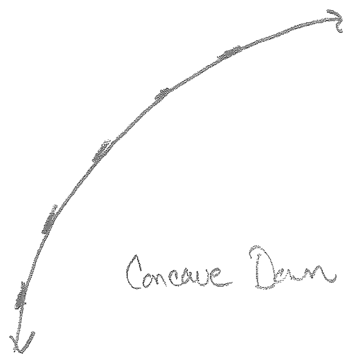
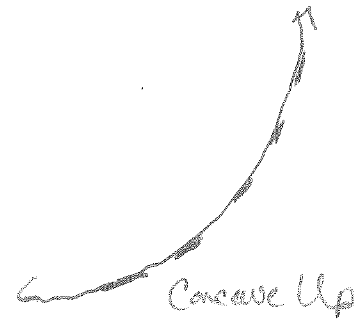
3.4 Concavity and the Second Derivative Test





Summary of Concavity

Concave Upward: $f'(x)$ increasing, $f''(x) > 0$
$f(x)$ lies above the tangent line

Concave Downward: $f'(x)$ decreasing, $f''(x) < 0$
$f(x)$ lies below the tangent line

Point of Inflection: A point on the graph where the concavity changes.

Determining Concavity and Points of Inflection

1. Find the $f''(x)$
2. Determine where $f''(x) = 0$ or $f''(x)$ undefined
3. Test intervals to determine where $f''(x)$ is positive or negative.

Examples: Determining Concavity and Points of Inflection

Determine the open intervals on which the graph of $f(x) = -4e^{\frac{-x^2}{8}}$ is concave upward or concave downward.

$f'(x) = -4e^{-\frac{x^2}{8}}\left(-\frac{1}{8}(2x)\right) = xe^{-\frac{x^2}{8}}$

$f''(x) = (x)\left(e^{-\frac{x^2}{8}}\right)\left(-\frac{1}{8}(2x)\right) + (1)e^{-\frac{x^2}{8}}$

$= e^{-\frac{x^2}{8}}\left(-\frac{1}{4}x^2 + 1\right)$

$-\frac{1}{4}x^2 = -1$

$x^2 = 4$

$x = \pm 2$

$f''(x)$: $-$ | $+$ | $-$ at -2, 2

$f(x)$ is concave up $(-2, 2)$ because $f''(x) > 0$

$f(x)$ is concave down $(-\infty, -2) \cup (2, \infty)$ because $f''(x) < 0$

Determine the open intervals on which the graph of $f(x) = \frac{x}{x^2-1}$ is concave upward or concave downward. Also identify any points of inflection.

$f'(x) = \frac{(x^2-1)(1) - (x)(2x)}{(x^2-1)^2} = \frac{x^2 - 1 - 2x^2}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}$

$f''(x) = \frac{(x^2-1)^2(-2x) - (-x^2-1)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{(2x)(x^2-1)\left[(x^2-1)(-1) - (-x^2-1)(2)\right]}{(x^2-1)^4}$

$= \frac{2x\left[-x^2+1+2x^2+2\right]}{(x^2-1)^3} = \frac{2x\left[x^2+3\right]}{(x^2-1)^3}$

$x = 0$

$x = \pm 1$

$f''(x)$: $-$ | $+$ | $-$ | $+$ at $\boxed{-1}$, 0, $\boxed{1}$

$f(x)$ is concave up $(-1, 0) \cup (1, \infty)$ because $f''(x) > 0$

$f(x)$ is concave down $(-\infty, -1) \cup (0, 1)$ because $f''(x) < 0$

$(0, 0)$ is a point of inflection

$f''(0) = 0$, $f''(x) > 0$ $(-1, 0)$, $f''(x) < 0$, $(0, 1)$

Determine the points of inflection and discuss the concavity of the graph of $f(x) = 2x^6 + 3x^5$.

Second Derivative Test

If $f'(c) = 0$ and $f''(c)$ exists:

If $f''(c) > 0$ 

$(c, f(c))$ is a relative minimum

If $f''(c) < 0$ 

$(c, f(c))$ is a relative maximum

If $f''(c) = 0$ The second derivative test failed

★ This does not mean it is neither

Example: Using the Second Derivative Test

Find the relative extrema of $f(x) = 2x^4 - 16x^2$ using the second derivative test.

$f'(x) = 8x^3 - 32x$

$0 = 8x(x^2 - 4)$ $\quad x = 0 \quad x = 2 \quad x = -2$

$f''(x) = 24x^2 - 32$

$f''(0) = -32$ $\quad (0, 0)$ relative maximum

$f''(2) = 24(4) - 32$ $\quad (2, -32)$ relative minimum

$f''(-2) = 24(4) - 32$ $\quad (-2, -32)$ is a relative minimum