5.2 Growth and Decay

Recall: When a differential equation involves both $x's$ and $y's$, rewrite the equation so that each variable occurs on only one side of the equation.

Write and solve the differential equation that models the verbal statement:

The rate of change of P with respect to t is proportional to $(10 - t)$.

$$\frac{dP}{dt} = K(10-t)$$
$$\int dP = \int K(10-t)\,dt$$
$$P = K\left(10t - \tfrac{1}{2}t^2\right) + C$$

OR

$$\frac{dP}{dt} = K(10-t)$$
$$\int dP = \int K(10-t)\,dt$$
$$P = K\frac{(10-t)^2}{2}(-1) + C$$
$$P = -\frac{K}{2}(10-t)^2 + C$$

Exponential Growth and Decay Model

If y is a differentiable function of t such that $y > 0$ and $y' = ky$, for some constant k, then:

$$\frac{dy}{dt} = ky$$
$$\frac{dy}{y} = k\,dt$$
$$\ln|y| = kt + C$$
$$e^{kt+C} = y$$
$$y = e^{kt} \cdot e^{C}$$
$$y = Ce^{kt}$$

- C is the initial amount
- k is the constant of proportionality / rate
- Exponential growth occurs when $k > 0$
- Exponential decay occurs when $k < 0$

Examples – Solving differential equations, exponential growth and decay

Solve the differential equation $\quad y' = \frac{\sqrt{x}}{2y}$

$$\frac{dy}{dx} = \frac{x^{1/2}}{2y}$$
$$\int 2y\,dy = \int x^{1/2}\,dx$$
$$2\left(\frac{y^2}{2}\right) = \frac{x^{3/2}}{3/2} + C$$
$$y^2 = \frac{2}{3}x^{3/2} + C$$

A sample contains 3 grams of Carbon 14 (C^{14}). How much carbon will remain after 100 years? (C^{14} has a half-life of 5730 years).

$\frac{1}{2} = e^{K(5730)}$

$\ln\frac{1}{2} = 5730K$

$\frac{\ln\frac{1}{2}}{5730} = K$

$y = 3e^{\left(\frac{\ln\frac{1}{2}}{5730}\right)(100)}$

$y = 2.963\ g$

or

$2.964\ g$

Solve the differential equation $y' = \frac{xy}{2lny}$

$\frac{dy}{dx} = \frac{xy}{2\ln y}$

$\int \frac{\ln y\,dy}{y} = \int \frac{1}{2}x\,dx$

$u = \ln y$

$du = \frac{1}{y}dy$

$\int u\,du = \frac{1}{2}x^2 + C$

$\frac{1}{2}u^2 = \frac{1}{2}x^2 + C$

$\frac{1}{2}(\ln y)^2 = \frac{1}{2}x^2 + C$

$(\ln y)^2 = x^2 + C$

Because of a slump in the economy, a company finds that its annual revenues have dropped from \$742,000 in 2013 to \$632,000 in 2015. If the revenue is following an exponential pattern of decline, what is the expected revenue for 2018?

$632 = 742e^{K(2)}$

$\frac{632}{742} = e^{2K}$

$\ln\left(\frac{632}{742}\right) = 2K$

$\frac{\ln\left(\frac{632}{742}\right)}{2} = K$

$y = 742e^{\left(\frac{\ln\left(\frac{632}{742}\right)}{2}\right)(5)}$

$y = \$496,806$

The rate of change of y is proportional to y. When $t = 0$, $y = 3$, and when $t = 1$, $y = 12$. What is the value of y when $t = 5$?

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln|y| = kt + C$$

$$e^{kt+C} = y$$

$$y = Ce^{kt}$$

$$3 = Ce^{k(0)}$$

$$3 = C$$

$$12 = 3e^{k(1)}$$

$$4 = e^{k}$$

$$\ln 4 = k$$

$$y = 3e^{(\ln 4)(5)}$$

$$y = 3072$$

Solve the differential equation: $xy + y' = 100x$.

$$y' = 100x - xy$$

$$\frac{dy}{dx} = 100x - xy$$

$$\frac{dy}{dx} = x(100-y)$$

$$\frac{dy}{100-y} = x\,dx$$

$$\int \frac{dy}{100-y} = \int x\,dx$$

$$-\ln|100-y| = \frac{1}{2}x^2 + C$$

$$\ln|100-y| = -\frac{1}{2}x^2 + C$$

$$e^{-\frac{1}{2}x^2+C} = 100-y$$

$$y = 100 - e^{-\frac{1}{2}x^2+C}$$

$$y = 100 - Ce^{-\frac{1}{2}x^2}$$

The rate of growth of a population of rabbits is modeled by the differential equation $y' = \frac{3}{2}y$

The original population (when $t = 0$) consisted of 40 rabbits. Approximately how many rabbits were there after three years?

$$\frac{dy}{dt} = \frac{3}{2}y$$

$$\frac{dy}{y} = \frac{3}{2}dt$$

$$\ln|y| = \frac{3}{2}t + C$$

$$e^{\frac{3}{2}t+C} = y$$

$$y = Ce^{\frac{3}{2}t}$$

$$y = 40e^{\frac{3}{2}t}$$

$$y = 40e^{\frac{3}{2}(3)}$$

$$y = 40e^{\frac{9}{2}} \approx 3600 \text{ rabbits}$$

or

3601 rabbits