Section 6.2 Volume: The Disk and Washer Methods

1. $V = \pi\int_0^1 (-x+1)^2\,dx = \pi\int_0^1 (x^2 - 2x + 1)\,dx = \pi\left[\frac{x^3}{3} - x^2 + x\right]_0^1 = \frac{\pi}{3}$

2. $V = \pi\int_0^2 (4 - x^2)^2\,dx = \pi\int_0^2 (x^4 - 8x^2 + 16)\,dx = \pi\left[\frac{x^5}{5} - \frac{8x^3}{3} + 16x\right]_0^2 = \frac{256\pi}{15}$

3. $V = \pi\int_1^4 \left(\sqrt{x}\right)^2\,dx = \pi\int_1^4 x\,dx = \pi\left[\frac{x^2}{2}\right]_1^4 = \frac{15\pi}{2}$

4. $V = \pi\int_0^3 \left(\sqrt{9 - x^2}\right)^2\,dx = \pi\int_0^3 (9 - x^2)\,dx = \pi\left[9x - \frac{x^3}{3}\right]_0^3 = 18\pi$

5. $V = \pi\int_0^1 \left[\left(x^2\right)^2 - \left(x^5\right)^2\right]dx = \pi\int_0^1 (x^4 - x^{10})\,dx = \pi\left[\frac{x^5}{5} - \frac{x^{11}}{11}\right]_0^1 = \pi\left(\frac{1}{5} - \frac{1}{11}\right) = \frac{6\pi}{55}$

6. $2 = 4 - \frac{x^2}{4}$

$8 = 16 - x^2$

$x^2 = 8$

$x = \pm 2\sqrt{2}$

$$V = \pi\int_{-2\sqrt{2}}^{2\sqrt{2}}\left[\left(4 - \frac{x^2}{4}\right)^2 - (2)^2\right]dx$$

$$= 2\pi\int_0^{2\sqrt{2}}\left[\frac{x^4}{16} - 2x^2 + 12\right]dx$$

$$= 2\pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + 12x\right]_0^{2\sqrt{2}}$$

$$= 2\pi\left[\frac{128\sqrt{2}}{80} - \frac{32\sqrt{2}}{3} + 24\sqrt{2}\right]$$

$$= \frac{448\sqrt{2}\pi}{15} \approx 132.69$$

7. $y = x^2 \Rightarrow x = \sqrt{y}$

$$V = \pi\int_0^4\left(\sqrt{y}\right)^2 dy = \pi\int_0^4 y\,dy$$

$$= \pi\left[\frac{y^2}{2}\right]_0^4 = 8\pi$$

8. $y = \sqrt{16 - x^2} \Rightarrow x = \sqrt{16 - y^2}$

$$V = \pi\int_0^4\left(\sqrt{16 - y^2}\right)^2 dy = \pi\int_0^4\left(16 - y^2\right)dy$$

$$= \pi\left[16y - \frac{y^3}{3}\right]_0^4 = \frac{128\pi}{3}$$

9. $y = x^{2/3} \Rightarrow x = y^{3/2}$

$$V = \pi\int_0^1\left(y^{3/2}\right)^2 dy = \pi\int_0^1 y^3\,dy = \pi\left[\frac{y^4}{4}\right]_0^1 = \frac{\pi}{4}$$

10. $V = \pi\int_1^4\left(-y^2 + 4y\right)^2 dy = \pi\int_1^4\left(y^4 - 8y^3 + 16y^2\right)dy$

$$= \pi\left[\frac{y^5}{5} - 2y^4 + \frac{16y^3}{3}\right]_1^4 = \frac{459\pi}{15} = \frac{153\pi}{5}$$

11. Because the axis of revolution is vertical, the integral should be found with respect to y, using

$y = 4e^x \Rightarrow x = \ln\frac{4}{y}$ and $x = 1$ on the interval $[4, 4e]$.

$$V = \pi\int_4^{4e}\left[1 - \left(\ln\frac{y}{4}\right)^2\right]dy$$

12. Because you are integrating with respect to y, the limits of integration should be

$y = 4e^0 = 4$ and $y = 4e^1 = 4e$.

$$V = \pi\int_4^{4e}\left[1 - \ln\left(\frac{y}{4}\right)^2\right]dy$$

13. $y = \sqrt{x}, y = 0, x = 3$

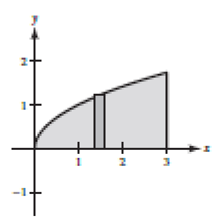
(a) $R(x) = \sqrt{x}, r(x) = 0$

$$V = \pi\int_0^3\left(\sqrt{x}\right)^2 dx$$

$$= \pi\int_0^3 x\,dx$$

$$= \pi\left[\frac{x^2}{2}\right]_0^3$$

$$= \frac{9\pi}{2}$$



(b) $R(y) = 3, r(y) = y^2$

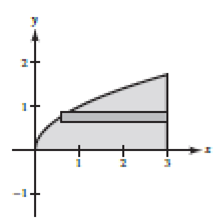
$$V = \pi\int_0^{\sqrt{3}}\left[3^2 - \left(y^2\right)^2\right]dy$$

$$= \pi\int_0^{\sqrt{3}}\left(9 - y^4\right)dy$$

$$= \pi\left[9y - \frac{y^5}{5}\right]_0^{\sqrt{3}}$$

$$= \pi\left[9\sqrt{3} - \frac{9}{5}\sqrt{3}\right]$$

$$= \frac{36\sqrt{3}\pi}{5}$$

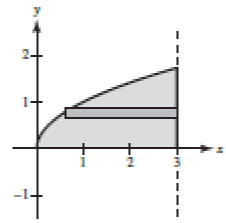


(c) $R(y) = 3 - y^2, r(y) = 0$

$$V = \pi\int_0^{\sqrt{3}} \left(3 - y^2\right)^2 dy = \pi\int_0^{\sqrt{3}} \left(9 - 6y^2 + y^4\right) dy$$

$$= \pi\left[9y - 2y^3 + \frac{y^5}{5}\right]_0^{\sqrt{3}} = \pi\left[9\sqrt{3} - 6\sqrt{3} + \frac{9\sqrt{3}}{5}\right]$$

$$= \frac{24\sqrt{3}\pi}{5}$$

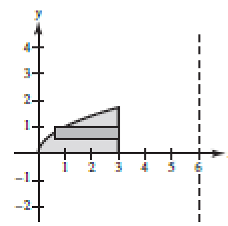


(d) $R(y) = 3 + \left(3 - y^2\right) = 6 - y^2, r(y) = 3$

$$V = \pi\int_0^{\sqrt{3}} \left[\left(6 - y^2\right)^2 - 3^2\right] dy = \pi\int_0^{\sqrt{3}} \left(y^4 - 12y^2 + 27\right) dy$$

$$= \pi\left[\frac{y^5}{5} - 4y^3 + 27y\right]_0^{\sqrt{3}} = \pi\left[\frac{9\sqrt{3}}{5} - 12\sqrt{3} + 27\sqrt{3}\right]$$

