Section 4.2 Area

1. $\displaystyle\sum_{i=1}^{6}(3i+2) = 3\sum_{i=1}^{6} i + \sum_{i=1}^{6} 2 = 3(1+2+3+4+5+6)+12 = 75$

2. $\displaystyle\sum_{k=3}^{9}\left(k^2+1\right) = \left(3^2+1\right)+\left(4^2+1\right)+\ldots+\left(9^2+1\right) = 287$

3. $\displaystyle\sum_{k=0}^{4}\frac{1}{k^2+1} = 1+\frac{1}{2}+\frac{1}{5}+\frac{1}{10}+\frac{1}{17} = \frac{158}{85}$

4. $\displaystyle\sum_{j=4}^{6}\frac{3}{j} = \frac{3}{4}+\frac{3}{5}+\frac{3}{6} = \frac{37}{20}$

5. $\displaystyle\sum_{k=1}^{4} c = c+c+c+c = 4c$

6. $\displaystyle\sum_{i=1}^{4}\left[(i-1)^2+(i+1)^3\right] = (0+8)+(1+27)+(4+64)+(9+125) = 238$

7. $\displaystyle\sum_{i=1}^{11}\frac{1}{5i}$

8. $\displaystyle\sum_{i=1}^{14}\frac{9}{1+i}$

9. $\displaystyle\sum_{j=1}^{6}\left[7\left(\frac{j}{6}\right)+5\right]$

10. $\displaystyle\sum_{j=1}^{4}\left[1-\left(\frac{j}{4}\right)^2\right]$

11. $\frac{2}{n}\sum_{i=1}^{n}\left[\left(\frac{2i}{n}\right)^3 - \left(\frac{2i}{n}\right)\right]$

12. $\frac{3}{n}\sum_{i=1}^{n}\left[2\left(1 + \frac{3i}{n}\right)^2\right]$

13. $\sum_{i=1}^{12} 7 = 7(12) = 84$

14. $\sum_{i=1}^{30} (-18) = (-18)(30) = -540$

15. $\sum_{i=1}^{24} 4i = 4\sum_{i=1}^{24} i = 4\left[\frac{24(25)}{2}\right] = 1200$

16. $\sum_{i=1}^{16} (5i - 4) = 5\sum_{i=1}^{16} i - 4(16) = 5\left[\frac{16(17)}{2}\right] - 64 = 616$

17. $\sum_{i=1}^{20} (i-1)^2 = \sum_{i=1}^{19} i^2 = \left[\frac{19(20)(39)}{6}\right] = 2470$

18. $\sum_{i=1}^{10} (i^2 - 1) = \sum_{i=1}^{10} i^2 - \sum_{i=1}^{10} 1 = \left[\frac{10(11)(21)}{6}\right] - 10 = 375$

19.
$$\begin{aligned}\sum_{i=1}^{15} i(i-1)^2 &= \sum_{i=1}^{15} i^3 - 2\sum_{i=1}^{15} i^2 + \sum_{i=1}^{15} i \\ &= \frac{15^2(16)^2}{4} - 2\frac{15(16)(31)}{6} + \frac{15(16)}{2} \\ &= 14{,}400 - 2480 + 120 = 12{,}040\end{aligned}$$

20.
$$\begin{aligned}\sum_{i=1}^{25} (i^3 - 2i) &= \sum_{i=1}^{25} i^3 - 2\sum_{i=1}^{25} i \\ &= \frac{(25)^2(26)^2}{4} - 2\frac{25(26)}{2} \\ &= 105{,}625 - 650 \\ &= 104{,}975\end{aligned}$$

21. $\sum_{i=1}^{n} \frac{2i+1}{n^2} = \frac{1}{n^2}\sum_{i=1}^{n}(2i+1) = \frac{1}{n^2}\left[2\frac{n(n+1)}{2} + n\right] = \frac{n+2}{n} = 1 + \frac{2}{n} = S(n)$

$$\begin{aligned}S(10) &= \frac{12}{10} = 1.2 \\ S(100) &= 1.02 \\ S(1000) &= 1.002 \\ S(10{,}000) &= 1.0002\end{aligned}$$

22.
$$\begin{aligned}\sum_{j=1}^{n} \frac{7j+4}{n^2} &= \frac{1}{n^2}\sum_{j=1}^{n}(7j+4) \\ &= \frac{1}{n^2}\left[7\frac{n(n+1)}{2} + 4n\right] \\ &= \frac{7n^2 + 7n}{2n^2} + \frac{4n}{n^2} = \frac{7n+15}{2n} = S(n)\end{aligned}$$

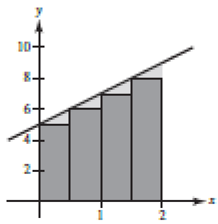
$$\begin{aligned}S(10) &= \tfrac{17}{4} = 4.25 \\ S(100) &= 3.575 \\ S(1000) &= 3.5075 \\ S(10{,}000) &= 3.50075\end{aligned}$$

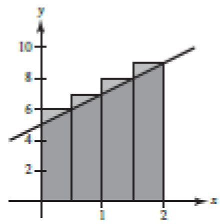
23.
$$\begin{aligned}\sum_{k=1}^{n} \frac{6k(k-1)}{n^3} &= \frac{6}{n^3}\sum_{k=1}^{n}(k^2 - k) = \frac{6}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right] \\ &= \frac{6}{n^2}\left[\frac{2n^2 + 3n + 1 - 3n - 3}{6}\right] = \frac{1}{n^2}\left[2n^2 - 2\right] = 2 - \frac{2}{n^2} = S(n)\end{aligned}$$

$$\begin{aligned}S(10) &= 1.98 \\ S(100) &= 1.9998 \\ S(1000) &= 1.999998 \\ S(10{,}000) &= 1.99999998\end{aligned}$$

24. $$\sum_{i=1}^{n}\frac{2i^3-3i}{n^4}=\frac{1}{n^4}\sum_{i=1}^{n}\left(2i^3-3i\right)$$
$$=\frac{1}{n^4}\left[2\frac{n^2(n+1)^2}{4}-3\frac{n(n+1)}{2}\right]$$
$$=\frac{(n+1)^2}{2n^2}-\frac{3(n+1)}{2n^3}=\frac{n^3+2n^2-2n-3}{2n^3}=S(n)$$
$$S(10)=0.5885$$
$$S(100)=0.5098985$$
$$S(1000)=0.5009989985$$
$$S(10{,}000)=0.50009999$$

25.





