2.5 ~~The Chain Rule~~ Implicit Differentiation

Explicitly Defined:

$y = \frac{1}{x}$

Implicitly Defined:

$xy = 1$

Examples: Differentiating with respect to x

$\frac{d}{dx}[7x^3] = 7 \cdot 3x^2 = 21x^2$

$\frac{d}{dx}[7y^3] \cdot 7 \cdot 3y^2 \cdot \frac{dy}{dx}$

$21y^2 \frac{dy}{dx}$

$\frac{d}{dx}[x^4 - 2y] = 4x^3 - 2\frac{dy}{dx}$

$\frac{d}{dx}\left[\frac{x^2}{y^3}\right] = \frac{(y^3)(2x) - (x^2)(3y^2)\frac{dy}{dx}}{y^6}$

Steps for Implicit Differentiation

1. Differentiate both sides with respect to x. *
2. Gather all terms with $\frac{dy}{dx}$ to one side
3. Factor out $\frac{dy}{dx}$
4. Solve for $\frac{dy}{dx}$

Examples: Implicit Differentiation

Find $\frac{dy}{dx}$ given $2y^4 - y^3x^2 + 4x^3 = 5$.

① $8y^3 \frac{dy}{dx} - [y^3 \cdot 2x + 3y^2 \frac{dy}{dx} \cdot x^2] + 12x^2 = 0$

② $8y^3 \frac{dy}{dx} - 2xy^3 - 3x^2y^2 \frac{dy}{dx} + 12x^2 = 0$

$8y^3 \frac{dy}{dx} - 3x^2y^2 \frac{dy}{dx} = 2xy^3 - 12x^2$

③ $\frac{dy}{dx}[8y^3 - 3x^2y^2] = 2xy^3 - 12x^2$

④ $\frac{dy}{dx} = \frac{2xy^3 - 12x^2}{8y^3 - 3x^2y^2}$

Determine the slope of the tangent line to the graph of $\frac{1}{4}x^2 + y^2 = 6$ at $(2, \sqrt{5})$

$$\frac{1}{2}x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -\frac{1}{2}x$$
$$\frac{dy}{dx} = \frac{-x}{4y}$$

$$\frac{dy}{dx}(2,\sqrt{5}) = \frac{-2}{4(\sqrt{5})} = -\frac{1}{2\sqrt{5}}$$

Determine the slope of the graph of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 5$ at the point $(8,1)$.

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}} \cdot \frac{dy}{dx} = 0$$
$$\frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = -\frac{2}{3}x^{-\frac{1}{3}}$$
$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = \frac{-y^{1/3}}{x^{1/3}}$$

$$\frac{dy}{dx}(8,1) = -\frac{1}{2}$$

Find the tangent line to the graph of $(x+2)^2 + (y-3)^2 = 37$ at the point $(4,4)$.

$$y - 4 = -6(x-4)$$

Example: Finding a Second Derivative with Implicit Differentiation

Given $(x-5)^2 + y^2 = 36$ find $\frac{d^2y}{dx^2}$

$$2(x-5)(1) + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2(x-5)$$
$$\frac{dy}{dx} = \frac{-2(x-5)}{2y} = \frac{-(x-5)}{y} = \frac{-x+5}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y(-1) - \left(\frac{dy}{dx}\right)(-x+5)}{y^2} = \frac{-y - \left(\frac{-x+5}{y}\right)(-x+5)}{y^2} \cdot \frac{y}{y} = \frac{-y^2 - (-x+5)^2}{y^3} = -\frac{36}{y^3}$$