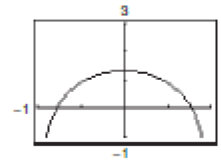
Section 7.7 Indeterminate Forms and L'Hôpital's Rule

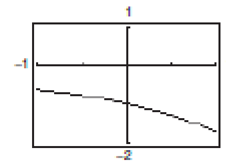
1. $\lim_{x \to 0} \frac{\sin 4x}{\sin 3x} \approx 1.3333 \quad \left(\text{exact: } \frac{4}{3}\right)$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	1.3177	1.3332	1.3333	1.3333	1.3332	1.3177



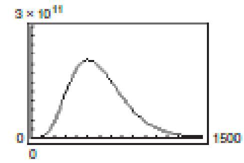
2. $\lim_{x \to 0} \frac{1 - e^x}{x} \approx -1$

x	-0.1	-0.01	-0.001	0.001	0.01	0.1
$f(x)$	-0.9516	-0.9950	-0.9995	-1.0005	-1.0050	-1.0517



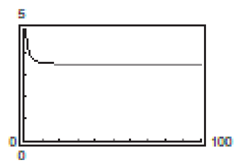
3. $\lim_{x\to\infty} x^5 e^{-x/100} \approx 0$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	0.9900	90,484	3.7×10^9	4.5×10^{10}	0	0



4. $\lim_{x\to\infty} \dfrac{6x}{\sqrt{3x^2 - 2x}} \approx 3.4641 \quad \left(\text{exact: } \dfrac{6}{\sqrt{3}}\right)$

x	1	10	10^2	10^3	10^4	10^5
$f(x)$	6	3.5857	3.4757	3.4653	3.4642	3.4641



5. (a) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2-16} = \lim_{x\to 4} \dfrac{3(x-4)}{(x-4)(x+4)} = \lim_{x\to 4} \dfrac{3}{x+4} = \dfrac{3}{8}$

(b) $\lim_{x\to 4} \dfrac{3(x-4)}{x^2-16} = \lim_{x\to 4} \dfrac{d/dx[3(x-4)]}{d/dx[x^2-16]} = \lim_{x\to 4} \dfrac{3}{2x} = \dfrac{3}{8}$

6. (a) $\lim_{x\to -4} \dfrac{2x^2+13x+20}{x+4} = \lim_{x\to -4} \dfrac{(x+4)(2x+5)}{x+4} = \lim_{x\to -4}(2x+5) = -8+5 = -3$

(b) $\lim_{x\to -4} \dfrac{2x^2+13x+20}{x+4} = \lim_{x\to -4} \dfrac{d/dx[2x^2+13x+20]}{d/dx[x+4]} = \lim_{x\to -4} \dfrac{4x+13}{1} = -3$

7. (a) $\lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} = \lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} \cdot \dfrac{\sqrt{x+10}+4}{\sqrt{x+10}+4} = \lim_{x\to 6} \dfrac{(x+10)-16}{(x-6)\left(\sqrt{x+10}+4\right)} = \lim_{x\to 6} \dfrac{1}{\sqrt{x+10}+4} = \dfrac{1}{8}$

(b) $\lim_{x\to 6} \dfrac{\sqrt{x+10}-4}{x-6} = \lim_{x\to 6} \dfrac{d/dx\left[\sqrt{x+10}-4\right]}{d/dx[x-6]} = \lim_{x\to 6} \dfrac{\frac{1}{2}(x+10)^{-1/2}}{1} = 1/8$

8. (a) $\lim_{x\to 0} \dfrac{\sin 6x}{4x} = \lim_{x\to 0}\left(\dfrac{3}{2} \cdot \dfrac{\sin 6x}{6x}\right) = \dfrac{3}{2}(1) = \dfrac{3}{2}$

(b) $\lim_{x\to 0} \dfrac{\sin 6x}{4x} = \lim_{x\to 0} \dfrac{d/dx[\sin 6x]}{d/dx[4x]} = \lim_{x\to 0} \dfrac{6\cos 6x}{4} = \dfrac{3}{2}$

9. (a) $\lim_{x\to\infty} \dfrac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty} \dfrac{5-(3/x)+\left(1/x^2\right)}{3-\left(5/x^2\right)} = \dfrac{5}{3}$

(b) $\lim_{x\to\infty} \dfrac{5x^2-3x+1}{3x^2-5} = \lim_{x\to\infty} \dfrac{(d/dx)\left[5x^2-3x+1\right]}{(d/dx)\left[3x^2-5\right]} = \lim_{x\to\infty} \dfrac{10x-3}{6x} = \lim_{x\to\infty} \dfrac{(d/dx)[10x-3]}{(d/dx)[6x]} = \lim_{x\to\infty} \dfrac{10}{6} = \dfrac{5}{3}$

10. (a) $\lim_{x\to\infty} \dfrac{4x-3}{5x^2+1} = \lim_{x\to\infty} \dfrac{(4/x)-\left(3/x^2\right)}{5+\left(1/x^2\right)} = 0$

(b) $\lim_{x\to\infty} \dfrac{4x-3}{5x^2+1} = \lim_{x\to\infty} \dfrac{(d/dx)[4x-3]}{(d/dx)\left[5x^2+1\right]} = \lim_{x\to\infty} \dfrac{4}{10x} = 0$

11. $\lim_{x\to 3} \dfrac{x^2-2x-3}{x-3} = \lim_{x\to 3} \dfrac{2x-2}{1} = 4$

12. $\lim_{x\to -2} \dfrac{x^2-3x-10}{x+2} = \lim_{x\to -2} \dfrac{2x-3}{1} = -7$

13. $\lim_{x\to 0} \frac{\sqrt{25 - x^2} - 5}{x} = \lim_{x\to 0} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 0} \frac{-x}{\sqrt{25 - x^2}} = 0$

14. $\lim_{x\to 5^-} \frac{\sqrt{25 - x^2}}{x - 5} = \lim_{x\to 5^-} \frac{\frac{1}{2}(25 - x^2)^{-1/2}(-2x)}{1}$

$= \lim_{x\to 5^-} \frac{-x}{\sqrt{25 - x^2}} = -\infty$

15. $\lim_{x\to 0^+} \frac{e^x - (1 + x)}{x^3} = \lim_{x\to 0^+} \frac{e^x - 1}{3x^2} = \lim_{x\to 0^+} \frac{e^x}{6x} = \infty$

16. $\lim_{x\to 1} \frac{\ln x^3}{x^2 - 1} = \lim_{x\to 1} \frac{3 \ln x}{x^2 - 1} = \lim_{x\to 1} \frac{3/x}{2x} = \frac{3}{2}$