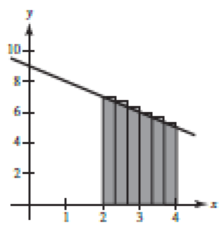
$$\Delta x=\frac{2-0}{4}=\frac{1}{2}$$

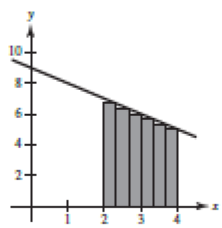
Left endpoints: Area $\approx \frac{1}{2}[5+6+7+8]=\frac{26}{2}=13$

Right endpoints: Area $\approx \frac{1}{2}[6+7+8+9]=\frac{30}{2}=15$

$13<\text{Area}<15$

26.



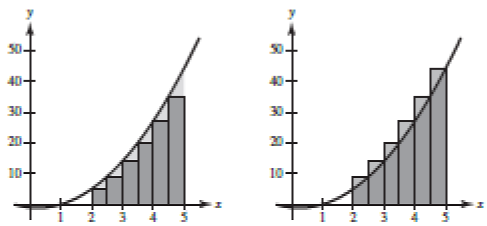


$$\Delta x=\frac{4-2}{6}=\frac{1}{3}$$

Left endpoints: Area $\approx \frac{1}{3}\left[7+\frac{20}{3}+\frac{19}{3}+6+\frac{17}{3}+\frac{16}{3}\right]=\frac{37}{3}\approx 12.333$

Right endpoints: Area $\approx \frac{1}{3}\left[\frac{20}{3}+\frac{19}{3}+6+\frac{17}{3}+\frac{16}{3}+\frac{15}{3}\right]=\frac{35}{3}\approx 11.667$

$\frac{35}{3}<\text{Area}<\frac{37}{3}$

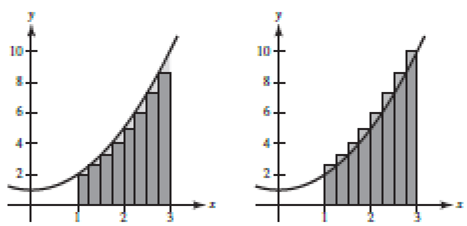
27. 

$$\Delta x = \frac{5-2}{6} = \frac{1}{2}$$

Left endpoints: Area $\approx \frac{1}{2}[5 + 9 + 14 + 20 + 27 + 35] = 55$

Right endpoints: Area $\approx \frac{1}{2}[9 + 14 + 20 + 27 + 35 + 44] = \frac{149}{2} = 74.5$

$55 < \text{Area} < 74.5$

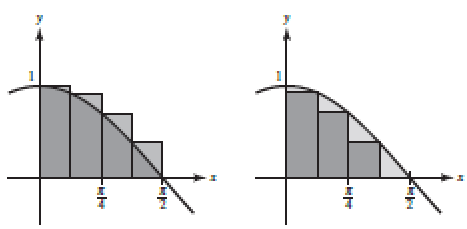
28. 

$$\Delta x = \frac{3-1}{8} = \frac{1}{4}$$

Left endpoints: Area $\approx \frac{1}{4}\left[2 + \frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16}\right] = \frac{155}{16} = 9.6875$

Right endpoint: Area $\approx \frac{1}{4}\left[\frac{41}{16} + \frac{13}{4} + \frac{65}{16} + 5 + \frac{97}{16} + \frac{29}{4} + \frac{137}{16} + 10\right] = 11.6875$

$9.6875 < \text{Area} < 11.6875$

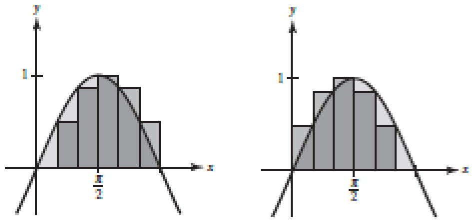
29. 

$$\Delta x = \frac{\frac{\pi}{2} - 0}{4} = \frac{\pi}{8}$$

Left endpoints: Area $\approx \frac{\pi}{8}\left[\cos(0) + \cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right)\right] \approx 1.1835$

Right endpoints: Area $\approx \frac{\pi}{8}\left[\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{3\pi}{8}\right) + \cos\left(\frac{\pi}{2}\right)\right] \approx 0.7908$

$0.7908 < \text{Area} < 1.1835$

30. 

$$\Delta x = \frac{\pi - 0}{6} = \frac{\pi}{6}$$

Left endpoints: Area $\approx \frac{\pi}{6}\left[\sin 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6}\right] \approx 1.9541$

Right endpoints: Area $\approx \frac{\pi}{6}\left[\sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin \pi\right] \approx 1.9541$

By symmetry, the answers are the same. The exact area (2) is larger.

31. $S = \left[3 + 4 + \frac{9}{2} + 5\right](1) = \frac{33}{2} = 16.5$

$s = \left[1 + 3 + 4 + \frac{9}{2}\right](1) = \frac{25}{2} = 12.5$

32. $S = [5 + 5 + 4 + 2](1) = 16$

$s = [4 + 4 + 2 + 0](1) = 10$

33. $S(4) = \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) + \sqrt{1}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3} + 2}{8} \approx 0.768$

$s(4) = 0\left(\frac{1}{4}\right) + \sqrt{\frac{1}{4}}\left(\frac{1}{4}\right) + \sqrt{\frac{1}{2}}\left(\frac{1}{4}\right) + \sqrt{\frac{3}{4}}\left(\frac{1}{4}\right) = \frac{1 + \sqrt{2} + \sqrt{3}}{8} \approx 0.518$

34. $S(4) = 4\left(e^{-0} + e^{-0.5} + e^{-1} + e^{-1.5}\right)\frac{1}{2} \approx 4.395$

