Section 3.1 Extrema on an Interval

1. $f(x) = \dfrac{x^2}{x^2 + 4}$

$f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2)(2x)}{(x^2 + 4)^2} = \dfrac{8x}{(x^2 + 4)^2}$

$f'(0) = 0$

2. $f(x) = \cos \dfrac{\pi x}{2}$

$f'(x) = -\dfrac{\pi}{2} \sin \dfrac{\pi x}{2}$

$f'(0) = 0$

$f'(2) = 0$

3. $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2}$

$f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3}$

$f'(2) = 0$

4. $f(x) = -3x\sqrt{x + 1}$

$f'(x) = -3x\left[\frac{1}{2}(x + 1)^{-1/2}\right] + \sqrt{x + 1}(-3)$

$= -\frac{3}{2}(x + 1)^{-1/2}[x + 2(x + 1)]$

$= -\frac{3}{2}(x + 1)^{-1/2}(3x + 2)$

$f'\left(-\frac{2}{3}\right) = 0$

5. $f(x) = (x + 2)^{2/3}$

$f'(x) = \frac{2}{3}(x + 2)^{-1/3}$

$f'(-2)$ is undefined.

6. Using the limit definition of the derivative,

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{(4 - |x|) - 4}{x} = 1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{(4 - |x|) - 4}{x - 0} = -1$$

$f'(0)$ does not exist, because the one-sided derivatives are not equal.

7. Critical number: $x = 2$

$x = 2$: absolute maximum (and relative maximum)

8. Critical number: $x = 0$

$x = 0$: neither

9. Critical numbers: $x = 1, 2, 3$

$x = 1, 3$: absolute maxima (and relative maxima)

$x = 2$: absolute minimum (and relative minimum)

10. Critical numbers: $x = 2, 5$

$x = 2$: neither

$x = 5$: absolute maximum (and relative maximum)

11. $f(x) = 4x^2 - 6x$

$f'(x) = 8x - 6 = 2(4x - 3)$

Critical number: $x = \frac{3}{4}$

12. $g(x) = x^4 - 8x^2$

$g'(x) = 4x^3 - 16x = 4x(x^2 - 4)$

Critical numbers: $x = 0, -2, 2$

13. $g(t) = t\sqrt{4 - t}, \quad t < 3$

$g'(t) = t\left[\dfrac{1}{2}(4 - t)^{-1/2}(-1)\right] + (4 - t)^{1/2}$

$= \dfrac{1}{2}(4 - t)^{-1/2}[-t + 2(4 - t)]$

$= \dfrac{8 - 3t}{2\sqrt{4 - t}}$

Critical number: $t = \dfrac{8}{3}$

14. $f(x) = \dfrac{4x}{x^2 + 1}$

$f'(x) = \dfrac{(x^2 + 1)(4) - (4x)(2x)}{(x^2 + 1)^2} = \dfrac{4(1 - x^2)}{(x^2 + 1)^2}$

Critical numbers: $x = \pm 1$

15. $h(x) = \sin^2 x + \cos x, \quad 0 < x < 2\pi$

$h'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1)$

Critical numbers in $(0, 2\pi)$: $x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

16. $f(\theta) = 2\sec\theta + \tan\theta,\ \ 0 < \theta < 2\pi$

$$f'(\theta) = 2\sec\theta\tan\theta + \sec^2\theta$$
$$= \sec\theta(2\tan\theta + \sec\theta)$$
$$= \sec\theta\left[2\left(\frac{\sin\theta}{\cos\theta}\right) + \frac{1}{\cos\theta}\right]$$
$$= \sec^2\theta(2\sin\theta + 1)$$

Critical numbers in $(0, 2\pi)$: $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