17. $\lim_{x\to 1} \frac{x^{11} - 1}{x^4 - 1} = \lim_{x\to 1} \frac{11x^{10}}{4x^3} = \frac{11}{4}$

18. $\lim_{x\to 1} \frac{x^a - 1}{x^b - 1} = \lim_{x\to 1} \frac{ax^{a-1}}{bx^{b-1}} = \frac{a}{b}$

19. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x} = \lim_{x\to 0} \frac{3 \cos 3x}{5 \cos 5x} = \frac{3}{5}$

20. $\lim_{x\to 0} \frac{\sin ax}{\sin bx} = \lim_{x\to 0} \frac{a \cos ax}{b \cos bx} = \frac{a}{b}$

21. $\lim_{x\to 0} \frac{\arcsin x}{x} = \lim_{x\to 0} \frac{1/\sqrt{1 - x^2}}{1} = 1$

22. $\lim_{x\to 1} \frac{\arctan x - (\pi/4)}{x - 1} = \lim_{x\to 1} \frac{1/(1 + x^2)}{1} = \frac{1}{2}$

23. $\lim_{x\to \infty} \frac{5x^2 + 3x - 1}{4x^2 + 5} = \lim_{x\to \infty} \frac{10x + 3}{8x} = \lim_{x\to \infty} \frac{10}{8} = \frac{5}{4}$

24. $\lim_{x\to \infty} \frac{5x + 3}{x^3 - 6x + 2} = \lim_{x\to \infty} \frac{5}{3x^2 - 6} = 0$

25. $\lim_{x\to \infty} \frac{x^2 + 4x + 7}{x - 6} = \lim_{x\to \infty} \frac{2x + 4}{1} = \infty$

26. $\lim_{x\to \infty} \frac{x^3}{x + 1} = \lim_{x\to \infty} \frac{3x^2}{1} = \infty$

27. $\lim_{x\to \infty} \frac{x^3}{e^{x/2}} = \lim_{x\to \infty} \frac{3x^2}{(1/2)e^{x/2}}$

$= \lim_{x\to \infty} \frac{6x}{(1/4)e^{x/2}} = \lim_{x\to \infty} \frac{6}{(1/8)e^{x/2}} = 0$

28. $\lim_{x\to \infty} \frac{x^3}{e^{x^2}} = \lim_{x\to \infty} \frac{3x^2}{2xe^{x^2}}$

$= \lim_{x\to \infty} \frac{6x}{(4x^2 + 2)e^{x^2}}$

$= \lim_{x\to \infty} \frac{6}{4x(2x^2 + 3)e^{x^2}} = 0$

29. $\lim_{x\to \infty} \frac{x}{\sqrt{x^2 + 1}} = \lim_{x\to \infty} \frac{1}{\sqrt{1 + (1/x^2)}} = 1$

Note: L'Hôpital's Rule does not work on this limit. See Exercise 83.

30. $\lim_{x\to \infty} \frac{x^2}{\sqrt{x^2 + 1}} = \lim_{x\to \infty} \frac{x}{\sqrt{1 + (1/x)^2}} = \infty$

31. $\lim_{x\to \infty} \frac{\cos x}{x} = 0$ by Squeeze Theorem

$\left(\frac{\cos x}{x} \le \frac{1}{x}, \text{ for } x > 0\right)$

32. $\lim_{x\to \infty} \frac{\sin x}{x - \pi} = 0$

Note: Use the Squeeze Theorem for $x > \pi$.

$-\frac{1}{x - \pi} \le \frac{\sin x}{x - \pi} \le \frac{1}{x - \pi}$

33. $\lim_{x\to \infty} \frac{\ln x}{x^2} = \lim_{x\to \infty} \frac{1/x}{2x} = \lim_{x\to \infty} \frac{1}{2x^2} = 0$

34. $\lim_{x\to \infty} \frac{\ln x^4}{x^3} = \lim_{x\to \infty} \frac{4 \ln x}{x^3} = \lim_{x\to \infty} \frac{4/x}{3x^2} = \lim_{x\to \infty} \frac{4}{3x^3} = 0$

35. $\lim_{x\to \infty} \frac{e^x}{x^4} = \lim_{x\to \infty} \frac{e^x}{4x^3}$

$= \lim_{x\to \infty} \frac{e^x}{12x^2}$

$= \lim_{x\to \infty} \frac{e^x}{24x}$

$= \lim_{x\to \infty} \frac{e^x}{24} = \infty$

36. $\lim_{x\to \infty} \frac{e^{x/2}}{x} = \lim_{x\to \infty} \frac{(1/2)e^{x/2}}{1} = \infty$

37. $\lim_{x\to 0} \frac{\sin 5x}{\tan 9x} = \lim_{x\to 0} \frac{5 \cos 5x}{9 \sec^2 9x} = \frac{5}{9}$

38. $\lim_{x\to 1} \frac{\ln x}{\sin \pi x} = \lim_{x\to 1} \frac{1/x}{\pi \cos \pi x} = -\frac{1}{\pi}$

39. $\lim\limits_{x\to 0} \dfrac{\arctan x}{\sin x} = \lim\limits_{x\to 0} \dfrac{1/(1+x^2)}{\cos x} = 1$

40. $\lim\limits_{x\to 0} \dfrac{x}{\arctan 2x} = \lim\limits_{x\to 0} \dfrac{1}{2/(1+4x^2)} = 1/2$

41. $\lim\limits_{x\to\infty} \dfrac{\int_1^x \ln\left(e^{4t-1}\right) dt}{x} = \lim\limits_{x\to\infty} \dfrac{\int_1^x (4t-1)\, dt}{x}$

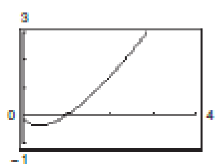
$= \lim\limits_{x\to\infty} \dfrac{4x-1}{1} = \infty$

42. $\lim\limits_{x\to 1^+} \dfrac{\int_1^x \cos\theta\, d\theta}{x-1} = \lim\limits_{x\to 1^+} \dfrac{\cos x}{1} = \cos(1)$

43. (a) $\lim\limits_{x\to\infty} x \ln x$, not indeterminate

(b) $\lim\limits_{x\to\infty} x \ln x = (\infty)(\infty) = \infty$

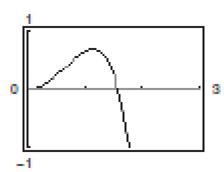
(c)



44. (a) $\lim\limits_{x\to 0^+} x^3 \cot x = (0)(\infty)$

(b) $\lim\limits_{x\to 0^+} x^3 \cot x = \lim\limits_{x\to 0^+} \dfrac{x^3}{\tan x} = \lim\limits_{x\to 0^+} \dfrac{3x^2}{\sec^2 x} = 0$

(c)



