Section 3.6 Optimization Problems

1. (a)

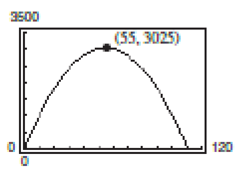
First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$

(b)

First Number, x	Second Number	Product, P
10	$110 - 10$	$10(110 - 10) = 1000$
20	$110 - 20$	$20(110 - 20) = 1800$
30	$110 - 30$	$30(110 - 30) = 2400$
40	$110 - 40$	$40(110 - 40) = 2800$
50	$110 - 50$	$50(110 - 50) = 3000$
60	$110 - 60$	$60(110 - 60) = 3000$
70	$110 - 70$	$70(110 - 70) = 2800$
80	$110 - 80$	$80(110 - 80) = 2400$
90	$110 - 90$	$90(110 - 90) = 1800$
100	$110 - 100$	$100(110 - 100) = 1000$

The maximum is attained near $x = 50$ and 60.

(c) $P = x(110 - x) = 110x - x^2$

(d) 

The solution appears to be $x = 55$.

(e) $\dfrac{dP}{dx} = 110 - 2x = 0$ when $x = 55$.

$$\frac{d^2P}{dx^2} = -2 < 0$$

P is a maximum when $x = 110 - x = 55$. The two numbers are 55 and 55.

2. (a)

Height, x	Length & Width	Volume
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$
3	$24 - 2(3)$	$3[24 - 2(3)]^2 = 972$
4	$24 - 2(4)$	$4[24 - 2(4)]^2 = 1024$
5	$24 - 2(5)$	$5[24 - 2(5)]^2 = 980$
6	$24 - 2(6)$	$6[24 - 2(6)]^2 = 864$

The maximum is attained near $x = 4$.

(b) $V = x(24 - 2x)^2, 0 < x < 12$

(c) $\dfrac{dV}{dx} = 2x(24 - 2x)(-2) + (24 - 2x)^2 = (24 - 2x)(24 - 6x)$

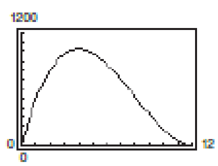
$= 12(12 - x)(4 - x) = 0$ when $x = 12, 4$ (12 is not in the domain).

$\dfrac{d^2V}{dx^2} = 12(2x - 16)$

$\dfrac{d^2V}{dx^2} < 0$ when $x = 4$.

When $x = 4, V = 1024$ is maximum.

(d)



The maximum volume seems to be 1024.

3. Let x and y be two positive numbers such that $x + y = S$.

$P = xy = x(S - x) = Sx - x^2$

$\dfrac{dP}{dx} = S - 2x = 0$ when $x = \dfrac{S}{2}$.

$\dfrac{d^2P}{dx^2} = -2 < 0$ when $x = \dfrac{S}{2}$.

P is a maximum when $x = y = S/2$.

4. Let x and y be two positive numbers such that $xy = 185$.

$S = x + y = x + \dfrac{185}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{185}{x^2} = 0$ when $x = \sqrt{185}$.

$\dfrac{d^2S}{dx^2} = \dfrac{370}{x^3} > 0$ when $x = \sqrt{185}$

S is a minimum when $x = y = \sqrt{185}$.

5. Let x and y be two positive numbers such that $xy = 147$.

$S = x + 3y = \dfrac{147}{y} + 3y$

$\dfrac{dS}{dy} = 3 - \dfrac{147}{y^2} = 0$ when $y = 7$.

$\dfrac{d^2S}{dy^2} = \dfrac{294}{y^3} > 0$ when $y = 7$.

S is minimum when $y = 7$ and $x = 21$.

6. Let x be a positive number.

$S = x + \dfrac{1}{x}$

$\dfrac{dS}{dx} = 1 - \dfrac{1}{x^2} = 0$ when $x = 1$.

$\dfrac{d^2S}{dx^2} = \dfrac{2}{x^3} > 0$ when $x = 1$.

The sum is a minimum when $x = 1$ and $1/x = 1$.

7. Let x and y be two positive numbers such that $x + 2y = 108$.

$$P = xy = y(108 - 2y) = 108y - 2y^2$$

$$\frac{dP}{dy} = 108 - 4y = 0 \text{ when } y = 27.$$

$$\frac{d^2P}{dy^2} = -4 < 0 \text{ when } y = 27.$$

P is a maximum when $x = 54$ and $y = 27$.

8. Let x and y be two positive numbers such that $x^2 + y = 54$.

$$P = xy = x\left(54 - x^2\right) = 54x - x^3$$

$$\frac{dP}{dx} = 54 - 3x^2 = 0 \text{ when } x = 3\sqrt{2}.$$

$$\frac{d^2P}{dx^2} = -6x < 0 \text{ when } x = 3\sqrt{2}.$$

The product is a maximum when $x = 3\sqrt{2}$ and $y = 36$.

9. Let x be the length and y the width of the rectangle.

$$2x + 2y = 80$$

$$y = 40 - x$$

$$A = xy = x(40 - x) = 40x - x^2$$

$$\frac{dA}{dx} = 40 - 2x = 0 \text{ when } x = 20.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = 20.$$

A is maximum when $x = y = 20$ meters.

10. Let x be the length and y the width of the rectangle.

$$2x + 2y = P$$

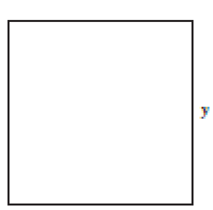
$$y = \frac{P - 2x}{2} = \frac{P}{2} - x$$

$$A = xy = x\left(\frac{P}{2} - x\right) = \frac{P}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{P}{2} - 2x = 0 \text{ when } x = \frac{P}{4}.$$

$$\frac{d^2A}{dx^2} = -2 < 0 \text{ when } x = \frac{P}{4}.$$

A is maximum when $x = y = P/4$ units. (A square!)



11. Let x be the length and y be the width of the rectangle.

$$xy = 49 \Rightarrow y = \frac{49}{x}$$

$$P = 2x + 2y$$

$$= 2x + 2\left(\frac{49}{x}\right) = 2x + \frac{98}{x}$$

$$\frac{dP}{dx} = 2 - \frac{98}{x^2} = 0$$

$$2x^2 = 98$$

$$x = \pm\sqrt{49} = 7$$

$$\frac{d^2P}{dx^2} = \frac{196}{x^3} > 0 \text{ when } x = 7.$$

When $x = 7, y = \frac{49}{7} = 7$.

