2.4 The Chain Rule

Example: Interpreting the Chain Rule

A person who weighs 125 pounds burns approximately 170 calories by running 2 miles in 20 minutes. Let y be the number of calories burned, u be the number of minutes spent running and x be the number of miles run.

Find and interpret each of the following:

$\frac{dy}{du}$ The number of calories burned per minutes spent running

$$\frac{170}{20} = 8.5 \text{ calories per minute}$$

$\frac{du}{dx}$

$\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy_{cal}}{du_{min}} \cdot \frac{du_{min}}{dx_{miles}}$$

$$8.5 \cdot 10 = 85$$

The Chain Rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then $y = f((g(x))$ is a differentiable function of x and:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \quad \underline{\text{OR}} \quad \frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Examples: Comparing Strategies

Find the derivative of the following first by simplifying, and then by using the Chain Rule

$y = (x + 2)^3$

$y = x^3 + (3)(x^2)(2) + (3)(x)(2^2) + 2^3$

$y = x^3 + 6x^2 + 12x + 8$

$y' = 3x^2 + 12x + 12$

$y = (x + 2)^3$

$y' = 3(x+2)^2(1)$

$y = sin2x$

$y = 2\sin x \cos x$

$y' = 2\sin x(-\sin x) + 2(\cos x)(\cos x)$

$= 2(\cos^2 x - \sin^2 x)$

$y = sin2x$

$y' = \cos 2x \cdot 2$

$= 2\cos 2x$

Examples: Identifying Functions

For each function $y = f(g(x))$, identify $u = g(x)$ and $y = f(u)$

A. $y = (x^2 - 1)^3$ $\quad u = x^2 - 1$ $\quad y = u^3$

B. $y = sinx^2$ $\quad u = x^2$ $\quad y = \sin u$

$(\sin x)^2 \neq \sin^2 x$

C. $y = \frac{2}{(x+7)^4}$ $\quad u = x+7$ $\quad y = \frac{2}{u^4} = 2u^{-4}$

Examples: Using the Chain Rule

Find $\frac{dy}{dx}$ for $y = \frac{1}{(3x-5)^2} = (3x-5)^{-2}$

$\frac{dy}{dx} = -2(3x-5)^{-3}(3) = \frac{-6}{(3x-5)^3}$

Find the derivative of $f(x) = (5x^2 + 2x)^7$

$f'(x) = 7(10x+2)(5x^2+2x)^6$

Differentiate the function $g(t) = \frac{6}{(3t^2-5)^5}$ $= 6(3t^2-5)^{-5}$

$$g'(t) = \frac{-180t}{(3t^2-5)^6}$$

Find the derivative of $f(x) = 5x\sqrt[3]{(3x+1)^2}$ $= 5x(3x+1)^{2/3}$

$$f'(x) = (5x)\left(\tfrac{2}{3}\right)(3x+1)^{-\frac{1}{3}}(3) + 5(3x+1)^{2/3}$$

$$= 5(3x+1)^{-\frac{1}{3}}\left[(2x)(1) + (3x+1)\right]$$

$$= \frac{5[5x+1]}{(3x+1)^{+1/3}}$$

Find y' if $y = \frac{x^2}{\sqrt{x^2-3}}$ $= x^2(x^2-3)^{-\frac{1}{2}}$

$$y' = \frac{x(x^2-6)}{(x^2-3)^{3/2}}$$

Differentiate $\left(\frac{4x-1}{x^2-2}\right)^2$ $= (4x-1)^2(x^2-2)^{-2}$

$$(4x-1)^2(-2)(x^2-2)^{-3}(2x) + 2(4x-1)(4)(x^2-2)^{-2}$$

$$4(4x-1)(x^2-2)^{-3}\left[(-x)(4x-1)(1) + 2(1)(x^2-2)\right]$$

$$\frac{4(4x-1)[-4x^2+x+2x^2-4]}{(x^2-2)^3} = \frac{4(4x-1)[-2x^2+x-4]}{(x^2-2)^3}$$

Find $\frac{dy}{dx}$ if $y = tan(x^2+1)$

$$\frac{dy}{dx} = \sec^2(x^2+1)\cdot 2x$$

$$= 2x\sec^2(x^2+1)$$