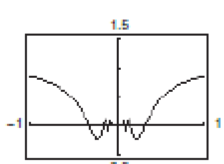
45. (a) $\lim\limits_{x\to\infty} \left(x \sin \dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim\limits_{x\to\infty} x \sin \dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\sin(1/x)}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{\left(-1/x^2\right)\cos(1/x)}{-1/x^2}$

$= \lim\limits_{x\to\infty} \cos \dfrac{1}{x} = 1$

(c)



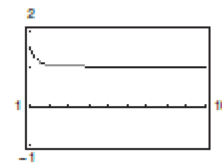
46. (a) $\lim\limits_{x\to\infty} \left(x \tan \dfrac{1}{x}\right) = (\infty)(0)$

(b) $\lim\limits_{x\to\infty} x \tan \dfrac{1}{x} = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x}$

$= \lim\limits_{x\to\infty} \dfrac{-\left(1/x^2\right)\sec^2(1/x)}{-\left(1/x^2\right)}$

$= \lim\limits_{x\to\infty} \sec^2 \dfrac{1}{x} = 1$

(c)



47. (a) $\lim\limits_{x\to 0^+} x^{1/x} = 0^\infty = 0$, not indeterminate

(See Exercise 105).

(b) Let $y = x^{1/x}$

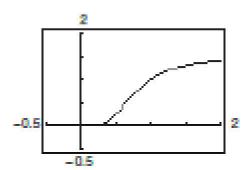
$\ln y = \ln x^{1/x} = \dfrac{1}{x} \ln x.$

Because $x \to 0^+$, $\dfrac{1}{x} \ln x \to (\infty)(-\infty) = -\infty$. So,

$\ln y \to -\infty \Rightarrow y \to 0^+$.

Therefore, $\lim\limits_{x\to 0^+} x^{1/x} = 0$.

(c)



48. (a) $\lim\limits_{x\to 0^+} \left(e^x + x\right)^{2/x} = 1^\infty$

(b) Let $y = \lim\limits_{x\to 0^+} \left(e^x + x\right)^{2/x}$.

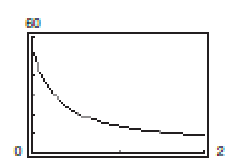
$\ln y = \lim\limits_{x\to 0^+} \dfrac{2\ln\left(e^x + x\right)}{x}$

$= \lim\limits_{x\to 0^+} \dfrac{2\left(e^x + 1\right)/\left(e^x + x\right)}{1} = 4$

So, $\ln y = 4 \Rightarrow y = e^4 \approx 54.598$. Therefore,

$\lim\limits_{x\to 0^+} \left(e^x + x\right)^{2/x} = e^4$.

(c)



49. (a) $\lim_{x \to \infty} x^{1/x} = \infty^0$

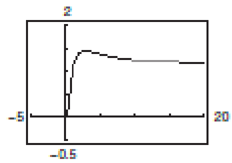
(b) Let $y = \lim_{x \to \infty} x^{1/x}$.

$$\ln y = \lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1/x}{1} = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$. Therefore,

$\lim_{x \to \infty} x^{1/x} = 1$.

(c)



50. (a) $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = 1^\infty$

(b) Let $y = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

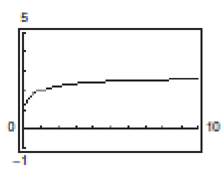
$$\ln y = \lim_{x \to \infty} \left[x \ln\left(1 + \frac{1}{x}\right)\right] = \lim_{x \to \infty} \frac{\ln[1 + (1/x)]}{1/x}$$

$$= \lim_{x \to \infty} \frac{\left[\dfrac{(-1/x^2)}{1 + (1/x)}\right]}{(-1/x^2)} = \lim_{x \to \infty} \frac{1}{1 + (1/x)} = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$. Therefore,

$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$.

(c)



51. (a) $\lim_{x \to 0^+} (1 + x)^{1/x} = 1^\infty$

(b) Let $y = \lim_{x \to 0^+} (1 + x)^{1/x}$.

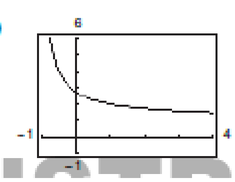
$$\ln y = \lim_{x \to 0^+} \frac{\ln(1 + x)}{x}$$

$$= \lim_{x \to 0^+} \left(\frac{1/(1 + x)}{1}\right) = 1$$

So, $\ln y = 1 \Rightarrow y = e^1 = e$.

Therefore, $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

(c)



52. (a) $\lim_{x \to \infty} (1 + x)^{1/x} = \infty^0$

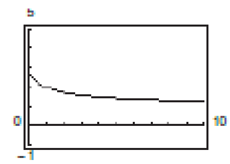
(b) Let $y = \lim_{x \to \infty} (1 + x)^{1/x}$.

$$\ln y = \lim_{x \to \infty} \frac{\ln(1 + x)}{x} = \lim_{x \to \infty} \left(\frac{1/(1 + x)}{1}\right) = 0$$

So, $\ln y = 0 \Rightarrow y = e^0 = 1$.

Therefore, $\lim_{x \to \infty} (1 + x)^{1/x} = 1$.

(c)



53. (a) $\lim_{x \to 0^+} \left[3(x)^{x/2}\right] = 0^0$

(b) Let $y = \lim_{x \to 0^+} 3(x)^{x/2}$.

$$\ln y = \lim_{x \to 0^+} \left[\ln 3 + \frac{x}{2} \ln x\right]$$

$$= \lim_{x \to 0^+} \left[\ln 3 + \frac{\ln x}{2/x}\right]$$

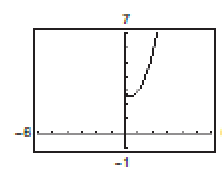
$$= \lim_{x \to 0^+} \ln 3 + \lim_{x \to 0^+} \frac{1/x}{-2/x^2}$$

$$= \lim_{x \to 0^+} \ln 3 - \lim_{x \to 0^+} \frac{x}{2}$$

$$= \ln 3$$

So, $\lim_{x \to 0^+} 3(x)^{x/2} = 3$.

(c)



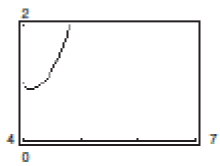
54. (a) $\lim_{x\to 4^+} [3(x-4)]^{x-4} = 0^0$

(b) Let $y = \lim_{x\to 4^+} [3(x-4)]^{x-4}$.

$$\ln y = \lim_{x\to 4^+} (x-4)\ln[3(x-4)]$$
$$= \lim_{x\to 4^+} \frac{\ln[3(x-4)]}{1/(x-4)}$$
$$= \lim_{x\to 4^+} \frac{1/(x-4)}{-1/(x-4)^2}$$
$$= \lim_{x\to 4^+} [-(x-4)] = 0$$

So, $\lim_{x\to 4^+} [3(x-4)]^{x-4} = 1$.

