Section 2.1 The Derivative and the Tangent Line Problem

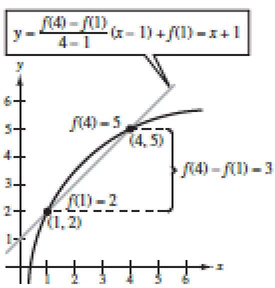
1. At (x_1, y_1), slope $= 0$.

At (x_2, y_2), slope $= \frac{5}{2}$.

2. At (x_1, y_1), slope $= \frac{2}{3}$.

At (x_2, y_2), slope $= -\frac{2}{5}$.

3. (a), (b)



(c) $y = \dfrac{f(4) - f(1)}{4 - 1}(x - 1) + f(1)$

$= \dfrac{3}{3}(x - 1) + 2$

$= 1(x - 1) + 2$

$= x + 1$

4. (a) $\dfrac{f(4) - f(1)}{4 - 1} = \dfrac{5 - 2}{3} = 1$

$\dfrac{f(4) - f(3)}{4 - 3} \approx \dfrac{5 - 4.75}{1} = 0.25$

So, $\dfrac{f(4) - f(1)}{4 - 1} > \dfrac{f(4) - f(3)}{4 - 3}$.

(b) The slope of the tangent line at $(1, 2)$ equals $f'(1)$. This slope is steeper than the slope of the line through $(1, 2)$ and $(4, 5)$. So, $\dfrac{f(4) - f(1)}{4 - 1} < f'(1)$.

5. $f(x) = 3 - 5x$ is a line. Slope $= -5$

6. $g(x) = \frac{3}{2}x + 1$ is a line. Slope $= \frac{3}{2}$

7. Slope at $(2, 5)$ $= \lim_{\Delta x \to 0} \dfrac{f(2 + \Delta x) - f(2)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{2(2 + \Delta x)^2 - 3 - \left[2(2)^2 -\right.}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{2\left[4 + 4\Delta x + (\Delta x)^2\right] - 3 -}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{8\Delta x + 2(\Delta x)^2}{\Delta x}$

$= \lim_{\Delta x \to 0} (8 + 2\Delta x) = 8$

8. Slope at $(3, -4)$ $= \lim_{\Delta x \to 0} \dfrac{f(3 + \Delta x) - f(3)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{5 - (3 + \Delta x)^2 - (-4)}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{5 - 9 - 6(\Delta x) - (\Delta x)^2 +}{\Delta x}$

$= \lim_{\Delta x \to 0} \dfrac{-6(\Delta x) - (\Delta x)^2}{\Delta x}$

$= \lim_{\Delta x \to 0} (-6 - \Delta x) = -6$

9. Slope at $(0, 0)$ $= \lim_{\Delta t \to 0} \dfrac{f(0 + \Delta t) - f(0)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{3(\Delta t) - (\Delta t)^2 - 0}{\Delta t}$

$= \lim_{\Delta t \to 0} (3 - \Delta t) = 3$

10. Slope at $(1, 5)$ $= \lim_{\Delta t \to 0} \dfrac{h(1 + \Delta t) - h(1)}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{(1 + \Delta t)^2 + 4(1 + \Delta t) - 5}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{1 + 2(\Delta t) + (\Delta t)^2 + 4 + 4(\Delta t}{\Delta t}$

$= \lim_{\Delta t \to 0} \dfrac{6(\Delta t) + (\Delta t)^2}{\Delta t}$

$= \lim_{\Delta t \to 0} (6 + \Delta t) = 6$

11. $f(x) = 7$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7 - 7}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 0 = 0$$

12. $g(x) = -3$

$$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-3 - (-3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

13. $f(x) = -5x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-5(x + \Delta x) - (-5x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-5x - 5\Delta x + 5x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-5\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (-5) = -5$$

14. $f(x) = 7x - 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7(x + \Delta x) - 3 - (7x - 3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7x + 7\Delta x - 3 - 7x + 3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{7(\Delta x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} 7 = 7$$

15. $h(s) = 3 + \frac{2}{3}s$

$$h'(s) = \lim_{\Delta s \to 0} \frac{h(s + \Delta s) - h(s)}{\Delta s}$$
$$= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}(s + \Delta s) - \left(3 + \frac{2}{3}s\right)}{\Delta s}$$
$$= \lim_{\Delta s \to 0} \frac{3 + \frac{2}{3}s + \frac{2}{3}\Delta s - 3 - \frac{2}{3}s}{\Delta s}$$
$$= \lim_{\Delta s \to 0} \frac{\frac{2}{3}\Delta s}{\Delta s} = \frac{2}{3}$$

16. $f(x) = 5 - \frac{2}{3}x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}(x + \Delta x) - \left(5 - \frac{2}{3}x\right)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{5 - \frac{2}{3}x - \frac{2}{3}\Delta x - 5 + \frac{2}{3}x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{-\frac{2}{3}(\Delta x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(-\frac{2}{3}\right) = -\frac{2}{3}$$

17. $f(x) = x^2 + x - 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 + (x + \Delta x) - 3 - (x^2 + x - 3)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 + x + \Delta x - 3 - x^2 - x + 3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2 + \Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x + 1) = 2x + 1$$

18. $f(x) = x^2 - 5$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - 5 - (x^2 - 5)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - 5 - x^2 + 5}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x(\Delta x) + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x
\end{aligned}$$

19. $f(x) = x^3 - 12x$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 - 12(x + \Delta x)\right] - \left[x^3 - 12x\right]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12x - 12\,\Delta x - x^3 + 12x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - 12\,\Delta x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2 - 12\right) = 3x^2 - 12
\end{aligned}$$