$s(4) = 4\left(e^{-0.5} + e^{-1} + e^{-1.5} + e^{-2}\right)\frac{1}{2} \approx 2.666$

35. $S(5) = 1\left(\frac{1}{5}\right) + \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) = \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} \approx 0.746$

$s(5) = \frac{1}{6/5}\left(\frac{1}{5}\right) + \frac{1}{7/5}\left(\frac{1}{5}\right) + \frac{1}{8/5}\left(\frac{1}{5}\right) + \frac{1}{9/5}\left(\frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} \approx 0.646$

36. $S(5) = 1\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right)$

$= \frac{1}{5}\left[1 + \frac{\sqrt{24}}{5} + \frac{\sqrt{21}}{5} + \frac{\sqrt{16}}{5} + \frac{\sqrt{9}}{5}\right] \approx 0.859$

$s(5) = \sqrt{1 - \left(\frac{1}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{2}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{3}{5}\right)^2}\left(\frac{1}{5}\right) + \sqrt{1 - \left(\frac{4}{5}\right)^2}\left(\frac{1}{5}\right) + 0 \approx 0.659$

37. $\lim_{n\to\infty} \sum_{i=1}^{n}\left(\frac{24i}{n^2}\right) = \lim_{n\to\infty} \frac{24}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{24}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty}\left[12\left(\frac{n^2 + n}{n^2}\right)\right] = 12 \lim_{n\to\infty}\left(1 + \frac{1}{n}\right) = 12$

38. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3i}{n}\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty}\frac{9}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\frac{9}{n^2}\left[\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\frac{9}{2}\left(\frac{n+1}{n}\right) = \frac{9}{2}$

39. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n^3}(i-1)^2 = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n-1} i^2 = \lim_{n\to\infty}\frac{1}{n^3}\left[\frac{(n-1)(n)(2n-1)}{6}\right]$

$$= \lim_{n\to\infty}\frac{1}{6}\left[\frac{2n^3 - 3n^2 + n}{n^3}\right] = \lim_{n\to\infty}\left[\frac{1}{6}\left(\frac{2-(3/n)+(1/n^2)}{1}\right)\right] = \frac{1}{3}$$

40. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2i}{n}\right)^2\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n^3}\sum_{i=1}^{n}(n+2i)^2$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[\sum_{i=1}^{n} n^2 + 4n\sum_{i=1}^{n} i + 4\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty}\frac{2}{n^3}\left[n^3 + (4n)\left(\frac{n(n+1)}{2}\right) + \frac{4(n)(n+1)(2n+1)}{6}\right]$$

$$= 2\lim_{n\to\infty}\left[1 + 2 + \frac{2}{n} + \frac{4}{3} + \frac{2}{n} + \frac{2}{3n^2}\right] = 2\left(1 + 2 + \frac{4}{3}\right) = \frac{26}{3}$$

41. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{i}{n}\right)\left(\frac{2}{n}\right) = 2\lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n} 1 + \frac{1}{n}\sum_{i=1}^{n} i\right] = 2\lim_{n\to\infty}\frac{1}{n}\left[n + \frac{1}{n}\left(\frac{n(n+1)}{2}\right)\right] = 2\lim_{n\to\infty}\left[1 + \frac{n^2+n}{2n^2}\right] = 2\left(1+\frac{1}{2}\right) = 3$

42. $\lim_{n\to\infty}\sum_{i=1}^{n}\left(2+\frac{3i}{n}\right)^3\left(\frac{3}{n}\right) = \lim_{n\to\infty}\frac{3}{n}\sum_{i=1}^{n}\left[\frac{2n+3i}{n}\right]^3$

$$= \lim_{n\to\infty}\frac{3}{n^4}\sum_{i=1}^{n}\left(8n^3 + 36n^2 i + 54ni^2 + 27i^3\right)$$

$$= \lim_{n\to\infty}\frac{3}{n^4}\left(8n^4 + 36n^2\frac{n(n+1)}{2} + 54n\frac{n(n+1)(2n+1)}{6} + 27\frac{n^2(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty}3\left(8 + \frac{18(n+1)}{n} + \frac{9(n+1)(2n+1)}{n^2} + \frac{27}{4}\cdot\frac{(n+1)^2}{n^2}\right)$$

$$= 3\left(8 + 18 + 18 + \frac{27}{4}\right) = \frac{609}{4} = 152.25$$

43. The last term in the first step should be $-\sum_{i=1}^{10} 24$.

$$\sum_{i=1}^{10} 3(i-2)^3 = 3\sum_{i=1}^{10} i^3 - 18\sum_{i=1}^{10} i^2 + 36\sum_{i=1}^{10} i - \sum_{i=1}^{10} 24$$

$$= 3\left[\frac{10^2(11)^2}{4}\right] - 18\left[\frac{(10)(11)(21)}{6}\right] + 36\left[\frac{10(11)}{2}\right] - 24(10)$$

$$= 9075 - 6930 + 1980 - 240$$

$$= 3885$$

44. $\lim_{n\to\infty}\frac{n+1}{n^2} = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^2}\right) = 0$

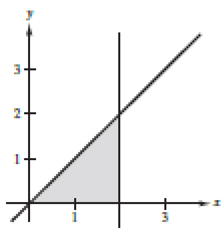
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2}\left(\frac{4}{n}\right) = \lim_{n\to\infty}\frac{4}{n^3}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty}\frac{4}{n^3}\left[\frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} 2\left(\frac{n+1}{2}\right)$$

$$= 2\cdot 0 = 0$$

45. (a)



$$A = \frac{1}{2}bh$$

$$= \frac{1}{2}(2)(2)$$

$$= 2 \text{ units}^2$$

(b) $\Delta x = \frac{2-0}{n} = \frac{2}{n}$

Endpoints: $0 < 1\left(\frac{2}{n}\right) < 2\left(\frac{2}{n}\right) < \ldots < (n-1)\left(\frac{2}{n}\right) < n\left(\frac{2}{n}\right) = 2$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} f\left(\frac{2i-2}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