(c)



55. (a) $\lim_{x\to 1^+} (\ln x)^{x-1} = 0^0$

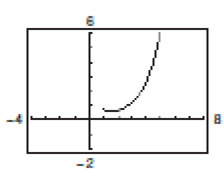
(b) Let $y = (\ln x)^{x-1}$.

$$\ln y = \ln\left[(\ln x)^{x-1}\right] = (x-1)\ln(\ln x)$$
$$= \frac{\ln(\ln x)}{(x-1)^{-1}}$$

$$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln(\ln x)}{(x-1)^{-1}}$$
$$= \lim_{x\to 1^+} \frac{1/(x\ln x)}{-(x-1)^{-2}}$$
$$= \lim_{x\to 1^+} \frac{-(x-1)^2}{x\ln x}$$
$$= \lim_{x\to 1^+} \frac{-2(x-1)}{1+\ln x} = 0$$

Because $\lim_{x\to 1^+} \ln y = 0$, $\lim_{x\to 1^+} y = 1$.

(c)



56. $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

(a) $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = \lim_{x\to 0^+} [\sin x]^x = 0^0$

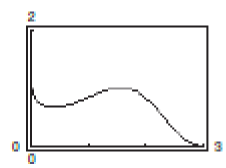
(b) Let $y = (\sin x)^x$

$$\ln y = x\ln(\sin x) = \frac{\ln(\sin x)}{1/x}$$

$$\lim_{x\to 0^+} \frac{\ln(\sin x)}{1/x} = \lim_{x\to 0^+} \frac{\cos x/\sin x}{-1/x^2}$$
$$= \lim_{x\to 0^+} \frac{-x^2\cos x}{\sin x}$$
$$= \lim_{x\to 0^+} \frac{x}{\sin x}\left(\frac{-x\cos x}{1}\right)$$
$$= 0$$

So, $\lim_{x\to 0^+} \left[\cos\left(\frac{\pi}{2} - x\right)\right]^x = 1$.

(c)

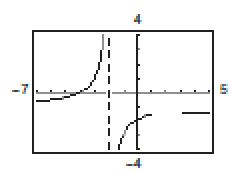


57. (a) $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \infty - \infty$

(b) $\lim_{x\to 2^+} \left(\frac{8}{x^2-4} - \frac{x}{x-2}\right) = \lim_{x\to 2^+} \frac{8 - x(x+2)}{x^2-4}$

$$= \lim_{x\to 2^+} \frac{(2-x)(4+x)}{(x+2)(x-2)}$$
$$= \lim_{x\to 2^+} \frac{-(x+4)}{x+2} = -\frac{3}{2}$$

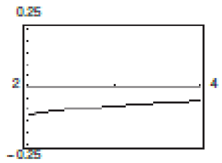
(c)



58. (a) $\displaystyle\lim_{x\to 2^+}\left(\frac{1}{x^2-4}-\frac{\sqrt{x-1}}{x^2-4}\right)=\infty-\infty$

(b) $$\lim_{x\to 2^+}\left(\frac{1}{x^2-4}-\frac{\sqrt{x-1}}{x^2-4}\right)=\lim_{x\to 2^+}\frac{1-\sqrt{x-1}}{x^2-4}$$
$$=\lim_{x\to 2^+}\frac{-1/\left(2\sqrt{x-1}\right)}{2x}$$
$$=\lim_{x\to 2^+}\frac{-1}{4x\sqrt{x-1}}=-\frac{1}{8}$$

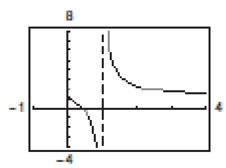
(c)



59. (a) $\displaystyle\lim_{x\to 1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right)=\infty-\infty$

(b) $$\lim_{x\to 1^+}\left(\frac{3}{\ln x}-\frac{2}{x-1}\right)=\lim_{x\to 1^+}\frac{3x-3-2\ln x}{(x-1)\ln x}$$
$$=\lim_{x\to 1^+}\frac{3-(2/x)}{\left[(x-1)/x\right]+\ln x}=\infty$$

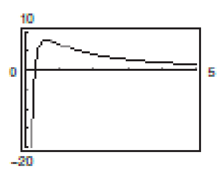
(c)



60. (a) $\displaystyle\lim_{x\to 0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right)=\infty-\infty$

(b) $\displaystyle\lim_{x\to 0^+}\left(\frac{10}{x}-\frac{3}{x^2}\right)=\lim_{x\to 0^+}\left(\frac{10x-3}{x^2}\right)=-\infty$

(c)



61. $\dfrac{0}{0}, \dfrac{\infty}{\infty}, 0\cdot\infty, 1^{\infty}, 0^0, \infty-\infty, \infty^0$

62. See Theorem 7.4.

63. (a) Let $f(x)=x^2-25$ and $g(x)=x-5$.

(b) Let $f(x)=(x-5)^2$ and $g(x)=x^2-25$.

(c) Let $f(x)=x^2-25$ and $g(x)=(x-5)^3$.

(Answers will vary.)

64. Let $f(x)=x+25$ and $g(x)=x$.

(Answers will vary.)

65. (a) Yes: $\dfrac{0}{0}$

(b) No: $\dfrac{0}{-1}$

(c) Yes: $\dfrac{\infty}{\infty}$

(d) Yes: $\dfrac{0}{0}$

(e) No: $\dfrac{-1}{0}$

(f) Yes: $\dfrac{0}{0}$

66. (a) From the graph, $\displaystyle\lim_{x\to 1^-}f(x)=\infty$.

(b) From the graph, $\displaystyle\lim_{x\to 1^+}f(x)=-\infty$.

(c) From the graph, $\displaystyle\lim_{x\to 1}f(x)$ does not exist.