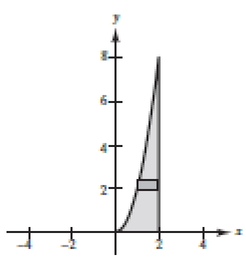
$$= \frac{84\sqrt{3}\pi}{5}$$



14. $y = 2x^2, y = 0, x = 2$

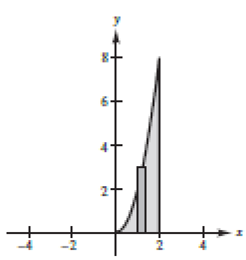
(a) $R(y) = 2, r(y) = \sqrt{y/2}$

$$V = \pi\int_0^8 \left(4 - \frac{y}{2}\right) dy = \pi\left[4y - \frac{y^2}{4}\right]_0^8 = 16\pi$$



(b) $R(x) = 2x^2, r(x) = 0$

$$V = \pi\int_0^2 4x^4\, dx = \pi\left[\frac{4x^5}{5}\right]_0^2 = \frac{128\pi}{5}$$



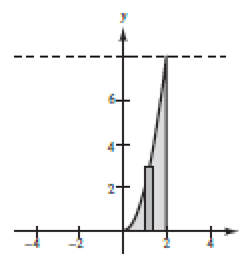
(c) $R(x) = 8, r(x) = 8 - 2x^2$

$$V = \pi\int_0^2 \left[64 - \left(64 - 32x^2 + 4x^4\right)\right] dx$$

$$= \pi\int_0^2 \left(32x^2 - 4x^4\right) dx = 4\pi\int_0^2 \left(8x^2 - x^4\right) dx$$

$$= 4\pi\left[\frac{8}{3}x^3 - \frac{1}{5}x^5\right]_0^2$$

$$= \frac{896\pi}{15}$$

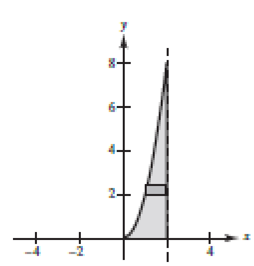


(d) $R(y) = 2 - \sqrt{y/2}, r(y) = 0$

$$V = \pi\int_0^8 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy$$

$$= \pi\int_0^8 \left(4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\right) dy$$

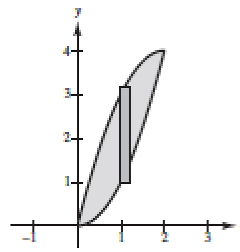
$$= \pi\left[4y - \frac{4\sqrt{2}}{3}y^{3/2} + \frac{y^2}{4}\right]_0^8 = \frac{16\pi}{3}$$



15. $y = x^2$, $y = 4x - x^2$ intersect at $(0, 0)$ and $(2, 4)$.

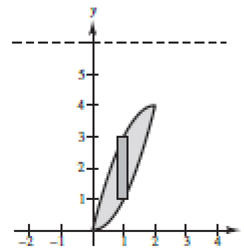
(a) $R(x) = 4x - x^2, r(x) = x^2$

$$\begin{aligned} V &= \pi\int_0^2 \left[\left(4x - x^2\right)^2 - x^4\right] dx \\ &= \pi\int_0^2 \left(16x^2 - 8x^3\right) dx \\ &= \pi\left[\frac{16}{3}x^3 - 2x^4\right]_0^2 = \frac{32\pi}{3} \end{aligned}$$



(b) $R(x) = 6 - x^2, r(x) = 6 - \left(4x - x^2\right)$

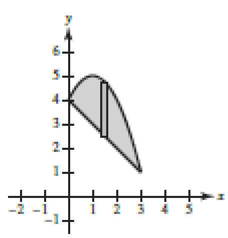
$$\begin{aligned} V &= \pi\int_0^2 \left[\left(6 - x^2\right)^2 - \left(6 - 4x + x^2\right)^2\right] dx \\ &= 8\pi\int_0^2 \left(x^3 - 5x^2 + 6x\right) dx \\ &= 8\pi\left[\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right]_0^2 = \frac{64\pi}{3} \end{aligned}$$



16. $y = 4 + 2x - x^2$, $y = 4 - x$ intersect at $(0, 4)$ and $(3, 1)$.

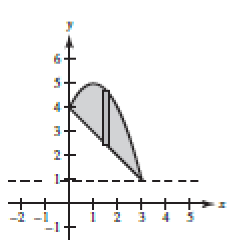
(a) $R(x) = 4 + 2x - x^2,\ r(x) = 4 - x$

$$\begin{aligned} V &= \pi\int_0^3 \left[\left(4 + 2x - x^2\right)^2 - (4 - x)^2\right] dx \\ &= \pi\int_0^3 \left(x^4 - 4x^3 - 5x^2 + 24x\right) dx \\ &= \pi\left[\frac{x^5}{5} - x^4 - \frac{5x^3}{3} + 12x^2\right]_0^3 = \frac{153\pi}{5} \end{aligned}$$



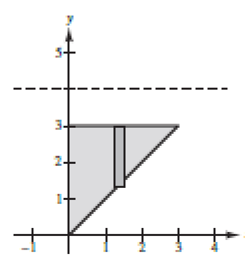
(b) $R(x) = \left(4 + 2x - x^2\right) - 1, r(x) = (4 - x) - 1$

$$\begin{aligned} V &= \pi\int_0^3 \left[\left(3 + 2x - x^2\right)^2 - (3 - x)^2\right] dx \\ &= \pi\int_0^3 \left(x^4 - 4x^3 - 3x^2 + 18x\right) dx \\ &= \pi\left[\frac{x^5}{5} - x^4 - x^3 + 9x^2\right]_0^3 = \frac{108\pi}{5} \end{aligned}$$



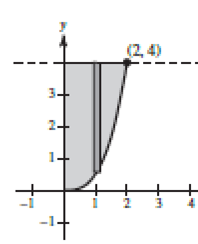
17. $R(x) = 4 - x, r(x) = 1$

$$\begin{aligned} V &= \pi\int_0^3 \left[(4 - x)^2 - (1)^2\right] dx \\ &= \pi\int_0^3 \left(x^2 - 8x + 15\right) dx \\ &= \pi\left[\frac{x^3}{3} - 4x^2 + 15x\right]_0^3 \\ &= 18\pi \end{aligned}$$



18. $R(x) = 4 - \dfrac{x^3}{2}, r(x) = 0$

$$\begin{aligned} V &= \pi\int_0^1 \left(4 - \frac{x^3}{2}\right)^2 dx \\ &= \pi\int_0^2 \left[16 - 4x^3 + \frac{x^6}{4}\right] dx \\ &= \pi\left[16x - x^4 + \frac{x^7}{28}\right]_0^2 \\ &= \pi\left(32 - 16 + \frac{128}{28}\right) \\ &= \frac{144}{7}\pi \end{aligned}$$



