4.6 The Natural Logarithmic Function: Integration

Log Rule for Integration

Let u be a differentiable function of x.

$\int \frac{1}{x} dx = \ln|x| + C$

$\int \frac{1}{u} du = ln|u| + C$ Or $\int \frac{u'}{u} dx = ln|u| + C$

$\int \frac{x^2}{3 - x^3} dx$

$u = 3 - x^3$

$du = -3x^2 dx$

$dx = \frac{du}{-3x^2}$

$\int x^2 \cdot \frac{1}{u} \cdot \frac{du}{-3x^2} = -\frac{1}{3} \int \frac{1}{u} du$

$= -\frac{1}{3} \ln|3 - x^3| + C$

Some Other Things we can try

1. "Double" Substitution, substitute for both the function and the variable
2. Use long division to separate the terms
3. Add and subtract a constant

$\int \frac{1}{1+\sqrt{2x}} dx$

$u = 1 + (2x)^{\frac{1}{2}}$ $(2x)^{\frac{1}{2}} = u - 1$

$du = \frac{1}{2}(2x)^{-\frac{1}{2}}(2) dx$

$dx = (2x)^{\frac{1}{2}} du$

$\int \frac{1}{u} \cdot (u-1) du$

$\int \frac{u-1}{u} du$

$\int 1 - \frac{1}{u} du$

$u - \ln|u| + C$

$(1 + (2x)^{\frac{1}{2}}) - \ln|1 + (2x)^{\frac{1}{2}}| + C$

$\int \frac{2x}{(x+1)^2} dx$

$u = x + 1$ $x = u - 1$

$du = dx$

$\int \frac{2(u-1)}{u^2} du$

$\int \frac{2u - 2}{u^2} du = \int \frac{2}{u} - \frac{2}{u^2} du$

$= 2\ln|u| - 2\left(\frac{u^{-1}}{-1}\right) + C$

$= 2\ln|x+1| + \frac{2}{(x+1)} + C$

Trigonometry Integration Rules

$\int sinu\, du =$ $-\cos u + C$

$\int cosu\, du =$ $\sin u + C$

$\int \tan u\, du =$ $-\ln|\cos u| + C$

$\int cotu\, du =$ $\ln|\sin u| + C$

$\int \sec u\, du =$ $\ln|\sec u + \tan u| + C$

$\int cscu\, du =$ $-\ln|\csc u + \cot u| + C$

$\int \tan u\, du = \int \frac{\sin u}{\cos u} du$

$w = \cos u$
$dw = -\sin u\, du$
$du = \frac{dw}{-\sin u}$

$= \int \sin u \cdot \frac{1}{u} \cdot \frac{dw}{-\sin u}$

$= -1\int \frac{1}{u} du = -\ln|u| + C = -\ln|\cos u| + C$

Examples – Integration

$\int \frac{1}{x-5} dx$

$\ln|x-5| + C$

$\int \frac{x}{\sqrt{9-x^2}} dx$

$u = 9 - x^2$
$du = -2x\, dx$
$dx = \frac{du}{-2x}$

$\int \cancel{x} \cdot u^{-\frac{1}{2}} \cdot \frac{du}{-2\cancel{x}}$

$-\frac{1}{2}\int u^{-\frac{1}{2}} du = -\frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) + C$

$= -(9-x^2)^{1/2} + C$

$\int \frac{1}{xln(x^2)} dx$

$u = \ln x^2$
$du = \frac{1}{x^2} \cdot 2x\, dx$
$du = \frac{2}{x} dx$
$dx = \frac{x\, du}{2}$