67.

x	10	10^2	10^4	10^6	10^8	10^{10}
$\dfrac{(\ln x)^4}{x}$	2.811	4.498	0.720	0.036	0.001	0.000

68.

x	1	5	10	20	30	40	50	100
$\dfrac{e^x}{x^5}$	2.718	0.047	0.220	151.614	4.40×10^5	2.30×10^9	1.66×10^{13}	2.69×10^{33}

69. $\displaystyle \lim_{x\to\infty} \frac{x^2}{e^{5x}} = \lim_{x\to\infty} \frac{2x}{5e^{5x}} = \lim_{x\to\infty} \frac{2}{25e^{5x}} = 0$

70. $\displaystyle \lim_{x\to\infty} \frac{x^3}{e^{2x}} = \lim_{x\to\infty} \frac{3x^2}{2e^{2x}} = \lim_{x\to\infty} \frac{6x}{4e^{2x}} = \lim_{x\to\infty} \frac{6}{8e^{2x}} = 0$

71. $$\begin{aligned}\lim_{x\to\infty} \frac{(\ln x)^3}{x} &= \lim_{x\to\infty} \frac{3(\ln x)^2(1/x)}{1}\\ &= \lim_{x\to\infty} \frac{3(\ln x)^2}{x}\\ &= \lim_{x\to\infty} \frac{6(\ln x)(1/x)}{1}\\ &= \lim_{x\to\infty} \frac{6(\ln x)}{x} = \lim_{x\to\infty} \frac{6}{x} = 0\end{aligned}$$

72. $$\begin{aligned}\lim_{x\to\infty} \frac{(\ln x)^2}{x^3} &= \lim_{x\to\infty} \frac{(2\ln x)/x}{3x^2}\\ &= \lim_{x\to\infty} \frac{2\ln x}{3x^3}\\ &= \lim_{x\to\infty} \frac{2/x}{9x^2} = \lim_{x\to\infty} \frac{2}{9x^3} = 0\end{aligned}$$

73. $$\begin{aligned}\lim_{x\to\infty} \frac{(\ln x)^n}{x^m} &= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}/x}{mx^{m-1}}\\ &= \lim_{x\to\infty} \frac{n(\ln x)^{n-1}}{mx^m}\\ &= \lim_{x\to\infty} \frac{n(n-1)(\ln x)^{n-2}}{m^2x^m}\\ &= \cdots = \lim_{x\to\infty} \frac{n!}{m^nx^m} = 0\end{aligned}$$

74. $$\begin{aligned}\lim_{x\to\infty} \frac{x^m}{e^{nx}} &= \lim_{x\to\infty} \frac{mx^{m-1}}{ne^{nx}}\\ &= \lim_{x\to\infty} \frac{m(m-1)x^{m-2}}{n^2e^{nx}}\\ &= \cdots = \lim_{x\to\infty} \frac{m!}{n^me^{nx}} = 0\end{aligned}$$

75. $y = x^{1/x}$, $x > 0$

Horizontal asymptote: $y = 1$ (See Exercise 49.)

$$\ln y = \frac{1}{x}\ln x$$

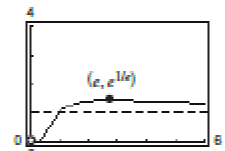
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = \frac{1}{x}\left(\frac{1}{x}\right) + (\ln x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = x^{1/x}\left(\frac{1}{x^2}\right)(1-\ln x) = x^{(1/x)-2}(1-\ln x) = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $(e, e^{1/e})$



76. $y = x^x$, $x > 0$

$\displaystyle \lim_{x\to\infty} x^x = \infty$ and $\displaystyle \lim_{x\to 0^+} x^x = 1$

No horizontal asymptotes

$$\ln y = x\ln x$$

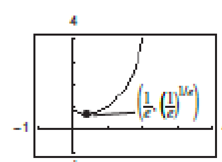
$$\left(\frac{1}{y}\right)\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x$$

$$\frac{dy}{dx} = x^x(1+\ln x) = 0$$

Critical number: $x = e^{-1}$

Intervals:	$(0, e^{-1})$	$(e^{-1}, 0)$
Sign of dy/dx:	$-$	$+$
$y = f(x)$:	Decreasing	Increasing

Relative maximum: $\left(e^{-1}, (e^{-1})^{e^{-1}}\right) = \left(\frac{1}{e}, \left(\frac{1}{e}\right)^{1/e}\right)$



77. $y = 2xe^{-x}$

$$\lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$$

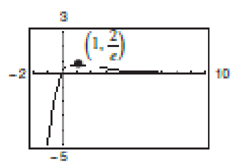
Horizontal asymptote: $y = 0$

$$\frac{dy}{dx} = 2x\left(-e^{-x}\right) + 2e^{-x}$$
$$= 2e^{-x}(1 - x) = 0$$

Critical number: $x = 1$

Intervals:	$(-\infty, 1)$	$(1, \infty)$
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(1, \dfrac{2}{e}\right)$



78. $y = \dfrac{\ln x}{x}$

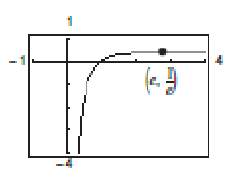
Horizontal asymptote: $y = 0$ (See Example 2.)

$$\frac{dy}{dx} = \frac{x(1/x) - (\ln x)(1)}{x^2} = \frac{1 - \ln x}{x^2} = 0$$

Critical number: $x = e$

Intervals:	$(0, e)$	(e, ∞)
Sign of dy/dx:	$+$	$-$
$y = f(x)$:	Increasing	Decreasing

Relative maximum: $\left(e, \dfrac{1}{e}\right)$



79. $\displaystyle\lim_{x\to\infty} \frac{e^{-x}}{1 + e^{-x}} = \frac{0}{1 + 0} = 0$

Limit is not of the form $0/0$ or ∞/∞.

L'Hôpital's Rule does not apply.

80. $\displaystyle\lim_{x\to\infty} x \cos\frac{1}{x} = \infty(1) = \infty$

Limit is not of the form $0/0$ or ∞/∞.

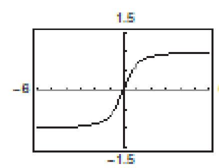
L'Hôpital's Rule does not apply.

81. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{x^2+1}}$$
$$= \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$
$$= \lim_{x\to\infty} \frac{x/\sqrt{x^2+1}}{1}$$
$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

(b) $$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/x}{\sqrt{x^2+1}/x}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + 1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

(c)



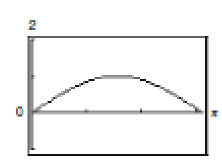
82. (a) Applying L'Hôpital's Rule twice results in the original limit, so L'Hôpital's Rule fails:

$\displaystyle\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$ is indeterminant: $\dfrac{\infty}{\infty}$

$$\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\sec^2 x}{\sec x \tan x}$$
$$= \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x} \quad \left(\frac{\infty}{\infty}\right)$$
$$= \lim_{x\to(\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x}$$
$$= \lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x}$$

(b) $$\lim_{x\to(\pi/2)^-} \frac{\tan x}{\sec x} = \lim_{x\to(\pi/2)^-} \frac{\sin x}{\cos x}(\cos x)$$
$$= \lim_{x\to(\pi/2)^-} \sin x = 1$$

(c)



83. $f(x) = \sin 3x,\ g(x) = \sin 4x$

$f'(x) = 3\cos 3x,\ g'(x) = 4\cos 4x$

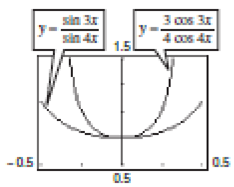
$y_1 = \dfrac{f(x)}{g(x)} = \dfrac{\sin 3x}{\sin 4x},$

