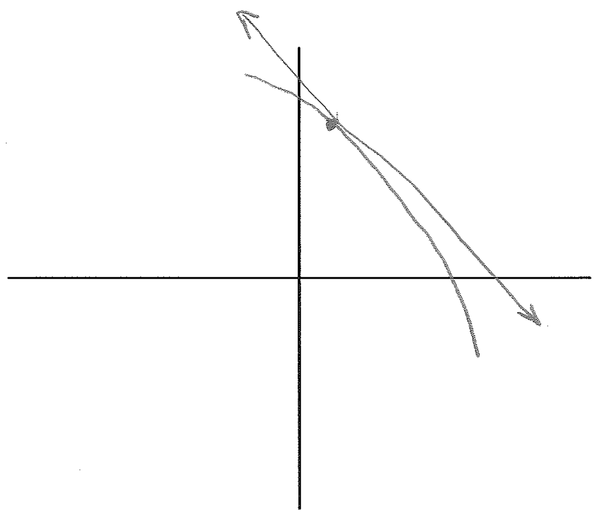
3.7 Differentials

Blast From the Past

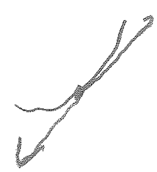
What was Newton's Method?



Used tangent line to approximate zeroes

Examples – Tangent Line Approximations

Find the equation of the tangent line to the function $f(x) = 1 + sinx$ at the point (0,1). Use linear approximation to complete the table and compare this to the actual values of the function.



$f'(x) = \cos x$

$f'(0) = \cos(0) = 1$

$y - 1 = 1(x - 0)$

$y = x + 1$

x	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
$y = 1 + sinx$	.52057	.90017	.99	1	1.01	1.0998	1.4794
$y =$ x+1	.5	.9	.99	1	1.01	1.1	1.5

Use tangent line approximation to find 1.99^5

$f(x) = x^5$ (2, 32)

$f'(x) = 5x^4$

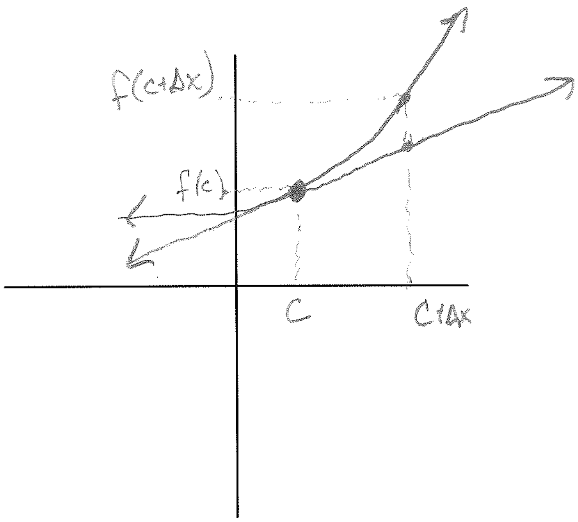
$f'(2) = 5(16) = 80$

$y - 32 = 80(x - 2)$

$y = 80x - 128$

$y = 80(1.99) - 128 \approx 31.2$

Differentials



$\Delta y = y_2 - y_1$

$f'(c)\Delta x$

$dy =$

Δx: change in x

Δy: change in y

dx: differential of x

dy: differential of y

*As Δx becomes smaller and smaller, what happens to the relationship between Δy and dy?

Example: Comparing Δy and dy

Compare Δy and dy for each of the functions below.

$y = 1 - 2x^2,\ x = 0,\ \Delta x = dx = -0.1$ $\qquad y' = -4x$

$\Delta y = (1-2(0)^2) - (1-2(-.1)^2) = 1 - .98 = .02$

$dy = (-4(0))(-.1) = 0$

$y = \sqrt{x},\ x = 4,\ \Delta x = dx = 0.1$ $\qquad y' = \frac{1}{2}x^{-\frac{1}{2}}$

$\Delta y = \sqrt{4.1} - \sqrt{4} = 2.02 - 2 = .02$

$dy = \left[\frac{1}{2}(4)^{-\frac{1}{2}}\right]\left[.01\right] = .025$

Calculating Differentials

$$y = x^n$$
$$\frac{dy}{dx} = nx^{n-1} \qquad dy = (nx^{n-1})dx$$

$$y = u \cdot v$$
$$\frac{dy}{dx} = u \cdot v' + u' \cdot v \qquad dy = (u \cdot v' + u'v)dx$$

Examples – Finding Differentials

Find the differential, dy, of each function.

$y = 2x^{\frac{3}{2}}$ $\qquad$ $y = xsinx$ $\qquad$ $y = sec3x^2$

$dy = 3x^{\frac{1}{2}}dx$ $\qquad$ $dy = (x\cos x + \sin x)\,dx$ $\qquad$ $dy = (6x\sec 3x^2 \tan 3x^2)dx$

Using Differentials to Approximate Function Values

$$\Delta y = f(x + \Delta x) - f(x) \approx dy$$

$\therefore f(x + \Delta x) \approx f(x) + dy$

Since $dy = f'(x)dx$

$f(x + \Delta x) \approx f(x) + f'(x)dx$

Examples – Using Differentials to approximate values

Use differentials to approximate each of the values below

$\sqrt[3]{8.7}$ $\quad f(x)=\sqrt[3]{x} \qquad x=8 \qquad \Delta x=.7$

$f'(x)=\frac{1}{3}x^{-2/3}$

$f(x)+f'(x)(\Delta x)$

$\sqrt[3]{8}+\frac{1}{3}(8)^{-2/3}\left(\frac{7}{10}\right) = 2+\left(\frac{1}{3}\right)\left(\frac{1}{4}\right)\left(\frac{7}{10}\right) = 2+\frac{7}{120} = 2\frac{7}{120}$

$(2.99)^3$ $\quad f(x)=x^3 \qquad x=3 \qquad \Delta x=-.01$

$f'(x)=3x^2$

$f(3)+f'(3)(-.01) = 27+27\left(-\frac{1}{100}\right) \approx 26.73$

Error Propagation

If x represents the measured value and $x+\Delta x$ represents the exact value then Δx is the error in measurement.

If x is then used in a calculation to compute $f(x)$, then the difference between $f(x+\Delta x)$ and $f(x)$ is the propagated error.

$$\Delta y = f(x+\Delta x) - f(x) \approx dy$$

Relative Error is found by comparing dy with y, this ratio is usually given as a percent error.

Examples – Error Propagation

The measured length of one side of a wooden cube is 4 inches. The measurement is correct to within 0.02 inch. Estimate the propagated error in the volume of the cube.

$x=4 \qquad \Delta x = \pm .02$

$V = x^3$

$dV = 3x^2 dx$

$dV = 3(4)^2(\pm .02) \approx \pm .96 \text{ in}^3$

Find the relative error in the calculation of the volume.

$\frac{.96}{64} \approx .015 \approx 1.5\%$

Estimate the propagated error in the surface area of the cube.

$x=4 \qquad dx = \pm .02$

$S = 6x^2$

$dS = 12x\,dx$

$dS = 12(4)(\pm .02) = \pm 0.96 \text{ in}^2$

Find the relative error in the calculation of the surface area.

$\frac{.96}{96} \approx .01 = 1\%$

In which calculation did any possible error in measurement seem to have the greatest affect? Is this what you would expect? Why or why not?