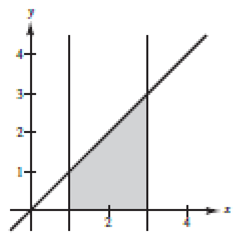
(e)

x	5	10	50	100
$s(n)$	1.6	1.8	1.96	1.98
$S(n)$	2.4	2.2	2.04	2.02

(f) $\lim_{n\to\infty}\sum_{i=1}^{n}\left[(i-1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n}(i-1) = \lim_{n\to\infty}\frac{4}{n^2}\left[\frac{n(n+1)}{2} - n\right] = \lim_{n\to\infty}\left[\frac{2(n+1)}{n} - \frac{4}{n}\right] = 2$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty}\left(\frac{4}{n^2}\right)\frac{n(n+1)}{2} = \lim_{n\to\infty}\frac{2(n+1)}{n} = 2$$

46. (a)



$$A = \frac{1}{2}(b_1 + b_2)h$$
$$= \frac{1}{2}(1 + 3)(2)$$
$$= 4 \text{ units}^2$$

(b) $\Delta x = \dfrac{3 - 1}{n} = \dfrac{2}{n}$

Endpoints:

$$1 < 1 + \frac{2}{n} < 1 + \frac{4}{n} < \cdots < 1 + \frac{2n}{n} = 3$$

$$1 < 1 + 1\left(\frac{2}{n}\right) < 1 + 2\left(\frac{2}{n}\right) < \cdots < 1 + (n - 1)\left(\frac{2}{n}\right) < 1 + n\left(\frac{2}{n}\right)$$

(c) Because $y = x$ is increasing, $f(m_i) = f(x_{i-1})$ on $[x_{i-1}, x_i]$.

$$s(n) = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x = \sum_{i=1}^{n} f\left[1 + (i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + (i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(d) $f(M_i) = f(x_i)$ on $[x_{i-1}, x_i]$

$$S(n) = \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} f\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right)$$

(e)

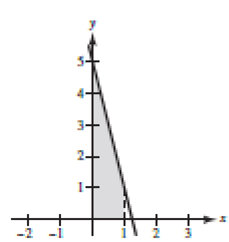
x	5	10	50	100
$s(n)$	3.6	3.8	3.96	3.98
$S(n)$	4.4	4.2	4.04	4.02

(f) $$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + (i - 1)\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\left(\frac{2}{n}\right)\left[n + \frac{2}{n}\left(\frac{n(n+1)}{2} - n\right)\right] = \lim_{n\to\infty}\left[2 + \frac{2n + 2}{n} - \frac{4}{n}\right] = \lim_{n\to\infty}\left[4 - \frac{2}{n}\right] = 4$$

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[1 + i\left(\frac{2}{n}\right)\right]\left(\frac{2}{n}\right) = \lim_{n\to\infty}\frac{2}{n}\left[n + \left(\frac{2}{n}\right)\frac{n(n+1)}{2}\right] = \lim_{n\to\infty}\left[2 + \frac{2(n+1)}{n}\right] = \lim_{n\to\infty}\left[4 + \frac{2}{n}\right] = 4$$

47. $y = -4x + 5$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[-4\left(\frac{i}{n}\right) + 5\right]\left(\frac{1}{n}\right)$$
$$= -\frac{4}{n^2}\sum_{i=1}^{n} i + 5$$
$$= -\frac{4}{n^2}\frac{n(n+1)}{2} + 5$$
$$= -2\left(1 + \frac{1}{n}\right) + 5$$



Area $= \lim_{n\to\infty} s(n) = 3$

48. $y = 3x - 2$ on $[2, 5]$. $\left(\textbf{Note: } \Delta x = \dfrac{5-2}{n} = \dfrac{3}{n}\right)$

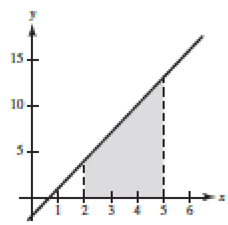
$$S(n) = \sum_{i=1}^{n} f\left(2 + \frac{3i}{n}\right)\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} \left[3\left(2 + \frac{3i}{n}\right) - 2\right]\left(\frac{3}{n}\right)$$

$$= 18 + 3\left(\frac{3}{n}\right)^2 \sum_{i=1}^{n} i - 6$$

$$= 12 + \frac{27}{n^2}\left(\frac{(n+1)n}{2}\right) = 12 + \frac{27}{2}\left(1 + \frac{1}{n}\right)$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 12 + \frac{27}{2} = \frac{51}{2}$$



49. $y = x^2 + 2$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1}{n}\right)$

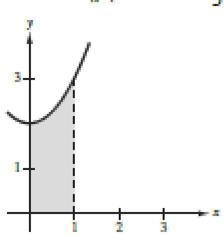
$$S(n) = \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n} \left[\left(\frac{i}{n}\right)^2 + 2\right]\left(\frac{1}{n}\right)$$

$$= \left[\frac{1}{n^3}\sum_{i=1}^{n} i^2\right] + 2$$

$$= \frac{n(n+1)(2n+1)}{6n^3} + 2 = \frac{1}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = \frac{7}{3}$$



50. $y = 3x^2 + 1$ on $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

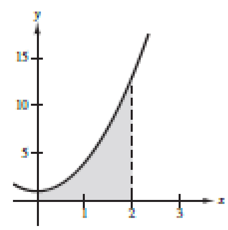
$$S(n) = \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n} \left[3\left(\frac{2i}{n}\right)^2 + 1\right]\left(\frac{2}{n}\right)$$

$$= \frac{24}{n^3}\sum_{i=1}^{n} i^2 + \frac{2}{n}\sum_{i=1}^{n} 1$$

$$= \frac{24}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{2}{n}(n)$$

$$= \frac{4(n+1)(2n+1)}{n^2} + 2$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 8 + 2 = 10$$



51. $y = 25 - x^2$ on $[1, 4]$. $\left(\textbf{Note: } \Delta x = \dfrac{3}{n}\right)$