17. $f(t) = te^{-2t}$

$f'(t) = e^{-2t} - 2te^{-2t} = e^{-2t}(1 - 2t)$

Critical number: $t = \frac{1}{2}$

18. $g(x) = 4x^2(3^x)$

$g'(x) = 8x(3^x) + 4x^2 3^x \ln 3 = 4x(3^x)(2 + x\ln 3)$

Critical numbers: $x = 0, -1.82$

19. $f(x) = x^2\log_2(x^2 + 1) = x^2\frac{\ln(x^2 + 1)}{\ln 2}$

$$f'(x) = 2x\frac{\ln(x^2 + 1)}{\ln 2} + x^2\frac{2x}{(\ln 2)(x^2 + 1)}$$
$$= \frac{2x}{\ln 2}\left[\ln(x^2 + 1) + \frac{x^2}{x^2 + 1}\right]$$

Critical number: $x = 0$

20. $g(t) = 2t\ln t$

$g'(t) = 2\ln t + 2t\left(\frac{1}{t}\right) = 2\ln t + 2$

Critical number: $t = \frac{1}{e}$

21. $f(x) = 3 - x,\ \ [-1, 2]$

$f'(x) = -1 \Rightarrow$ no critical numbers

Left endpoint: $(-1, 4)$ Maximum

Right endpoint: $(2, 1)$ Minimum

22. $f(x) = \frac{3}{4}x + 2, [0,4]$

$f'(x) = \frac{3}{4} \Rightarrow$ no critical numbers

Left endpoint: $(0, 2)$ Minimum

Right endpoint: $(4, 5)$ Maximum

23. $h(x) = 5 - 2x^2, [-3, 1]$

$h'(x) = -4x$

Critical number: $x = 0$

Left endpoint: $(-3, -13)$ Minimum

Critical number: $(0, 5)$ Maximum

Right endpoint: $(1, 3)$

24. $g(x) = 2x^2 - 8x,\ [0, 6]$

$g'(x) = 4x - 8 = 4(x - 2)$

Critical number: $x = 2$

Left endpoint: $(0, 0)$

Critical number: $(2, -8)$ Minimum

Right endpoint: $(6, 24)$ Maximum

25. $f(x) = x^3 - \frac{3}{2}x^2, [-1, 2]$

$f'(x) = 3x^2 - 3x = 3x(x - 1)$

Left endpoint: $\left(-1, -\frac{5}{2}\right)$ Minimum

Right endpoint: $(2, 2)$ Maximum

Critical number: $(0, 0)$

Critical number: $\left(1, -\frac{1}{2}\right)$

26. $f(x) = 2x^3 - 6x, [0, 3]$

$f'(x) = 6x^2 - 6 = 6(x^2 - 1)$

Critical number: $x = 1$ ($x = -1$ not in interval.)

Left endpoint: $(0, 0)$

Critical number: $(1, -4)$ Minimum

Right endpoint: $(3, 36)$ Maximum

27. $f(x) = 3x^{2/3} - 2x,\ \ [-1, 1]$

$f'(x) = 2x^{-1/3} - 2 = \frac{2(1 - \sqrt[3]{x})}{\sqrt[3]{x}}$

Left endpoint: $(-1, 5)$ Maximum

Critical number: $(0, 0)$ Minimum

Right endpoint: $(1, 1)$

28. $g(x) = \sqrt[3]{x} = x^{1/3}, [-8, 8]$

$g'(x) = \dfrac{1}{3x^{2/3}}$

Critical number: $x = 0$

Left endpoint: $(-8, -2)$ Minimum

Critical number: $(0, 0)$

Right endpoint: $(8, 2)$ Maximum

29. $h(s) = \dfrac{1}{s-2} = (s-2)^{-1}, [0, 1]$

$h'(s) = \dfrac{-1}{(s-2)^2}$

Left endpoint: $\left(0, -\dfrac{1}{2}\right)$ Maximum

Right endpoint: $(1, -1)$ Minimum

30. $h(t) = \dfrac{t}{t+3}, [-1, 6]$

$h'(t) = \dfrac{(t+3)(1) - t(1)}{(t+3)^2} = \dfrac{3}{(t+3)^2}$

No critical numbers

Left endpoint: $\left(-1, -\dfrac{1}{2}\right)$ Minimum

Right endpoint: $\left(6, \dfrac{2}{3}\right)$ Maximum

31. $y = 3 - |t - 3|, \quad [-1, 5]$

For $x < 3, y = 3 + (t - 3) = t$

and $y' = 1 \neq 0$ on $[-1, 3)$

For $x > 3, y = 3 - (t - 3) = 6 - t$

and $y' = -1 \neq 0$ on $(3, 5]$

So, $x = 3$ is the only critical number.

