4.4 The Fundamental Theorem of Calculus, Day 1

The Fundamental Theorem of Calculus

If a function f is continuous on the closed interval $[a, b]$ and F is an antiderivative of f on the interval $[a, b]$, then $$\int_a^b f(x)dx = F(b) - F(a)$$

?? Do we need the C?

No. $\int_a^b f(x)\,dx = (F(b)+C) - (F(a)+C) \Rightarrow$ C's will cancel

Evaluate the definite integral $\int_1^9 \frac{1}{\sqrt{x}}\,dx$ $= \int_1^9 x^{-1/2}dx$

$\left.\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right|_1^9 = \left.2x^{\frac{1}{2}}\right|_1^9 = 2(9)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}} = 6-2=4$

Evaluate the definite integral $\int_0^4 |x^2 - 4x + 3|\,dx$ $(x-3)(x-1)$

$\xleftrightarrow{\;+\;|_1\;-\;|_3\;+\;}$

$\begin{cases} x^2-4x+3 & x<1 \\ -x^2+4x-3 & 1<x<3 \\ x^2-4x+3 & x>3 \end{cases}$

$\int_0^1 x^2-4x+3dx + \int_1^3 -x^2+4x-3dx + \int_3^4 x^2-4x+3\,dx$

$\left.\frac{1}{3}x^3-2x^2+3x\right|_0^1 + \left.-\frac{1}{3}x^3+2x^2-3x\right|_1^3 + \left.\frac{1}{3}x^3-2x^2+3x\right|_3^4$

$\Rightarrow (\frac{1}{3}-2+3)+(-9+18-9)-(-\frac{1}{3}+2-3)$
$+(\frac{64}{3}-32+12)-(9-18+9)$
$=4$

Examples – Evaluating Definite Integrals WITHOUT a calculator

$\int_2^5 (-3x+4)dx$

$\left.-\frac{3}{2}x^2+4x\right|_2^5$

$\left(-\frac{3}{2}(25)+20\right) - \left[\left(-\frac{3}{2}\right)(4)+8\right]$

$=-19.5 = -\frac{39}{2}$

$\int_{-1}^1 (x^3 - 9x)dx$

$\left.\frac{1}{4}x^4 - \frac{9}{2}x^2\right|_{-1}^1$

$\left[\left(\frac{1}{4}\right)-\left(\frac{9}{2}\right)\right] - \left[\left(\frac{1}{4}\right)-\left(\frac{9}{2}\right)\right]$

$=0$

$$\int_{-8}^{-1} \frac{x-x^2}{2\sqrt[3]{x}}\,dx = \int_{-8}^{-1} \frac{1}{2}x^{2/3} - \frac{1}{2}x^{5/3}\,dx$$

$$\left(\frac{1}{2}\right)\left(\frac{x^{5/3}}{5/3}\right) - \left(\frac{1}{2}\right)\left(\frac{x^{8/3}}{8/3}\right)\Bigg|_{-8}^{-1}$$

$$\frac{3}{10}x^{5/3} - \frac{3}{16}x^{8/3}\Bigg|_{-8}^{-1}$$

$$\left[\frac{3}{10}(-1) - \frac{3}{16}(1)\right] - \left[\frac{3}{10}(-32) - \frac{3}{16}(256)\right]$$

$$= -\frac{3}{10} - \frac{3}{16} + \frac{96}{10} + 48 = \frac{4569}{80} = 57.1125$$

$$\int_{\pi}^{2\pi} -\sin x\,dx$$

$$-(-\cos x)\Big|_{\pi}^{2\pi} = \cos x\Big|_{\pi}^{2\pi}$$

$$\cos(2\pi) - \cos(\pi)$$

$$1 - (-1) = 2$$

$$\int_{1}^{3} |x^2 - 4|\,dx$$

$$(x-2)(x+2)$$

$$\overset{+}{\longleftarrow}\underset{-2}{+}\overset{-}{\longrightarrow}\underset{2}{+}\overset{+}{\longrightarrow}$$

$$\begin{cases} x^2-4 & x < -2 \\ -x^2+4 & -2 < x < 2 \\ x^2-4 & x > 2 \end{cases}$$

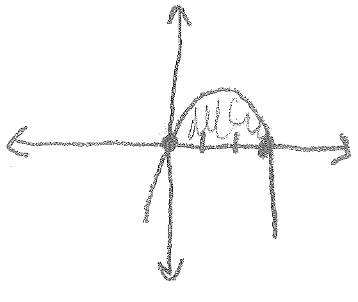
$$\int_{1}^{2} -x^2+4\,dx + \int_{2}^{3} x^2-4\,dx = -\frac{1}{3}x^3+4x\Big|_{1}^{2} + \frac{1}{3}x^3-4x\Big|_{2}^{3}$$

$$\left[\left(-\frac{8}{3}+8\right) - \left(-\frac{1}{3}+4\right)\right] + \left[(9-12) - \left(\frac{8}{3}-8\right)\right] = 4$$

Examples – Area with Definite Integrals

Find the area of the region bounded by the graphs of the equations $y = -x^2 + 3x$ and $y = 0$.