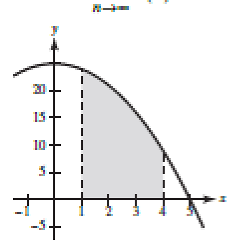
$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{3i}{n}\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \left[25 - \left(1 + \frac{3i}{n}\right)^2\right]\left(\frac{3}{n}\right)$$

$$= \frac{3}{n}\sum_{i=1}^{n} \left[24 - \frac{9i^2}{n^2} - \frac{6i}{n}\right]$$

$$= \frac{3}{n}\left[24n - \frac{9}{n^2}\,\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\,\frac{n(n+1)}{2}\right]$$

$$= 72 - \frac{9}{2n^2}(n+1)(2n+1) - \frac{9}{n}(n+1)$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 72 - 9 - 9 = 54$$

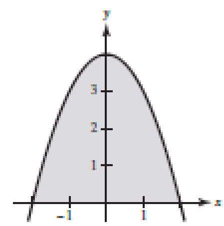


52. $y = 4 - x^2$ on $[-2, 2]$. Find area of region over the interval $[0, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2}{n}\right)$

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n}\left[4 - \left(\frac{2i}{n}\right)^2\right]\left(\frac{2}{n}\right) \\ &= 8 - \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\ &= 8 - \frac{8n(n+1)(2n+1)}{6n^3} = 8 - \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \end{aligned}$$

$$\frac{1}{2}\text{Area} = \lim_{n\to\infty} s(n) = 8 - \frac{8}{3} = \frac{16}{3}$$

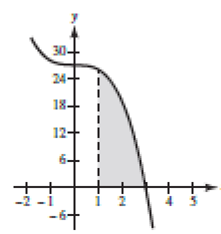
$$\text{Area} = \frac{32}{3}$$



53. $y = 27 - x^3$ on $[1, 3]$. $\left(\textbf{Note: } \Delta x = \dfrac{3-1}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned} s(n) &= \sum_{i=1}^{n} f\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[27 - \left(1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right) \\ &= \frac{2}{n}\sum_{i=1}^{n}\left[26 - \frac{8i^3}{n^3} - \frac{12i^2}{n^2} - \frac{6i}{n}\right] \\ &= \frac{2}{n}\left[26n - \frac{8}{n^3}\frac{n^2(n+1)^2}{4} - \frac{12}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{6}{n}\frac{n(n+1)}{2}\right] \\ &= 52 - \frac{4}{n^2}(n+1)^2 - \frac{4}{n^2}(n+1)(2n+1) - \frac{6n+1}{n} \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} s(n) = 52 - 4 - 8 - 6 = 34$$

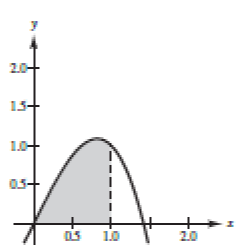


54. $y = 2x - x^3$ on $[0, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-0}{n} = \dfrac{1}{n}\right)$

Because y both increases and decreases on $[0, 1]$, $T(n)$ is neither an upper nor lower sum.

$$\begin{aligned} T(n) &= \sum_{i=1}^{n} f\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(\frac{i}{n}\right) - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) \\ &= \frac{2}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^4}\sum_{i=1}^{n} i^3 = \frac{n(n+1)}{n^2} - \frac{1}{n^4}\left[\frac{n^2(n+1)^2}{4}\right] = 1 + \frac{1}{n} - \frac{1}{4} - \frac{2}{4n} - \frac{1}{4n^2} \end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 1 - \frac{1}{4} = \frac{3}{4}$$



55. $y = x^2 - x^3$ on $[-1, 1]$. $\left(\textbf{Note: } \Delta x = \dfrac{1-(-1)}{n} = \dfrac{2}{n}\right)$

Because y both increases and decreases on $[-1, 1]$, $T(n)$ is neither an upper nor a lower sum.

$$T(n) = \sum_{i=1}^{n} f\left(-1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \sum_{i=1}^{n}\left[\left(-1 + \frac{2i}{n}\right)^2 - \left(-1 + \frac{2i}{n}\right)^3\right]\left(\frac{2}{n}\right)$$

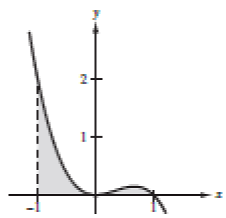
$$= \sum_{i=1}^{n}\left[\left(1 - \frac{4i}{n} + \frac{4i^2}{n^2}\right) - \left(-1 + \frac{6i}{n} - \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right)\right]\left(\frac{2}{n}\right)$$

$$= \sum_{i=1}^{n}\left[2 - \frac{10i}{n} + \frac{16i^2}{n^2} - \frac{8i^3}{n^3}\right]\left(\frac{2}{n}\right) = \frac{4}{n}\sum_{i=1}^{n} 1 - \frac{20}{n^2}\sum_{i=1}^{n} i + \frac{32}{n^3}\sum_{i=1}^{n} i^2 - \frac{16}{n^4}\sum_{i=1}^{n} i^3$$

$$= \frac{4}{n}(n) - \frac{20}{n^2}\cdot\frac{n(n+1)}{2} + \frac{32}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} - \frac{16}{n^4}\cdot\frac{n^2(n+1)^2}{4}$$

$$= 4 - 10\left(1 + \frac{1}{n}\right) + \frac{16}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) - 4\left(1 + \frac{2}{n} + \frac{1}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} T(n) = 4 - 10 + \frac{32}{3} - 4 = \frac{2}{3}$$



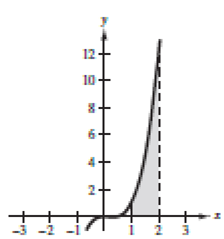
56. $y = 2x^3 - x^2$ on $[1, 2]$. $\left(\textbf{Note: } \Delta x = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$s(n) = \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=1}^{n}\left[2\left(1 + \frac{i}{n}\right)^3 - \left(1 + \frac{i}{n}\right)^2\right]\left(\frac{1}{n}\right)$$

$$= \sum_{i=1}^{n}\left(\frac{2i^3}{n^3} + \frac{5i^2}{n^2} + \frac{4i}{n} + 1\right)\left(\frac{1}{n}\right)$$

$$= \frac{2}{n^4}\cdot\frac{n^2(n+1)^2}{4} + \frac{5}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} + \frac{4}{n^2}\cdot\frac{n(n+1)}{2} + 1$$