19. $R(x) = 4,\ r(x) = 4 - \dfrac{3}{1+x}$

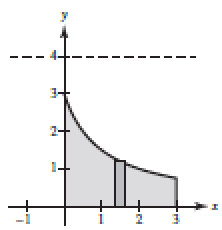
$$V = \pi \int_0^3 \left[4^2 - \left(4 - \frac{3}{1+x}\right)^2\right] dx$$

$$= \pi \int_0^3 \left[\frac{24}{1+x} - \frac{9}{(1+x)^2}\right] dx$$

$$= \pi\left[24 \ln|1+x| + \frac{9}{1+x}\right]_0^3$$

$$= \pi\left[\left(24 \ln 4 + \frac{9}{4}\right) - 9\right]$$

$$= \left(48 \ln 2 - \frac{27}{4}\right)\pi \approx 83.318$$



20. $R(x) = 4,\ r(x) = 4 - \sec x$

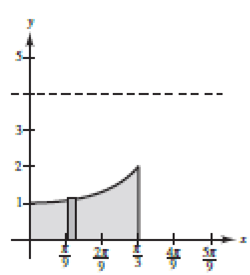
$$V = \pi \int_0^{\pi/3} \left[(4)^2 - (4 - \sec x)^2\right] dx$$

$$= \pi \int_0^{\pi/3} (8 \sec x - \sec^2 x)\, dx$$

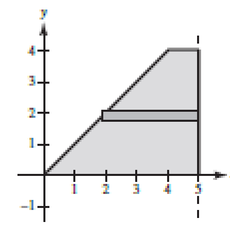
$$= \pi\left[8 \ln|\sec x + \tan x| - \tan x\right]_0^{\pi/3}$$

$$= \pi\left[\left(8 \ln\left|2 + \sqrt{3}\right| - \sqrt{3}\right) - (8 \ln|1 + 0| - 0)\right]$$

$$= \pi\left[8 \ln\left(2 + \sqrt{3}\right) - \sqrt{3}\right] \approx 27.66$$



21. $R(y) = 5 - y,\ r(y) = 0$



$$V = \pi \int_0^4 (5 - y)^2\, dy$$

$$= \pi \int_0^4 (25 - 10y + y^2)\, dy$$

$$= \pi\left[25y - 5y^2 + \frac{y^3}{3}\right]_0^4$$

$$= \pi\left[100 - 80 + \frac{64}{3}\right]$$

$$= \frac{124\pi}{3}$$

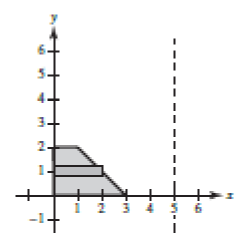
22. $y = 3 - x,\ x = 3 - y$

$R(y) = 5,\ r(y) = 5 - (3 - y) = 2 + y$

$$V = \pi \int_0^2 \left[5^2 - (2 + y)^2\right] dy$$

$$= \pi \int_0^2 (-y^2 - 4y + 21)\, dy$$

$$= \pi\left[\frac{-y^3}{3} - 2y^2 + 21y\right]_0^2 = \frac{94\pi}{3}$$



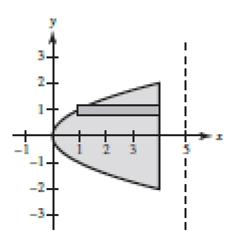
23. $R(y) = 5 - y^2,\ r(y) = 1$

$$V = \pi \int_{-2}^2 \left[(5 - y^2)^2 - 1\right] dy$$

$$= 2\pi \int_0^2 \left[y^4 - 10y^2 + 24\right] dy$$

$$= 2\pi\left[\frac{y^5}{5} - \frac{10y^3}{3} + 24y\right]_0^2$$

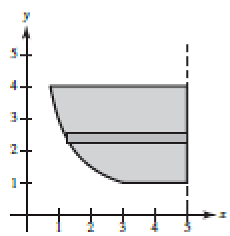
$$= 2\pi\left[\frac{32}{5} - \frac{80}{3} + 48\right] = \frac{832\pi}{15}$$



24. $xy = 3,\ x = \dfrac{3}{y}$

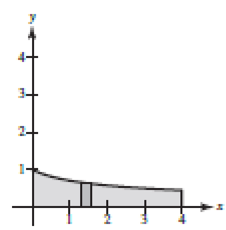
$$R(y) = 5 - \frac{3}{y},\ r(y) = 0$$

$$\begin{aligned} V &= \pi\int_1^4 \left(5 - \frac{3}{y}\right)^2 dy \\ &= \pi\int_1^4 \left(25 + \frac{9}{y^2} - \frac{30}{y}\right) dy \\ &= \pi\left[25y - \frac{9}{y} - 30\ln y\right]_1^4 \\ &= \pi\left[\left(100 - \frac{9}{4} - 30\ln 4\right) - (25 - 9)\right] \\ &= \pi\left(\frac{327}{4} - 30\ln 4\right) \approx 126.17 \end{aligned}$$



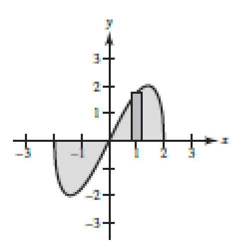
25. $R(x) = \dfrac{1}{\sqrt{x+1}},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_0^4 \left(\frac{1}{\sqrt{x+1}}\right)^2 dx \\ &= \pi\int_0^4 \frac{1}{x+1}\, dx \\ &= \pi\Big[\ln|x+1|\Big]_0^4 \\ &= \pi \ln 5 \end{aligned}$$



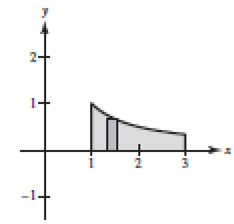
26. $R(x) = x\sqrt{4 - x^2},\ r(x) = 0$

$$\begin{aligned} V &= 2\pi\int_0^2 \left(x\sqrt{4 - x^2}\right)^2 dx \\ &= 2\pi\int_0^2 \left(4x^2 - x^4\right) dx \\ &= 2\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 \\ &= 2\pi\left[\frac{32}{3} - \frac{32}{5}\right] = \frac{128\pi}{15} \end{aligned}$$



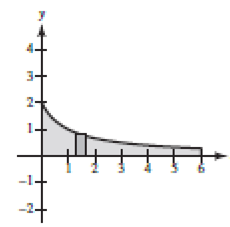
27. $R(x) = \dfrac{1}{x},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_1^3 \left(\frac{1}{x}\right)^2 dx \\ &= \pi\left[-\frac{1}{x}\right]_1^3 \\ &= \pi\left[-\frac{1}{3} + 1\right] = \frac{2}{3}\pi \end{aligned}$$



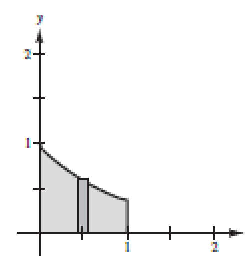
28. $R(x) = \dfrac{2}{x+1},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_0^6 \left(\frac{2}{x+1}\right)^2 dx \\ &= 4\pi\int_0^6 (x+1)^{-2}\, dx \\ &= 4\pi\left[\frac{-1}{x+1}\right]_0^6 \\ &= 4\pi\left[-\frac{1}{7} + 1\right] = \frac{24\pi}{7} \end{aligned}$$