P is minimum when the length and width are 7 feet.

12. Let x be the length and y the width of the rectangle.

$$xy = A$$

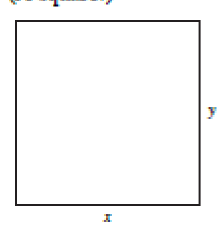
$$y = \frac{A}{x}$$

$$P = 2x + 2y = 2x + 2\left(\frac{A}{x}\right) = 2x + \frac{2A}{x}$$

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2} = 0 \text{ when } x = \sqrt{A}.$$

$$\frac{d^2P}{dx^2} = \frac{4A}{x^3} > 0 \text{ when } x = \sqrt{A}.$$

P is minimum when $x = y = \sqrt{A}$ centimeters. (A square!)



13. $d = \sqrt{(x-2)^2 + \left[x^2 - (1/2)\right]^2}$

$$= \sqrt{x^4 - 4x + (17/4)}$$

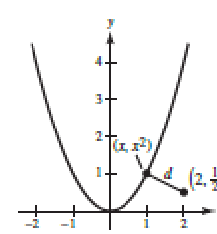
Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^4 - 4x + \tfrac{17}{4}.$$

$$f'(x) = 4x^3 - 4 = 0$$

$$x = 1$$

By the First Derivative Test, the point nearest to $\left(2, \frac{1}{2}\right)$ is $(1, 1)$.



14. $f(x) = (x-1)^2, (-5, 3)$

$$d = \sqrt{(x+5)^2 + \left[(x-1)^2 - 3\right]^2}$$
$$= \sqrt{(x^2+10x+25) + (x^2-2x-2)^2}$$
$$= \sqrt{(x^2+10x+25) + (x^4-4x^3+8x+4)}$$
$$= \sqrt{x^4-4x^3+x^2+18x+29}$$

Because d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^4 - 4x^3 + x^2 + 18x + 29$$
$$g'(x) = 4x^3 - 12x^2 + 2x + 18$$
$$= 2(x+1)(2x^2 - 8x + 9) = 0$$
$$x = -1$$

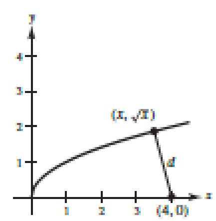
By the First Derivative Test, $x = -1$ yields a minimum. So, $(-1, 4)$ is closest to $(-5, 3)$.

15. $d = \sqrt{(x-4)^2 + (\sqrt{x} - 0)^2}$

$$= \sqrt{x^2 - 7x + 16}$$

Because d is smallest when the expression inside the radical is smallest, you need only find the critical numbers of

$$f(x) = x^2 - 7x + 16.$$
$$f'(x) = 2x - 7 = 0$$
$$x = \tfrac{7}{2}$$



By the First Derivative Test, the point nearest to $(4, 0)$ is $\left(7/2, \sqrt{7/2}\right)$.

16. $f(x) = \sqrt{x-8}, (12, 0)$

$$d = \sqrt{(x-12)^2 + \left(\sqrt{x-8} - 0\right)^2}$$
$$= \sqrt{x^2 - 24x + 144 + x - 8}$$
$$= \sqrt{x^2 - 23x + 136}$$

Because d is smallest when the expression inside the radical is smallest, you need to find the critical numbers of

$$g(x) = x^2 - 23x + 136$$
$$g'(x) = 2x - 23 = 0 \text{ when } x = \frac{23}{2}$$
$$g''(x) = 2 > 0 \text{ at } x = \frac{23}{2}$$

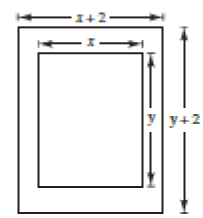
The point nearest to $(12, 0)$ is

$$\left(\frac{23}{2}, f\left(\frac{23}{2}\right)\right) = \left(\frac{23}{2}, \frac{\sqrt{14}}{2}\right)$$

17. $xy = 30 \Rightarrow y = \dfrac{30}{x}$

$$A = (x+2)\left(\frac{30}{x} + 2\right) \text{ (see figure)}$$
$$\frac{dA}{dx} = (x+2)\left(\frac{-30}{x^2}\right) + \left(\frac{30}{x} + 2\right)$$
$$= \frac{2(x^2 - 30)}{x^2} = 0 \text{ when } x = \sqrt{30}.$$
$$y = \frac{30}{\sqrt{30}} = \sqrt{30}$$

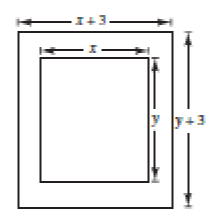
By the First Derivative Test, the dimensions $(x+2)$ by $(y+2)$ are $\left(2+\sqrt{30}\right)$ by $\left(2+\sqrt{30}\right)$ (approximately 7.477 by 7.477). These dimensions yield a minimum area.



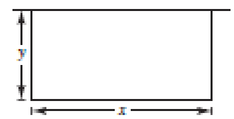
18. $xy = 36 \Rightarrow y = \dfrac{36}{x}$

$$A = (x+3)(y+3) = (x+3)\left(\frac{36}{x} + 3\right)$$
$$= 36 + \frac{108}{x} + 3x + 9$$
$$\frac{dA}{dx} = \frac{-108}{x^2} + 3 = 0 \Rightarrow 3x^2 = 108 \Rightarrow x = 6, y = 6$$

Dimensions: 9×9



19. $$xy = 245{,}000 \text{ (see figure)}$$



$$S = x + 2y$$

$$= \left(x + \frac{490{,}000}{x}\right) \text{ where } S \text{ is the length of fence needed.}$$

$$\frac{dS}{dx} = 1 - \frac{490{,}000}{x^2} = 0 \text{ when } x = 700.$$

$$\frac{d^2S}{dx^2} = \frac{980{,}000}{x^3} > 0 \text{ when } x = 700.$$

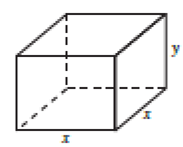
S is a minimum when $x = 700$ meters and $y = 350$ meters.

20. $$S = 2x^2 + 4xy = 337.5$$

$$y = \frac{337.5 - 2x^2}{4x}$$

$$V = x^2y = x^2\left[\frac{337.5 - 2x^2}{4x}\right] = 84.375x - \frac{1}{2}x^3$$

$$\frac{dV}{dx} = 84.375 - \frac{3}{2}x^2 = 0 \Rightarrow x^2 = 56.25 \Rightarrow x = 7.5 \text{ and } y = 7.5.$$



$$\frac{d^2V}{dx^2} = -3x < 0 \text{ for } x = 7.5.$$

The maximum value occurs when $x = y = 7.5$ centimeters.