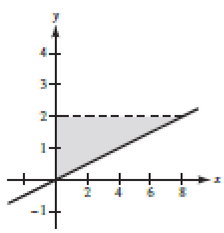
$$\text{Area} = \lim_{n\to\infty} s_n = \frac{1}{2} + \frac{5}{3} + 2 + 1 = \frac{31}{6}$$



57. $f(y) = 4y,\ 0 \le y \le 2 \quad \left(\textbf{Note: } \Delta y = \dfrac{2-0}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f(m_i)\Delta y \\ &= \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} 4\left(\frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \frac{16}{n^2}\sum_{i=1}^{n} i \\ &= \left(\frac{16}{n^2}\right)\cdot\frac{n(n+1)}{2} = \frac{8(n+1)}{n} = 8 + \frac{8}{n} \end{aligned}$$

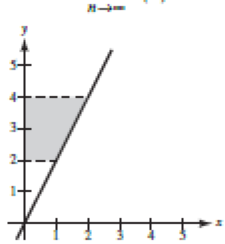
$$\text{Area} = \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(8 + \frac{8}{n}\right) = 8$$



58. $g(y) = \dfrac{1}{2}y,\ 2 \le y \le 4. \quad \left(\textbf{Note: } \Delta y = \dfrac{4-2}{n} = \dfrac{2}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} g\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\ &= \sum_{i=1}^{n} \frac{1}{2}\left(2 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) = \frac{2}{n}\sum_{i=1}^{n}\left(1 + \frac{i}{n}\right) \\ &= \frac{2}{n}\left[n + \frac{1}{n}\frac{n(n+1)}{2}\right] = 2 + \frac{n+1}{n} \end{aligned}$$

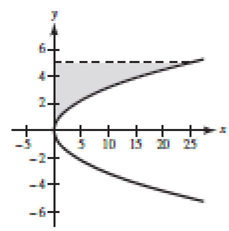
$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + 1 = 3$$



59. $f(y) = y^2,\ 0 \le y \le 5 \quad \left(\textbf{Note: } \Delta y = \dfrac{5-0}{n} = \dfrac{5}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f\left(\frac{5i}{n}\right)\left(\frac{5}{n}\right) \\ &= \sum_{i=1}^{n}\left(\frac{5i}{n}\right)^2\left(\frac{5}{n}\right) \\ &= \frac{125}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{125}{n^3}\cdot\frac{n(n+1)(2n+1)}{6} \\ &= \frac{125}{n^2}\left(\frac{2n^2+3n+1}{6}\right) = \frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2} \end{aligned}$$

$$\text{Area } \lim_{n\to\infty} S(n) = \lim_{n\to\infty}\left(\frac{125}{3} + \frac{125}{2n} + \frac{125}{6n^2}\right) = \frac{125}{3}$$

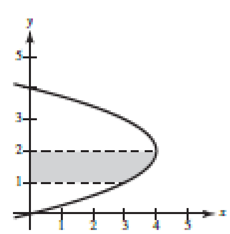


60. $f(y) = 4y - y^2,\ 1 \le y \le 2.$

$\left(\textbf{Note: } \Delta y = \dfrac{2-1}{n} = \dfrac{1}{n}\right)$

$$\begin{aligned} S(n) &= \sum_{i=1}^{n} f\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left[4\left(1 + \frac{i}{n}\right) - \left(1 + \frac{i}{n}\right)^2\right] \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(4 + \frac{4i}{n} - 1 - \frac{2i}{n} - \frac{i^2}{n^2}\right) \\ &= \frac{1}{n}\sum_{i=1}^{n}\left(3 + \frac{2i}{n} - \frac{i^2}{n^2}\right) \\ &= \frac{1}{n}\left[3n + \frac{2}{n}\frac{n(n+1)}{2} - \frac{1}{n^2}\frac{n(n+1)(2n+1)}{6}\right] \\ &= 3 + \frac{n+1}{n} - \frac{(n+1)(2n+1)}{6n^2} \end{aligned}$$

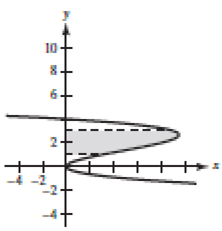
$$\text{Area} = \lim_{n\to\infty} S(n) = 3 + 1 - \frac{1}{3} = \frac{11}{3}$$



61. $g(y) = 4y^2 - y^3,\ 1 \le y \le 3.$ $\left(\textbf{Note: } \Delta y = \frac{3-1}{n} = \frac{2}{n}\right)$

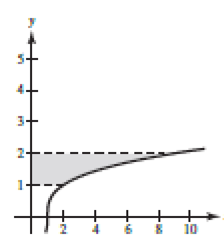
$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} g\left(1 + \frac{2i}{n}\right)\left(\frac{2}{n}\right) \\
&= \sum_{i=1}^{n}\left[4\left(1+\frac{2i}{n}\right)^2 - \left(1+\frac{2i}{n}\right)^3\right]\frac{2}{n} \\
&= \frac{2}{n}\sum_{i=1}^{n} 4\left[1 + \frac{4i}{n} + \frac{4i^2}{n^2}\right] - \left[1 + \frac{6i}{n} + \frac{12i^2}{n^2} + \frac{8i^3}{n^3}\right] \\
&= \frac{2}{n}\sum_{i=1}^{n}\left[3 + \frac{10i}{n} + \frac{4i^2}{n^2} - \frac{8i^3}{n^3}\right] \\
&= \frac{2}{n}\left[3n + \frac{10}{n}\frac{n(n+1)}{2} + \frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - \frac{8}{n^2}\frac{n^2(n+1)^2}{4}\right]
\end{aligned}$$

$$\text{Area} = \lim_{n\to\infty} S(n) = 6 + 10 + \frac{8}{3} - 4 = \frac{44}{3}$$