Left endpoint: $(-1, -1)$ Minimum

Right endpoint: $(5, 1)$

Critical number: $(3, 3)$ Maximum

32. $g(x) = |x + 4|, \ [-7, 1]$

g is the absolute value function shifted 4 units to the left. So, the critical number is $x = -4$.

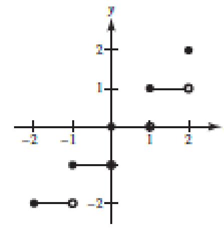
Left endpoint: $(-7, 3)$

Critical number: $(-4, 0)$ Minimum

Right endpoint: $(1, 5)$ Maximum

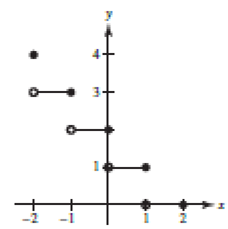
33. $f(x) = [\![x]\!], \quad [-2, 2]$

From the graph of f, you see that the maximum value of f is 2 for $x = 2$, and the minimum value is -2 for $-2 \le x < -1$.



34. $h(x) = [\![2 - x]\!], \quad [-2, 2]$

From the graph you see that the maximum value of h is 4 at $x = -2$, and the minimum value is 0 for $1 < x \le 2$.



35. $f(x) = \sin x, \left[\dfrac{5\pi}{6}, \dfrac{11\pi}{6}\right]$

$f'(x) = \cos x$

Critical number: $x = \dfrac{3\pi}{2}$

Left endpoint: $\left(\dfrac{5\pi}{6}, \dfrac{1}{2}\right)$ Maximum

Critical number: $\left(\dfrac{3\pi}{2}, -1\right)$ Minimum

Right endpoint: $\left(\dfrac{11\pi}{6}, -\dfrac{1}{2}\right)$

36. $g(x) = \sec x, \quad \left[-\dfrac{\pi}{6}, \dfrac{\pi}{3}\right]$

$g'(x) = \sec x \tan x$

Left endpoint: $\left(-\dfrac{\pi}{6}, \dfrac{2}{\sqrt{3}}\right) \approx \left(-\dfrac{\pi}{6}, 1.1547\right)$

Right endpoint: $\left(\dfrac{\pi}{3}, 2\right)$ Maximum

Critical number: $(0, 1)$ Minimum

37. $y = 3 \cos x, \quad [0, 2\pi]$

$y' = -3 \sin x$

Critical number in $(0, 2\pi)$: $x = \pi$

Left endpoint: $(0, 3)$ Maximum

Critical number: $(\pi, -3)$ Minimum

Right endpoint: $(2\pi, 3)$ Maximum

38. $y = \tan\left(\frac{\pi x}{8}\right), \quad [0, 2]$

$y' = \frac{\pi}{8}\sec^2\left(\frac{\pi x}{8}\right) \neq 0$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(2, 1)$ Maximum

39. $f(x) = \arctan x^2, [-2, 1]$

$f'(x) = \frac{2x}{1 + x^4}$

Critical number: $x = 0$

Left endpoint: $(-2, \arctan 4) \approx (-2, 1.326)$ Maximum

Right endpoint: $(1, \arctan 1) = \left(1, \frac{\pi}{4}\right) \approx (1, 0.785)$

Critical number: $(0, 0)$ Minimum

40. $g(x) = \frac{\ln x}{x}, [1, 4]$

$g'(x) = \frac{x\left(\frac{1}{x}\right) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$