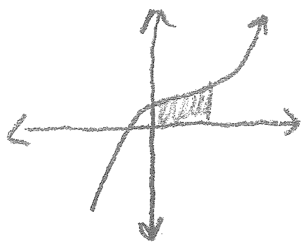
$$y = -x(x-3)$$



$$\int_{0}^{3} -x^2+3x\,dx$$

$$-\frac{1}{3}x^3 + \frac{3}{2}x^2\Big|_{0}^{3} = -9 + \frac{27}{2} = \frac{9}{2}$$

Find the area of the region bounded by the graph of $y = \frac{x^3+2}{4}$, the $x-axis$, and the vertical lines $x = 0$ and $x = 2$.



$$\frac{1}{4}\int_0^2 x^3+2\,dx = \frac{1}{4}\left(\frac{1}{4}x^4\right) + \frac{1}{2}x \Big|_0^2$$

$$= \frac{1}{16}x^4 + \frac{1}{2}x \Big|_0^2 = 1+1 = 2$$

Net Change Theorem

If $F'(x)$ is the rate of change of a quantity $F(x)$, then the definite integral of $F'(x)$ from a to b gives the total change, or net change of $F(x)$ on the interval $[a, b]$.

$$\int_a^b F'(x)dx = F(b) - F(a) = net\ change\ in\ F\ from\ a\ to\ b.$$

The velocity (in feet per second) of a particle moving along a line is $v(t) = t^3 - 14t^2 + 56t - 64$, where t is the time in seconds.

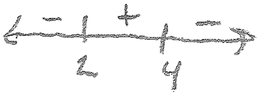
a. What is the displacement of the particle on the time interval $2 \leq t \leq 8$?

$$\int_2^8 t^3 - 14t^2 + 56t - 64\,dt = \frac{1}{4}t^4 - \frac{14}{3}t^3 + 28t^2 - 64t \Big|_2^8$$

$$= \left[\frac{1}{4}(4096) - \frac{14}{3}(512) + 28(64) - 64(8)\right] - \left[\frac{1}{4}(16) - \frac{14}{3}(8) + 28(4) - 64(2)\right]$$

$$1024 - \frac{7168}{3} + 1792 - 512 - 4 + \frac{112}{3} - 112 + 128 = -36$$

b. What is the total distance traveled by the particle on the time interval $2 \leq t \leq 8$?



$$\int_2^4 t^3 - 14t^2 + 56t - 64\,dt + -\int_4^8 t^3 - 14t^2 + 56t - 64\,dt$$

$$\frac{1}{4}t^4 - \frac{14}{3}t^3 + 28t^2 - 64t \Big|_2^4 - \left[\frac{1}{4}t^4 - \frac{14}{3}t^3 + 28t^2 - 64t\right] \Big|_4^8$$

$$\frac{20}{3} - \left(-\frac{128}{3}\right) = \frac{148}{3}$$

Examples – Net Change Application Problems

Water is being drained from a tank at a rate of $(100 + 4t)$ liters per minute, where t is the time in minutes and $0 \le t \le 60$.

a. Find the amount of water that has been drained from the tank in the first 5 minutes.

$$\int_0^5 100 + 4t \, dt = 100t + 2t^2 \Big|_0^5 = (500 + 50) - 0 = 550 \text{ liters}$$

b. If the tank had 720 liters before it started draining, write an expression how many liters does it now have in the tank?

$$720 - \int_0^5 100 + 4t \, dt$$

The velocity of a particle moving along the x-axis is $v(t) = \cos(t)$, where t is the time in seconds and $t \ge 0$. When $t = 0$, the position s of the particle is $s = 2$.

a. What is the displacement of the particle on the time interval $0 \le t \le \frac{\pi}{2}$?

$$\int_0^{\pi/2} \cos t \, dt = \sin t \Big|_0^{\pi/2} = \sin \tfrac{\pi}{2} - (\sin 0)$$

$$= 1 - 0 = 1$$

b. What is the position of the particle at $t = \frac{\pi}{2}$?

$$2 + \int_0^{\pi/2} \cos t \, dt = 2 + \sin t \Big|_0^{\pi/2}$$

$$= 2 + 1 = 3$$

Using your calculator to evaluate definite integrals