20. $f(x) = x^3 + x^2$

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 + (x + \Delta x)^2\right] - \left[x^3 + x^2\right]}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + x^2 + 2x\,\Delta x + (\Delta x)^2 - x^3 - x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{3x^2\,\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 + 2x\,\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2 + 2x + (\Delta x)\right) = 3x^2 + 2x
\end{aligned}$$

21. $f(x) = \dfrac{1}{x-1}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{1}{x+\Delta x-1} - \dfrac{1}{x-1}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x-1) - (x+\Delta x-1)}{\Delta x(x+\Delta x-1)(x-1)} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x+\Delta x-1)(x-1)} \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x+\Delta x-1)(x-1)} \\
&= -\frac{1}{(x-1)^2}
\end{aligned}
$$

22. $f(x) = \dfrac{1}{x^2}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{1}{(x+\Delta x)^2} - \dfrac{1}{x^2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2 - (x+\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2} \\
&= \lim_{\Delta x \to 0} \frac{-2x\,\Delta x - (\Delta x)^2}{\Delta x(x+\Delta x)^2 x^2} \\
&= \lim_{\Delta x \to 0} \frac{-2x - \Delta x}{(x+\Delta x)^2 x^2} \\
&= \frac{-2x}{x^4} \\
&= -\frac{2}{x^3}
\end{aligned}
$$

23. $f(x) = \sqrt{x+4}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x+4} - \sqrt{x+4}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x+4} + \sqrt{x+4}}{\sqrt{x+\Delta x+4} + \sqrt{x+4}} \right) \\
&= \lim_{\Delta x \to 0} \frac{(x+\Delta x+4) - (x+4)}{\Delta x\left[\sqrt{x+\Delta x+4} + \sqrt{x+4}\right]} \\
&= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x+4} + \sqrt{x+4}} = \frac{1}{\sqrt{x+4} + \sqrt{x+4}} = \frac{1}{2\sqrt{x+4}}
\end{aligned}
$$

24. $f(x) = \dfrac{4}{\sqrt{x}}$

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{4}{\sqrt{x+\Delta x}} - \dfrac{4}{\sqrt{x}}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\Delta x\sqrt{x}\sqrt{x+\Delta x}} \cdot \left(\frac{\sqrt{x} + \sqrt{x+\Delta x}}{\sqrt{x} + \sqrt{x+\Delta x}} \right) \\
&= \lim_{\Delta x \to 0} \frac{4x - 4(x+\Delta x)}{\Delta x\sqrt{x}\sqrt{x+\Delta x}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \\
&= \lim_{\Delta x \to 0} \frac{-4}{\sqrt{x}\sqrt{x+\Delta x}\left(\sqrt{x} + \sqrt{x+\Delta x}\right)} \\
&= \frac{-4}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)} = \frac{5-2}{x\sqrt{x}}
\end{aligned}
$$

25. (a) $f(x) = x^2 + 3$

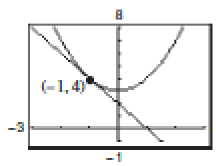
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 3\right] - \left(x^2 + 3\right)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 3 - x^2 - 3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

At $(-1, 4)$, the slope of the tangent line is $m = 2(-1) = -2$.

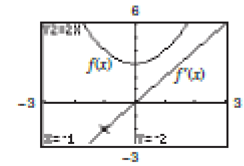
The equation of the tangent line is

$$y - 4 = -2(x + 1)$$
$$y - 4 = -2x - 2$$
$$y = -2x + 2.$$

(b)

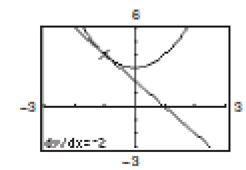


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(-1, 4)$ is $m = -2$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = -1$.



The graphing utility confirms that $dy/dx = -2$ at $(-1, 4)$.

26. (a) $f(x) = x^2 + 2x - 1$

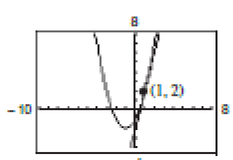
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 + 2(x + \Delta x) - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[x^2 + 2x\Delta x + (\Delta x)^2 + 2x + 2\Delta x - 1\right] - \left[x^2 + 2x - 1\right]}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2 + 2\Delta x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x + 2) = 2x + 2$$

At $(1, 2)$ the slope of the tangent line is $m = 2(1) + 2 = 4$.

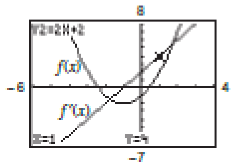
The equation of the tangent line is

$$y - 2 = 4(x - 1)$$
$$y - 2 = 4x - 4$$
$$y = 4x - 2.$$

(b)

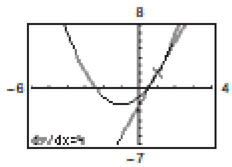


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(1, 2)$ is $m = 4$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = 1$.



The graphing utility confirms that $dy/dx = 4$ at $(1, 2)$.

27. (a) $f(x) = x^3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\,\Delta x + (\Delta x)^2\right) = 3x^2$$

At $(2, 8)$, the slope of the tangent is $m = 3(2)^2 = 12$.

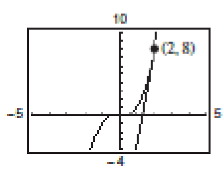
The equation of the tangent line is

$y - 8 = 12(x - 2)$

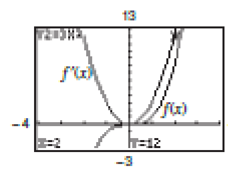
$y - 8 = 12x - 24$

$y = 12x - 16.$

(b)

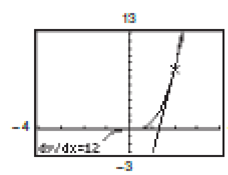


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(2, 8)$ is $m = 12$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = 2$.



The graphing utility confirms that $dy/dx = 12$ at $(2, 8)$.