21. $$16 = 2y + x + \pi\left(\frac{x}{2}\right)$$

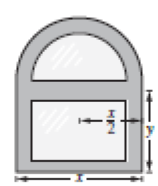
$$32 = 4y + 2x + \pi x$$

$$y = \frac{32 - 2x - \pi x}{4}$$

$$A = xy + \frac{\pi}{2}\left(\frac{x}{2}\right)^2 = \left(\frac{32 - 2x - \pi x}{4}\right)x + \frac{\pi x^2}{8} = 8x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\frac{dA}{dx} = 8 - x - \frac{\pi}{2}x + \frac{\pi}{4}x = 8 - x\left(1 + \frac{\pi}{4}\right) = 0 \text{ when } x = \frac{8}{1 + (\pi/4)} = \frac{32}{4 + \pi}.$$

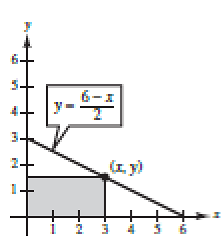
$$\frac{d^2A}{dx^2} = -\left(1 + \frac{\pi}{4}\right) < 0 \text{ when } x = \frac{32}{4 + \pi}.$$



$$y = \frac{32 - 2[32/(4 + \pi)] - \pi[32/(4 + \pi)]}{4} = \frac{16}{4 + \pi}$$

The area is maximum when $y = \dfrac{16}{4 + \pi}$ feet and $x = \dfrac{32}{4 + \pi}$ feet.

22. You can see from the figure that $A = xy$ and $y = \dfrac{6 - x}{2}$.



$$A = x\left(\frac{6 - x}{2}\right) = \frac{1}{2}(6x - x^2).$$

$$\frac{dA}{dx} = \frac{1}{2}(6 - 2x) = 0 \text{ when } x = 3.$$

$$\frac{d^2A}{dx^2} = -1 < 0 \text{ when } x = 3.$$

A is a maximum when $x = 3$ and $y = 3/2$.

23. (a) $\dfrac{y-2}{0-1} = \dfrac{0-2}{x-1}$

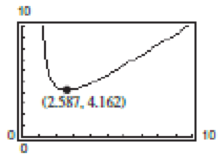
$$y = 2 + \frac{2}{x-1}$$

$$L = \sqrt{x^2 + y^2}$$

$$= \sqrt{x^2 + \left(2 + \frac{2}{x-1}\right)^2}$$

$$= \sqrt{x^2 + 4 + \frac{8}{x-1} + \frac{4}{(x-1)^2}}, \quad x > 1$$

(b)



L is minimum when $x \approx 2.587$ and $L \approx 4.162$.

(c) Area $= A(x) = \dfrac{1}{2}xy = \dfrac{1}{2}x\left(2 + \dfrac{2}{x-1}\right) = x + \dfrac{x}{x-1}$

$$A'(x) = 1 + \frac{(x-1) - x}{(x-1)^2} = 1 - \frac{1}{(x-1)^2} = 0$$

$$(x-1)^2 = 1$$

$$x - 1 = \pm 1$$

$$x = 0, 2 \text{ (select } x = 2)$$

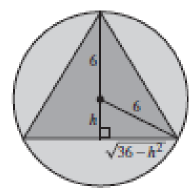
Then $y = 4$ and $A = 4$.

Vertices: $(0, 0)$, $(2, 0)$, $(0, 4)$

24. (a) $A = \dfrac{1}{2}\text{base} \times \text{height} = \dfrac{1}{2}\left(2\sqrt{36 - h^2}\right)(6 + h) = \sqrt{36 - h^2}(6 + h)$

$$\frac{dA}{dh} = \frac{1}{2}(36 - h^2)^{-1/2}(-2h)(6 + h) + (36 - h^2)^{1/2}$$

$$= (36 - h^2)^{-1/2}\left[-h(6 + h) + (36 - h^2)\right] = \frac{-2(h^2 + 3h - 18)}{\sqrt{36 - h^2}} = \frac{-2(h + 6)(h - 3)}{\sqrt{36 - h^2}}$$



$\dfrac{dA}{dh} = 0$ when $h = 3$, which is a maximum by the First Derivative Test.

So, the sides are $2\sqrt{36 - h^2} = 6\sqrt{3}$, an equilateral triangle.

Area $= 27\sqrt{3}$ sq. units

(b) $\cos\alpha = \dfrac{6 + h}{2\sqrt{3}\sqrt{6 + h}} = \dfrac{\sqrt{6 + h}}{2\sqrt{3}}$

$$\tan\alpha = \frac{\sqrt{36 - h^2}}{6 + h}$$

$$\text{Area} = 2\left(\frac{1}{2}\right)\left(\sqrt{36 - h^2}\right)(6 + h) = (6 + h)^2 \tan\alpha = 144\cos^4\alpha \tan\alpha$$

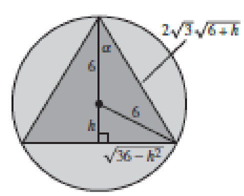
$$A'(\alpha) = 144\left[\cos^4\alpha \sec^2\alpha + 4\cos^3(-\sin\alpha)\tan\alpha\right] = 0$$

$$\Rightarrow \cos^4\alpha \sec^2\alpha = 4\cos^3\alpha \sin\alpha \tan\alpha$$

$$1 = 4\cos\alpha \sin\alpha \tan\alpha$$

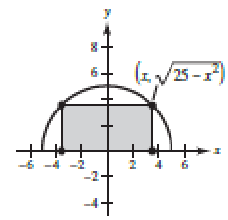
$$\frac{1}{4} = \sin^2\alpha$$

$$\sin\alpha = \frac{1}{2} \Rightarrow \alpha = 30^\circ \text{ and } A = 27\sqrt{3}$$



(c) Equilateral triangle

25. $A = 2xy = 2x\sqrt{25 - x^2}$ (see figure)



$$\frac{dA}{dx} = 2x\left(\frac{1}{2}\right)\left(\frac{-2x}{\sqrt{25 - x^2}}\right) + 2\sqrt{25 - x^2} = 2\left(\frac{25 - 2x^2}{\sqrt{25 - x^2}}\right) = 0 \text{ when } x = y = \frac{5\sqrt{2}}{2} \approx 3.54.$$

By the First Derivative Test, the inscribed rectangle of maximum area has vertices

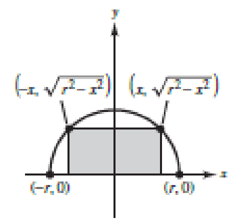
$$\left(\pm\frac{5\sqrt{2}}{2}, 0\right), \left(\pm\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right).$$

Width: $\frac{5\sqrt{2}}{2}$; Length: $5\sqrt{2}$

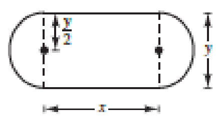
26. $A = 2xy = 2x\sqrt{r^2 - x^2}$ (see figure)

$$\frac{dA}{dx} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} = 0 \text{ when } x = \frac{\sqrt{2}r}{2}.$$

By the First Derivative Test, A is maximum when the rectangle has dimensions $\sqrt{2}r$ by $(\sqrt{2}r)/2$.