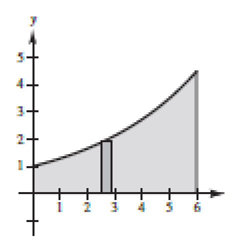
29. $R(x) = e^{-x},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_0^1 \left(e^{-x}\right)^2 dx \\ &= \pi\int_0^1 e^{-2x}\, dx \\ &= \left[-\frac{\pi}{2}e^{-2x}\right]_0^1 \\ &= \frac{\pi}{2}\left(1 - e^{-2}\right) \approx 1.358 \end{aligned}$$



30. $R(x) = e^{x/4},\ r(x) = 0$

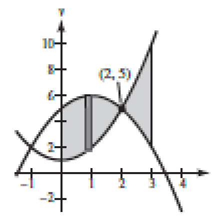
$$\begin{aligned} V &= \pi\int_0^6 \left(e^{x/4}\right)^2 dx \\ &= \pi\int_0^6 e^{x/2}\, dx \\ &= \pi\left[2e^{x/2}\right]_0^6 \\ &= \pi\left(2e^3 - 2\right) \approx 119.92 \end{aligned}$$



31. $$\begin{aligned} x^2 + 1 &= -x^2 + 2x + 5 \\ 2x^2 - 2x - 4 &= 0 \\ x^2 - x - 2 &= 0 \\ (x - 2)(x + 1) &= 0 \end{aligned}$$

The curves intersect at $(-1, 2)$ and $(2, 5)$.

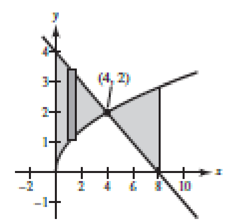
$$\begin{aligned} V &= \pi\int_0^2\left[\left(5 + 2x - x^2\right)^2 - \left(x^2 + 1\right)^2\right] dx + \pi\int_2^3\left[\left(x^2 + 1\right)^2 - \left(5 + 2x - x^2\right)^2\right] dx \\ &= \pi\int_0^2\left(-4x^3 - 8x^2 + 20x + 24\right) dx + \pi\int_2^3\left(4x^3 + 8x^2 - 20x - 24\right) dx \\ &= \pi\left[-x^4 - \frac{8}{3}x^3 + 10x^2 + 24x\right]_0^2 + \pi\left[x^4 + \frac{8}{3}x^3 - 10x^2 - 24x\right]_2^3 \\ &= \pi\frac{152}{3} + \pi\frac{125}{3} = \frac{277\pi}{3} \end{aligned}$$



32. $$\begin{aligned} \sqrt{x} &= -\frac{1}{2}x + 4 \\ x &= \frac{1}{4}x^2 - 4x + 16 \\ 0 &= x^2 - 20x + 64 \\ 0 &= (x - 4)(x - 16) \end{aligned}$$

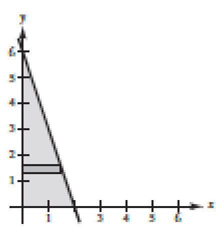
The curves intersect at $(4, 2)$. (Note $x = 16$ is an extraneous root.)

$$\begin{aligned} V &= \pi\int_0^4\left[\left(4 - \frac{1}{2}x\right)^2 - \left(\sqrt{x}\right)^2\right] dx + \pi\int_4^8\left[\left(\sqrt{x}\right)^2 - \left(4 - \frac{1}{2}x\right)^2\right] dx \\ &= \pi\int_0^4\left(\frac{x^2}{4} - 5x + 16\right) dx + \pi\int_4^8\left(-\frac{x^2}{4} + 5x - 16\right) dx \\ &= \pi\left[\frac{x^3}{12} - \frac{5x^2}{2} + 16x\right]_0^4 + \pi\left[-\frac{x^3}{12} + \frac{5x^2}{2} - 16x\right]_4^8 \\ &= \frac{88}{3}\pi + \frac{56}{3}\pi = 48\pi \end{aligned}$$



33. $y = 6 - 3x \Rightarrow x = \frac{1}{3}(6 - y)$

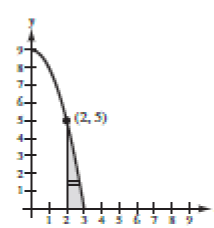
$$\begin{aligned} V &= \pi\int_0^6\left[\frac{1}{3}(6 - y)\right]^2 dy \\ &= \frac{\pi}{9}\int_0^6\left[36 - 12y + y^2\right] dy \\ &= \frac{\pi}{9}\left[36y - 6y^2 + \frac{y^3}{3}\right]_0^6 \\ &= \frac{\pi}{9}\left[216 - 216 + \frac{216}{3}\right] \\ &= 8\pi \end{aligned}$$



34. $y = 9 - x^2, y = 0, x = 2, x = 3$

$x = \sqrt{9 - y}$

$$\begin{aligned} V &= \pi\int_0^5\left[\left(\sqrt{9 - y}\right)^2 - 2^2\right] dy \\ &= \pi\int_0^5 (5 - y)\, dy \\ &= \pi\left[5y - \frac{y^2}{2}\right]_0^5 = \pi\left(25 - \frac{25}{2}\right) = \frac{25\pi}{2} \end{aligned}$$

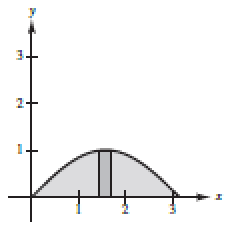


35. $V = \pi\int_0^{\pi}(\sin x)^2\,dx$

$$= \pi\int_0^{\pi}\frac{1-\cos 2x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\pi} = \frac{\pi}{2}[\pi] = \frac{\pi^2}{2}$$

Numerical approximation: 4.9348



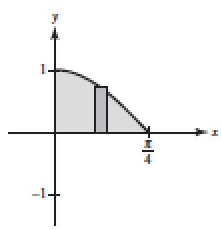
36. $V = \pi\int_0^{\pi/4}\cos^2 2x\,dx$

$$= \pi\int_0^{\pi/4}\frac{1+\cos 4x}{2}\,dx$$

$$= \frac{\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\pi/4}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

