5.3 Separation of Variables

Separable Equations: Equations in which all x terms can be collected with dx and all y terms can be collected with dy. A solution can then be obtained through integration.

**It is important to remember on the AP exam that if you do not begin with separation of variables (and show this in your work) you will often not receive any additional points on the problem.

*Remember that we learned in Section 5.1 how we can check our solutions when we solve differential equations.

Examples – Solving Differential Equations

Solve each of the following and be sure to show all work!

Find the general solution of the differential equation $\sqrt{1-4x^2}\,y' = x$

$\sqrt{1-4x^2}\,\frac{dy}{dx} = x$

$\int dy = \int \frac{x}{\sqrt{1-4x^2}}\,dx$

$u = 1-4x^2$

$du = -8x\,dx$

$dx = \frac{du}{-8x}$

$y = \int x \cdot u^{-\frac{1}{2}}\,\frac{du}{-8x}$

$y = -\frac{1}{8}\int u^{-\frac{1}{2}}\,du$

$y = -\frac{1}{8}\left(\frac{u^{1/2}}{1/2}\right) + C$

$y = -\frac{1}{4}(1-4x^2)^{1/2} + C$

Find the general solution of the differential equation $yy' - 2e^x = 0$

$y\frac{dy}{dx} - 2e^x = 0$

$y\frac{dy}{dx} = 2e^x$

$\int y\,dy = \int 2e^x\,dx$

$\frac{1}{2}y^2 = 2e^x + C$

Find the general solution of $(y\cos x)\frac{dy}{dx} = \sec x$

$$y \cos x \frac{dy}{dx} = \sec x$$

$$y\,dy = \frac{\sec x}{\cos x}\,dx$$

$$\int y\,dy = \int \sec^2 x\,dx$$

$$\frac{1}{2}y^2 = \tan x + C$$

Find the particular solution of the differential equation than satisfies the initial condition $y(0) = 1$.

$$y\sqrt{1-x^2}\,y' - x\sqrt{1-y^2} = 0$$

$$y(1-x^2)^{\frac{1}{2}} \frac{dy}{dx} = x(1-y^2)^{\frac{1}{2}}$$

$$\int \frac{y}{(1-y^2)^{1/2}}\,dy = \int \frac{x}{(1-x^2)^{1/2}}\,dx$$

$$u = 1-y^2$$

$$du = -2y\,dy$$

$$dy = \frac{du}{-2y}$$

$$-\frac{1}{2}\frac{(1-y^2)^{\frac{1}{2}}}{\frac{1}{2}} = -\frac{1}{2}\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$-(1-y^2)^{\frac{1}{2}} = -(1-x^2)^{\frac{1}{2}} + C$$

$$-(1-1)^{\frac{1}{2}} = -(1-0)^{\frac{1}{2}} + C$$

$$0 = -1 + C$$

$$1 = C$$

$$-(1-y^2)^{\frac{1}{2}} = -(1-x^2)^{\frac{1}{2}} + 1$$

$$(1-y^2)^{\frac{1}{2}} = (1-x^2)^{\frac{1}{2}} - 1$$

Find the particular solution of the differential equation that satisfies the initial condition $u(0) = 1$.

$$\frac{du}{dv} uvsinv^2 \qquad \frac{du}{dv} = uvsinv^2$$

$$\int \frac{du}{u} = \int vsinv^2 dv \qquad \begin{array}{l} w = v^2 \\ dw = 2v\,dv \\ dv = \frac{dw}{2v} \end{array}$$

$$\ln|u| = \frac{1}{2}(-\cos v^2) + C$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C \qquad u = e^{\frac{1}{2}} \cdot e^{-\frac{1}{2}\cos v^2}$$

$$e^{-\frac{1}{2}\cos v^2 + C} = u \qquad u = e^{\frac{1}{2}(1-\cos v^2)}$$

$$u = Ce^{-\frac{1}{2}\cos v^2}$$

$$1 = Ce^{-\frac{1}{2}\cos 0}$$

$$1 = Ce^{-\frac{1}{2}}$$

$$e^{\frac{1}{2}} = C$$

The rate of change in the number of foxes $N(t)$ in a population is directly proportional to $550 - N(t)$, where t is the time in years. When $t = 0$, the population is 250, and when $t = 3$, the population has increased to 500. Find the population when $t = 5$.

$$\frac{dN}{dt} = K(550 - N) \qquad 250 = 550 - Ce^{-K(0)}$$

$$-300 = -C$$

$$\int \frac{dN}{550-N} = \int K dt \qquad 300 = C$$

$$500 = 550 - 300e^{-3K}$$

$$y = 550 - 300e^{-K(5)} \qquad -\ln|550-N| = Kt + C \qquad -50 = -300e^{-3K}$$

$$\ln|550-N| = -Kt - C \qquad \frac{1}{6} = e^{-3K}$$

$$y \approx 535 \text{ foxes}$$

$$e^{-Kt-C} = 550 - N \qquad \frac{1}{6} = e^{-3K}$$

$$\ln\left(\tfrac{1}{6}\right) = -3K$$

$$N = 550 - Ce^{-Kt} \qquad \frac{\ln\left(\frac{1}{6}\right)}{-3} = K$$

A new product is introduced through an advertising campaign to a population of 1.5 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 6 months, 375,000 people have heard of the product. How many will have heard of it by the end of 1 year?

$$\frac{dP}{dt} = K(1.5 - P)$$

$$\frac{dP}{1.5 - P} = K\,dt$$

$$\int \frac{dP}{1.5-P} = \int K\,dt$$

$$-\ln|1.5 - P| = Kt + C$$

$$\ln|1.5 - P| = -Kt - C$$

$$1.5 - P = e^{-Kt-C}$$

$$1.5 - Ce^{-Kt} = P$$

$$0 = 1.5 - Ce^{-K(0)}$$

$$0 = 1.5 - C$$

$$C = 1.5$$

$$.375 = 1.5 - 1.5e^{-K(\frac{1}{2})}$$

$$-1.125 = -1.5e^{-\frac{1}{2}K}$$

$$.75 = e^{-\frac{1}{2}K}$$

$$\ln(.75) = -\tfrac{1}{2}K$$

$$-2\ln(.75) = K$$

$$P = 1.5 - 1.5e^{2\ln(.75)t}$$

$$P = 1.5 - 1.5e^{2\ln(.75)(1)}$$

$P = 656{,}250$ people

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 80 grams of A is present, and after 1 hour ($t = 1$), only 25 grams of A remain. How much of A is present after 3 hours?

$$\frac{dA}{dt} = KA^2$$

$$\int \frac{dA}{A^2} = \int K\,dt$$

$$\int A^{-2}\,dA = \int K\,dt$$

$$-A^{-1} = Kt + C$$

$$-\frac{1}{A} = Kt + C$$

$$-\frac{1}{Kt + C} = A$$

$$-\frac{1}{C} = 80$$

$$C = -\frac{1}{80}$$

$$A = \frac{-1}{Kt - \frac{1}{80}}$$

$$25 = \frac{-1}{K(1) - \frac{1}{80}}$$

$$25 = \frac{-1}{K - \frac{1}{80}}$$

$$25\left(K - \tfrac{1}{80}\right) = -1$$

$$K - \frac{1}{80} = -\frac{1}{25}$$

$$K = -\frac{1}{25} + \frac{1}{80} = \frac{-11}{400}$$

$$A = \frac{-1}{\frac{-11}{400}t - \frac{1}{80}}$$

$$A = \frac{-1}{\frac{-11}{400}(3) - \frac{1}{80}} = 10.526\text{ g}$$