4.2, 4.3 Area and Reimann Sums Day 3

Pulling it all together

Find the limit of $s(n)$ as $n \to \infty$. Hint: Remember the formulas from a couple of days ago.

$$s(n) = \frac{64}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$\lim_{n\to\infty} \frac{64}{6}\left[\frac{n(n+1)(2n+1)}{n^3}\right] = \lim_{n\to\infty} \frac{32}{3}\left[\frac{n(2n^2+2n+n+1)}{n^3}\right] = \lim_{n\to\infty} \frac{32}{3}\left[\frac{2n^3+3n^2+n}{n^3}\right]$$

$$= \lim_{n\to\infty} \frac{32}{3}\left[2+\frac{3}{n}+\frac{1}{n^2}\right] = \left(\frac{32}{3}\right)(2) = \frac{64}{3}$$

Find a formula for the sum of n terms, then use the formula to find the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n}\right)\left(\frac{2}{n}\right)$$

$$\lim_{n\to\infty} \frac{4}{n^2}\sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{4}{n^2}\left(\frac{n(n+1)}{2}\right) = \lim_{n\to\infty} (2)\left(\frac{n^2+n}{n^2}\right) = \lim_{n\to\infty} (2)\left(1+\frac{1}{n}\right)$$

$$= (2)(1) = 2$$

Riemann Sum

A Reimann sum, S_n, for function f on the interval $[a, b]$ is a sum of the form

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x$$

Where the interval $[a, b]$ is partitioned into n subintervals of widths Δx (so $\Delta x = \frac{b-a}{n}$) and the numbers c_k are sample points, one in each subinterval (left hand, right hand, midpoint, or any other point in the subinterval)

- If each sample point is picked so that $f(c)$ is the lowest point in its respective subinterval, then each rectangle has an area that is less than the actual area. In this case the Riemann sum is called a lower sum.
- An upper sum is a Reimann sum with each sample point taken where $f(c)$ is the largest point in its respective subinterval
- A midpoint sum is formed by choosing each sample point at the midpoint of the respective subinterval

*An upper sum is an upper bound for the area of the region and a lower sum is a lower bound. The actual area must be somewhere between the two.

As n increases, L_n and R_n get closer to each other and to the actual area **. The exact area is** $\lim_{n\to\infty} R_n$ **or** $\lim_{n\to\infty} L_n$ **or** $\lim_{n\to\infty}$ (any point)

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x$$

Examples – Using the limit process to find the area under the curve.

Find the area under the curve $y = 2x^2 - x^3$ $[0,1]$ using the limit process.

$\Delta x = \frac{1-0}{n} = \frac{1}{n}$

$a + i\Delta x$

$0 + i(\frac{1}{n})$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[2\left(\frac{i}{n}\right)^2 - \left(\frac{i}{n}\right)^3\right]\left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i^2}{n^3} - \frac{i^3}{n^4}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n^3} \sum_{i=1}^{n} i^2 - \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \left(\frac{2}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) - \left(\frac{1}{n^4}\right)\left(\frac{n^2(n+1)^2}{4}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{n^3}\right)\left(\frac{2n^3 + 3n^2 + n}{6}\right) - \left(\frac{1}{n^4}\right)\left(\frac{n^2(n^2+2n+1)}{4}\right) = \lim_{n\to\infty} \left(\frac{2n^3 + 3n^2 + n}{3n^3}\right) - \left(\frac{1}{n^4}\right)\left(\frac{n^4 + 2n^3 + n^2}{4}\right)$$

$$= \lim_{n\to\infty} \left(\frac{2}{3} + \frac{1}{n} + \frac{1}{3n^2}\right) - \left(\frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}\right) = \frac{2}{3} - \frac{1}{4} = \frac{8}{12} - \frac{3}{12} = \frac{5}{12}$$

Find the area under the curve of $y = 3x - 4$ $[2,5]$ using the limit process.

$\frac{5-2}{n} = \frac{3}{n}$

$a + i\Delta x$

$2 + i(\frac{3}{n})$

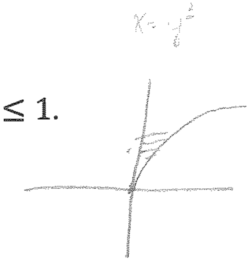
$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[3\left(2 + \frac{3i}{n}\right) - 4\right]\left[\frac{3}{n}\right]$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(6 + \frac{9i}{n}\right) - 4\right]\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{18}{n} + \frac{27i}{n^2} - \frac{12}{n}$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 6 + \lim_{n\to\infty} \frac{27}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \left(\frac{1}{n}\right)(6n) + \left(\frac{27}{n^2}\right)\left(\frac{n(n+1)}{2}\right)$$

$$= \lim_{n\to\infty} 6 + \frac{27}{2}\left(\frac{n^2+n}{n^2}\right) = 6 + \frac{27}{2} = \frac{12}{2} + \frac{27}{2} = \frac{39}{2}$$

$\frac{1-0}{n} = \frac{1}{n}$ $\quad 0 + \frac{i}{n}$



Find the area of the region bounded by the graph of $f(y) = y^3$ and the $y - axis$ for $0 \le y \le 1$.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^3 \left(\frac{1}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4} = \lim_{n\to\infty} \frac{1}{n^4} \sum_{i=1}^{n} i^3$$

$$= \lim_{n\to\infty} \left(\frac{1}{n^4}\right)\left(\frac{n^2(n+1)^2}{4}\right) = \lim_{n\to\infty} \left(\frac{n^2(n^2+2n+1)}{4n^4}\right) = \lim_{n\to\infty} \frac{n^2+2n+1}{4n^2}$$

$$= \frac{1}{4}$$

Finding the area under the curve of $y = 3x - 4$ $[2,5]$ can be represented by what definite integral?

$$\int_2^5 (3x-4)\,dx$$

Therefore...

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$

Examples – Evaluating Definite Integrals using the limit definition.

Evaluate using the limit definition

$$\int_0^3 (x^2+1)dx \qquad \frac{3-0}{n} = \frac{3}{n} \qquad 0 + \frac{3i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^2 + 1\right]\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{9i^2}{n^2} + 1\right)\left(\frac{3}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{27i^2}{n^3} + \frac{3}{n}$$

$$= \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 + \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 3 = \lim_{n\to\infty} \left(\frac{27}{n^3}\right)\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{3n}{n}$$

$$= \lim_{n\to\infty} \left(\frac{27}{n^3}\right)\left(\frac{2n^3+3n^2+n}{6}\right) + 3 = \lim_{n\to\infty} \left(\frac{9}{2}\right)\left(\frac{2n^3+3n^2+n}{n^3}\right) + 3$$

$$= \lim_{n\to\infty} 9 + \frac{27}{2n} + \frac{9}{2n^2} + 3 = 12$$