$y_2 = \dfrac{f'(x)}{g'(x)} = \dfrac{3\cos 3x}{4\cos 4x}$

As $x \to 0,\ y_1 \to 0.75$ and $y_2 \to 0.75$

By L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x} = \lim_{x\to 0}\frac{3\cos 3x}{4\cos 4x} = \frac{3}{4}$$



84. $f(x) = e^{3x} - 1,\ g(x) = x$

$f'(x) = 3e^{3x},\ g'(x) = 1$

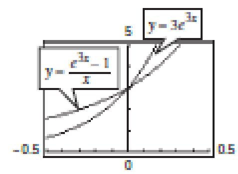
$y_1 = \dfrac{f(x)}{g(x)} = \dfrac{e^{3x} - 1}{x},$

$y_2 = \dfrac{f'(x)}{g'(x)} = 3e^{3x}$

As $x \to 0,\ y_1 \to 3$ and $y_2 \to 3$

By L'Hôpital's Rule,

$$\lim_{x\to 0}\frac{e^{3x} - 1}{x} = \lim_{x\to 0}\frac{3e^{3x}}{1} = 3$$



85. $$\lim_{k\to 0}\frac{32\left(1 - e^{-kt} + \dfrac{v_0ke^{-kt}}{32}\right)}{k} = \lim_{k\to 0}\frac{32\left(1 - e^{-kt}\right)}{k} + \lim_{k\to 0}\left(v_0e^{-kt}\right) = \lim_{k\to 0}\frac{32\left(0 + te^{-kt}\right)}{1} + \lim_{k\to 0}\left(\frac{v_0}{e^{kt}}\right) = 32t + v_0$$

86. $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$

$$\ln A = \ln P + nt\ln\left(1 + \frac{r}{n}\right) = \ln P + \frac{\ln\left(1 + \dfrac{r}{n}\right)}{\dfrac{1}{nt}}$$

$$\lim_{n\to\infty}\left[\frac{\ln\left(1 + \dfrac{r}{n}\right)}{\dfrac{1}{nt}}\right] = \lim_{n\to\infty}\left[\frac{-\dfrac{r}{n^2}\left(\dfrac{1}{1 + (r/n)}\right)}{-\left(\dfrac{1}{n^2t}\right)}\right] = \lim_{n\to\infty}\left[rt\left(\frac{1}{1 + \dfrac{r}{n}}\right)\right] = rt$$

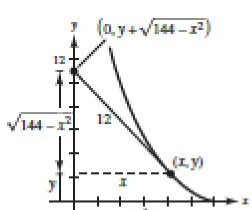
Because $\lim\limits_{n\to\infty}\ln A = \ln P + rt$, you have $\lim\limits_{n\to\infty}A = e^{(\ln P + rt)} = e^{\ln P}e^{rt} = Pe^{rt}$. Alternatively,

$$\lim_{n\to\infty}A = \lim_{n\to\infty}P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{n\to\infty}P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = Pe^{rt}.$$

87. Let N be a fixed value for n. Then

$$\lim_{x\to\infty}\frac{x^{N-1}}{e^x} = \lim_{x\to\infty}\frac{(N-1)x^{N-2}}{e^x} = \lim_{x\to\infty}\frac{(N-1)(N-2)x^{N-3}}{e^x} = \cdots = \lim_{x\to\infty}\left[\frac{(N-1)!}{e^x}\right] = 0. \quad \text{(See Exercise 74.)}$$

88. (a) $m = \dfrac{dy}{dx} = \dfrac{y - \left(y + \sqrt{144 - x^2}\right)}{x - 0} = -\dfrac{\sqrt{144 - x^2}}{x}$



(b) $y = -\int \frac{\sqrt{144 - x^2}}{x}\,dx$

Let $x = 12\sin\theta$, $dx = 12\cos\theta\,d\theta$, $\sqrt{144 - x^2} = 12\cos\theta$.

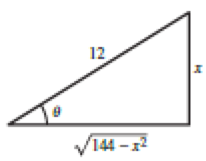
$$y = -\int \frac{12\cos\theta}{12\sin\theta} 12\cos\theta\,d\theta = -12\int \frac{1 - \sin^2\theta}{\sin\theta}\,d\theta$$

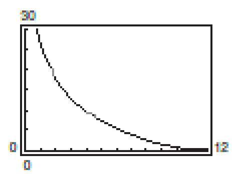
$$= -12\int(\csc\theta - \sin\theta)\,d\theta = -12\ln\left|\csc\theta - \cot\theta\right| - 12\cos\theta + C$$

$$= -12\ln\left|\frac{12}{x} - \frac{\sqrt{144 - x^2}}{x}\right| - 12\left(\frac{\sqrt{144 - x^2}}{12}\right) + C = -12\ln\left|\frac{12 - \sqrt{144 - x^2}}{x}\right| - \sqrt{144 - x^2} + C$$

When $x = 12$, $y = 0 \Rightarrow C = 0$. So, $y = -12\ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$.

Note: $\frac{12 - \sqrt{144 - x^2}}{x} > 0$ for $0 < x \le 12$





(c) Vertical asymptote: $x = 0$

(d) $y + \sqrt{144 - x^2} = 12 \Rightarrow y = 12 - \sqrt{144 - x^2}$

So,
$$12 - \sqrt{144 - x^2} = -12\ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right) - \sqrt{144 - x^2}$$
$$-1 = \ln\left(\frac{12 - \sqrt{144 - x^2}}{x}\right)$$
$$xe^{-1} = 12 - \sqrt{144 - x^2}$$
$$\left(xe^{-1} - 12\right)^2 = \left(-\sqrt{144 - x^2}\right)^2$$
$$x^2e^{-2} - 24xe^{-1} + 144 = 144 - x^2$$
$$x^2\left(e^{-2} + 1\right) - 24xe^{-1} = 0$$
$$x\left[x\left(e^{-2} + 1\right) - 24e^{-1}\right] = 0$$
$$x = 0 \text{ or } x = \frac{24e^{-1}}{e^{-2} + 1} \approx 7.77665.$$

Therefore, $s = \int_{7.77665}^{12} \sqrt{1 + \left(-\frac{\sqrt{144 - x^2}}{x}\right)^2}\,dx = \int_{7.77665}^{12} \sqrt{\frac{x^2 + \left(144 - x^2\right)}{x^2}}\,dx$

$$= \int_{7.77665}^{12} \frac{12}{x}\,dx = \Big[12\ln|x|\Big]_{7.77665}^{12} = 12(\ln 12 - \ln 7.77665) \approx 5.2 \text{ meters.}$$