28. (a) $f(x) = x^3 + 1$

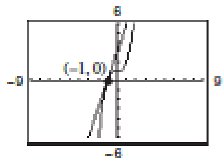
$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^3 + 1\right] - (x^3 + 1)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 + 1 - x^3 - 1}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left[3x^2 + 3x(\Delta x) + (\Delta x)^2\right] = 3x^2$$

At $(-1, 0)$, the slope of the tangent line is $m = 3(-1)^2 = 3$. The equation of the tangent line is

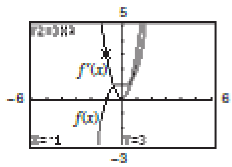
$y - 0 = 3(x + 1)$

$y = 3x + 3.$

(b)

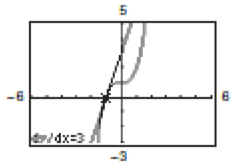


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(-1, 0)$ is $m = 3$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = -1$.



The graphing utility confirms that $dy/dx = 3$ at $(-1, 0)$.

29. (a) $f(x) = \sqrt{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

At $(1, 1)$, the slope of the tangent line is

$$m = \frac{1}{2\sqrt{1}} = \frac{1}{2}.$$

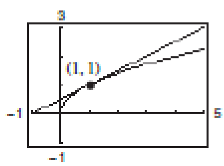
The equation of the tangent line is

$$y - 1 = \frac{1}{2}(x - 1)$$

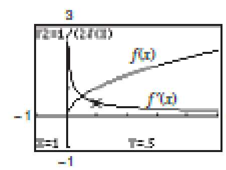
$$y - 1 = \frac{1}{2}x - \frac{1}{2}$$

$$y = \frac{1}{2}x + \frac{1}{2}.$$

(b)

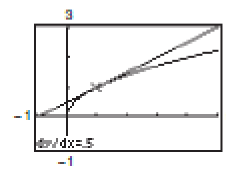


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(1, 1)$ is $m = \frac{1}{2}$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = 1$.



The graphing utility confirms that $dy/dx = \frac{1}{2}$ at $(1, 1)$.

30. (a) $f(x) = \sqrt{x-1}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x-1} - \sqrt{x-1}}{\Delta x} \cdot \left(\frac{\sqrt{x+\Delta x-1} + \sqrt{x-1}}{\sqrt{x+\Delta x-1} + \sqrt{x-1}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x-1) - (x-1)}{\Delta x\left(\sqrt{x+\Delta x-1} + \sqrt{x-1}\right)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}}$$

At $(5, 2)$, the slope of the tangent line is $m = \dfrac{1}{2\sqrt{5-1}} = \dfrac{1}{4}$.

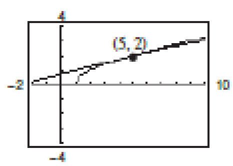
The equation of the tangent line is

$$y - 2 = \frac{1}{4}(x-5)$$

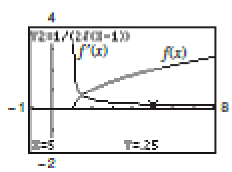
$$y - 2 = \frac{1}{4}x - \frac{5}{4}$$

$$y = \frac{1}{4}x + \frac{3}{4}.$$

(b)

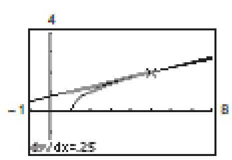


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(5, 2)$ is $m = \frac{1}{4}$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = 5$.



The graphing utility confirms that $dy/dx = \frac{1}{4}$ at $(5, 2)$.

31. (a) $f(x) = x + \dfrac{4}{x}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x) + \dfrac{4}{x + \Delta x} - \left(x + \dfrac{4}{x}\right)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x(x + \Delta x)(x + \Delta x) + 4x - x^2(x + \Delta x) - 4(x + \Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^3 + 2x^2(\Delta x) + x(\Delta x)^2 - x^3 - x^2(\Delta x) - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2(\Delta x) + x(\Delta x)^2 - 4(\Delta x)}{x(\Delta x)(x + \Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{x^2 + x(\Delta x) - 4}{x(x + \Delta x)}$$

$$= \frac{x^2 - 4}{x^2} = 1 - \frac{4}{x^2}$$

At $(-4, -5)$, the slope of the tangent line is $m = 1 - \dfrac{4}{(-4)^2} = \dfrac{3}{4}$.

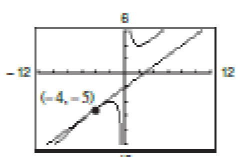
The equation of the tangent line is

$$y + 5 = \frac{3}{4}(x + 4)$$

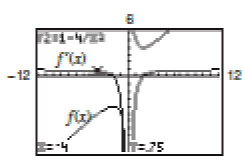
$$y + 5 = \frac{3}{4}x + 3$$

$$y = \frac{3}{4}x - 2.$$

(b)

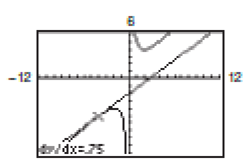


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(-4, -5)$ is $m =$

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = -4$.



The graphing utility confirms that $dy/dx = \frac{3}{4}$ at $(-4, -5)$.

32. (a) $f(x) = x + \dfrac{6}{x+2}$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{6}{(x+\Delta x)+2} - \dfrac{6}{x+2}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{6x + 12 - 6(x + \Delta x + 2)}{\Delta x(x + \Delta x + 2)(x+2)}$$

$$= \lim_{\Delta x \to 0} \frac{6x + 12 - 6x - 6\Delta x - 12}{\Delta x(x + \Delta x + 2)(x+2)}$$

$$= \lim_{\Delta x \to 0} \frac{-6\Delta x}{\Delta x(x + \Delta x + 2)(x+2)}$$

$$= \frac{-6}{(x+2)^2}$$

At $(0, 3)$, the slope of the tangent line is $m = -\frac{6}{4} = -\frac{3}{2}$.