Critical number: $x = e$

Left endpoint: $(1, 0)$ Minimum

Right endpoint: $\left(4, \frac{\ln 4}{4}\right) \approx (4, 0.347)$

Critical number: $\left(e, \frac{1}{e}\right) \approx (2.718, 0.368)$ Maximum

41. $h(x) = 5e^x - e^{2x}, [-1, 2]$

$h'(x) = 5e^x - 2e^{2x} = e^x(5 - 2e^x)$

$5 - 2e^x = 0 \Rightarrow e^x = \frac{5}{2} \Rightarrow x = \ln\left(\frac{5}{2}\right) \approx 0.916$

Critical number: $x = \ln\left(\frac{5}{2}\right)$

Left endpoint: $\left(-1, \frac{5}{e} - \frac{1}{e^2}\right) \approx (-1, 1.704)$

Right endpoint: $(2, 5e^2 - e^4) \approx (2, -17.653)$ Minimum

Critical number: $\left(\ln\left(\frac{5}{2}\right), \frac{25}{4}\right)$ Maximum

Note: $h\left(\ln\left(\frac{5}{2}\right)\right) = 5e^{\ln(5/2)} - e^{2\ln(5/2)}\ 5$

$= 5\left(\frac{5}{2}\right) - \left(\frac{5}{2}\right)^2 = \frac{25}{4}$

42. $y = x^2 - 8\ln x, [1, 5]$

$y' = 2x - \frac{8}{x}$

$2x - \frac{8}{x} = 0 \Rightarrow 2x^2 = 8 \Rightarrow x = 2$

$(x = -2$ not in domain$)$

Critical number: $x = 2$

Left endpoint: $(1, 1)$

Right endpoint: $(5, 25 - 8\ln 5) \approx (5, 12.124)$ Maximum

Critical number: $(2, 4 - 8\ln 2) \approx (2, -1.545)$ Minimum

43. $y = e^x \sin x, [0, \pi]$

$y' = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$

Left endpoint: $(0, 0)$ Minimum

Critical number: $\left(\frac{3\pi}{4}, \frac{\sqrt{2}}{2}e^{3\pi/4}\right) \approx \left(\frac{3\pi}{4}, 7.46\right)$

Maximum

Right endpoint: $(\pi, 0)$ Minimum

44. $y = x\ln(x + 3), [0, 3]$

$y' = x\left(\frac{1}{x + 3}\right) + \ln(x + 3)$

Left endpoint: $(0, 0)$ Minimum

Right endpoint: $(3, 3\ln 6) \approx (3, 5.375)$ Maximum

45. The function also needs to be a evaluated at $x = -2$ and $x = 3$, which are the endpoints of the interval $[-2, 3]$. $x = 0$ is a critical number, but not a relative extremum.

$f'(x) = -4x^3 - 6x^2$ and $-4x^3 - 6x^2 = 0$ when $x = -\frac{3}{2}$ and $x = 0$.

$f\left(-\frac{3}{2}\right) = \frac{27}{16}, f(0) = 0, f(-2) = 0,$ and $f(3) = -135$

So, f has extrema at $x = -\frac{3}{2}$ and $x = 3$.

46. Because $x = -\frac{\sqrt{6}}{3} \approx -0.8165$, this critical number is not in the domain of $[0.5, 5]$.

$g'(x) = 6x - \frac{4}{x}$ and $6x - \frac{4}{x} = 0$ when $x = \pm\frac{\sqrt{6}}{3}$ $\left(\textbf{Note: } x = -\frac{\sqrt{6}}{3} \text{ is not in the domain}\right)$.

$f\left(\frac{\sqrt{6}}{3}\right) = 2 - 4\ln\frac{\sqrt{6}}{3} \approx 2.811$, $f(0.5) = \frac{3}{4} - \ln 0.5 \approx 3.523$, and $f(5) = 75 - 4\ln 5 \approx 68.562$

So, g has extrema at $x = \frac{\sqrt{6}}{3}$ and $x = 5$.