89. $f(x) = x^3, g(x) = x^2 + 1, [0, 1]$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{f(1) - f(0)}{g(1) - g(0)} = \frac{3c^2}{2c}$$

$$\frac{1}{1} = \frac{3c}{2}$$

$$c = \frac{2}{3}$$

90. $f(x) = \frac{1}{x}, g(x) = x^2 - 4, [1, 2]$

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{-1/2}{3} = \frac{-1/c^2}{2c}$$

$$-\frac{1}{6} = -\frac{1}{2c^3}$$

$$2c^3 = 6$$

$$c = \sqrt[3]{3}$$

91. $f(x) = \sin x, g(x) = \cos x, \left[0, \frac{\pi}{2}\right]$

$$\frac{f(\pi/2) - f(0)}{g(\pi/2) - g(0)} = \frac{f'(c)}{g'(c)}$$

$$\frac{1}{-1} = \frac{\cos c}{-\sin c}$$

$$-1 = -\cot c$$

$$c = \frac{\pi}{4}$$

92. $f(x) = \ln x, g(x) = x^3, [1, 4]$

$$\frac{f(4) - f(1)}{g(4) - g(1)} = \frac{f'(c)}{g'(c)}$$

$$\frac{\ln 4}{63} = \frac{1/c}{3c^2} = \frac{1}{3c^3}$$

$$3c^3 \ln 4 = 63$$

$$c^3 = \frac{21}{\ln 4}$$

$$c = \sqrt[3]{\frac{21}{\ln 4}} \approx 2.474$$

93. False. L'Hôpital's Rule does not apply because

$\lim_{x \to 0} \left(x^2 + x + 1\right) \neq 0.$

$$\lim_{x \to 0^+} \frac{x^2 + x + 1}{x} = \lim_{x \to 0^+} \left(x + 1 + \frac{1}{x}\right) = 1 + \infty = \infty$$

94. False. If $y = e^x/x^2$, then

$$y' = \frac{x^2e^x - 2xe^x}{x^4} = \frac{xe^x(x - 2)}{x^4} = \frac{e^x(x - 2)}{x^3}.$$

95. True

96. False. Let $f(x) = x$ and $g(x) = x + 1$. Then

$\lim_{x \to \infty} \frac{x}{x + 1} = 1$, but $\lim_{x \to \infty} \left[x - (x + 1)\right] = -1$.

97. Area of triangle: $\frac{1}{2}(2x)(1 - \cos x) = x - x\cos x$

Shaded area: Area of rectangle $-$ Area under curve

$$2x(1 - \cos x) - 2\int_0^x (1 - \cos t)\,dt = 2x(1 - \cos x) - 2\big[t - \sin t\big]_0^x$$
$$= 2x(1 - \cos x) - 2(x - \sin x)$$
$$= 2\sin x - 2x\cos x$$

Ratio: $\lim_{x \to 0} \frac{x - x\cos x}{2\sin x - 2x\cos x} = \lim_{x \to 0} \frac{1 + x\sin x - \cos x}{2\cos x + 2x\sin x - 2\cos x}$

$$= \lim_{x \to 0} \frac{1 + x\sin x - \cos x}{2x\sin x}$$

$$= \lim_{x \to 0} \frac{x\cos x + \sin x + \sin x}{2x\cos x + 2\sin x}$$

$$= \lim_{x \to 0} \frac{x\cos x + 2\sin x}{2x\cos x + 2\sin x} \cdot \frac{1/\cos x}{1/\cos x} = \lim_{x \to 0} \frac{x + 2\tan x}{2x + 2\tan x} = \lim_{x \to 0} \frac{1 + 2\sec^2 x}{2 + 2\sec^2 x} = \frac{3}{4}$$

98. (a) $\sin\theta = BD$

$\cos\theta = DO \Rightarrow AD = 1 - \cos\theta$

$$\text{Area } \Delta ABD = \frac{1}{2}bh = \frac{1}{2}(1 - \cos\theta)\sin\theta = \frac{1}{2}\sin\theta - \frac{1}{2}\sin\theta\cos\theta$$

(b) Area of sector: $\frac{1}{2}\theta$

$$\text{Shaded area: } \frac{1}{2}\theta - \text{Area } \Delta OBD = \frac{1}{2}\theta - \frac{1}{2}(\cos\theta)(\sin\theta) = \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta$$

(c) $$R = \frac{(1/2)\sin\theta - (1/2)\sin\theta\cos\theta}{(1/2)\theta - (1/2)\sin\theta\cos\theta} = \frac{\sin\theta - \sin\theta\cos\theta}{\theta - \sin\theta\cos\theta}$$

(d) $$\lim_{\theta\to0} R = \lim_{\theta\to0}\frac{\sin\theta - (1/2)\sin 2\theta}{\theta - (1/2)\sin 2\theta} = \lim_{\theta\to0}\frac{\cos\theta - \cos 2\theta}{1 - \cos 2\theta} = \lim_{\theta\to0}\frac{-\sin\theta + 2\sin 2\theta}{2\sin 2\theta} = \lim_{\theta\to0}\frac{-\cos\theta + 4\cos 2\theta}{4\cos 2\theta} = \frac{3}{4}$$

99. $$\lim_{x\to0}\frac{4x - 2\sin 2x}{2x^3} = \lim_{x\to0}\frac{4 - 4\cos 2x}{6x^2} = \lim_{x\to0}\frac{8\sin 2x}{12x} = \lim_{x\to0}\frac{16\cos 2x}{12} = \frac{16}{12} = \frac{4}{3}$$

Let $c = \frac{4}{3}$.

100. Let $y = \left(e^x + x\right)^{1/x}$.

$$\ln y = \frac{1}{x}\ln\left(e^x + x\right) = \frac{\ln\left(e^x + x\right)}{x}$$

$$\lim_{x\to0}\frac{\ln\left(e^x + x\right)}{x} = \lim_{x\to0}\frac{e^x + 1}{e^x + x} = \frac{2}{1} = 2$$

So, $\lim_{x\to0}\left(e^x + x\right)^{1/x} = e^2$.

Let $c = e^2 \approx 7.389$.

101. $$\lim_{x\to0}\frac{a - \cos bx}{x^2} = 2$$

Near $x = 0$, $\cos bx \approx 1$ and $x^2 \approx 0 \Rightarrow a = 1$.

Using L'Hôpital's Rule,

$$\lim_{x\to0}\frac{1 - \cos bx}{x^2} = \lim_{x\to0}\frac{b\sin bx}{2x} = \lim_{x\to0}\frac{b^2\cos bx}{2} = 2.$$

So, $b^2 = 4$ and $b = \pm2$.