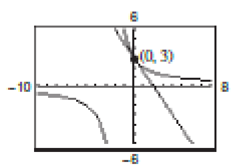
The equation of the tangent line is

$$y - 3 = -\frac{3}{2}(x - 0)$$

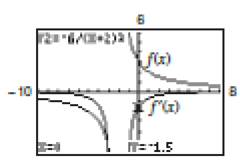
$$y - 3 = -\frac{3}{2}x$$

$$y = -\frac{3}{2}x + 3.$$

(b)

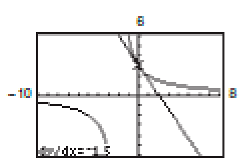


(c) To confirm part (a), use the *derivative* feature of a graphing utility to graph $f(x)$ and $f'(x)$.



The graphing utility confirms that the slope of the tangent line of $f(x)$ at $(0, 3)$ is $m = -\frac{3}{2}$.

To confirm part (b), use the *tangent* feature of a graphing utility to graph $f(x)$ and its tangent line at $x = 0$.



The graphing utility confirms that $dy/dx = -\frac{3}{2}$ at $(0, 3)$.

33. Using the limit definition of a derivative, $f'(x) = -\frac{1}{2}x$

Because the slope of the given line is -1, you have

$$-\tfrac{1}{2}x = -1$$

$$x = 2.$$

At the point $(2, -1)$, the tangent line is parallel to $x + y = 0$. The equation of this line is

$$y - (-1) = -1(x - 2)$$

$$y = -x + 1.$$

34. Using the limit definition of a derivative, $f'(x) = 4x$.

Because the slope of the given line is -4, you have

$$4x = -4$$

$$x = -1.$$

At the point $(-1, 2)$, the tangent line is parallel to $4x + y + 3 = 0$. The equation of this line is

$$y - 2 = -4(x + 1)$$

$$y = -4x - 2.$$

35. Using the limit definition of a derivative, $f'(x) = 3x^2$.

Because the slope of the given line is 3, you have

$$3x^2 = 3$$

$$x = \pm 1.$$

Therefore, at the points $(1, 1)$ and $(-1, -1)$ the tangent lines are parallel to $3x - y + 1 = 0$.

These lines have equations

$$y - 1 = 3(x - 1) \quad \text{and} \quad y + 1 = 3(x + 1)$$

$$y = 3x - 2 \qquad\qquad y = 3x + 2.$$

36. Using the limit definition of a derivative,

$$f'(x) = \frac{-1}{2(x-1)^{3/2}}.$$

Because the slope of the given line is $-\frac{1}{2}$, you have

$$\frac{-1}{2(x-1)^{3/2}} = -\frac{1}{2}$$

$$1 = (x - 1)^{3/2}$$

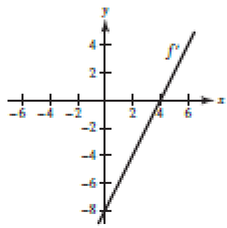
$$1 = x - 1 \Rightarrow x = 2.$$

At the point $(2, 1)$, the tangent line is parallel to $x + 2y + 7 = 0$. The equation of the tangent line is

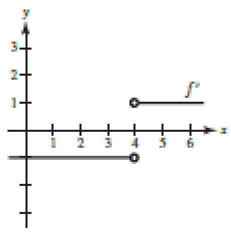
$$y - 1 = -\frac{1}{2}(x - 2)$$

$$y = -\frac{1}{2}x + 2.$$

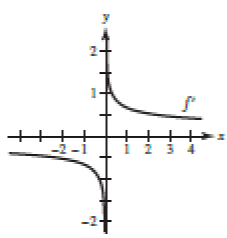
37. The slope of the graph of f is negative for $x < 4$, positive for $x > 4$, and 0 at $x = 4$.



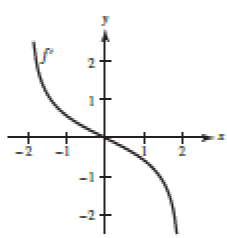
38. The slope of the graph of f is -1 for $x < 4$, 1 for $x > 4$, and undefined at $x = 4$.



39. The slope of the graph of f is negative for $x < 0$ and positive for $x > 0$. The slope is undefined at $x = 0$.

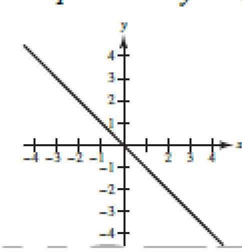


40. The slope is positive for $-2 < x < 0$ and negative for $0 < x < 2$. The slope is undefined at $x = \pm 2$, and 0 at $x = 0$.



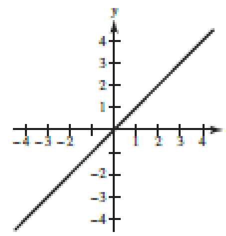
41. Answers will vary.

