4.5 Integration by Substitution

Antidifferentiation of a Composite Function

Let g be a function whose range is an interval I, and let f be a function that is continuous on I. If g is differentiable on its domain and F is an antiderivative of f on I, then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

Letting $u = g(x)$ gives $du = g'(x)dx$ and

$$\int f(u)du = F(u) + C$$

Change of Variables for Indefinite Integrals

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C$$

Change of Variables for Definite Integrals

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du = F(g(b)) - F(g(a))$$

$\int \frac{2}{\sqrt[3]{3x}}dx$ $\quad u = 3x$, $du = 3dx$, $dx = \frac{du}{3}$

$2\int u^{-\frac{1}{3}} \cdot \frac{du}{3} = \frac{2}{3}\int u^{-\frac{1}{3}}du = \frac{2}{3}\left(\frac{u^{2/3}}{2/3}\right) + C$

$= u^{2/3} + C = (3x)^{2/3} + C$

$$\int \frac{2}{\sqrt[3]{3x}}dx$$

Examples – Using Substitution (Change of Variables)

$\int (x^2 - 1)^3(2x)dx$

$u = x^2 - 1$, $du = 2x\,dx$, $dx = \frac{du}{2x}$

$\int u^3 (\cancel{2x}) \frac{du}{\cancel{2x}} = \int u^3 du$

$= \frac{u^4}{4} + C = \frac{1}{4}(x^2-1)^4 + C$

$\int (1 - 2x^2)^3(-4x)dx$

$u = 1 - 2x^2$, $du = -4x\,dx$, $dx = \frac{du}{-4x}$

$\int u^3 (\cancel{-4x}) \frac{du}{\cancel{-4x}}$

$\int u^3 du = \frac{1}{4}u^4 + C$

$= \frac{1}{4}(1-2x^2)^4 + C$

$\int x^3\sqrt{x^4+2}\,dx$ $\quad u = x^4+2$, $du = 4x^3\,dx$, $dx = \frac{du}{4x^3}$

$\int \cancel{x^3}\, u^{\frac{1}{2}} \frac{du}{4\cancel{x^3}}$

$\frac{1}{4}\int u^{\frac{1}{2}}du = \frac{1}{4}\left(\frac{u^{3/2}}{3/2}\right)+C = \frac{1}{6}(x^4+2)^{3/2}+C$

$\int x(4x^2+3)^4dx$ $\quad u = 4x^2+3$, $du = 8x\,dx$, $dx = \frac{du}{8x}$

$\int \cancel{x}\, u^4 \frac{du}{8x}$

$\frac{1}{8}\int u^4 du = \frac{1}{8}\left(\frac{u^5}{5}\right)+C$
$= \frac{1}{40}(4x^2+3)^5+C$

$\int(5\cos 5x)dx$ $\quad u = 5x$, $du = 5dx$, $dx = \frac{du}{5}$

$\cancel{5}\int \cos u \frac{du}{\cancel{5}}$

$\int \cos u\, du = \sin u + C = \sin 5x + C$

$\int \frac{x^2}{(16-x^3)^2}dx$ $\quad u = 16-x^3$, $du = -3x^2dx$, $dx = \frac{du}{-3x^2}$

$\int \cancel{x^2}\cdot u^{-2}\frac{du}{-3x^2}$

$-\frac{1}{3}\int u^{-2}du = -\frac{1}{3}\left(\frac{u^{-1}}{-1}\right)+C$
$= \frac{1}{3}(16-x^3)^{-1}+C$

A $\int \frac{10x^2}{\sqrt{1+x^3}}dx$ $\quad u = 1+x^3$, $du = 3x^2dx$, $dx = \frac{du}{3x^2}$

$\int 10x^2\cdot u^{-\frac{1}{2}}\frac{du}{3\cancel{x^2}}$

$\frac{10}{3}\int u^{-\frac{1}{2}}du = \frac{10}{3}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right)+C = \frac{20}{3}(1+x^3)^{\frac{1}{2}}+C$