47. $f(x) = 2x - 3$

(a) Minimum: $(0, -3)$

Maximum: $(2, 1)$

(b) Minimum: $(0, -3)$

(c) Maximum: $(2, 1)$

(d) No extrema

48. $f(x) = \sqrt{4 - x^2}$

(a) Minima: $(-2, 0)$ and $(2, 0)$

Maximum: $(0, 2)$

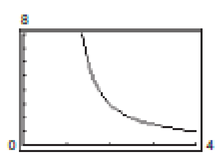
(b) Minimum: $(-2, 0)$

(c) Maximum: $(0, 2)$

(d) Maximum: $\left(1, \sqrt{3}\right)$

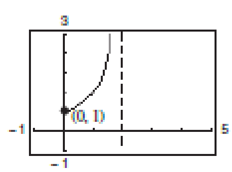
49. $f(x) = \frac{3}{x - 1}$, $(1, 4]$

Right endpoint: $(4, 1)$ Minimum

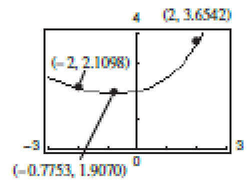


50. $f(x) = \frac{2}{2 - x}$, $[0, 2)$

Left endpoint: $(0, 1)$ Minimum



51. $f(x) = \sqrt{x + 4}\,e^{x^2/10}$, $[-2, 2]$

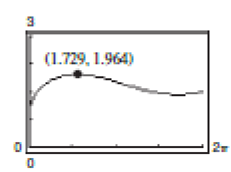


$$f'(x) = \frac{(2x^2 + 8x + 5)e^{x^2/10}}{10\sqrt{x + 4}}$$

Right endpoint: $(2, 3.6542)$ Maximum

Critical point: $(-0.7753, 1.9070)$ Minimum

52. $f(x) = \sqrt{x} + \cos\frac{x}{2}$, $[0, 2\pi]$

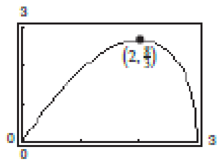


$$f'(x) = \frac{1}{2\sqrt{x}} - \frac{1}{2}\sin\frac{x}{2}$$

Left endpoint: $(0, 1)$ Minimum

Critical point: $(1.729, 1.964)$ Maximum

53. (a)



Maximum: $\left(2, \frac{8}{3}\right)$

(b) $f(x) = \frac{4}{3}x\sqrt{3 - x}, \quad [0, 3]$

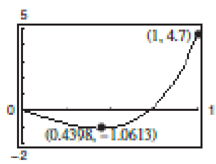
$f'(x) = \frac{4}{3}\left[x\left(\frac{1}{2}\right)(3 - x)^{-1/2}(-1) + (3 - x)^{1/2}(1)\right] = \frac{4}{3}(3 - x)^{-1/2}\left(\frac{1}{2}\right)\left[-x + 2(3 - x)\right] = \frac{2(6 - 3x)}{3\sqrt{3 - x}} = \frac{6(2 - x)}{3\sqrt{3 - x}} = \frac{2(2 -}{\sqrt{3 -}}$

Left endpoint: $(0, 0)$ Minimum

Critical point: $\left(2, \frac{8}{3}\right)$ Maximum

Right endpoint: $(3, 0)$ Minimum

54. (a)



Minimum: $(0.4398, -1.0613)$

(b)
$$f(x) = 3.2x^5 + 5x^3 - 3.5x, \quad [0, 1]$$
$$f'(x) = 16x^4 + 15x^2 - 3.5$$

$16x^4 + 15x^2 - 3.5 = 0$

$$x^2 = \frac{-15 \pm \sqrt{(15)^2 - 4(16)(-3.5)}}{2(16)} = \frac{-15 \pm \sqrt{449}}{32}$$

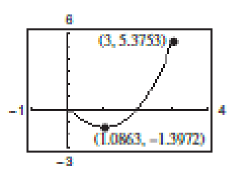
$$x = \sqrt{\frac{-15 + \sqrt{449}}{32}} \approx 0.4398$$

Left endpoint: $(0, 0)$

Critical point: $(0.4398, -1.0613)$ Minimum

Right endpoint: $(1, 4.7)$ Maximum

55. (a)