27. (a) $P = 2x + 2\pi r = 2x + 2\pi\left(\frac{y}{2}\right) = 2x + \pi y = 200 \Rightarrow y = \frac{200 - 2x}{\pi} = \frac{2}{\pi}(100 - x)$



(b)

Length, x	Width, y	Area, xy
10	$\frac{2}{\pi}(100 - 10)$	$(10)\frac{2}{\pi}(100 - 10) \approx 573$
20	$\frac{2}{\pi}(100 - 20)$	$(20)\frac{2}{\pi}(100 - 20) \approx 1019$
30	$\frac{2}{\pi}(100 - 30)$	$(30)\frac{2}{\pi}(100 - 30) \approx 1337$
40	$\frac{2}{\pi}(100 - 40)$	$(40)\frac{2}{\pi}(100 - 40) \approx 1528$
50	$\frac{2}{\pi}(100 - 50)$	$(50)\frac{2}{\pi}(100 - 50) \approx 1592$
60	$\frac{2}{\pi}(100 - 60)$	$(60)\frac{2}{\pi}(100 - 60) \approx 1528$

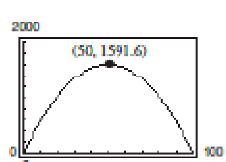
The maximum area of the rectangle is approximately 1592 square meters.

(c) $A = xy = x\frac{2}{\pi}(100 - x) = \frac{2}{\pi}(100x - x^2)$

(d) $A' = \frac{2}{\pi}(100 - 2x)$. $A' = 0$ when $x = 50$.

The maximum value is approximately 1592 when the length is 50 meters and the width is $\frac{100}{\pi}$ meters.

(e)



The maximum area is approximately 1591.55 square meters ($x = 50$ meters).

28. $V = \pi r^2 h = 22$ cubic inches or $h = \dfrac{22}{\pi r^2}$

(a)

Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$

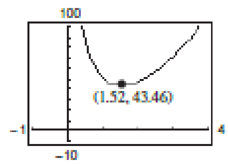
(b)

Radius, r	Height	Surface Area
0.2	$\dfrac{22}{\pi(0.2)^2}$	$2\pi(0.2)\left[0.2 + \dfrac{22}{\pi(0.2)^2}\right] \approx 220.3$
0.4	$\dfrac{22}{\pi(0.4)^2}$	$2\pi(0.4)\left[0.4 + \dfrac{22}{\pi(0.4)^2}\right] \approx 111.0$
0.6	$\dfrac{22}{\pi(0.6)^2}$	$2\pi(0.6)\left[0.6 + \dfrac{22}{\pi(0.6)^2}\right] \approx 75.6$
0.8	$\dfrac{22}{\pi(0.8)^2}$	$2\pi(0.8)\left[0.8 + \dfrac{22}{\pi(0.8)^2}\right] \approx 59.0$
1.0	$\dfrac{22}{\pi(1.0)^2}$	$2\pi(1.0)\left[1.0 + \dfrac{22}{\pi(1.0)^2}\right] \approx 50.3$
1.2	$\dfrac{22}{\pi(1.2)^2}$	$2\pi(1.2)\left[1.2 + \dfrac{22}{\pi(1.2)^2}\right] \approx 45.7$
1.4	$\dfrac{22}{\pi(1.4)^2}$	$2\pi(1.4)\left[1.4 + \dfrac{22}{\pi(1.4)^2}\right] \approx 43.7$
1.6	$\dfrac{22}{\pi(1.6)^2}$	$2\pi(1.6)\left[1.6 + \dfrac{22}{\pi(1.6)^2}\right] \approx 43.6$
1.8	$\dfrac{22}{\pi(1.8)^2}$	$2\pi(1.8)\left[1.8 + \dfrac{22}{\pi(1.8)^2}\right] \approx 44.8$
2.0	$\dfrac{22}{\pi(2.0)^2}$	$2\pi(2.0)\left[2.0 + \dfrac{22}{\pi(2.0)^2}\right] \approx 47.1$

The minimum seems to the about 43.6 for $r = 1.6$.

(c) $S = 2\pi r^2 + 2\pi rh$

$$= 2\pi r(r + h) = 2\pi r\left[r + \frac{22}{\pi r^2}\right] = 2\pi r^2 + \frac{44}{r}$$

(d) 

The minimum seems to be 43.46 for $r \approx 1.52$.

(e) $\frac{dS}{dr} = 4\pi r - \frac{44}{r^2} = 0$ when $r = \sqrt[3]{11/\pi} \approx 1.52$ in.

$h = \frac{22}{\pi r^2} \approx 3.04$ in.

Note: Notice that $h = \frac{22}{\pi r^2} = \frac{22}{\pi\left(11/\pi\right)^{2/3}} = 2\left(\frac{11^{1/3}}{\pi^{1/3}}\right) = 2r$.

29. Let x be the sides of the square ends and y the length of the package.

$$P = 4x + y = 108 \Rightarrow y = 108 - 4x$$

$$V = x^2y = x^2(108 - 4x) = 108x^2 - 4x^3$$

$$\frac{dV}{dx} = 216x - 12x^2$$

$$= 12x(18 - x) = 0 \text{ when } x = 18.$$

$$\frac{d^2V}{dx^2} = 216 - 24x = -216 < 0 \text{ when } x = 18.$$

The volume is maximum when $x = 18$ inches and $y = 108 - 4(18) = 36$ inches.

