4.2, 4.3 Area and Reimann Sums Day 1

Blast from the Past!

Sigma Notation is used as shorthand to write the sum of a sequence (or a very large number) of terms

$$\sum_{k=m}^{n}(a_k) = a_m + a_{(m+1)} + a_{(m+2)} + \cdots . a_{(n-1)} + a_n$$

We can also utilize sigma notation with functions:

$$\sum_{k=m}^{n}(f(k)) = f(m) + f(m+1) + f(m+2) + \cdots .. f(n-1) + f(n)$$

Evaluate each of the following:

$\sum_{n=1}^{6}(2n+11)$

$13 + 15 + 17 + 19 + 21 + 23$

$= 108$

$\sum_{k=7}^{11}(42-9k)$

$-21 + -30 + -39 + -48 + -57$

$= -195$

$\sum_{n=1}^{6} 2(-3)^{n-1}$

$2 + -6 + 18 + -54 + 162 + -486$

$= -364$

Some Properties of Summation

$\sum_{i=1}^{n}(\boldsymbol{k a_i}) = K\sum_{i=1}^{n} a_i$

$\sum_{i=1}^{n}(\boldsymbol{a_i \pm b_i}) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Commonly Used Summation Formulas ☆ must start with 1!

$\sum_{i=1}^{n}(\boldsymbol{k}) = Kn$

$\sum_{i=1}^{n}(\boldsymbol{i}) = \frac{n(n+1)}{2}$

$\sum_{i=1}^{n}(\boldsymbol{i^2}) = \frac{n(n+1)(2n+1)}{6}$

$\sum_{i=1}^{n}(\boldsymbol{i^3}) = \frac{n^2(n+1)^2}{4}$

Examples – Evaluating Summation Expressions using the Properties and Summation Formulas

$\sum_{i=1}^{5} 2i^2 - 3i + 6$

$2\sum_{i=1}^{5} i^2 - 3\sum_{i=1}^{5} i + \sum_{i=1}^{5} 6$

$2\left(\frac{5(5+1)(10+1)}{6}\right) - 3\left(\frac{5(6)}{2}\right) + 5(6)$

$2(55) - 3(15) + 30 = 95$

$\sum_{i=1}^{8} 3i(i-6) = \sum_{i=1}^{8} 3i^2 - 18i$

$3\sum_{i=1}^{8} i^2 - 18\sum_{i=1}^{8} i$

$3\left(\frac{8(9)(17)}{6}\right) - 18\left(\frac{8(9)}{2}\right)$

$3(204) - 18(36) = -36$

Checking them on your calculator

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Examples – Using Sigma Notation to write the sum

Write each of the following sums using sigma notation.

$\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots \frac{1}{3^8}$

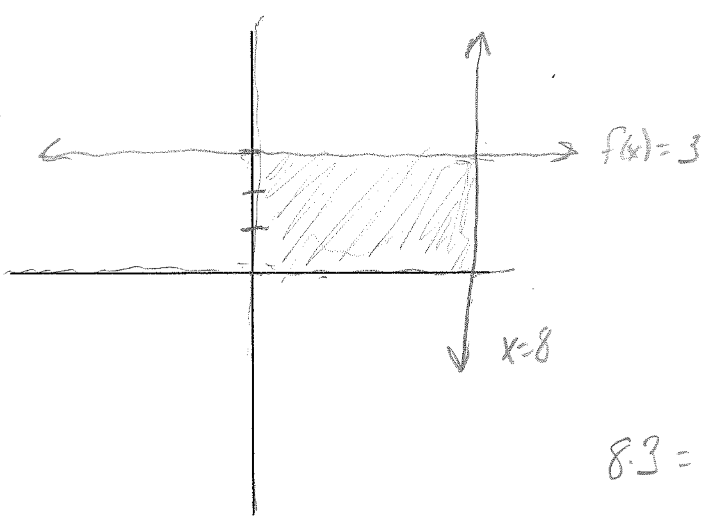
$\sum_{i=1}^{8} \frac{1}{3^i}$ or $\sum_{i=1}^{8} 3^{-i}$

$\frac{1}{1}[2 + (3)(1)] + \frac{1}{2}[2 + (3(2)] + \cdots \frac{1}{15}[2 + 3(15)]$

$\sum_{i=1}^{15} \frac{1}{i}\left[2 + (3)(i)\right]$

Examples – Finding Area (Using Geometry)

Find the area of the region bound by the graph of $f(x) = 3$, the x-axis, the y-axis and $x = 8$.