Sample answer: $y = -x$



42. Answers will vary.

Sample answer: $y = x$



43. $g(4) = 5$ because the tangent line passes through $(4, 5)$.

$$g'(4) = \frac{5 - 0}{4 - 7} = -\frac{5}{3}$$

44. $h(-1) = 4$ because the tangent line passes through $(-1, 4)$.

$$h'(-1) = \frac{6 - 4}{3 - (-1)} = \frac{2}{4} = \frac{1}{2}$$

45. $f(x) = 5 - 3x$ and $c = 1$

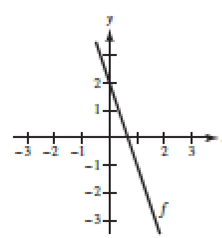
46. $f(x) = x^3$ and $c = -2$

47. $f(x) = -x^2$ and $c = 6$

48. $f(x) = 2\sqrt{x}$ and $c = 9$

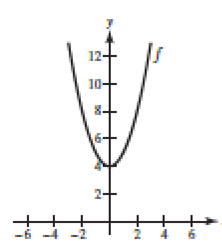
49. $f(0) = 2$ and $f'(x) = -3, -\infty < x < \infty$

$f(x) = -3x + 2$



50. $f(0) = 4, f'(0) = 0; f'(x) < 0$ for $x < 0, f'(x) > 0$ for $x > 0$

Answers will vary: *Sample answer:* $f(x) = x^2 + 4$



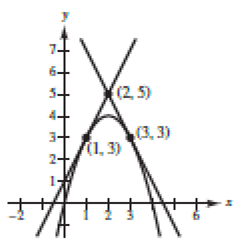
51. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 4 - 2x$. The slope of the line through $(2, 5)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{5 - y_0}{2 - x_0} = 4 - 2x_0$$
$$5 - y_0 = (2 - x_0)(4 - 2x_0)$$
$$5 - \left(4x_0 - x_0^2\right) = 8 - 8x_0 + 2x_0^2$$
$$0 = x_0^2 - 4x_0 + 3$$
$$0 = (x_0 - 1)(x_0 - 3) \Rightarrow x_0 = 1, 3$$

Therefore, the points of tangency are $(1, 3)$ and $(3, 3)$, and the corresponding slopes are 2 and -2. The equations of the tangent lines are:

$y - 5 = 2(x - 2)$ $\quad$ $y - 5 = -2(x - 2)$

$y = 2x + 1$ $\quad$ $y = -2x + 9$



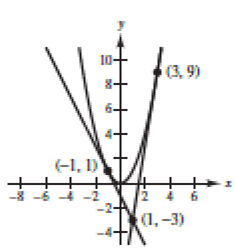
52. Let (x_0, y_0) be a point of tangency on the graph of f. By the limit definition for the derivative, $f'(x) = 2x$. The slope of the line through $(1, -3)$ and (x_0, y_0) equals the derivative of f at x_0:

$$\frac{-3 - y_0}{1 - x_0} = 2x_0$$
$$-3 - y_0 = (1 - x_0)2x_0$$
$$-3 - x_0^2 = 2x_0 - 2x_0^2$$
$$x_0^2 - 2x_0 - 3 = 0$$
$$(x_0 - 3)(x_0 + 1) = 0 \Rightarrow x_0 = 3, -1$$

Therefore, the points of tangency are $(3, 9)$ and $(-1, 1)$, and the corresponding slopes are 6 and -2. The equations of the tangent lines are:

$y + 3 = 6(x - 1)$ $\quad$ $y + 3 = -2(x - 1)$

$y = 6x - 9$ $\quad$ $y = -2x - 1$



53. (a) $f(x) = x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^2 + 2x(\Delta x) + (\Delta x)^2 - x^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (2x + \Delta x) = 2x$$

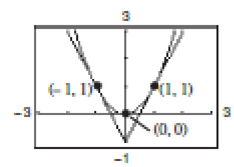
At $x = -1$, $f'(-1) = -2$ and the tangent line is

$y - 1 = -2(x + 1)$ or $y = -2x - 1$.

At $x = 0$, $f'(0) = 0$ and the tangent line is $y =$

At $x = 1$, $f'(1) = 2$ and the tangent line is

$y = 2x - 1$.



For this function, the slopes of the tangent lines ar always distinct for different values of x.

(b) $$g'(x) = \lim_{\Delta x \to 0} \frac{g(x + \Delta x) - g(x)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta x\left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x(\Delta x) + (\Delta x)^2\right) = 3x^2$$

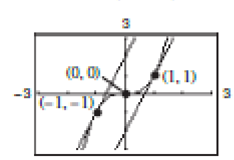
At $x = -1$, $g'(-1) = 3$ and the tangent line is

$y + 1 = 3(x + 1)$ or $y = 3x + 2$.

At $x = 0$, $g'(0) = 0$ and the tangent line is $y =$

At $x = 1$, $g'(1) = 3$ and the tangent line is

$y - 1 = 3(x - 1)$ or $y = 3x - 2$.



For this function, the slopes of the tangent lines ar sometimes the same.

54. (a) $g'(0) = -3$

(b) $g'(3) = 0$

(c) Because $g'(1) = -\frac{8}{3}$, g is decreasing (falling) at $x = 1$.

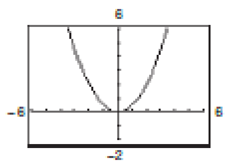
(d) Because $g'(-4) = \frac{7}{3}$, g is increasing (rising) at $x = -4$.

(e) Because $g'(4)$ and $g'(6)$ are both positive, $g(6)$ is greater than $g(4)$, and $g(6) - g(4) > 0$.

(f) No, it is not possible. All you can say is that g is decreasing (falling) at $x = 2$.

