'(x) \rightarrow dy = f'(x)dx$

$$\int dy = \int f'(x)dx$$

$$y = f(x) + C$$

Find the general solution of the differential equation $y' = \frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

$dy = \frac{1}{2}dx$

$\int dy = \int \frac{1}{2}dx$

$y = \frac{1}{2}x + C$

Find the equation of y given the derivative and the indicated point on the curve $\frac{dy}{dx} = 2(x-1)$, (3,2).

$dy = 2(x-1)dx$

$\int dy = 2\int x-1\,dx$

$y = 2\left(\frac{x^2}{2} - x\right) + C$

$y = x^2 - 2x + C$

$2 = (3)^2 - 2(3) + C$

$2 = 9 - 6 + C$

$2 = 3 + C$

$-1 = C$

$\boxed{y = x^2 - 2x - 1}$

Examples – Solving Differential Equations

Find the equation of y given the derivative and the indicated point on the curve $\frac{dy}{dx} = -\frac{1}{x^2}$ at (1,3).

$dy = -\frac{1}{x^2}dx$

$\int dy = \int -x^{-2}dx$

$y = -\left(\frac{x^{-1}}{-1}\right) + C$

$y = x^{-1} + C$

$3 = (1)^{-1} + C$

$2 = C$

$\boxed{y = \frac{1}{x} + 2}$

Find the general solution of the differential equation $\frac{dr}{d\theta} = \pi$.

$dr = \pi\,d\theta$

$\int dr = \int \pi\,d\theta$

$\boxed{r = \pi\theta + C}$