30. $$V = \pi r^2 x$$

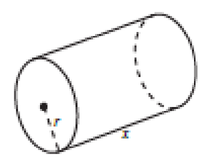
$$x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r \text{ (see figure)}$$

$$V = \pi r^2(108 - 2\pi r) = \pi\left(108r^2 - 2\pi r^3\right)$$

$$\frac{dV}{dr} = \pi\left(216r - 6\pi r^2\right) = 6\pi r(36 - \pi r)$$

$$= 0 \text{ when } r = \frac{36}{\pi} \text{ and } x = 36.$$

$$\frac{d^2V}{dr^2} = \pi(216 - 12\pi r) < 0 \text{ when } r = \frac{36}{\pi}.$$



Volume is maximum when $x = 36$ inches and $r = 36/\pi \approx 11.459$ inches.

31. No. The volume will change because the shape of the container changes when squeezed.

32. No, there is no minimum area. If the sides are x and y, then $2x + 2y = 20 \Rightarrow y = 10 - x$. The area is $A(x) = x(10 - x) = 10x - x^2$. This can be made arbitrarily small by selecting $x \approx 0$.

33. $$V = 14 = \frac{4}{3}\pi r^3 + \pi r^2 h$$

$$h = \frac{14 - (4/3)\pi r^3}{\pi r^2} = \frac{14}{\pi r^2} - \frac{4}{3}r$$

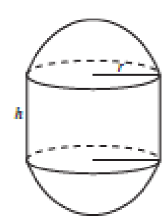
$$S = 4\pi r^2 + 2\pi rh = 4\pi r^2 + 2\pi r\left(\frac{14}{\pi r^2} - \frac{4}{3}r\right) = 4\pi r^2 + \frac{28}{r} - \frac{8}{3}\pi r^2 = \frac{4}{3}\pi r^2 + \frac{28}{r}$$

$$\frac{dS}{dr} = \frac{8}{3}\pi r - \frac{28}{r^2} = 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}} \approx 1.495 \text{ cm}$$

$$\frac{d^2S}{dr^2} = \frac{8}{3}\pi + \frac{56}{r^3} > 0 \text{ when } r = \sqrt[3]{\frac{21}{2\pi}}.$$

The surface area is minimum when $r = \sqrt[3]{\frac{21}{2\pi}}$ centimeters and $h = 0$.

The resulting solid is a sphere of radius $r \approx 1.495$ centimeters.



34. $V = 4000 = \frac{4}{3}\pi r^3 + \pi r^2 h$

$$h = \frac{4000}{\pi r^2} - \frac{4}{3}r$$

Let k = cost per square foot of the surface area of the sides, then $2k$ = cost per square foot of the hemispherical ends.

$$C = 2k(4\pi r^2) + k(2\pi rh) = k\left[8\pi r^2 + 2\pi r\left(\frac{4000}{\pi r^2} - \frac{4}{3}r\right)\right] = k\left[\frac{16}{3}\pi r^2 + \frac{8000}{r}\right]$$

$\frac{dC}{dr} = k\left[\frac{32}{3}\pi r - \frac{8000}{r^2}\right] = 0$ when $r = \sqrt[3]{\frac{750}{\pi}} \approx 6.204$ ft and $h \approx 24.814$ ft

By the Second Derivative Test, you have $\frac{d^2C}{dr^2} = k\left[\frac{32}{3}\pi + \frac{12{,}000}{r^3}\right] > 0$ when $r = \sqrt[3]{\frac{750}{\pi}}$.

The cost is minimum when $r = \sqrt[3]{\frac{750}{\pi}}$ feet and $h \approx 24.814$ feet.

35. Let x be the length of a side of the square and y the length of a side of the triangle.

$$4x + 3y = 10$$

$$A = x^2 + \frac{1}{2}y\left(\frac{\sqrt{3}}{2}y\right)$$

$$= \frac{(10 - 3y)^2}{16} + \frac{\sqrt{3}}{4}y^2$$

$$\frac{dA}{dy} = \frac{1}{8}(10 - 3y)(-3) + \frac{\sqrt{3}}{2}y = 0$$

$$-30 + 9y + 4\sqrt{3}y = 0$$

$$y = \frac{30}{9 + 4\sqrt{3}}$$

$$\frac{d^2A}{dy^2} = \frac{9 + 4\sqrt{3}}{8} > 0$$

A is minimum when $y = \frac{30}{9 + 4\sqrt{3}}$ and $x = \frac{10\sqrt{3}}{9 + 4\sqrt{3}}$.

36. (a) Let x be the side of the triangle and y the side of the square.

$$A = \frac{3}{4}\left(\cot\frac{\pi}{3}\right)x^2 + \frac{4}{4}\left(\cot\frac{\pi}{4}\right)y^2 \text{ where } 3x + 4y = 20$$

$$= \frac{\sqrt{3}}{4}x^2 + \left(5 - \frac{3}{4}x\right)^2, \ 0 \le x \le \frac{20}{3}.$$

$$A' = \frac{\sqrt{3}}{2}x + 2\left(5 - \frac{3}{4}x\right)\left(-\frac{3}{4}\right) = 0$$

$$x = \frac{60}{4\sqrt{3} + 9}$$

When $x = 0$, $A = 25$; when $x = 60/(4\sqrt{3} + 9)$, $A \approx 10.847$; and when $x = 20/3$, $A \approx 19.245$. Area is maximum when all 20 feet are used on the square.

(b) Let x be the side of the square and y the side of the pentagon.

$$A = \frac{4}{4}\left(\cot\frac{\pi}{4}\right)x^2 + \frac{5}{4}\left(\cot\frac{\pi}{5}\right)y^2 \text{ where } 4x + 5y = 20$$

$$= x^2 + 1.7204774\left(4 - \frac{4}{5}x\right)^2, 0 \le x \le 5.$$

$$A' = 2x - 2.75276384\left(4 - \frac{4}{5}x\right) = 0$$

$$x \approx 2.62$$

When $x = 0$, $A \approx 27.528$; when $x \approx 2.62$, $A \approx 13.102$; and when $x = 5$, $A \approx 25$. Area is maximum when all 20 feet are used on the pentagon.

(c) Let x be the side of the pentagon and y the side of the hexagon.

$$A = \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{6}{4}\left(\cot\frac{\pi}{6}\right)y^2 \text{ where } 5x + 6y = 20$$

$$= \frac{5}{4}\left(\cot\frac{\pi}{5}\right)x^2 + \frac{3}{2}\left(\sqrt{3}\right)\left(\frac{20 - 5x}{6}\right)^2, 0 \le x \le 4.$$

$$A' = \frac{5}{2}\left(\cot\frac{\pi}{5}\right)x + 3\sqrt{3}\left(-\frac{5}{6}\right)\left(\frac{20 - 5x}{6}\right) = 0$$

$$x \approx 2.0475$$

When $x = 0$, $A \approx 28.868$; when $x \approx 2.0475$, $A \approx 14.091$; and when $x = 4$, $A \approx 27.528$. Area is maximum when all 20 feet are used on the hexagon.