Numerical approximation: 1.2337



37. $V = \pi\int_1^2\left(e^{x-1}\right)^2\,dx$

$$= \pi\int_1^2 e^{2x-2}\,dx$$

$$= \frac{\pi}{2}e^{2x-2}\Big]_1^2$$

$$= \frac{\pi}{2}\left(e^2 - 1\right)$$

Numerical approximation: 10.0359

38. $V = \pi\int_{-1}^2\left[e^{x/2} + e^{-x/2}\right]^2\,dx$

$$= \pi\int_{-1}^2\left[e^x + e^{-x} + 2\right]dx$$

$$= \pi\left[e^x - e^{-x} + 2x\right]_{-1}^2$$

$$= \pi\left[\left(e^2 - e^{-2} + 4\right) - \left(e^{-1} - e - 2\right)\right]$$

$$= \pi\left(e^2 + e + 6 - e^{-2} - e^{-1}\right)$$

Numerical approximation: 49.0218

39. $V = \pi\int_0^2\left[e^{-x^2}\right]^2\,dx \approx 1.9686$

40. $V = \pi\int_1^3[\ln x]^2\,dx \approx 3.2332$

41. $V = \pi\int_0^5\left[2\arctan(0.2x)\right]^2\,dx \approx 15.4115$

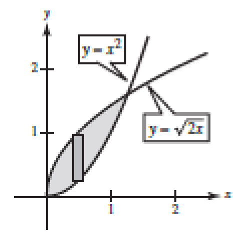
42. $x^2 = \sqrt{2x}$

$$x^4 = 2x$$

$$x^3 = 2$$

$$x = 2^{1/3} \approx 1.2599$$

$$V = \pi\int_0^{2^{1/3}}\left[\left(\sqrt{2x}\right)^2 - \left(x^2\right)^2\right]dx \approx 2.9922$$



43. $V = \pi\int_0^1 y^2\,dy = \pi\frac{y^3}{3}\Big]_0^1 = \frac{\pi}{3}$

44. $V = \pi\int_0^1\left[1^2 - (1-y)^2\right]dy$

$$= \pi\int_0^1\left[2y - y^2\right]dy$$

$$= \pi\left[y^2 - \frac{y^3}{3}\right]_0^1$$

$$= \pi\left(1 - \frac{1}{3}\right) = \frac{2}{3}\pi$$

45. $V = \pi\int_0^1\left(x^2 - x^4\right)dx$

$$= \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$

$$= \pi\left(\frac{1}{3} - \frac{1}{5}\right)$$

$$= \frac{2\pi}{15}$$

46. $V = \pi\int_0^1\left[\left(1 - x^2\right)^2 - \left(1 - x\right)^2\right] dx$

$= \pi\int_0^1\left[1 - 2x^2 + x^4 - 1 + 2x - x^2\right] dx$

$= \pi\int_0^1\left[2x - 3x^2 + x^4\right] dx$

$= \pi\left[x^2 - x^3 + \dfrac{x^5}{5}\right]_0^1$

$= \pi\left(\dfrac{1}{5}\right) = \dfrac{\pi}{5}$

47. $V = \pi\int_0^1(1 - y)\, dy$

$= \pi\left[y - \dfrac{y^2}{2}\right]_0^1 = \pi\left(1 - \dfrac{1}{2}\right) = \dfrac{\pi}{2}$

48. $V = \pi\int_0^1\left(1 - \sqrt{y}\right)^2 dy$

$= \pi\int_0^1\left(1 - 2\sqrt{y} + y\right) dy$

$= \pi\left[y - \dfrac{4}{3}y^{3/2} + \dfrac{y^2}{2}\right]_0^1$

$= \pi\left(1 - \dfrac{4}{3} + \dfrac{1}{2}\right)$

$= \dfrac{\pi}{6}$

49. $V = \pi\int_0^1\left(y - y^2\right) dy$

$= \pi\left[\dfrac{y^2}{2} - \dfrac{y^3}{3}\right]_0^1 = \pi\left(\dfrac{1}{2} - \dfrac{1}{3}\right) = \dfrac{\pi}{6}$

50. $V = \pi\int_0^1\left[(1 - y)^2 - \left(1 - \sqrt{y}\right)^2\right] dy$

$= \pi\int_0^1\left[1 - 2y + y^2 - 1 + 2\sqrt{y} - y\right] dy$

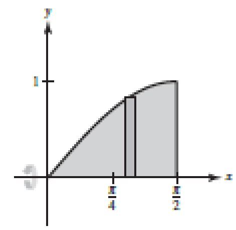
$= \pi\int_0^1\left[2\sqrt{y} - 3y + y^2\right] dy$

$= \pi\left[\dfrac{4}{3}y^{3/2} - \dfrac{3y^2}{2} + \dfrac{y^3}{3}\right]_0^1$

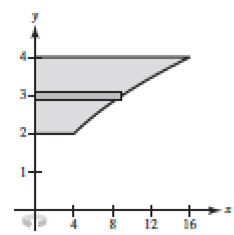
$= \pi\left(\dfrac{4}{3} - \dfrac{3}{2} + \dfrac{1}{3}\right)$

$= \dfrac{\pi}{6}$

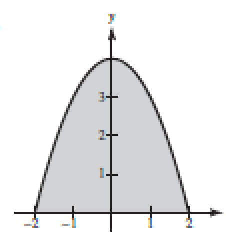
51. $\pi\int_0^{\pi/2}\sin^2 x\, dx$ represents the volume of the solid generated by revolving the region bounded by $y = \sin x, y = 0, x = 0, x = \pi/2$ about the x-axis.

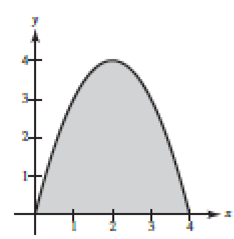


52. $\pi\int_2^4 y^4\, dy$ represents the volume of the solid generated by revolving the region bounded by $x = y^2, x = 0, y = 2$, and $y = 4$ about the y-axis.



53.





The volumes are the same because the solid has been translated horizontally. $\left(4x - x^2 = 4 - (x - 2)^2\right)$

54. (a) Matches (ii) because the axis of rotation is vertical, and this is the washer method.

(b) Matches (iv) because the axis of rotation is horizontal, and this is the washer method.

(c) Matches (i) because the axis of rotation is horizontal.

(d) Matches (iii) because the axis of rotation is vertical.

55. $V = \pi\int_0^4 \left(\sqrt{x}\right)^2\,dx = \pi\int_0^4 x\,dx = \left[\dfrac{\pi x^2}{2}\right]_0^4 = 8\pi$

Let $0 < c < 4$ and set

$$\pi\int_0^c x\,dx = \left[\frac{\pi x^2}{2}\right]_0^c = \frac{\pi c^2}{2} = 4\pi.$$

$$c^2 = 8$$

$$c = \sqrt{8} = 2\sqrt{2}$$

So, when $x = 2\sqrt{2}$, the solid is divided into two parts of equal volume.

56. Set $\pi\int_0^c x\,dx = \dfrac{8\pi}{3}$ (one third of the volume).

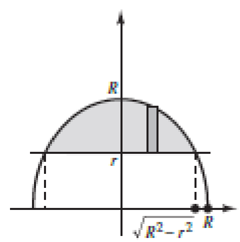
Then $\dfrac{\pi c^2}{2} = \dfrac{8\pi}{3}$, $c^2 = \dfrac{16}{3}$, $c = \dfrac{4}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$.

To find the other value, set $\pi\int_0^d x\,dx = \dfrac{16\pi}{3}$ (two thirds of the volume).

Then $\dfrac{\pi d^2}{2} = \dfrac{16\pi}{3}$, $d^2 = \dfrac{32}{3}$, $d = \dfrac{\sqrt{32}}{\sqrt{3}} = \dfrac{4\sqrt{6}}{3}$.

