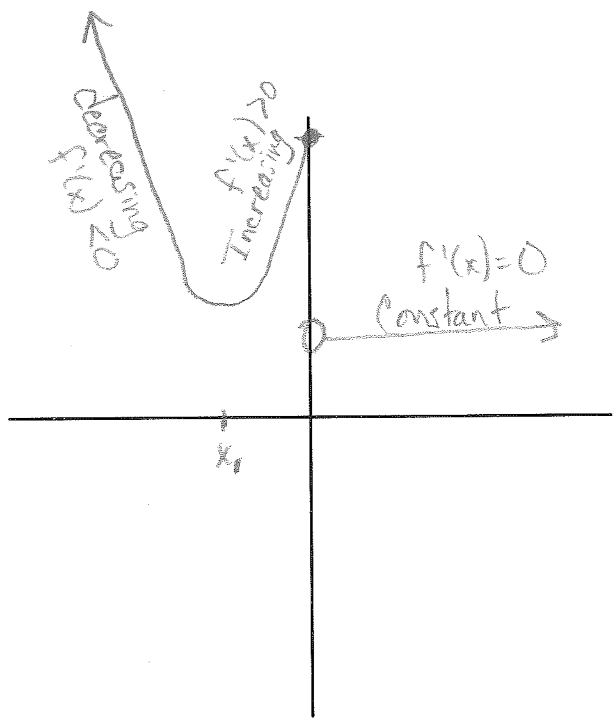
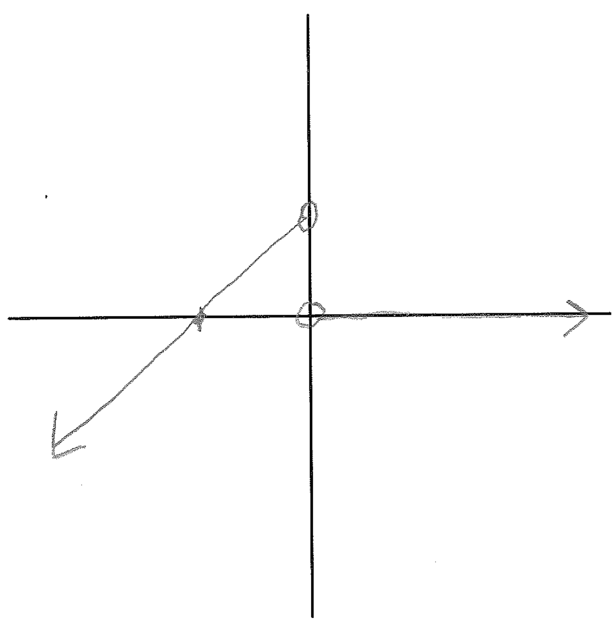
3.3 Increasing and Decreasing Functions and the First Derivative Test



Sketch a graph of the derivative of $f(x)$



Test for Increasing and Decreasing Functions (Theorem 3.5)

1. If $f'(x) > 0$ for all x in (a,b), then f is increasing on (a,b).
2. If $f'(x) < 0$ for all x in (a,b), then f is decreasing on (a,b).
3. If $f'(x) = 0$ for all x in (a,b), then f is constant on (a,b)

How To Determine Where a Function is Increasing/Decreasing

1. Locate critical numbers of f, use these to create intervals
2. Test the sign of $\underline{f'(x)}$ in each interval
3. Use theorem 3.5 to determine increasing/decreasing intervals. ☆ Never use "if" in your descriptions.

$$f(x) = x^3 - \frac{3}{2}x^2$$

① $f'(x) = 3x^2 - 3x$

$0 = 3x(x-1)$ $x=0$ $x=1$

② $f'(x)$: $+$ on $(-\infty, 0)$, $-$ on $(0, 1)$, $+$ on $(1, \infty)$

③ $f(x)$ is increasing $(-\infty, 0) \cup (1, \infty)$ because $f'(x) > 0$

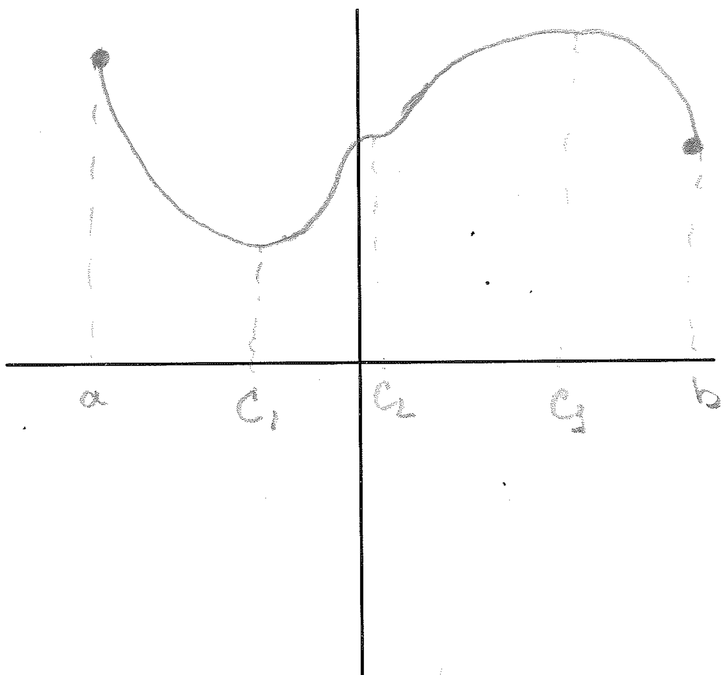
$f(x)$ is decreasing $(0,1)$ because $f'(x) < 0$

Strictly Monotonic: A function that is increasing or decreasing on its entire domain.