(d) Let x be the side of the hexagon and r the radius of the circle.

$$A = \frac{6}{4}\left(\cot\frac{\pi}{6}\right)x^2 + \pi r^2 \text{ where } 6x + 2\pi r = 20$$

$$= \frac{3\sqrt{3}}{2}x^2 + \pi\left(\frac{10}{\pi} - \frac{3x}{\pi}\right)^2, 0 \le x \le \frac{10}{3}.$$

$$A' = 3\sqrt{3} - 6\left(\frac{10}{\pi} - \frac{3x}{\pi}\right) = 0$$

$$x \approx 1.748$$

When $x = 0$, $A \approx 31.831$; when $x \approx 1.748$, $A \approx 15.138$; and when $x = 10/3$, $A \approx 28.868$. Area is maximum when all 20 feet are used on the circle.

In general, using all of the wire for the figure with more sides will enclose the most area.

37. Let S be the strength and k the constant of proportionality. Given

$h^2 + w^2 = 20^2, h^2 = 20^2 - w^2,$

$$S = kwh^2$$

$$S = kw\left(400 - w^2\right) = k\left(400w - w^3\right)$$

$$\frac{dS}{dw} = k\left(400 - 3w^2\right) = 0 \text{ when } w = \frac{20\sqrt{3}}{3} \text{ in.}$$

and $h = \dfrac{20\sqrt{6}}{3}$ in.

$$\frac{d^2S}{dw^2} = -6kw < 0 \text{ when } w = \frac{20\sqrt{3}}{3}.$$

These values yield a maximum.

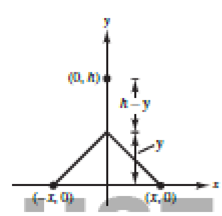
38. Let A be the amount of the power line.

$$A = h - y + 2\sqrt{x^2 + y^2}$$

$$\frac{dA}{dy} = -1 + \frac{2y}{\sqrt{x^2 + y^2}} = 0 \text{ when } y = \frac{x}{\sqrt{3}}.$$

$$\frac{d^2A}{dy^2} = \frac{2x^2}{\left(x^2 + y^2\right)^{3/2}} > 0 \text{ for } y = \frac{x}{\sqrt{3}}.$$

The amount of power line is minimum when $y = x/\sqrt{3}$.



39. 

$$C(x) = 2k\sqrt{x^2+4} + k(4-x)$$

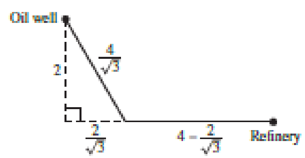
$$C'(x) = \frac{2xk}{\sqrt{x^2+4}} - k = 0$$

$$2x = \sqrt{x^2+4}$$

$$4x^2 = x^2 + 4$$

$$3x^2 = 4$$

$$x = \frac{2}{\sqrt{3}}$$



The path of the pipe should go underwater from the oil well to the coast following the hypotenuse of a right triangle with leg lengths of 2 kilometers and $2/\sqrt{3}$ kilometers for a distance of $4/\sqrt{3}$ kilometers. Then the pipe should go down the coast to the refinery for a distance of $\left(4 - 2/\sqrt{3}\right)$ kilometers.

40. $\sin\alpha = \dfrac{h}{s} \Rightarrow s = \dfrac{h}{\sin\alpha},\ 0 < \alpha < \dfrac{\pi}{2}$

$$\tan\alpha = \frac{h}{2} \Rightarrow h = 2\tan\alpha \Rightarrow s = \frac{2\tan\alpha}{\sin\alpha} = 2\sec\alpha$$

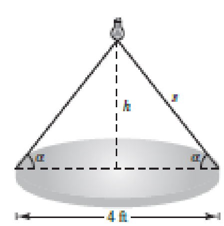
$$I = \frac{k\sin\alpha}{s^2} = \frac{k\sin\alpha}{4\sec^2\alpha} = \frac{k}{4}\sin\alpha\cos^2\alpha$$

$$\frac{dI}{d\alpha} = \frac{k}{4}\left[\sin\alpha(-2\sin\alpha\cos\alpha) + \cos^2\alpha(\cos\alpha)\right]$$

$$= \frac{k}{4}\cos\alpha\left[\cos^2\alpha - 2\sin^2\alpha\right]$$

$$= \frac{k}{4}\cos\alpha\left[1 - 3\sin^2\alpha\right]$$

$$= 0 \text{ when } \alpha = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ or when } \sin\alpha = \pm\frac{1}{\sqrt{3}}.$$



Because α is acute, you have $\sin\alpha = \dfrac{1}{\sqrt{3}} \Rightarrow h = 2\tan\alpha = 2\left(\dfrac{1}{\sqrt{2}}\right) = \sqrt{2}$ ft.

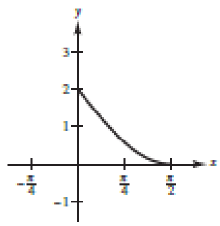
Because $\left(d^2I\right)/\left(d\alpha^2\right) = (k/4)\sin\alpha\left(9\sin^2\alpha - 7\right) < 0$ when $\sin\alpha = 1/\sqrt{3}$, this yields a maximum.

41. $p(t) = \dfrac{250}{1 + 4e^{-t/3}}$

$$p'(t) = \frac{1000}{3}\frac{e^{-t/3}}{\left(1 + 4e^{-t/3}\right)^2};\ p'(2) \approx 18.35 \text{ elk/month}$$

$$p''(t) = \frac{1000}{9}\frac{e^{-t/3}\left(4e^{-t/3} - 1\right)}{\left(1 + 4e^{-t/3}\right)^3} = 0 \text{ when } t \approx 4.16 \text{ months}$$

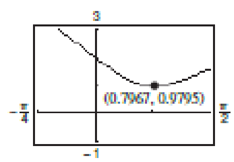
42. $f(x) = 2 - 2\sin x$



(a) Distance from origin to y-intercept is 2.

Distance from origin to x-intercept is $\pi/2 \approx 1.57$.

(b) $d = \sqrt{x^2 + y^2} = \sqrt{x^2 + (2 - 2\sin x)^2}$



Minimum distance $= 0.9795$ at $x = 0.7967$.