Minimum: $(1.0863, -1.3972)$

(b) $f(x) = \left(x^2 - 2x\right)\ln(x + 3), \quad [0, 3]$

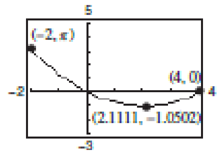
$$f'(x) = \left(x^2 - 2x\right) \cdot \frac{1}{x + 3} + (2x - 2)\ln(x + 3) = \frac{x^2 - 2x + \left(2x^2 + 4x - 6\right)\ln(x + 3)}{x + 3}$$

Left endpoint: $(0, 0)$

Critical point: $(1.0863, -1.3972)$ Minimum

Right endpoint: $(3, 5.3753)$ Maximum

56. (a)



Minimum: $(2.1111, -1.0502)$

(b) $f(x) = (x - 4)\arcsin\dfrac{x}{4}, \quad [-2, 4]$

$$f'(x) = (x - 4)\frac{\frac{1}{4}}{\sqrt{1 - \frac{x^2}{16}}} + \arcsin\frac{x}{4} = \frac{x - 4}{4\sqrt{1 - \frac{x^2}{16}}} + \arcsin\frac{x}{4}$$

Left endpoint: $(-2, \pi)$ Maximum

Critical point: $(2.1111, -1.0502)$ Minimum

Right endpoint: $(4, 0)$

57. $f(x) = (1 + x^3)^{1/2}, \quad [0, 2]$

$$f'(x) = \frac{3}{2}x^2(1 + x^3)^{-1/2}$$

$$f''(x) = \frac{3}{4}(x^4 + 4x)(1 + x^3)^{-3/2}$$

$$f'''(x) = -\frac{3}{8}(x^6 + 20x^3 - 8)(1 + x^3)^{-5/2}$$

Setting $f''' = 0$, you have $x^6 + 20x^3 - 8 = 0$.

$$x^3 = \frac{-20 \pm \sqrt{400 - 4(1)(-8)}}{2}$$

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1$$

In the interval $[0, 2]$, choose

$$x = \sqrt[3]{-10 \pm \sqrt{108}} = \sqrt{3} - 1 \approx 0.732.$$

$\left|f''\left(\sqrt[3]{-10 + \sqrt{108}}\right)\right| \approx 1.47$ is the maximum value.

58. $f(x) = \dfrac{1}{x^2 + 1}, \quad \left[\dfrac{1}{2}, 3\right]$

$$f'(x) = \frac{-2x}{(x^2 + 1)^2}$$

$$f''(x) = \frac{-2(1 - 3x^2)}{(x^2 + 1)^3}$$

$$f'''(x) = \frac{24x - 24x^3}{(x^2 + 1)^4}$$

Setting $f''' = 0$, you have $x = 0, \pm 1$.

$\left|f''(1)\right| = \dfrac{1}{2}$ is the maximum value.

59. $f(x) = e^{-x^2/2}, [0, 1]$

$$f'(x) = -xe^{-x^2/2}$$

$$f''(x) = -x\left(-xe^{-x^2/2}\right) - e^{-x^2/2}$$

$$= e^{-x^2/2}(x^2 - 1)$$

$$f'''(x) = e^{-x^2/2}(2x) + (x^2 - 1)\left(-xe^{-x^2/2}\right)$$

$$= xe^{-x^2/2}(3 - x^2)$$

$\left|f''(0)\right| = 1$ is the maximum value.

60. $f(x) = x\ln(x + 1), [0, 2]$

$$f'(x) = \frac{x}{(x + 1)} + \ln(x + 1)$$

$$f''(x) = \frac{x + 1 - x}{(x + 1)^2} + \frac{1}{x + 1}$$

$$= \frac{1}{(x + 1)^2} + \frac{1}{x + 1} = \frac{x + 2}{(x + 1)^2}$$

$$f'''(x) = \frac{(x + 1)^2 - (x + 2)2(x + 1)}{(x + 1)^4} = \frac{-x - 3}{(x + 1)^3}$$

$\left|f''(0)\right| = 2$ is the maximum value.