$\int \frac{1}{x} \cdot \frac{1}{u} \cdot \frac{x\, du}{2}$

$\frac{1}{2}\int \frac{1}{u} du$

$\frac{1}{2}\ln|\ln x^2| + C$

$\int \frac{1}{x^{\frac{2}{3}}\left(1+x^{\frac{1}{3}}\right)} dx$

$u = 1 + x^{\frac{1}{3}}$
$du = \frac{1}{3}x^{-2/3} dx$
$dx = 3x^{2/3} du$

$\int \frac{1}{x^{2/3}} \cdot \frac{1}{u} \cdot 3x^{2/3} du$

$3\int \frac{1}{u} du = 3\ln\left|1 + x^{\frac{1}{3}}\right| + C$

$\int \frac{2cos2x}{sin2x} dx$

$u = \sin 2x$
$du = \cos 2x \cdot 2\, dx$
$dx = \frac{du}{2\cos 2x}$

$\int 2\cos 2x \cdot \frac{1}{u} \cdot \frac{du}{2\cos 2x}$

$\ln|u| + C = \ln|\sin 2x| + C$

$\int \frac{2x^2-3}{2x^3-9x} dx$

$u = 2x^3 - 9x$
$du = 6x^2 - 9\, dx$
$dx = \frac{du}{3(2x^2-3)}$

$\int (2x^2-3) \cdot \frac{1}{u} \cdot \frac{du}{3(2x^2-3)}$

$\frac{1}{3}\int \frac{1}{u} du = \frac{1}{3}\ln|2x^3 - 9x| + C$

$\int \frac{x(x-2)}{(x-1)^3}dx = \int \frac{(x^2-2x+1)-1}{(x-1)^3}dx$

$= \int \frac{(x-1)^2-1}{(x-1)^3}dx = \int \frac{(x-1)^2}{(x-1)^3}dx - \int \frac{1}{(x-1)^3}dx$

$= \int \frac{1}{(x-1)}dx - \int \frac{1}{(x-1)^3}dx = \ln|x-1| - (-\frac{1}{2})(x-1)^{-2} + C$

$= \ln|x-1| + \frac{1}{2}(x-1)^{-2} + C$

$\int \tan\theta\, d\theta = -\ln|\cos\theta| + C$

$\int \frac{\sin x}{1+\cos x}dx$

$u = 1+\cos x$

$du = -\sin x\,dx$

$dx = \frac{du}{-\sin x}$

$\int \sin x \cdot \frac{1}{u}\cdot\frac{du}{-\sin x}$

$-1\int \frac{1}{u}du = -1\ln|1+\cos x| + C$

$\int \frac{x^2+x+1}{x^2+1}dx$

$$\begin{array}{r} 1 + \frac{x}{x^2+1} \\ x^2+1\,\overline{)\,x^2+x+1} \\ -(x^2 \quad\; +1) \\ \hline x \end{array}$$

$\int 1\,dx + \int \frac{x}{x^2+1}dx$

$u = x^2+1$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$x + \int x \cdot \frac{1}{u}\cdot\frac{du}{2x}$

$x + \frac{1}{2}\ln|x^2+1| + C$

$\int \sec\frac{x}{2}\,dx$

$u = \frac{1}{2}x$

$du = \frac{1}{2}dx$

$dx = 2du$

$\int \sec u\,(2du)$

$2\ln|\sec\frac{1}{2}x + \tan\frac{1}{2}x| + C$

$\int_0^1 \frac{x-1}{x+1}dx$

$$\begin{array}{r} 1 + \frac{-2}{x+1} \\ x+1\,\overline{)\,x-1} \\ -(x+1) \\ \hline -2 \end{array}$$

$\int_0^1 1\,dx + \int_0^1 \frac{-2}{x+1}dx$

$x\Big|_0^1 + -2\ln|x+1|\Big|_0^1 =$

$1 - 0 - 2\ln 2 + 2\ln 1 = 1 - 2\ln 2$

$\int_1^2 \frac{1-\cos x}{x-\sin x}dx$

$u = x - \sin x$

$du = 1-\cos x\,dx$

$dx = \frac{du}{1-\cos x}$

$u(2) = 2-\sin 2$

$u(1) = 1-\sin 1$

$\int_{1-\sin 1}^{2-\sin 2}(1-\cos x)\cdot\frac{1}{u}\cdot\frac{du}{1-\cos x}$

$\ln|u|\Big|_{1-\sin 1}^{2-\sin 2} = \ln|2-\sin 2| - \ln|1-\sin 1|$

$\int_0^1 \ln e^{x^2}dx = \int_0^1 x^2\,dx$

$\frac{1}{3}x^3\Big|_0^1 = \frac{1}{3} - 0 = \frac{1}{3}$

Find the average value of $f(x) = \csc x$ on the interval $\left[\frac{\pi}{6}, \frac{\pi}{4}\right]$.

$\frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\int_{\pi/6}^{\pi/4}\csc x\,dx = \frac{1}{\frac{\pi}{12}}\left(-\ln|\csc x + \cot x|\right)\Big|_{\pi/6}^{\pi/4}$

$\frac{12}{\pi}\left(-\ln(\sqrt{2}+1) - (-\ln(2+\sqrt{3}))\right)$

$\frac{12}{\pi}\left(-\ln(\sqrt{2}+1) + \ln(2+\sqrt{3})\right) = \frac{12}{\pi}\ln\frac{(2+\sqrt{3})}{(\sqrt{2}+1)}$