55. $f(x) = \frac{1}{2}x^2$

(a)

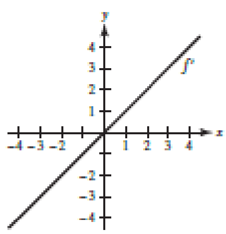


$f'(0) = 0, f'(1/2) = 1/2, f'(1) = 1, f'(2) = 2$

(b) By symmetry:

$f'(-1/2) = -1/2, f'(-1) = -1, f'(-2) = -2$

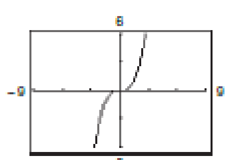
(c)



(d)
$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}(x+\Delta x)^2 - \frac{1}{2}x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\left(x^2 + 2x(\Delta x) + (\Delta x)^2\right) - \frac{1}{2}x^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left(x + \frac{\Delta x}{2}\right) = x \end{aligned}$$

56. $f(x) = \frac{1}{3}x^3$

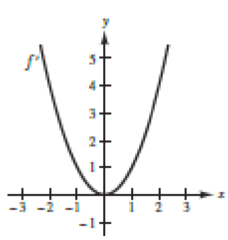
(a)



$f'(0) = 0, f'(1/2) = 1/4, f'(1) = 1,$
$f'(2) = 4, f'(3) = 9$

(b) By symmetry: $f'(-1/2) = 1/4, f'(-1) = 1,$
$f'(-2) = 4, f'(-3) = 9$

(c)

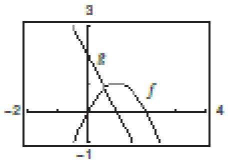


(d)
$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{3}(x+\Delta x)^3 - \frac{1}{3}x^3}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\frac{1}{3}\left(x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3\right) - \frac{1}{3}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \left[x^2 + x(\Delta x) + \frac{1}{3}(\Delta x)^2\right] = x^2 \end{aligned}$$

57. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left[2(x + 0.01) - (x + 0.01)^2 - 2x + x^2\right]100$

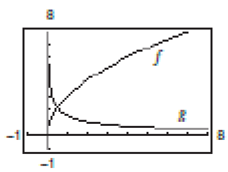
$= 2 - 2x - 0.01$



The graph of $g(x)$ is approximately the graph of $f'(x) = 2 - 2x$.

58. $g(x) = \dfrac{f(x + 0.01) - f(x)}{0.01}$

$= \left(3\sqrt{x + 0.01} - 3\sqrt{x}\right)100$



The graph of $g(x)$ is approximately the graph of $f'(x) = \dfrac{3}{2\sqrt{x}}$.

59. $f(2) = 2(4 - 2) = 4$, $f(2.1) = 2.1(4 - 2.1) = 3.99$

$f'(2) \approx \dfrac{3.99 - 4}{2.1 - 2} = -0.1 \quad \left[\text{Exact: } f'(2) = 0\right]$

60. $f(2) = \dfrac{1}{4}(2^3) = 2$, $f(2.1) = 2.31525$

$f'(2) \approx \dfrac{2.31525 - 2}{2.1 - 2} = 3.1525\left[\text{Exact: } f'(2) = 3\right]$

61. $f(x) = x^2 - 5$, $c = 3$

$$f'(3) = \lim_{x\to3} \frac{f(x) - f(3)}{x - 3} = \lim_{x\to3} \frac{x^2 - 5 - (9 - 5)}{x - 3} = \lim_{x\to3} \frac{(x - 3)(x + 3)}{x - 3} = \lim_{x\to3} (x + 3) = 6$$

62. $g(x) = x^2 - x$, $c = 1$

$$g'(1) = \lim_{x\to1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to1} \frac{x^2 - x - 0}{x - 1} = \lim_{x\to1} \frac{x(x - 1)}{x - 1} = \lim_{x\to1} x = 1$$

63. $f(x) = x^3 + 2x^2 + 1$, $c = -2$

$$f'(-2) = \lim_{x\to-2} \frac{f(x) - f(-2)}{x + 2} = \lim_{x\to-2} \frac{(x^3 + 2x^2 + 1) - 1}{x + 2} = \lim_{x\to-2} \frac{x^2(x + 2)}{x + 2} = \lim_{x\to-2} x^2 = 4$$

64. $f(x) = x^3 + 6x$, $c = 2$

$$f'(2) = \lim_{x\to2} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2} \frac{(x^3 + 6x) - 20}{x - 2} = \lim_{x\to2} \frac{(x - 2)(x^2 + 2x + 10)}{x - 2} = \lim_{x\to2} (x^2 + 2x + 10) = 18$$

65. $g(x) = \sqrt{|x|}$, $c = 0$

$g'(0) = \lim_{x\to0} \dfrac{g(x) - g(0)}{x - 0} = \lim_{x\to0} \dfrac{\sqrt{|x|}}{x}$. Does not exist.

As $x \to 0^-$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{-1}{\sqrt{|x|}} \to -\infty$.

As $x \to 0^+$, $\dfrac{\sqrt{|x|}}{x} = \dfrac{1}{\sqrt{x}} \to \infty$.

Therefore $g(x)$ is not differentiable at $x = 0$.

66. $f(x) = \frac{3}{x}, c = 4$

$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4}$$
$$= \lim_{x \to 4} \frac{\frac{3}{x} - \frac{3}{4}}{x - 4}$$
$$= \lim_{x \to 4} \frac{12 - 3x}{4x(x - 4)}$$
$$= \lim_{x \to 4} \frac{-3(x - 4)}{4x(x - 4)}$$
$$= \lim_{x \to 4} -\frac{3}{4x} = -\frac{3}{16}$$

67. $f(x) = (x - 6)^{2/3}, c = 6$

$$f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6}$$
$$= \lim_{x \to 6} \frac{(x - 6)^{2/3} - 0}{x - 6} = \lim_{x \to 6} \frac{1}{(x - 6)^{1/3}}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

68. $g(x) = (x + 3)^{1/3}, c = -3$

$$g'(-3) = \lim_{x \to -3} \frac{g(x) - g(-3)}{x - (-3)}$$
$$= \lim_{x \to -3} \frac{(x + 3)^{1/3} - 0}{x + 3} = \lim_{x \to -3} \frac{1}{(x + 3)^{2/3}}.$$

Does not exist.

Therefore $g(x)$ is not differentiable at $x = -3$.

69. $h(x) = |x + 7|, c = -7$

$$h'(-7) = \lim_{x \to -7} \frac{h(x) - h(-7)}{x - (-7)}$$
$$= \lim_{x \to -7} \frac{|x + 7| - 0}{x + 7} = \lim_{x \to -7} \frac{|x + 7|}{x + 7}.$$

Does not exist.