61. $f(x) = \tan x$

f is continuous on $[0, \pi/4]$ but not on $[0, \pi]$.

$$\lim_{x \to (\pi/2)^-} \tan x = \infty.$$

62. A: absolute minimum

B: relative maximum

C: neither

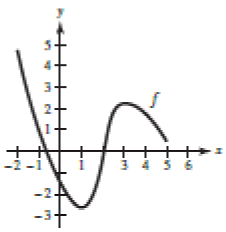
D: relative minimum

E: relative maximum

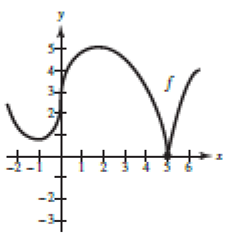
F: relative minimum

G: neither

63.



64.



65. (a) Yes

(b) No

66. (a) No

(b) Yes

67. $P = VI - RI^2 = 12I - 0.5I^2, 0 \le I \le 15$

$P = 0$ when $I = 0$.

$P = 67.5$ when $I = 15$.

$P' = 12 - I = 0$

Critical number: $I = 12$ amps

When $I = 12$ amps, $P = 72$, the maximum output.

No, a 20-amp fuse would not increase the power output. P is decreasing for $I > 12$.

68. $x = \dfrac{v^2 \sin 2\theta}{32}, 45° \le \theta \le 135°$

$\dfrac{d\theta}{dt}$ is constant.

$\dfrac{dx}{dt} = \dfrac{dx}{d\theta}\dfrac{d\theta}{dt}$ (by the Chain Rule) $= \dfrac{v^2 \cos 2\theta}{16}\dfrac{d\theta}{dt}$

In the interval $[45°, 135°]$, $\theta = 45°$ and $\theta = 135°$ indicate minimums for dx/dt and $\theta \neq 90°$ indicates a maximum for dx/dt. This implies that the sprinkler waters longest when $\theta = 45°$ and $\theta = 135°$.

So, the lawn farthest from the sprinkler gets the most water.

69. $$S = 6hs + \frac{3s^2}{2}\left(\frac{\sqrt{3} - \cos\theta}{\sin\theta}\right), \frac{\pi}{6} \le \theta \le \frac{\pi}{2}$$

$$\frac{dS}{d\theta} = \frac{3s^2}{2}\left(-\sqrt{3}\csc\theta\cot\theta + \csc^2\theta\right)$$

$$= \frac{3s^2}{2}\csc\theta\left(-\sqrt{3}\cot\theta + \csc\theta\right) = 0$$

$$\csc\theta = \sqrt{3}\cot\theta$$

$$\sec\theta = \sqrt{3}$$

$$\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553 \text{ radians}$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\frac{\pi}{6}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{3}\right)$$

$$S\left(\operatorname{arcsec}\sqrt{3}\right) = 6hs + \frac{3s^2}{2}\left(\sqrt{2}\right)$$

S is minimum when $\theta = \operatorname{arcsec}\sqrt{3} \approx 0.9553$ radian.

70. $f(x) = ax^3 + bx^2 + cx + d, \quad a \ne 0$

$f'(x) = 3ax^2 + 2bx + c$

The quadratic polynomial can have zero, one, or two zeros.

$$x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a} = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}$$

Zero critical numbers: $b^2 < 3ac$.

Example: $(a = b = c = 1, d = 0) f(x) = x^3 + x^2 +$ has no critical numbers.

One critical number: $b^2 = 3ac$.

Example: $(a = 1, b = c = d = 0) f(x) = x^3$ has one critical number, $x = 0$.

Two critical numbers: $b^2 > 3ac$.

Example:

$(a = c = 1, b = 2, d = 0) f(x) = x^3 + 2x^2 + x$ has two critical numbers: $x = -1, -\dfrac{1}{3}$.

71. True

72. False. Let $f(x) = x^2$. $x = 0$ is a critical number of f.

$g(x) = f(x - k) = (x - k)^2$

$x = k$ is a critical number of g.

73. If f has a maximum value at $x = c$, then $f(c) \ge f(x)$ for all x in I. So, $-f(c) \le -f(x)$ for all x in I. So, $-f$ has a minimum value at $x = c$.