(c) Let $f(x) = d^2(x) = x^2 + (2 - 2\sin x)^2$.

$f'(x) = 2x + 2(2 - 2\sin x)(-2\cos x)$

Setting $f'(x) = 0$, you obtain $x \approx 0.7967$, which corresponds to $d = 0.9795$.

43. $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\sqrt{144 - r^2}$

$$\frac{dV}{dr} = \frac{1}{3}\pi\left[r^2\left(\frac{1}{2}\right)(144 - r^2)^{-1/2}(-2r) + 2r\sqrt{144 - r^2}\right]$$

$$= \frac{1}{3}\pi\left[\frac{288r - 3r^3}{\sqrt{144 - r^2}}\right]$$

$$= \pi\left[\frac{r(96 - r^2)}{\sqrt{144 - r^2}}\right] = 0 \text{ when } r = 0, 4\sqrt{6}.$$

By the First Derivative Test, V is maximum when $r = 4\sqrt{6}$ and $h = 4\sqrt{3}$.

Area of circle: $A = \pi(12)^2 = 144\pi$

Lateral surface area of cone:

$$S = \pi\left(4\sqrt{6}\right)\sqrt{\left(4\sqrt{6}\right)^2 + \left(4\sqrt{3}\right)^2} = 48\sqrt{6}\pi$$

Area of sector:

$$144\pi - 48\sqrt{6}\pi = \frac{1}{2}\theta r^2 = 72\theta$$

$$\theta = \frac{144\pi - 48\sqrt{6}\pi}{72}$$

$$= \frac{2\pi}{3}\left(3 - \sqrt{6}\right) \approx 1.153 \text{ radians or } 66°$$

44. (a)

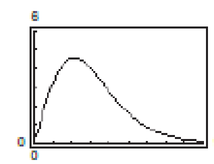
$$f(c) = f(c + x)$$
$$10ce^{-c} = 10(c + x)e^{-(c+x)}$$
$$\frac{c}{e^c} = \frac{c + x}{e^{c+x}}$$
$$ce^{c+x} = (c + x)e^c$$
$$ce^x = c + x$$
$$ce^x - c = x$$
$$c = \frac{x}{e^x - 1}$$

(b) $A(x) = xf(c)$

$$= x\left[10\left(\frac{x}{e^x - 1}\right)e^{-x/(e^x - 1)}\right]$$

$$= \frac{10x^2}{e^x - 1}e^{x/(1 - e^x)}$$

(c) $A(x) = \dfrac{10x^2}{e^x - 1}e^{x/(1 - e^x)}$

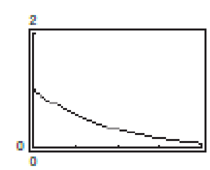


The maximum area is 4.591 for $x = 2.118$ and $f(x) = 2.547$.

(d) $c = \dfrac{x}{e^x - 1}$

$\lim_{x \to 0^+} c = 1$

$\lim_{x \to \infty} c = 0$



45. Let d be the amount deposited in the bank, i be the interest rate paid by the bank, and P be the profit.

$P = (0.12)d - id$

$d = ki^2$ (because d is proportional to i^2)

$P = (0.12)(ki^2) - i(ki^2) = k(0.12i^2 - i^3)$

$\frac{dP}{di} = k(0.24i - 3i^2) = 0$ when $i = \frac{0.24}{3} = 0.08$.

$\frac{d^2P}{di^2} = k(0.24 - 6i) < 0$ when $i = 0.08$ (**Note:** $k > 0$

The profit is a maximum when $i = 8\%$.

46. (a) The profit is increasing on $(0, 40)$.

(b) The profit is decreasing on $(40, 60)$.

(c) In order to yield a maximum profit, the company should spend about \$40 thousand.

(d) The point of diminishing returns is the point where the concavity changes, which in this case is $x = 20$ thousand dollars.

47. $y = \dfrac{L}{1 + ae^{-x/b}}, a > 0, b > 0, L > 0$

$$y' = \frac{-L\left(-\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^2} = \frac{\frac{aL}{b}e^{-x/b}}{\left(1 + ae^{-x/b}\right)^2}$$

$$y'' = \frac{\left(1 + ae^{-x/b}\right)^2\left(\frac{-aL}{b^2}e^{-x/b}\right) - \left(\frac{aL}{b}e^{-x/b}\right)2\left(1 + ae^{-x/b}\right)\left(\frac{-a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^4}$$

$$= \frac{\left(1 + ae^{-x/b}\right)\left(\frac{-aL}{b^2}e^{-x/b}\right) + 2\left(\frac{aL}{b}e^{-x/b}\right)\left(\frac{a}{b}e^{-x/b}\right)}{\left(1 + ae^{-x/b}\right)^3} = \frac{Lae^{-x/b}\left(ae^{-x/b} - 1\right)}{\left(1 + ae^{-x/b}\right)^3 b^2}$$

$$y'' = 0 \text{ if } ae^{-x/b} = 1 \Rightarrow \frac{-x}{b} = \ln\left(\frac{1}{a}\right) \Rightarrow x = b \ln a$$

$$y(b \ln a) = \frac{L}{1 + ae^{-(b \ln a)/b}} = \frac{L}{1 + a(1/a)} = \frac{L}{2}$$

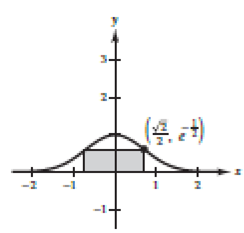
Therefore, the y-coordinate of the inflection point is $L/2$.

48. $A = (\text{base})(\text{height}) = 2xe^{-x^2}$

$$\frac{dA}{dx} = -4x^2e^{-x^2} + 2e^{-x^2}$$

$$= 2e^{-x^2}\left(1 - 2x^2\right) = 0 \text{ when } x = \frac{\sqrt{2}}{2}$$

$$A = \sqrt{2}e^{-1/2}$$



49. $S_1 = (4m - 1)^2 + (5m - 6)^2 + (10m - 3)^2$

$$\frac{dS_1}{dm} = 2(4m - 1)(4) + 2(5m - 6)(5) + 2(10m - 3)(10)$$

$$= 282m - 128 = 0 \text{ when } m = \frac{64}{141}$$

Line: $y = \dfrac{64}{141}x$

$$S = \left|4\left(\frac{64}{141}\right) - 1\right| + \left|5\left(\frac{64}{141}\right) - 6\right| + \left|10\left(\frac{64}{141}\right) - 3\right|$$

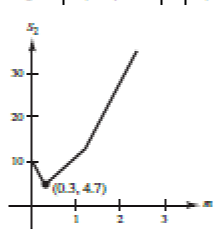
$$= \left|\frac{256}{141} - 1\right| + \left|\frac{320}{141} - 6\right| + \left|\frac{640}{141} - 3\right| = \frac{858}{141} \approx 6.1 \text{ mi}$$

50. $S_2 = |4m - 1| + |5m - 6| + |10m - 3|$

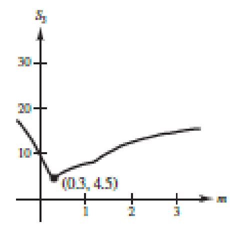
Using a graphing utility, you can see that the minimum occurs when $m = 0.3$.