62. $h(y) = y^3 + 1,\ 1 \le y \le 2$ $\left(\textbf{Note: } \Delta y = \frac{1}{n}\right)$

$$\begin{aligned}
S(n) &= \sum_{i=1}^{n} h\left(1 + \frac{i}{n}\right)\left(\frac{1}{n}\right) \\
&= \sum_{i=1}^{n}\left[\left(1+\frac{i}{n}\right)^3 + 1\right]\frac{1}{n} \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(2 + \frac{i^3}{n^3} + \frac{3i^2}{n^2} + \frac{3i}{n}\right) \\
&= \frac{1}{n}\left[2n + \frac{1}{n^3}\frac{n^2(n+1)^2}{4} + \frac{3}{n^2}\frac{n(n+1)(2n+1)}{6} + \frac{3}{n}\frac{3n(n+1)}{2n}\right] \\
&= 2 + \frac{(n+1)^2}{n^2 4} + \frac{1}{2}\frac{(n+1)(2n+1)}{n^2} + \frac{3(n+1)}{2n}
\end{aligned}$$



$$\text{Area} = \lim_{n\to\infty} S(n) = 2 + \frac{1}{4} + 1 + \frac{3}{2} = \frac{19}{4}$$

63. $f(x) = x^2 + 3,\ 0 \le x \le 2,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \frac{1}{2},\ c_1 = \frac{1}{4},\ c_2 = \frac{3}{4},\ c_3 = \frac{5}{4},\ c_4 = \frac{7}{4}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 3\right]\left(\frac{1}{2}\right) = \frac{1}{2}\left[\left(\frac{1}{16}+3\right) + \left(\frac{9}{16}+3\right) + \left(\frac{25}{16}+3\right) + \left(\frac{49}{16}+3\right)\right] = \frac{69}{8}$$

64. $f(x) = x^2 + 4x,\ 0 \le x \le 4,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = 1,\ c_1 = \dfrac{1}{2},\ c_2 = \dfrac{3}{2},\ c_3 = \dfrac{5}{2},\ c_4 = \dfrac{7}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\,\Delta x = \sum_{i=1}^{4}\left[c_i^2 + 4c_i\right](1) = \left[\left(\frac{1}{4} + 2\right) + \left(\frac{9}{4} + 6\right) + \left(\frac{25}{4} + 10\right) + \left(\frac{49}{4} + 14\right)\right] = 53$$

65. $f(x) = \tan x,\ 0 \le x \le \dfrac{\pi}{4},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2}$.

$\Delta x = \dfrac{\pi}{16},\ c_1 = \dfrac{\pi}{32},\ c_2 = \dfrac{3\pi}{32},\ c_3 = \dfrac{5\pi}{32},\ c_4 = \dfrac{7\pi}{32}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}(\tan c_i)\left(\frac{\pi}{16}\right) = \frac{\pi}{16}\left(\tan\frac{\pi}{32} + \tan\frac{3\pi}{32} + \tan\frac{5\pi}{32} + \tan\frac{7\pi}{32}\right) \approx 0.345$$

66. $f(x) = \cos x,\ 0 \le x \le \dfrac{\pi}{2},\ n = 4$

Let $c_i = \dfrac{x_i + x_{i+1}}{2}$.

$\Delta x = \dfrac{\pi}{8},\ c_1 = \dfrac{\pi}{16},\ c_2 = \dfrac{3\pi}{16},\ c_3 = \dfrac{5\pi}{16},\ c_4 = \dfrac{7\pi}{16}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\cos(ci)\left(\frac{\pi}{8}\right) = \frac{\pi}{8}\left(\cos\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos\frac{5\pi}{16} + \cos\frac{7\pi}{16}\right) \approx 1.006$$

67. $f(x) = \ln x,\ 1 \le x \le 5,\ n = 4$

Let $c_i = \dfrac{x_i + x_{i-1}}{2},\ \Delta x = 1$

$c_1 = \dfrac{3}{2},\ c_2 = \dfrac{5}{2},\ c_3 = \dfrac{7}{2},\ c_4 = \dfrac{9}{2}$

$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[\ln(c_i)\right](1) \approx 0.40547 + 0.91629 + 1.25276 + 1.50408 \approx 4.0786$$

68. $f(x) = xe^x,\ 0 \le x \le 2,\ n = 4$

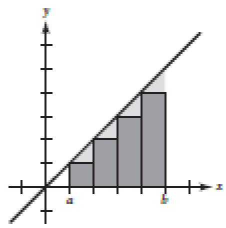
Let $c_i = \dfrac{x_i + x_{i-1}}{2},\ \Delta x = \dfrac{1}{2}$

$c_1 = \dfrac{1}{4},\ c_2 = \dfrac{3}{4},\ c_3 = \dfrac{5}{4},\ c_4 = \dfrac{7}{4}$

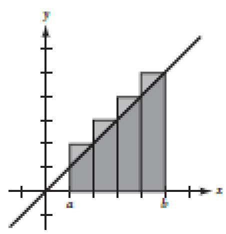
$$\text{Area} \approx \sum_{i=1}^{n} f(c_i)\Delta x = \sum_{i=1}^{4}\left[\ln\left(c_i e^{ci}\right)\right]\left(\frac{1}{2}\right) \approx [0.32101 + 1.58775 + 4.36293 + 10.07055]$$

$$\left(\frac{1}{2}\right) \approx (16.34224)\left(\frac{1}{2}\right) \approx 8.1711$$

69. You can use the line $y = x$ bounded by $x = a$ and $x = b$. The sum of the areas of these inscribed rectangles is the lower sum.



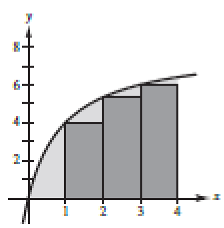
The sum of the areas of these circumscribed rectangles is the upper sum.



You can see that the rectangles do not contain all of the area in the first graph and the rectangles in the second graph cover more than the area of the region. The exact value of the area lies between these two sums.