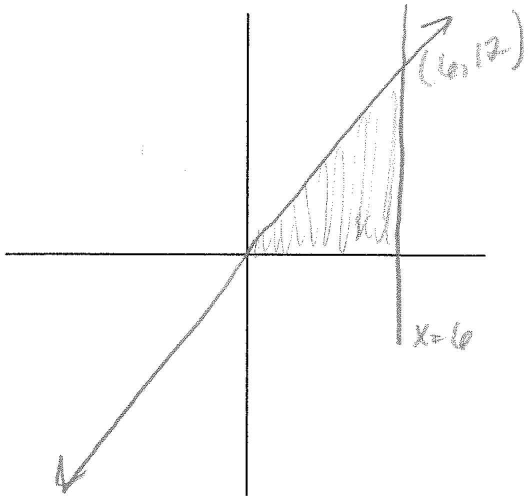


Definite Integral:

$\int_0^8 3\,dx$

$8 \cdot 3 = 24$

Find the area of the region bound by the graph of $f(x) = 2x$, the x-axis, the y-axis and $x = 6$.

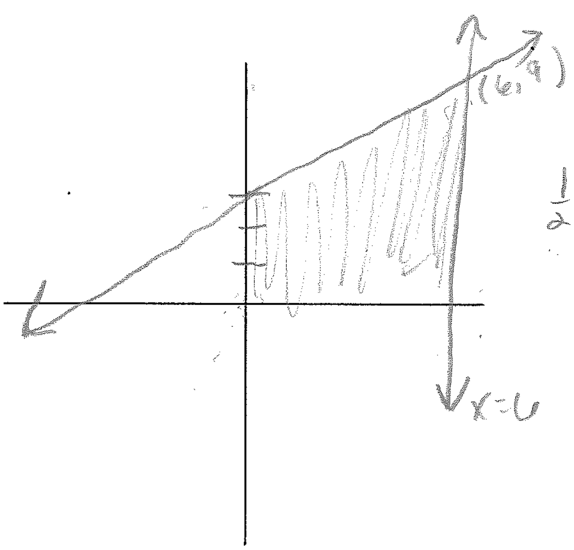


$\left(\frac{1}{2}\right)(6)(12) = 36$

Definite Integral:

$\int_0^6 2x\,dx$

Find the area of the region bound by $f(x) = x + 3$, the x-axis, the y-axis and $x = 6$.



$\frac{1}{2}(6)(3+9)$

$= 36$

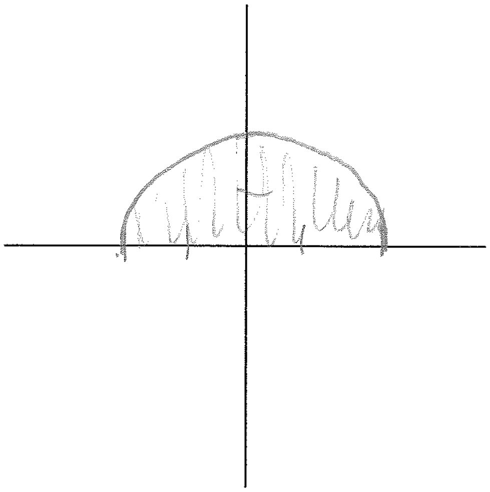
Definite Integral:

$\int_0^6 (x+3)\,dx$

Find the area of the region bound by $f(x) = \sqrt{4 - x^2}$ and the x-axis.

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$



$\frac{1}{2}\pi(2)^2 = 2\pi$

Definite Integral:

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx$$

The Integral

Integral

Integrand

$y = \int f(x)dx$ ← differential

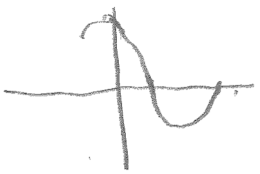
The Definite Integral as the Area of a Region

If f is continuous and non-negative on the closed interval $[a, b]$ then the area of the region bounded by the graph of f, the x-axis and the vertical lines $x = a$ and $x = b$ is given by:

$$\int_a^b f(x)\, dx$$

- The definite integral gives an **area** for a region only if the function is above the x-axis for the entire interval
- Otherwise, the integral will be positive when the area above the x-axis is greater than the area below the x-axis
- OR the integral will be negative when the area above the x-axis is less than the area below the x-axis
- OR the integral will be 0 when the area above the x-axis is equal to the area below the x-axis

Based on the above descriptions what conclusions can you draw about calculations of definite integrals when the graph lies above the x-axis? Below the x-axis?



