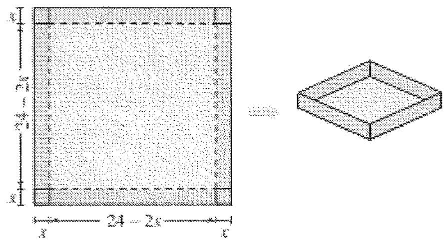
3.6: Optimization

An open box of maximum volume is to be made from a square piece of material, 24 in on each side, by cutting equal squares from the corners and turning up the sides.



Domain: $0 < x < 12$

a) Fill in the following chart below and guess the maximum volume

x: Height	Side Length	Volume
1	24-2(1)	484
2	24-2(2)	800
3	24-2(3)	972
4	24-2(4)	1024
5	24-2(5)	980
6	24-2(6)	864

b) Write the volume as a function of x.

$$V(x) = x(24-2x)(24-2x)$$

c) Use calculus to find the critical number of the function in part b.

$$V(x) = 576x - 96x^2 + 4x^3$$
$$V'(x) = 576 - 192x + 12x^2$$
$$0 = 12(x^2 - 16x + 48)$$

$$0 = 12(x-12)(x-4)$$
$$x = 4$$

d) Use the candidate's test to find the maximum volume. Verify using your graphing calculator.

4 is the only candidate

$$V''(x) = 24x - 192$$
$$V''(4) = 24(4) - 192 = -96$$

$\therefore x = 4$ is a maximum

Dimensions: 16 in x 16 in x 4 in   Max Volume: 1024 in^3

Problem Solving Strategy for Applied Maximum and Minimum Problems

1) Assign symbols to all given quantities to be determined. When possible, make a sketch to visualize the problem.
2) Write a primary equation for the quantity that is to be optimized (maximized or minimized).
3) Reduce the primary equation to one having a single independent variable. This may involve the use of secondary equations relating the independent variables of the primary equation.
4) Determine the domain of the primary equation. That is, determine the values for which the stated problem makes sense.
5) Determine the desired maximum or minimum value by the calculus techniques discussed in Sections 3.1 and 3.3.
   Note: When working on a **closed** interval, compare values of $f(x)$ at critical numbers **and** the endpoints. (Candidate's Test)
6) Find values of all variables and make sure you answer the original question.

Examples – Solving Optimization Problems

The sum of one positive number, x, and two times a second positive number, y, is 90. Find x and y if the product of x and the square of y is to be a maximum.

$x + 2y = 90$

$x = 90 - 2y$

$P(x,y) = xy^2$

$P(y) = (90-2y)y^2$

$P(y) = 90y^2 - 2y^3$

$P'(y) = 180y - 6y^2$

$0 = 6y(30 - y)$

$y = 0 \quad y = 30$

$P''(y) = 180 - 12y$

$P''(30) = 180 - 12(30) = -180$

$\therefore y = 30$ is a max.

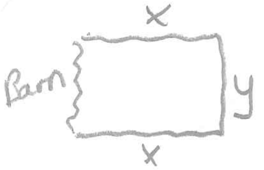
D: $(0, 45)$

$x = 90 - 2(30)$

$\boxed{x = 30 \quad y = 30}$

A farmer wishes to fence a rectangular enclosure adjacent to a barn such that the barn acts as one side of the enclosure.

a) If he has 2000 ft. of fencing for the other three sides, what is the maximum area he can enclose?



D: $0 < x < 1000$

$2x + y = 2000$

$y = 2000 - 2x$

$A(x,y) = x \cdot y$

$A(x) = x(2000 - 2x)$

$A(x) = 2000x - 2x^2$

$A'(x) = 2000 - 4x$

$0 = 2000 - 4x$

$4x = 2000$

$x = 500$

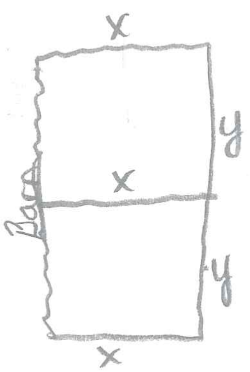
$A''(x) = -4$

$\therefore$ 500 is a max.

$A(500) = 500(2000 - 1000)$

$\boxed{= 500{,}000 \text{ ft}^2}$

b) Suppose the farmer instead wanted to use the 2000 ft of fencing to enclose two adjacent rectangular corrals (still using the barn along one side of the corrals.) What is the maximum area he can now enclose?