The First Derivative Test (Theorem 3.6)

Let c be a critical number of a function f that is continuous on the open interval I containing c. If f is differentiable on the interval, except possibly at c, then $f(c)$ can be classified as follows:

1. If $f'(x)$ changes from negative to positive at c, then f has a relative minimum at c
2. If $f'(x)$ changes from positive to negative at c, then f has a relative maximum at c.
3. If $f'(x)$ does not change sign at c it is neither a relative maximum or minimum.



f has a relative minimum at c_1, $f'(x)<0$ (a, c_1) $f'(x)>0$ (c_1, c_2), $f'(c_1)=0$

f has a relative maximum at c_3, $f'(x)>0$ (c_2, c_3), $f'(x)<0$, (c_3, b), $f'(c_3)=0$

Examples: Increasing/Decreasing and First Derivative Test

Find the open intervals on which $f(x) = x^4 - 8x^2$ is increasing or decreasing.

$f'(x) = 4x^3 - 16x$

$0 = 4x(x^2-4)$

$0 = 4x(x-2)(x+2)$ $x=0$ $x=2$ $x=-2$

$f'(x)$ − + − +

−2 0 2

$f(x)$ is increasing $(-2,0) \cup (2,\infty)$, $f'(x) > 0$

$f(x)$ is decreasing $(-\infty,-2) \cup (0,2)$, $f'(x) < 0$

Determine if the following functions are strictly monotonic.

a. $f(x) = e^{-x}$, $(-\infty, \infty)$

$f'(x) = -e^{-x}$, no critical numbers, $f'(x) < 0$, strictly monotonic

b. $f(x) = \frac{1}{x^2}$, $(-2,2)$

$f'(x) = -2x^{-3} = \frac{-2}{x^3}$

+ −

−2 0 2

Not strictly monotonic

c. $f(x) = \ln(x+4)$, $(-4, \infty)$

$f'(x) = \frac{1}{x+4}$

+

−4

Strictly monotonic

Find the relative extrema of $f(x) = 4cos\left(\frac{3}{2}x\right)$ on the interval $(0,2\pi)$.

$f'(x) = 4(-\sin(\frac{3}{2}x))(\frac{3}{2}) = -6\sin\frac{3}{2}x$

$0 = -6\sin\frac{3}{2}x$

$\frac{3}{2}x = 0$ $\frac{3}{2}x = \pi$

$\frac{3}{2}x = 0 + 2\pi n$ $\frac{3}{2}x = \pi + 2\pi n$

$x = \frac{2}{3}\pi + \frac{4\pi n}{3}$

$x=0$ $x = \frac{4\pi}{3}$ $x = \frac{2}{3}\pi$ $x = 2\pi$

− + −

0 $\frac{2}{3}\pi$ $\frac{4\pi}{3}$ 2π

$(\frac{2\pi}{3}, -4)$ Rel. min

$f'(\frac{2\pi}{3}) = 0$, $f'(x) < 0$ $(0, \frac{2}{3}\pi)$; $f'(x) > 0$, $(\frac{2}{3}\pi, \frac{4\pi}{3})$

$(\frac{4\pi}{3}, 4)$ Rel Max

$f'(\frac{4\pi}{3}) = 0$, $f'(x) > 0$ $(\frac{2}{3}\pi, \frac{4\pi}{3})$; $f'(x) < 0$ $(\frac{4\pi}{3}, 2\pi)$

Find the relative extrema of $f(x) = -(x^4-1)^{\frac{2}{5}}$

$f'(x) = -\frac{2}{5}(x^4-1)^{-3/5}(4x^3)$

$= \frac{-8x^3}{5(x^4-1)^{3/5}}$ $\quad x=0$ $\quad x=1,-1$

$+ \quad -1 \quad - \quad 0 \quad + \quad 1 \quad -$

• $(-1,0)$ Rel Max

$f'(-1)$ is undefined, $f'(x)>0$, $(-\infty,-1)$

$f'(x)<0$, $(-1,0)$

• $(0,-1)$ Rel Min

$f'(0)=0$, $f'(x)<0$, $(-1,0)$

$f'(x)>0$ $(0,1)$

• $(1,0)$ Rel Max

$f'(1)$ is undefined, $f'(x)>0$, $(0,1)$

$f'(x)<0$, $(1,\infty)$

Find the relative extrema of $f(x) = \frac{x^4}{x^2-1}$

$f'(x) = \frac{(x^2-1)(4x^3)-(x^4)(2x)}{(x^2-1)^2} = \frac{2x^3(2x^2-2-x^2)}{(x^2-1)^2} = \frac{2x^3(x^2-2)}{(x^2-1)^2}$ $\quad x=0, x=\pm\sqrt{2}$ $\quad x=\pm 1$

$- \quad -\sqrt{2} \quad + \quad -1 \quad + \quad 0 \quad - \quad 1 \quad - \quad \sqrt{2} \quad +$

• Rel Min $(-\sqrt{2},4)$, $f'(-\sqrt{2})=0$, $f'(x)<0$ $(-\infty,-\sqrt{2})$

$f'(x)>0$, $(-\sqrt{2},-1)$

• Rel Max $(0,0)$, $f'(0)=0$, $f'(x)>0$, $(-1,0)$, $f'(x)<0$ $(0,1)$

• Rel Min $(\sqrt{2},4)$, $f'(\sqrt{2})=0$, $f'(x)<0$, $(1,\sqrt{2})$

$f'(x)>0$, $(\sqrt{2},\infty)$

The derivative of $f(x)$ is $f'(x) = (x-2)(x-3)^2$ identify the x-values of any relative extrema.

$- \quad 2 \quad + \quad 3 \quad +$

Rel Min $x=2$, $f'(2)=0$, $f'(x)<0$, $(-\infty,2)$

$f'(x)>0$, $(2,3)$