Integrals will be positive when it lies above the x-axis

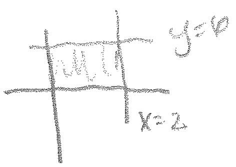
Integrals will be negative when it lies below the x-axis

Examples – Writing Definite Integrals

Go back and write the definite integral for each of the areas that we calculated earlier

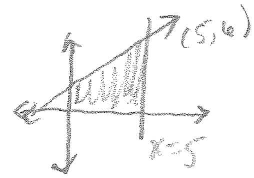
Sketch a representation of the following integral and then evaluate

$\int_0^2 6dx$



$= 6 \cdot 2 = 12$

$\int_0^5 (x+1)dx$



$= \frac{1}{2}(5)(1+6) = \frac{35}{2}$

Continuity Implies Integrability

If a function is continuous on the closed interval $[a, b]$ then it can be integrated on $[a, b,]$

From 2.1 we found the differentiability implied continuity

$\therefore$ Differentiability implies Integrability

** Continuity does not imply differentiability and integrability does not imply continuity

Properties of Definite Integrals

1. If f is defined at $x = a$, then $\int_a^a f(x)dx = 0$

2. If f is integrable on $[a, b]$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. If f is integrable on the three closed intervals determined by $a, b,$ and c, then

   $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ (Additive Interval Property)

4. If f and g are integrable on $[a.b]$ and k is a constant then the functions of $k \cdot f$ and $f \pm g$ are integrable on $[a, b]$ and

   $\int kf(x)dx = k\int f(x)dx$ $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

5. If f is integrable and non-negative on the closed interval $[a, b]$, then

$$\int_a^b f(x)dx > 0$$

6. If f and g are integrable on the closed interval $[a, b]$ and $f(x) \leq g(x)$ for every x in $[a, b]$ then

$$\int_a^b f(x)dx \leq \int_a^b g(x)\,dx$$

Examples – Evaluating Definite Integrals

Given $\int_1^3 f(x)dx = 5$ $\qquad$ $\int_1^3 [f(x)]^2 dx = 30$ $\qquad$ $\int_1^3 dx = 2$

Evaluate: $\int_1^1 [f(x)]^2 dx = 0$

Evaluate $\int_1^3 (2(f(x)^2 - 3f(x) + 4)dx$

$$2\int_1^3 [f(x)]^2 dx - 3\int_1^3 f(x)dx + 4\int_1^3 dx$$

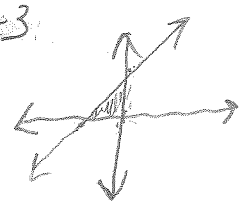
$$2(30) - 3(5) + 4(2) = 53$$

Example: Evaluating an integral that involves absolute value

Evaluate the definite integral $\int_{-5}^{0} |x + 3|dx$

$$= \begin{cases} x+3 & x \geq -3 \\ -x-3 & x < -3 \end{cases}$$

$$\int_{-5}^{-3} (-x-3)\,dx + \int_{-3}^{0} (x+3)\,dx$$

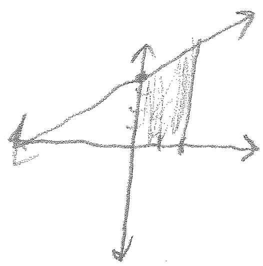


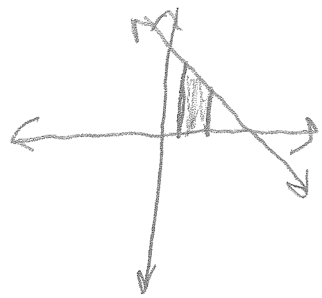
$$\left(\tfrac{1}{2}\right)(2)(2) + \left(\tfrac{1}{2}\right)(3)(3) = 2 + \frac{9}{2} = \frac{13}{2}$$

Example: Evaluating an integral that involves a piecewise function

Evaluate $\int_0^4 f(x)dx$ $\qquad f(x) = \begin{cases} x+4 & x<2 \\ -2x+16 & x \geq 2 \end{cases}$

$$\int_0^2 (x+4)dx + \int_2^4 (-2x+16)dx$$





$$\frac{1}{2}(2)(4+6) + \frac{1}{2}(2)(12+8)$$

$$10 + 20 = 30$$