70. See the definition of the area of a region in the plane on page 292.

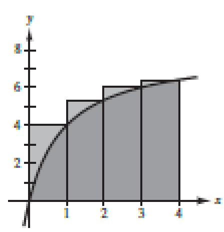
71. (a)



Lower sum:

$s(4) = 0 + 4 + 5\frac{1}{3} + 6 = 15\frac{1}{3} = \frac{46}{3} \approx 15.333$

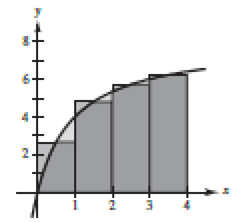
(b)



Upper sum:

$S(4) = 4 + 5\frac{1}{3} + 6 + 6\frac{2}{5} = 21\frac{11}{15} = \frac{326}{15} \approx 21.733$

(c)



Midpoint Rule:

$M(4) = 2\frac{2}{3} + 4\frac{4}{5} + 5\frac{5}{7} + 6\frac{2}{9} = \frac{6112}{315} \approx 19.403$

(d) In each case, $\Delta x = 4/n$. The lower sum uses left end-points, $(i - 1)(4/n)$. The upper sum uses right endpoints, $(i)(4/n)$. The Midpoint Rule uses midpoints, $\left(i - \frac{1}{2}\right)(4/n)$.

(e)

N	4	8	20	100	200
$s(n)$	15.333	17.368	18.459	18.995	19.06
$S(n)$	21.733	20.568	19.739	19.251	19.188
$M(n)$	19.403	19.201	19.137	19.125	19.125

(f) $s(n)$ increases because the lower sum approaches the exact value as n increases. $S(n)$ decreases because the upper sum approaches the exact value as n increases. Because of the shape of the graph, the lower sum is always smaller than the exact value, whereas the upper sum is always larger.

72. (a) Left endpoint of first subinterval is 1.

Left endpoint of last subinterval is $4 - \frac{1}{4} = \frac{15}{4}$.

(b) Right endpoint of first subinterval is $1 + \frac{1}{4} = \frac{5}{4}$.

Right endpoint of second subinterval is $1 + \frac{1}{2} = \frac{3}{2}$.

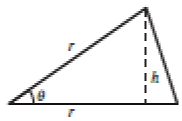
(c) The rectangles lie above the graph.

73. Suppose there are n rows and $n + 1$ columns in the figure. The stars on the left total $1 + 2 + \cdots + n$, as do the stars on the right. There are $n(n + 1)$ stars in total, so

$$2[1 + 2 + \cdots + n] = n(n + 1)$$
$$1 + 2 + \cdots + n = \frac{1}{2}(n)(n + 1).$$

74. (a) $\theta = \frac{2\pi}{n}$

(b) $\sin\theta = \frac{h}{r}$



$h = r\sin\theta$

$$A = \frac{1}{2}bh = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$$

(c) $A_n = n\left(\frac{1}{2}r^2\sin\frac{2\pi}{n}\right)$

$$= \frac{r^2 n}{2}\sin\frac{2\pi}{n} = \pi r^2\left(\frac{\sin(2\pi/n)}{2\pi/n}\right)$$

Let $x = 2\pi/n$. As $n \to \infty$, $x \to 0$.

$$\lim_{n\to\infty} A_n = \lim_{x\to 0}\pi r^2\left(\frac{\sin x}{x}\right) = \pi r^2(1) = \pi r^2$$

75. $f(x) = \sin\frac{\pi x}{4}$, $[0, 4]$

Use the Midpoint Rule with $n = 8$ to approximate the area of the region, where $\Delta x = 0.5$

The midpoints of the subregions are 0.25, 0.75, 1.25, 1.75, 2.25, 2.75, 3.25, and 3.75.

$$\begin{aligned}\text{Area} &\approx \sum_{i=1}^{n} f(c_i)\Delta x\\ &= \sum_{i=1}^{8}\left(\sin\frac{c_i\pi}{4}\right)(0.5)\\ &= 0.5\left(\sin\frac{0.25\pi}{4} + \sin\frac{0.75\pi}{4} + \cdots + \sin\frac{3.75\pi}{4}\right)\\ &\approx 0.5(5.126)\\ &= 2.563\end{aligned}$$

So, the answer is B.

76.

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}\right)\left[8\left(\frac{i}{n}\right) + 3\right] &= \lim_{n\to\infty}\left[\frac{8}{n^2}\sum_{i=1}^{n} i + \frac{1}{n}\sum_{i=1}^{n} 3\right]\\ &= \lim_{n\to\infty}\left[\frac{8}{n^2}\left(\frac{n(n+1)}{2} + \frac{1}{n}(3n)\right)\right]\\ &= \lim_{n\to\infty}\left[\frac{4(n+1)}{n} + 3\right]\\ &= 4 + 0 + 3\\ &= 7\end{aligned}$$

So, the answer is C.

77. $f(x) = e^{-2x} + 1$, $[0, 4]$

Use the Midpoint Rule with $n = 8$ to approximate the area of the region, where $\Delta x = 0.5$.

The midpoints of the subregions are 0.25, 0.75, 1.25, 1.75, 2.25, 2.75, 3.25, and 3.75.

$$\begin{aligned}\text{Area} &\approx \sum_{i=1}^{n} f(c_i)\Delta x\\ &= \sum_{i=1}^{8}\left(e^{-2c_i} + 1\right)(0.5)\\ &= 0.5\left[\left(e^{-2\cdot 0.25} + 1\right) + \left(e^{-2\cdot 0.75} + 1\right) + \cdots + \left(e^{-2\cdot 3.75} + 1\right)\right]\\ &\approx 0.5(8.960)\\ &= 4.480\end{aligned}$$

So, the answer is B.

78. $f(x) = \int f'(x)dx$

$= \int (6x^2 - 7)dx$

$= 2x^3 - 7x + C$

Use $f(-2) = 25$ to find C.

$f(-2) = 2(-2)^3 - 7(-2) + C = 25$

$C = 27$

$f(x) = 2x^3 - 7x + 27$

$f(1) = 2(1)^3 - 7(1) + 27$

$= 22$

So, the answer is C.