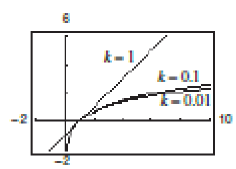
Answer: $a = 1, b = \pm2$

102. $$f(x) = \frac{x^k - 1}{k}$$

$k = 1,\quad f(x) = x - 1$

$k = 0.1,\quad f(x) = \frac{x^{0.1} - 1}{0.1} = 10\left(x^{0.1} - 1\right)$

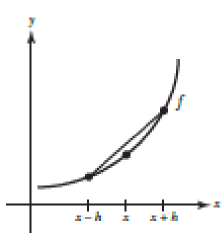
$k = 0.01,\quad f(x) = \frac{x^{0.01} - 1}{0.01} = 100\left(x^{0.01} - 1\right)$



$$\lim_{k\to0^+}\frac{x^k - 1}{k} = \lim_{k\to0^+}\frac{x^k(\ln x)}{1} = \ln x$$

103. (a) $$\lim_{h\to0}\frac{f(x+h) - f(x-h)}{2h} = \lim_{h\to0}\frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \lim_{h\to0}\left[\frac{f'(x+h) + f'(x-h)}{2}\right] = \frac{f'(x) + f'(x)}{2} = f'(x)$$

(b)



Graphically, the slope of the line joining $(x - h, f(x - h))$ and $(x + h, f(x + h))$ is approximately $f'(x)$. So, $\lim_{h\to0}\frac{f(x+h) - f(x-h)}{2h} = f'(x)$.

104. $\displaystyle\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = \lim_{h\to 0}\frac{f'(x+h)(1)+f'(x-h)(-1)}{2h}$

$$= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h}$$

$$= \lim_{h\to 0}\frac{f''(x+h)(1)-f''(x-h)(-1)}{2}$$

$$= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2}$$

$$= \frac{f''(x)+f''(x)}{2} = f''(x)$$

105. $\displaystyle\lim_{x\to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln f(x)$

$\displaystyle\lim_{x\to a} g(x)\ln f(x) = (\infty)(-\infty) = -\infty$

As $x \to a$, $\ln y \Rightarrow -\infty$, and therefore $y = 0$.

So, $\displaystyle\lim_{x\to a} f(x)^{g(x)} = 0$.

106. $\displaystyle\lim_{x\to a} f(x)^{g(x)}$

$y = f(x)^{g(x)}$

$\ln y = g(x)\ln f(x)$

$\displaystyle\lim_{x\to a} g(x)\ln f(x) = (-\infty)(-\infty) = \infty$

As $x \to a$, $\ln y \Rightarrow \infty$, and therefore $y = \infty$.

So, $\displaystyle\lim_{x\to a} f(x)^{g(x)} = \infty$.

107. $\displaystyle f'(a)(b-a) - \int_a^b f''(t)(t-b)\,dt = f'(a)(b-a) - \left\{\Big[f'(t)(t-b)\Big]_a^b - \int_a^b f'(t)\,dt\right\}$

$$= f'(a)(b-a) + f'(a)(a-b) + \Big[f(t)\Big]_a^b = f(b) - f(a)$$

$dv = f''(t)\,dt \Rightarrow \quad v = f'(t)$

$u = t - b \quad \Rightarrow du = dt$

108. (a) $\displaystyle\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)}$ is of form 0^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$\displaystyle\ln y = \frac{\ln 2}{1+\ln x}\ln x$

$\displaystyle\lim_{x\to 0^+}\ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$

So, $\displaystyle\lim_{x\to 0^+} x^{(\ln 2)/(1+\ln x)} = 2.$

(b) $\displaystyle\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)}$ is of form ∞^0.

Let $y = x^{(\ln 2)/(1+\ln x)}$

$\displaystyle\ln y = \frac{\ln 2}{1+\ln x}\ln x$

$\displaystyle\lim_{x\to\infty}\ln y = \frac{\ln 2(1/x)}{1/x} = \ln 2.$

So, $\displaystyle\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = 2.$

(c) $\displaystyle\lim_{x\to 0}(x+1)^{(\ln 2)/(x)}$ is of form 1^∞.

Let $y = (x+1)^{(\ln 2)/(x)}$

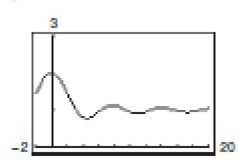
$\displaystyle\ln y = \frac{\ln 2}{x}\ln(x+1)$

$\displaystyle\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{(\ln 2)1/(x+1)}{1} = \ln 2.$

So, $\displaystyle\lim_{x\to 0}(x+1)^{(\ln 2)/(x)} = 2.$

109. (a) $\displaystyle h(x) = \frac{x+\sin x}{x}$

$\displaystyle\lim_{x\to\infty} h(x) = 1$



(b) $\displaystyle h(x) = \frac{x+\sin x}{x} = \frac{x}{x} + \frac{\sin x}{x} = 1 + \frac{\sin x}{x}, x$

So, $\displaystyle\lim_{x\to\infty} h(x) = \lim_{x\to\infty}\left[1 + \frac{\sin x}{x}\right] = 1 + 0 = 1.$

(c) No. $h(x)$ is not an indeterminate form.

110. (a) $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x + x\sin x}{x^2 - 4} = \lim_{x\to\infty} \frac{1+\sin x}{x - 4/x} = 0$

(Because $|1 + \sin x| \le 1$ and $x \to \infty$.)

(b) $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x(1 + \sin x) = \infty$

$\displaystyle\lim_{x\to\infty} g(x) = \lim_{x\to\infty} (x^2 - 4) = \infty$

(c) $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1 + \sin x + x\cos x}{2x}$ undefined

(d) No. If $\displaystyle\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then you cannot assume anything about $\displaystyle\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

111. $\displaystyle\lim_{x\to 0} \frac{4e^x - \sin x - 4}{x^2 + 4x} = \lim_{x\to 0} \frac{4e^x - \cos x}{2x + 4}$ (By L'Hôpital's Rule)

$\displaystyle= \frac{4-1}{4} = \frac{3}{4}$

So, the answer is B.

112. $\displaystyle\lim_{x\to 0^+} (-x\ln x) = \lim_{x\to 0^+} -\frac{\ln x}{\frac{1}{x}}$

$\displaystyle= \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}}$

$\displaystyle= \lim_{x\to 0^+} x$

$= 0$

So, the answer is A.

113. $\displaystyle\lim_{x\to 2} \frac{\int_2^x e^{t/2}\,dt}{x^3 - 8} = \lim_{x\to 2} \frac{e^{x/2}}{3x^2}$

$\displaystyle= \frac{e^1}{3(2)^2}$

$\displaystyle= \frac{e}{12}$

So, the answer is C.