D: $0 < x < 666\frac{2}{3}$

$3x + 2y = 2000$

$2y = 2000 - 3x$

$y = 1000 - \frac{3}{2}x$

$A(x,y) = x(2y)$

$A(x) = x(2(1000 - \frac{3}{2}x))$

$A(x) = x(2000 - 3x)$

$A(x) = 2000x - 3x^2$

$A'(x) = 2000 - 6x$

$0 = 2000 - 6x$

$6x = 2000$

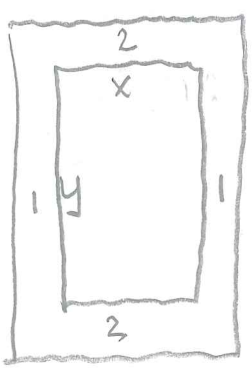
$x = 333\frac{1}{3}$

$A''(x) = -6$

$\therefore x = 333\frac{1}{3}$ is a max

$A(333\frac{1}{3}) = \boxed{333{,}333\frac{1}{3} \text{ ft}^2}$

A printed page contains 200 in^2 and must have 2 inch margins at the top and bottom as well as 1 inch side margins. What are the dimensions of the rectangular page that will allow for the most print?



D: $0 < x < 48$

$(x+2)(y+4) = 200$

$y + 4 = \frac{200}{x+2}$

$y = \frac{200}{x+2} - 4$

$P(x,y) = xy$

$P(x) = x\left(\frac{200}{x+2} - 4\right)$

$P(x) = \frac{200x}{x+2} - 4x$

$P'(x) = \frac{(x+2)(200) - 200x}{(x+2)^2} - 4$

$P'(x) = \frac{200x + 400 - 200x}{(x+2)^2} - 4$

$P'(x) = \frac{400 - 4(x+2)^2}{(x+2)^2}$

$P'(x) = 400 - 4(x^2 + 4x + 4)$

$P'(x) = 400 - 4x^2 - 16x - 16$

$0 = -4x^2 - 16x + 384$

$0 = -4(x^2 + 4x - 96)$

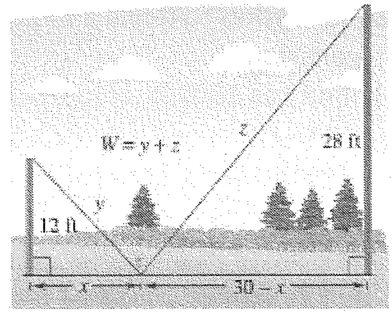
$0 = -4(x-8)(x+12)$

$x = 8$ ~~$x = -12$~~

$P'(8) = 0$, $P'(x) > 0$ for $(0, 8)$; $P'(x) < 0$ for $(8, 48)$

$\boxed{\text{Dimensions: } 10\text{in} \times 20\text{in}}$

Two posts, one 12 ft. high and the other 28 ft. high stand 30 ft. apart. They are to be stayed by two wires, attached to a single stake, running from ground level to the top of each post. Where should the stake be placed to use the least amount of wire?



$y = \sqrt{12^2 + x^2}$

$z = \sqrt{28^2 + (30-x)^2}$

$W = \sqrt{144 + x^2} + \sqrt{x^2 - 60x + 1684}$

$W' = \frac{1}{2}(144+x^2)^{-1/2}(2x) + \frac{1}{2}(x^2 - 60x + 1684)^{-\frac{1}{2}}(2x-60)$

$0 = \frac{1}{2}(144+x^2)^{-\frac{1}{2}}(2x) + \frac{1}{2}(x^2 - 60x + 1684)^{-\frac{1}{2}}(2x-60)$

$x = 9$ (used calculator)

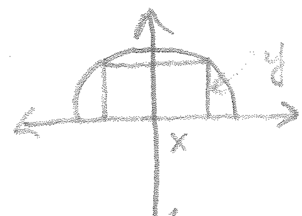
$W(0) = 53.087$

$W(9) = 50$

$W(30) = 60.311$

The stake should be placed 9 feet from the base of the 12 ft. pole

A rectangle is bounded by the x-axis and the semicircle $y = \sqrt{36 - x^2}$. What length and width should the rectangle have so that its area is maximized?