The x-values that divide the solid into three parts of equal volume are $x = \left(4\sqrt{3}\right)/3$ and $x = \left(4\sqrt{6}\right)/3$.

57.
$$\begin{aligned} V &= \pi\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \left[\left(\sqrt{R^2-x^2}\right)^2 - r^2\right] dx \\ &= 2\pi\int_0^{\sqrt{R^2-r^2}} \left(R^2 - r^2 - x^2\right) dx \\ &= 2\pi\left[\left(R^2-r^2\right)x - \frac{x^3}{3}\right]_0^{\sqrt{R^2-r^2}} \\ &= 2\pi\left[\left(R^2-r^2\right)^{3/2} - \frac{\left(R^2-r^2\right)^{3/2}}{3}\right] = \frac{4}{3}\pi\left(R^2-r^2\right)^{3/2} \end{aligned}$$

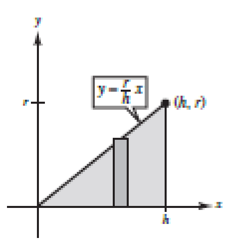


58. Let $R = 6$ in the previous Exercise.

$$\begin{aligned} \tfrac{4}{3}\pi\left(36 - r^2\right)^{3/2} &= \tfrac{1}{2}\left(\tfrac{4}{3}\right)\pi(6)^3 \\ \left(36 - r^2\right)^{3/2} &= 108 \\ 36 - r^2 &= (108)^{2/3} \\ r^2 &= 36 - 108^{2/3} \\ r &= \sqrt{36 - 108^{2/3}} \approx 3.65 \end{aligned}$$

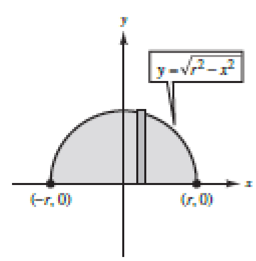
59. $R(x) = \dfrac{r}{h}x,\ r(x) = 0$

$$V = \pi\int_0^h \frac{r^2}{h^2}x^2\,dx = \left[\frac{r^2\pi}{3h^2}x^3\right]_0^h = \frac{r^2\pi}{3h^2}h^3 = \frac{1}{3}\pi r^2 h$$

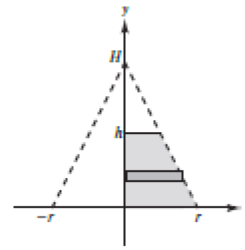


60. $R(x) = \sqrt{r^2 - x^2},\ r(x) = 0$

$$\begin{aligned} V &= \pi\int_{-r}^{r} \left(r^2 - x^2\right) dx \\ &= 2\pi\int_0^r \left(r^2 - x^2\right) dx \\ &= 2\pi\left[r^2x - \tfrac{1}{3}x^3\right]_0^r \\ &= 2\pi\left(r^3 - \tfrac{1}{3}r^3\right) = \tfrac{4}{3}\pi r^3 \end{aligned}$$

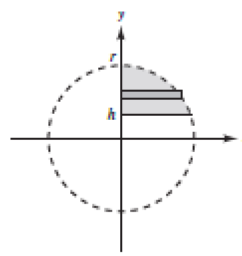


61. $x = r - \dfrac{r}{H}y = r\left(1 - \dfrac{y}{H}\right), R(y) = r\left(1 - \dfrac{y}{H}\right), r(y) = 0$



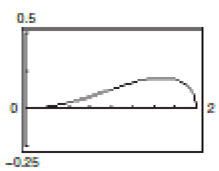
$$\begin{aligned} V &= \pi\int_0^h \left[r\left(1 - \frac{y}{H}\right)\right]^2 dy = \pi r^2 \int_0^h \left(1 - \frac{2}{H}y + \frac{1}{H^2}y^2\right) dy \\ &= \pi r^2\left[y - \frac{1}{H}y^2 + \frac{1}{3H^2}y^3\right]_0^h \\ &= \pi r^2\left(h - \frac{h^2}{H} + \frac{h^3}{3H^2}\right) = \pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right) \end{aligned}$$

62. $x = \sqrt{r^2 - y^2}, R(y) = \sqrt{r^2 - y^2}, r(y) = 0$



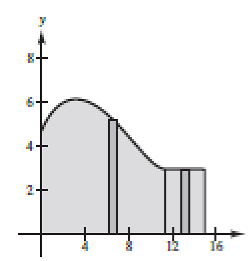
$$\begin{aligned} V &= \pi\int_h^r \left(\sqrt{r^2 - y^2}\right)^2 dy \\ &= \pi\int_h^r \left(r^2 - y^2\right) dy \\ &= \pi\left[r^2 y - \frac{y^3}{3}\right]_h^r \\ &= \pi\left[\left(r^3 - \frac{r^3}{3}\right) - \left(r^2 h - \frac{h^3}{3}\right)\right] \\ &= \pi\left(\frac{2r^3}{3} - r^2 h + \frac{h^3}{3}\right) = \frac{\pi}{3}\left(2r^3 - 3r^2 h + h^3\right) \end{aligned}$$

63.



$$V = \pi\int_0^2 \left(\frac{1}{8}x^2\sqrt{2 - x}\right)^2 dx = \frac{\pi}{64}\int_0^2 x^4(2 - x)\, dx = \frac{\pi}{64}\left[\frac{2x^5}{5} - \frac{x^6}{6}\right]_0^2 = \frac{\pi}{30}\text{ m}^3$$

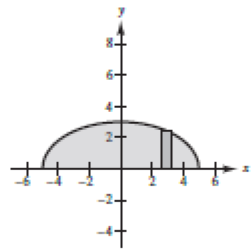
64. $y = \begin{cases} \sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}, & 0 \le x \le 11.5 \\ 2.95, & 11.5 < x \le 15 \end{cases}$



$$\begin{aligned} V &= \pi\int_0^{11.5} \left(\sqrt{0.1x^3 - 2.2x^2 + 10.9x + 22.2}\right)^2 dx + \pi\int_{11.5}^{15} 2.95^2\, dx \\ &= \pi\left[\frac{0.1x^4}{4} - \frac{2.2x^3}{3} + \frac{10.9x^2}{2} + 22.2x\right]_0^{11.5} + \pi\left[2.95^2 x\right]_{11.5}^{15} \\ &\approx 1031.9016 \text{ cubic centimeters} \end{aligned}$$

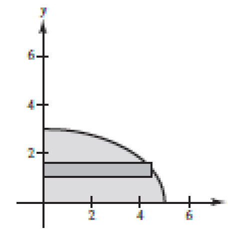
65. (a) $R(x) = \frac{3}{5}\sqrt{25 - x^2}, r(x) = 0$

$$V = \frac{9\pi}{25}\int_{-5}^{5}\left(25 - x^2\right) dx$$
$$= \frac{18\pi}{25}\int_{0}^{5}\left(25 - x^2\right) dx$$
$$= \frac{18\pi}{25}\left[25x - \frac{x^3}{3}\right]_0^5$$
$$= 60\pi$$