A $\int x\sin(x^2)dx$ $\quad u = x^2$, $du = 2x\,dx$, $dx = \frac{du}{2x}$

$\int \cancel{x}\sin u \frac{du}{2x}$

$\frac{1}{2}\int \sin u\, du = \frac{1}{2}(-\cos u)+C$
$= -\frac{1}{2}\cos x^2 + C$

$\int \sqrt{\cot x}\csc^2 x\,dx$ $\quad u = \cot x$, $du = -\csc^2 x\,dx$, $dx = \frac{du}{-\csc^2 x}$

$\int u^{\frac{1}{2}}\csc^2 x \frac{du}{-\csc^2 x}$

$-1\int u^{\frac{1}{2}}du = -1\left(\frac{u^{3/2}}{3/2}\right)+C$

$-\frac{2}{3}(\cot x)^{3/2}+C$

$\int \frac{e^{\frac{1}{x}}}{x^2}dx$ $\quad u = x^{-1}$, $du = -x^{-2}dx$, $dx = \frac{du}{-x^{-2}} = -x^2du$

$\int e^u\cdot\frac{1}{\cancel{x^2}}\cdot -x^2du$

$-1\int e^u du = -e^u+C = -e^{\frac{1}{x}}+C$

$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}}(\csc(2x)\cot(2x))dx$ $\quad u = 2x$, $du = 2dx$, $dx = \frac{du}{2}$ $\quad u\left(\frac{\pi}{12}\right) = 2\left(\frac{\pi}{12}\right) = \frac{\pi}{6}$, $u\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

$\int_{\frac{\pi}{6}}^{\pi/2}\csc u\cot u\,du =$

$-\csc u\Big|_{\pi/6}^{\pi/2} = -\csc\frac{\pi}{2} - \left(-\csc\frac{\pi}{6}\right)$
$= -1+2 = 1$

$\int_0^2 \frac{x}{\sqrt{1+2x^2}}dx$ $\quad u = 1+2x^2$, $du = 4x\,dx$, $dx = \frac{du}{4x}$ $\quad u(0) = 1$, $u(2) = 9$

$\int_1^9 x\cdot u^{-\frac{1}{2}}\frac{du}{4x} = \frac{1}{4}\int_1^9 u^{-1/2}du$

$= \left(\frac{1}{4}\right)\left(\frac{u^{1/2}}{1/2}\right)\Big|_1^9 = \frac{1}{2}u^{\frac{1}{2}}\Big|_1^9 = \frac{3}{2}-\frac{1}{2} = 1$

Double Substitution – Substitute for both the function and the variable

$\int x\sqrt{2x+1}\,dx$ $\quad u=2x+1 \quad x=\frac{u-1}{2}$

$du=2\,dx$

$dx=\frac{du}{2}$

$\int\left(\frac{u-1}{2}\right)u^{\frac{1}{2}}\frac{du}{2}$

$\frac{1}{4}\int u^{3/2}-u^{\frac{1}{2}}\,du=\frac{1}{4}\left(\frac{u^{5/2}}{5/2}-\frac{u^{3/2}}{3/2}\right)=\frac{1}{10}(2x+1)^{5/2}-\frac{1}{6}(2x+1)^{\frac{3}{2}}+C$

$\int\frac{2x-1}{\sqrt{x+3}}\,dx$ $\quad u=x+3 \quad x=u-3$

$du=dx$

$\int\frac{2(u-3)-1}{u^{1/2}}\,du$

$\int\frac{2u-7}{u^{1/2}}\,du=\int 2u^{1/2}-7u^{-1/2}\,du$

$=2\left(\frac{u^{3/2}}{3/2}\right)-7\left(\frac{u^{1/2}}{1/2}\right)+C=\frac{4}{3}(x+3)^{3/2}-14(x+3)^{\frac{1}{2}}+C$

Space

An even function is symmetric with respect to the y-axis. How could this help us evaluate $\int_{-a}^{a} f(x)dx$ if $f(x)$ is even?

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$

An odd function is symmetric with respect to the x-axis. How could this help us evaluate $\int_{-a}^{a} f(x)dx$ if $f(x)$ is odd?

origin

It will be 0.