Line: $y = 0.3x$

$$S_2 = |4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3| = 4.7 \text{ mi}$$



51. $S_3 = \dfrac{|4m - 1|}{\sqrt{m^2 + 1}} + \dfrac{|5m - 6|}{\sqrt{m^2 + 1}} + \dfrac{|10m - 3|}{\sqrt{m^2 + 1}}$

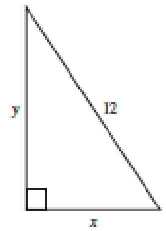


Using a graphing utility, you can see that the minimum occurs when $x \approx 0.3$.

Line: $y \approx 0.3x$

$$S_3 = \frac{|4(0.3) - 1| + |5(0.3) - 6| + |10(0.3) - 3|}{\sqrt{(0.3)^2 + 1}} \approx 4.5 \text{ mi}$$

52.



$$x^2 + y^2 = 12^2$$
$$y^2 = 144 - x^2$$
$$y = \pm\sqrt{144 - x^2} \quad \left(\text{Because } y > 0, \text{ use } y = \sqrt{144 - x^2}.\right)$$

Let A be the area to be maximized.

$$A = \frac{1}{2}xy$$
$$= \frac{1}{2}x\sqrt{144 - x^2},\ 0 < x < 12$$

$$\frac{dA}{dx} = \frac{1}{2}x\left[\frac{1}{2}(144 - x^2)^{-1/2} \cdot (-2x)\right] + \frac{1}{2}\sqrt{144 - x^2}$$
$$= \frac{\sqrt{144 - x^2}}{2} - \frac{x^2}{2\sqrt{144 - x^2}}$$

$$\frac{dA}{dx} = 0$$

$$\frac{\sqrt{144 - x^2}}{2} = \frac{x^2}{2\sqrt{144 - x^2}}$$
$$(144 - x^2) = x^2$$
$$2x^2 = 144$$
$$x = \pm\sqrt{72} = \pm 6\sqrt{2} \quad \left(\text{Because } x > 0, \text{ use } x = 6\sqrt{2}.\right)$$
$$y = \sqrt{144 - \left(6\sqrt{2}\right)^2} = \sqrt{72} = 6\sqrt{2}$$

The maximum area is $A = \frac{1}{2}\left(6\sqrt{2}\right)\left(6\sqrt{2}\right) = 36$ square units.

So, the answer is C.

53. (a) $r^2 + h^2 = 6^2$

$$h^2 = \sqrt{36 - r^2}$$

$$h = \pm\sqrt{36 - r^2} \quad \left(\text{Because } h > 0, \text{ use } h = \sqrt{36 - r^2}.\right)$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2\left(\sqrt{36 - r^2}\right)$$

So, the formula is $V = \frac{\pi}{3}r^2\sqrt{36 - r^2}$.

(b)

$$V = \frac{\pi}{3}r^2\sqrt{36 - r^2}$$

$$\frac{dV}{dr} = \frac{\pi}{3}r^2\left[\frac{1}{2}\left(36 - r^2\right)^{-1/2}(-2r)\right] + \frac{\pi}{3}(2r)\sqrt{36 - r^2} = \frac{2\pi r}{3}\sqrt{36 - r^2} - \frac{\pi r^3}{3\sqrt{36 - r^2}}$$

$$\frac{dV}{dr} = 0$$

$$\frac{2\pi r}{3}\sqrt{36 - r^2} = \frac{\pi r^3}{3\sqrt{36 - r^2}}$$

$$2\left(36 - r^2\right) = r^2$$

$$72 = 3r^2$$

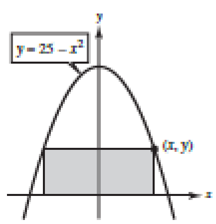
$$24 = r^2$$

$$\pm 2\sqrt{6} = r \quad \left(\text{Because } r > 0, \text{ use } r = 2\sqrt{6}.\right)$$

$$h = \sqrt{36 - r^2} = \sqrt{36 - \left(2\sqrt{6}\right)^2} = \sqrt{12} = 2\sqrt{3}$$

So, the volume of the cone is maximized when $r = 2\sqrt{6}$ units and $h = 2\sqrt{3}$ units.

54.



(a) Let the length be $2x$, the width be y, and A be the area to be maximized.

$$A = lw = 2xy$$

$$= 2x\left(25 - x^2\right)$$

$$= 50x - 2x^3$$

$$\frac{dA}{dx} = 50 - 6x^2$$

$$\frac{dA}{dx} = 0$$

$$50 - 6x^2 = 0$$

$$6x^2 = 50$$

$$x = \pm\frac{5\sqrt{3}}{3}$$

So, the length is $2x = 2\left(\frac{5\sqrt{3}}{3}\right) = \frac{10\sqrt{3}}{3}$ units and the width is $y = 25 - \left(\frac{5\sqrt{3}}{3}\right)^2 = 25 - \frac{25}{3} = \frac{50}{3}$ units.

The maximum area is $A = \left(\frac{10\sqrt{3}}{3}\right)\left(\frac{50}{3}\right) = \frac{500\sqrt{3}}{9}$ square units.

(b) $$\begin{aligned} P &= 2l + 2w \\ &= 4x + 2y \\ &= 4x + 2(25 - x^2) \\ &= -2x^2 + 4x + 50 \\ \frac{dP}{dx} &= -4x^2 + 4 \\ 0 &= -4x^2 + 4 \\ 4x^2 &= 4 \\ x &= 1 \end{aligned}$$

So, the length is $2x = 2(1) = 2$ units and the width is $y = 25 - (1)^2 = 24$ units.

The maximum perimeter is $P = 2(2) + 2(24) = 52$ units.