(b) $R(y) = \frac{5}{3}\sqrt{9 - y^2}, r(y) = 0, x \ge 0$

$$V = \frac{25\pi}{9}\int_{0}^{3}\left(9 - y^2\right) dy$$
$$= \frac{25\pi}{9}\left[9y - \frac{y^3}{3}\right]_0^3$$
$$= 50\pi$$



66. Total volume: $V = \frac{4\pi(50)^3}{3} = \frac{500{,}000\pi}{3}$ ft^3

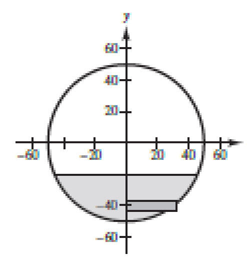
Volume of water in the tank:

$$\pi\int_{-50}^{y_0}\left(\sqrt{2500 - y^2}\right)^2 dy = \pi\int_{-50}^{y_0}\left(2500 - y^2\right) dy = \pi\left[2500y - \frac{y^3}{3}\right]_{-50}^{y_0} = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$

When the tank is one-fourth of its capacity:

$$\frac{1}{4}\left(\frac{500{,}000\pi}{3}\right) = \pi\left(2500y_0 - \frac{y_0^3}{3} + \frac{250{,}000}{3}\right)$$
$$125{,}000 = 7500y_0 - y_0^3 + 250{,}000$$
$$y_0^3 - 7500y_0 - 125{,}000 = 0$$
$$y_0 \approx -17.36$$

Depth: $-17.36 - (-50) = 32.64$ feet

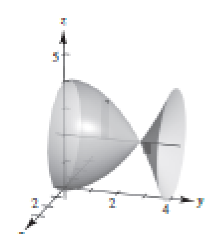


When the tank is three-fourths of its capacity the depth is $100 - 32.64 = 67.36$ feet.

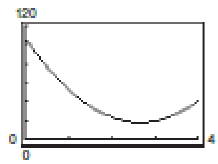
67. (a) First find where $y = b$ intersects the parabola:

$$b = 4 - \frac{x^2}{4}$$
$$x^2 = 16 - 4b = 4(4 - b)$$
$$x = 2\sqrt{4 - b}$$

$$V = \int_0^{2\sqrt{4-b}} \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx + \int_{2\sqrt{4-b}}^{4} \pi\left[b - 4 + \frac{x^2}{4}\right]^2 dx$$
$$= \int_0^4 \pi\left[4 - \frac{x^2}{4} - b\right]^2 dx$$
$$= \pi\int_0^4\left[\frac{x^4}{16} - 2x^2 + \frac{bx^2}{2} + b^2 - 8b + 16\right] dx$$
$$= \pi\left[\frac{x^5}{80} - \frac{2x^3}{3} + \frac{bx^3}{6} + b^2x - 8bx + 16x\right]_0^4$$
$$= \pi\left(\frac{64}{5} - \frac{128}{3} + \frac{32}{3}b + 4b^2 - 32b + 64\right) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$$



(b) Graph of $V(b) = \pi\left(4b^2 - \frac{64}{3}b + \frac{512}{15}\right)$



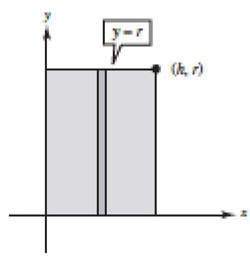
Minimum volume is17.87 for $b = 2.67$.

(c) $V'(b) = \pi\left(8b - \frac{64}{3}\right) = 0 \Rightarrow b = \frac{64/3}{8} = \frac{8}{3} = 2\frac{2}{3}$

$V''(b) = 8\pi > 0 \Rightarrow b = \frac{8}{3}$ is a relative minimum.

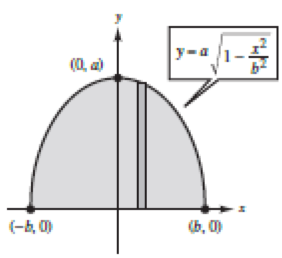
68. (a) $\pi\int_0^h r^2\,dx$ (ii)

is the volume of a right circular cylinder with radius r and height h.



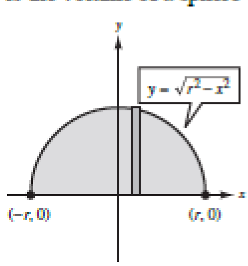
(b) $\pi\int_{-b}^{b}\left(a\sqrt{1 - \frac{x^2}{b^2}}\right)^2 dx$ (iv)

is the volume of an ellipsoid with axes $2a$ and $2b$.



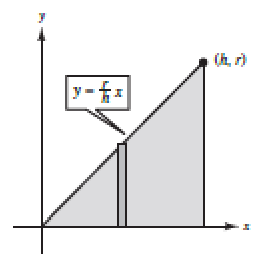
(c) $\pi\int_{-r}^{r}\left(\sqrt{r^2 - x^2}\right)^2 dx$ (iii)

is the volume of a sphere with radius r.



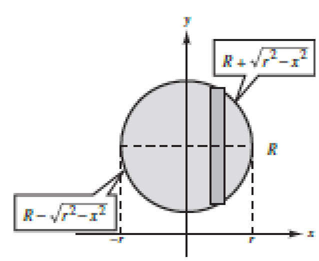
(d) $\pi\int_0^h\left(\frac{rx}{h}\right)^2 dx$ (i)

is the volume of a right circular cone with the radius of the base as r and height h.



(e) $\pi\int_{-r}^{r}\left[\left(R + \sqrt{r^2 - x^2}\right)^2 - \left(R - \sqrt{r^2 - x^2}\right)^2\right] dx$ (v)

is the volume of a torus with the radius of its circular cross section as r and the distance from the axis of the torus to the center of its cross section as R.

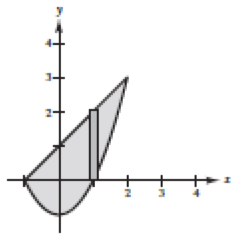


69. Let $A_1(x)$ and $A_2(x)$ equal the areas of the cross sections of the two solids for $a \le x \le b$. Because $A_1(x) = A_2(x)$, you have

$$V_1 = \int_a^b A_1(x)\,dx = \int_a^b A_2(x)\,dx = V_2.$$

So, the volumes are the same.

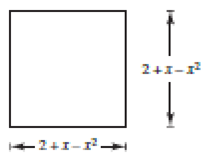
70.



Base of cross section $= (x+1) - (x^2 - 1) = 2 + x - x^2$

(a) $A(x) = b^2 = (2 + x - x^2)^2 = 4 + 4x - 3x^2 - 2x^3 + x^4$

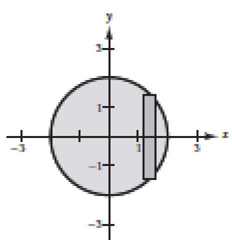
$$V = \int_{-1}^{2} (4 + 4x - 3x^2 - 2x^3 + x^4)\,dx = \left[4x + 2x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{81}{10}$$



(b) $A(x) = bh = (2 + x - x^2)1$

$$V = \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \frac{9}{2}$$

71.