74. The graph of h does not have a minimum on the open interval $(-1, 4]$. The graph of h has a maximum at $(2, 4)$.

"h has a maximum at $(2, 4)$" is the only true statement. So, the answer is C.

75.
$$\begin{aligned} f(x) &= x^2(3x-1)^3 \\ f'(x) &= x^2\left[3(3x-1)^2(3)\right] + (3x-1)^3(2x) \\ &= 9x^2(3x-1)^2 + 2x(3x-1)^3 \\ &= x(3x-1)^2\left[9x + 2(3x-1)\right] \\ &= x(3x-1)^2(15x-2) \end{aligned}$$

$$x = 0 \qquad 3x - 1 = 0 \qquad 15x - 2 = 0$$
$$x = \tfrac{1}{3} \qquad x = \tfrac{2}{15}$$

The critical numbers for $f(x)$ are $x = 0$, $x = \frac{2}{15}$, and $x = \frac{1}{3}$. So, the answer is D.

76. (a)
$$\begin{aligned} f(x) &= \frac{4\ln x}{x^3} \\ f'(x) &= \frac{x^3(4/x) - 4\ln x(3x^2)}{(x^3)^2} \\ &= \frac{4x^2 - 12x^2\ln x}{x^6} \\ &= \frac{4x^2(1 - 3\ln x)}{x^6} \\ &= \frac{4(1 - 3\ln x)}{x^4} \end{aligned}$$

(b)
$$\begin{aligned} f'(x) &= \frac{4(1 - 3\ln x)}{x^4} \\ f'(e) &= \frac{4\left[1 - 3\ln(e)\right]}{(e)^4} \\ &= \frac{4(1-3)}{e^4} \\ &= -8e^{-4} \end{aligned}$$

When $x = e$, $f(e) = \dfrac{4\ln(e)}{(e)^3} = 4e^{-3}$.

So, an equation of the tangent line is

$$\begin{aligned} y - 4e^{-3} &= -8e^{-4}(x - e) \\ y &= -8e^{-4}x + 8e^{-3} + 4e^{-3} \\ y &= -8e^{-4}x + 12e^{-3}. \end{aligned}$$

(c)
$$\begin{aligned} f'(x) &= \frac{4(1 - 3\ln x)}{x^4} \\ 0 &= \frac{4}{x^4} - \frac{12\ln x}{x^4} \\ \frac{12\ln x}{x^4} &= \frac{4}{x^4} \\ 12\ln x &= 4 \\ \ln x &= \frac{1}{3} \\ x &= e^{1/3} \quad \text{(critical number)} \end{aligned}$$

$$\begin{aligned} f\left(e^{1/3}\right) &= \frac{4\ln\left(e^{1/3}\right)}{(1/3)^3} \\ &= 27 \cdot 4\left(\frac{1}{3}\right) \\ &= 36 \end{aligned}$$

So, the point $\left(e^{1/3}, 36\right)$ is a relative maximum.

(d) $\displaystyle\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{4\ln x}{x^3} = -\infty$

77. (a) $g(x) = \sin x \cos x$

$$g'(x) = \sin x(-\sin x) + \cos x(\cos x)$$
$$= -\sin^2 x + \cos^2 x$$

$$g'\left(\frac{\pi}{3}\right) = -\sin^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right)$$
$$= -\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$
$$= -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2}$$

$$y - \frac{\sqrt{3}}{4} = -\frac{1}{2}\left(x - \frac{\pi}{3}\right)$$
$$y = -\frac{1}{2}x + \frac{\pi}{6} + \frac{\sqrt{3}}{4}$$

(b) $g'(x) = -\sin^2 x + \cos^2 x$

$$0 = -\sin^2 x + \cos^2 x$$
$$\sin^2 x = \cos^2 x$$
$$\sin x = \cos x$$
$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ and } \frac{7\pi}{4}$$

So, $x = \frac{\pi}{4}$ is a relative maximum, $x = \frac{3\pi}{4}$ is a
relative minimum, $x = \frac{5\pi}{4}$ is a relative maximu
and $x = \frac{7\pi}{4}$ is a relative minimum.