Therefore $h(x)$ is not differentiable at $x = -7$.

70. $f(x) = |x - 6|, c = 6$

$$f'(6) = \lim_{x \to 6} \frac{f(x) - f(6)}{x - 6}$$
$$= \lim_{x \to 6} \frac{|x - 6| - 0}{x - 6} = \lim_{x \to 6} \frac{|x - 6|}{x - 6}.$$

Does not exist.

Therefore $f(x)$ is not differentiable at $x = 6$.

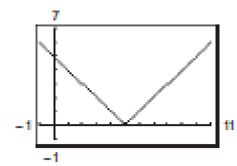
71. $f(x)$ is differentiable everywhere except at $x = \pm 2$. (Discontinuities)

72. $f(x)$ is differentiable everywhere except at $x = \pm 3$. (Sharp turns in the graph)

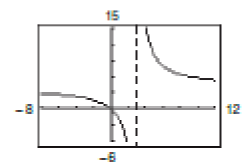
73. $f(x)$ is differentiable everywhere except at $x = -4$. (Sharp turn in the graph)

74. $f(x)$ is differentiable everywhere except at $x = 0$. (Discontinuity)

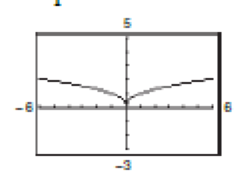
75. $f(x) = |x - 5|$ is differentiable everywhere except a $x = -5$. There is a sharp corner at $x = 5$.



76. $f(x) = \frac{4x}{x - 3}$ is differentiable everywhere except at $x = 3$. f is not defined at $x = 3$. (Vertical asymptot

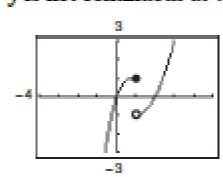


77. $f(x) = x^{2/5}$ is differentiable for all $x \neq 0$. There is sharp corner at $x = 0$.



78. f is differentiable for all $x \neq 1$.

f is not continuous at $x = 1$.



79. $f(x) = |x - 1|$

The derivative from the left is

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{|x - 1| - 0}{x - 1} = -1.$$

The derivative from the right is

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{|x - 1| - 0}{x - 1} = 1.$$

The one-sided limits are not equal. Therefore, f is not differentiable at $x = 1$.

80. $f(x) = \sqrt{1 - x^2}$

The derivative from the left does not exist because

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{\sqrt{1 - x^2} - 0}{x - 1}$$
$$= \lim_{x\to1^-} \frac{\sqrt{1 - x^2}}{x - 1} \cdot \frac{\sqrt{1 - x^2}}{\sqrt{1 - x^2}}$$
$$= \lim_{x\to1^-} -\frac{1 + x}{\sqrt{1 - x^2}} = -\infty.$$

(Vertical tangent)

The limit from the right does not exist since *f* is undefined for $x > 1$. Therefore, *f* is not differentiable at $x = 1$.

81. $f(x) = \begin{cases} (x - 1)^3, & x \le 1 \\ (x - 1)^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{(x - 1)^3 - 0}{x - 1}$$
$$= \lim_{x\to1^-} (x - 1)^2 = 0.$$

The derivative from the right is

$$\lim_{x\to1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^+} \frac{(x - 1)^2 - 0}{x - 1}$$
$$= \lim_{x\to1^+} (x - 1) = 0.$$

The one-sided limits are equal. Therefore, *f* is differentiable at $x = 1$. $\left(f'(1) = 0\right)$

82. $f(x) = \begin{cases} x, & x \le 1 \\ x^2, & x > 1 \end{cases}$

The derivative from the left is

$$\lim_{x\to1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^-} \frac{x - 1}{x - 1} = \lim_{x\to1^-} 1 = 1.$$

The derivative from the right is

$$\lim_{x\to1^+} = \frac{f(x) - f(1)}{x - 1} = \lim_{x\to1^+} \frac{x^2 - 1}{x - 1} = \lim_{x\to1^+} (x + 1) = 2.$$

The one-sided limits are not equal. Therefore, *f* is not differentiable at $x = 1$.

83. Note that *f* is continuous at $x = 2$.

$f(x) = \begin{cases} x^2 + 1, & x \le 2 \\ 4x - 3, & x > 2 \end{cases}$

The derivative from the left is

$$\lim_{x\to2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^-} \frac{(x^2 + 1) - 5}{x - 2}$$
$$= \lim_{x\to2^-} (x + 2) = 4.$$

The derivative from the right is

$$\lim_{x\to2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^+} \frac{(4x - 3) - 5}{x - 2} = \lim_{x\to2^+} 4 = 4$$

The one-sided limits are equal. Therefore, *f* is differentiable at $x = 2$. $\left(f'(2) = 4\right)$

84. Note that *f* is continuous at $x = 2$.

$f(x) = \begin{cases} \frac{1}{2}x + 1, & x < 2 \\ \sqrt{2x}, & x \ge 2 \end{cases}$

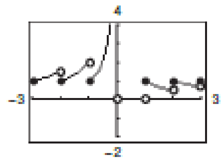
The derivative from the left is

$$\lim_{x\to2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^-} \frac{\left(\frac{1}{2}x + 1\right) - 2}{x - 2}$$
$$= \lim_{x\to2^-} \frac{\frac{1}{2}(x - 2)}{x - 2} = \frac{1}{2}.$$

The derivative from the right is

$$\lim_{x\to2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x\to2^+} \frac{\sqrt{2x} - 2}{x - 2} \cdot \frac{\sqrt{2x} + 2}{\sqrt{2x} + 2}$$
$$= \lim_{x\to2^+} \frac{2x - 4}{(x - 2)\left(\sqrt{2x} + 2\right)}$$
$$= \lim_{x\to2^+} \frac{2(x - 2)}{(x - 2)\left(\sqrt{2x} + 2\right)}$$
$$= \lim_{x\to2^+} \frac{2}{\sqrt{2x} + 2} = \frac{1}{2}.$$

The one-sided limits are equal. Therefore, *f* is differentiable at $x = 2$. $\left(f'(2) = \frac{1}{2}\right)$

85. 