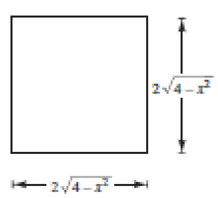
Base of cross section $= 2\sqrt{4 - x^2}$

(a) $A(x) = b^2 = \left(2\sqrt{4 - x^2}\right)^2$

$$V = \int_{-2}^{2} 4(4 - x^2)\,dx$$

$$= 4\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{128}{3}$$

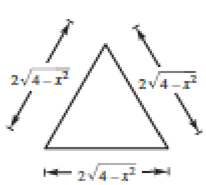


(b) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{3}\sqrt{4 - x^2}\right) = \sqrt{3}(4 - x^2)$

$$V = \sqrt{3}\int_{-2}^{2} (4 - x^2)\,dx$$

$$= \sqrt{3}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{32\sqrt{3}}{3}$$

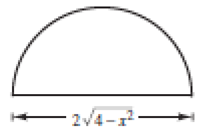


(c) $A(x) = \frac{1}{2}\pi r^2 = \frac{\pi}{2}\left(\sqrt{4 - x^2}\right)^2 = \frac{\pi}{2}\left(4 - x^2\right)$

$$V = \frac{\pi}{2}\int_{-2}^{2} \left(4 - x^2\right) dx$$

$$= \frac{\pi}{2}\left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{16\pi}{3}$$

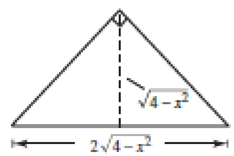


(d) $A(x) = \frac{1}{2}bh = \frac{1}{2}\left(2\sqrt{4 - x^2}\right)\left(\sqrt{4 - x^2}\right) = 4 - x^2$

$$V = \int_{-2}^{2} \left(4 - x^2\right) dx$$

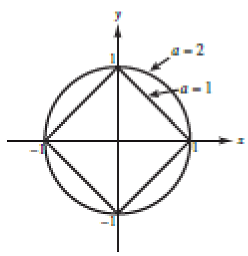
$$= \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \frac{32}{3}$$



72. (a) When $a = 1$: $|x| + |y| = 1$ represents a square.

When $a = 2$: $|x|^2 + |y|^2 = 1$ represents a circle.



(b) $|y| = \left(1 - |x|^a\right)^{1/a}$

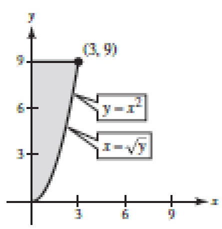
$$A = 2\int_{-1}^{1}\left(1 - |x|^a\right)^{1/a} dx = 4\int_{0}^{1}\left(1 - x^a\right)^{1/a} dx$$

Solve the equation for $|y|$, as shown below. Then form n slices, each of whose area is approximated by the integral below. Finally, sum the volumes of these n slices.

$|y| = \left(1 - |x|^a\right)^{1/a}$

$$A = 2\int_{-1}^{1}\left(1 - |x|^a\right)^{1/a} dx = 4\int_{0}^{1}\left(1 - x^a\right)^{1/a} dx$$

73. For each axis of revolution, find the respective volume of each region.



x-axis: $V = \pi\int_0^3\left[9^2 - \left(x^2\right)^2\right] dx = \pi\int_0^3\left(81 - x^4\right) dx = \pi\left[81x - \frac{1}{5}x^5\right]_0^3 = \frac{972}{5}\pi = 194.4\pi$

y-axis: $V = \pi\int_0^9\left(\sqrt{y}\right)^2 dy = \pi\int_0^9 y\, dy = \pi\left[\frac{1}{2}y^2\right]_0^9 = \frac{81}{2}\pi = 40.5\pi$

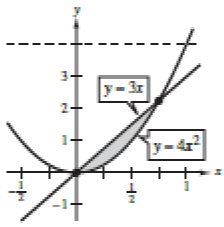
$x = 3$: $V = \pi\int_0^9\left[(3)^2 - \left(3 - \sqrt{y}\right)^2\right] dy = \pi\int_0^9\left[9 - \left(9 - 6\sqrt{y} + y\right)\right] dy$

$$= \pi\int_0^9\left(6\sqrt{y} - y\right) dy = \pi\left[4y^{3/2} - \frac{1}{2}y^2\right]_0^9 = \frac{135}{2}\pi \approx 67.5\,\pi$$

The resulting solids about each axis from least to greatest are y-axis, $x = 3$, and x-axis.

So, the answer is C.

74.



Because $y = 4$ is the axis of revolution, integrate with respect to x and the limits of integration are $x = 0$ and $x = \frac{3}{4}$.

$R(x) = 4 - 4x^2$ and $r(x) = 4 - 3x$

$$V = \int_0^{3/4} \left[\left(4 - x^2\right)^2 - \left(4 - 3x\right)^2\right] dx$$

So, the answer is B.

75. (a) The graphs intersect at $(0, 2)$, $(0.81, 1.4)$, and $(1.76, 2.72)$.

$$\begin{aligned} A &= \int_0^{0.81} \left[f(x) - g(x)\right] dx + \int_{0.81}^{1.76} \left[g(x) - f(x)\right] dx \\ &= \int_0^{0.81} \left[x^4 - 4x^2 + 2x + \sin(\pi x)\right] dx + \int_{0.81}^{1.76} \left[-\sin(\pi x) - x^4 + 4x^2 - 2x\right] dx \\ &= \left[\frac{1}{5}x^5 - \frac{4}{3}x^3 + x^2 - \frac{1}{\pi}\cos(\pi x)\right]_0^{0.81} + \left[\frac{1}{\pi}\cos(\pi x) - \frac{1}{5}x^5 + \frac{4}{3}x^3 - x^2\right]_{0.81}^{1.76} \\ &\approx 0.5988 + 1.3066 \\ &\approx 1.905 \end{aligned}$$

(b) Base of cross section: $g(x) - f(x) = \left[2 - \sin(\pi x)\right] - \left(x^4 - 4x^2 + 2x + 2\right)$

$$= -\sin(\pi x) - x^4 + 4x^2 - 2x$$

$$\begin{aligned} V &= \int_{0.807}^{1.757} \left[-\sin(\pi x) - x^4 + 4x^2 - 2x\right]^2 dx \\ &= 2.222 \end{aligned}$$

So, the volume of the solid is 2.222 units^3.

(c) $h(x) = g(x) - f(x)$

$$\frac{dh}{dx} = g'(x) - f'(x) = \left[-\pi\cos(\pi x)\right] - \left(4x^3 - 8x + 2\right) = -\pi\cos(\pi x) - 4x^3 + 8x - 2$$

So, $h'(1.2) = -\pi\cos(\pi \cdot 1.2) - 4(1.2)^3 + 8(1.2) - 2 \approx 3.230$.