$D: -6 < x < 6$

$A(x,y) = 2xy$

$A(x) = 2x(36-x^2)^{\frac{1}{2}}$

$A'(x) = 2x(\frac{1}{2})(36-x^2)^{-\frac{1}{2}}(-2x) + 2(36-x^2)^{\frac{1}{2}}$

$A'(x) = 2(36-x^2)^{-\frac{1}{2}}[-x^2 + 36 - x^2]$

$A'(x) = \frac{2(-2x^2+36)}{(36-x^2)^{-\frac{1}{2}}}$ $x = \pm 3\sqrt{2}$ $x \neq \pm 6$

Length: $6\sqrt{2}$

Width: $3\sqrt{2}$

$A'(3\sqrt{2}) = 0$, $A'(x) < 0$ $3\sqrt{2} < x < 6$, $A'(x) > 0$, $-3\sqrt{2} < x < 3\sqrt{2}$

The demand function for a product is modeled by $p = 56e^{-.000012x}$, where p is the price per unit (in dollars) and x is the number of units. What price will yield a maximum revenue?

$R = x(56e^{-.000012x})$

$R' = x(56)(-.000012)(e^{-.000012x}) + 56e^{-.000012x}$

$0 = 56e^{-.000012x}(-.000012x + 1)$

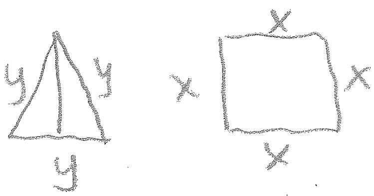
$x = 83,333\frac{1}{3}$

$p = 56e^{-.000012(83,333\frac{1}{3})}$

$\approx \$20.60$

$R'(83333\frac{1}{3}) = 0$, $R'(x) > 0$ $0 < x < 83,333\frac{1}{3}$, $R'(x) < 0$ $83,333\frac{1}{3} < x < \infty$

The combined perimeters of an equilateral triangle and a square is 10. Find the dimensions of the triangle and the square that produce the minimum total area.



$A = \frac{\sqrt{3}}{4} y^2$

$3y + 4x = 10$

$3y = 10 - 4x$

$y = \frac{10}{3} - \frac{4}{3}x$

$A = \frac{\sqrt{3}}{4} y^2 + x^2$

$A = \frac{\sqrt{3}}{4} \left(\frac{10}{3} - \frac{4}{3}x\right)^2 + x^2$

$A = \frac{\sqrt{3}}{4} \left(\frac{100}{9} - \frac{80}{9}x + \frac{16}{9}x^2\right) + x^2$

$A = \frac{100\sqrt{3}}{36} - \frac{80\sqrt{3}}{36}x + \frac{16\sqrt{3}}{36}x^2 + \frac{36}{36}x^2$

$A = \left(\frac{36 + 16\sqrt{3}}{36}\right)x^2 - \frac{80\sqrt{3}}{36}x + \frac{100\sqrt{3}}{36}$

$A' = \frac{36 + 16\sqrt{3}}{18}x - \frac{80\sqrt{3}}{36}$

$\frac{80\sqrt{3}}{36} = \frac{36 + 16\sqrt{3}}{18}x$

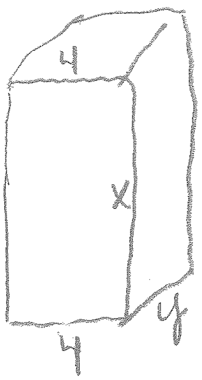
$x = \frac{80\sqrt{3}}{36} \cdot \frac{18}{36 + 16\sqrt{3}} \approx 1.087$

$A''(1.087) = \frac{36 + 16\sqrt{3}}{18} > 0$, minimum

$x \approx 1.087$

$y \approx 1.884$

A tank with a rectangular base and rectangular sides is open at the top. It is to be constructed so that its width is 4 meters and its volume is 36 cubic meters. If building the tank costs $10 per square meter for the base and $5 per square meter for the sides, what is the cost of the least expensive tank, and what are its dimensions?



D: $0 < x < \infty$

$36 = 4xy$

$y = \frac{36}{4x} = \frac{9}{x}$

$Cost = (4)\left(\frac{9}{x}\right)(10) + 2(4x)(5) + 2(x)\left(\frac{9}{x}\right)(5)$

$Cost = \frac{360}{x} + 40x + 90$

$C' = -360x^{-2} + 40$

$C' = \frac{-360 + 40x^2}{x^2}$

$0 = -360 + 40x^2$

$360 = 40x^2$

$9 = x^2$

$\pm 3 = x$ only consider 3

$c'(3) = 0$

$c'(x) < 0, \quad 0 < x < 3$

$c'(x) > 0, \quad 3 < x < \infty$

$C = 4\left(\frac{9}{3}\right)(10) + 2(12)(5) + 2(3)\left(\frac{9}{3}\right)(5)$

$= 120 + 120 + 90 = \$330$

Dimensions 3 m x 3 m x 4 m