Let $g(x) = \frac{[\![x]\!]}{x}$.

For $f(x) = [\![x]\!]$,

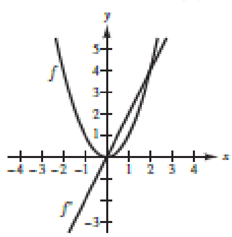
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{[\![x]\!] - 0}{x} = \lim_{x \to 0^-} \frac{[\![x]\!]}{x} = \lim_{x \to 0^-} [\![x]\!] \cdot \lim_{x \to 0^-} \frac{1}{x} = -1 \cdot \lim_{x \to 0^-} \frac{1}{x} = \lim_{x \to 0^-} \frac{-1}{x} = \infty.$$

On the other hand,

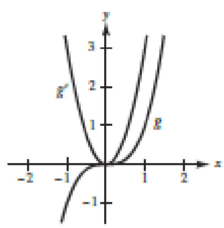
$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{[\![x]\!] - 0}{x} = \lim_{x \to 0^+} \frac{[\![x]\!]}{x} = \lim_{x \to 0^+} [\![x]\!] \cdot \lim_{x \to 0^+} \frac{1}{x} = 0 \cdot \lim_{x \to 0^+} \frac{1}{x} = 0.$$

So, f is not differentiable at $x = 0$ because $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist. f is differentiable for all $x \neq n$, n an integer.

86. (a) $f(x) = x^2$ and $f'(x) = 2x$



(b) $g(x) = x^3$ and $g'(x) = 3x^2$



(c) The derivative is a polynomial of degree 1 less than the original function. If $h(x) = x^n$, then $h'(x) = nx^{n-1}$.

(d) If $f(x) = x^4$, then

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^4 + 4x^3(\Delta x) + 6x^2(\Delta x)^2 + 4x(\Delta x)^3 + (\Delta x)^4 - x^4}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x\left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right)}{\Delta x} = \lim_{\Delta x \to 0} \left(4x^3 + 6x^2(\Delta x) + 4x(\Delta x)^2 + (\Delta x)^3\right) = 4x^3. \end{aligned}$$

So, if $f(x) = x^4$, then $f'(x) = 4x^3$, which is consistent with the conjecture. However, this is not a proof because you mus verify the conjecture for all integer values of n, $n \geq 2$.

87. $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Using the Squeeze Theorem, you have
$-|x| \leq x\sin(1/x) \leq |x|, x \neq 0.$

So, $\lim_{x\to 0} x\sin(1/x) = 0 = f(0)$ and f is continuous at $x = 0$.

Using the alternative form of the derivative, you have

$$\lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0}\frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0}\left(\sin\frac{1}{x}\right).$$

Because this limit does not exist ($\sin(1/x)$ oscillates between -1 and 1), the function is not differentiable at $x = 0$.

$$g(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Using the Squeeze Theorem again, you have
$-x^2 \leq x^2\sin(1/x) \leq x^2, x \neq 0.$

So, $\lim_{x\to 0} x^2\sin(1/x) = 0 = g(0)$ and g is continuous at $x = 0$. Using the alternative form of the derivative again, you have

$$\lim_{x\to 0}\frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0}\frac{x^2\sin(1/x) - 0}{x - 0}$$
$$= \lim_{x\to 0} x\sin\frac{1}{x} = 0.$$

Therefore, g is differentiable at $x = 0$, $g'(0) = 0$.

88. Using the alternative form of a derivative,

$$f'(c) = \lim_{x\to c}\frac{f(x) - f(c)}{x - c}$$

can be written as $\lim_{x\to 2}\frac{\ln(x + 4) - \ln 6}{x - 2}$, which is $f'(2)$, if $f(x) = \ln(x + 4)$.

So, the answer is B.

89. The function g has possible discontinuities at $x = -3$, $x = 0$, and $x = 2$. At $x = -3$, the graph has a vertical tangent line so it is continuous but not differentiable. At $x = 0$, the graph is not continuous, so it is not differentiable. At $x = 2$, the graph has a sharp turn, so it is continuous but not differentiable.

So, the answer is C.

90. The graph of $f'(x) = 2x - 4$ is a line. So, the slope of f at $(x, f(x)) = 2x - 4$ is as follows.

x	-2	-1	0	1	2	3
Slope at $(x, f(x))$	-8	-6	-4	-2	0	2

Using the slopes, the graph of $f(x)$ is a parabola that opens up with a minimum at $x = 2$. $f(x) = x^2 - 4x + 1$ is the only choice whose graph matches these characteristics. So, find $f'(x)$ of $f(x) = x^2 - 4x + 1$.

$$f'(x) = \lim_{\Delta x\to 0}\frac{f(x + \Delta x) - f(x)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{(x + \Delta x)^2 - 4(x + \Delta x) + 1 - (x^2 - 4x + 1)}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{x^2 + 2x\Delta x + (\Delta x)^2 - 4x - 4\Delta x + 1 - x^2 + 4x - 1}{\Delta x}$$
$$= \lim_{\Delta x\to 0}\frac{2x\Delta x + (\Delta x)^2 - 4\Delta x}{\Delta x}$$
$$= \lim_{\Delta x\to 0} 2x + \Delta x - 4$$
$$= 2x - 4$$

So, the answer is D.