Section 2.5 Implicit Differentiation

1. $x^2 + y^2 = 9$

$2x + 2yy' = 0$

$y' = -\dfrac{x}{y}$

2. $x^2 - y^2 = 25$

$2x - 2yy' = 0$

$y' = \dfrac{x}{y}$

3. $x^{1/2} + y^{1/2} = 16$

$\dfrac{1}{2}x^{-1/2} + \dfrac{1}{2}y^{-1/2}y' = 0$

$y' = -\dfrac{x^{-1/2}}{y^{-1/2}}$

$= -\sqrt{\dfrac{y}{x}}$

4. $2x^3 + 3y^3 = 64$

$$6x^2 + 9y^2y' = 0$$
$$9y^2y' = -6x^2$$
$$y' = \frac{-6x^2}{9y^2} = -\frac{2x^2}{3y^2}$$

5. $x^3 - xy + y^2 = 7$

$$3x^2 - xy' - y + 2yy' = 0$$
$$(2y - x)y' = y - 3x^2$$
$$y' = \frac{y - 3x^2}{2y - x}$$

6. $x^2y + y^2x = -2$

$$x^2y' + 2xy + y^2 + 2yxy' = 0$$
$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$
$$y' = \frac{-y(y + 2x)}{x(x + 2y)}$$

7. $x^3y^3 - y - x = 0$

$$3x^3y^2y' + 3x^2y^3 - y' - 1 = 0$$
$$(3x^3y^2 - 1)y' = 1 - 3x^2y^3$$
$$y' = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

8. $\sqrt{xy} = x^2y + 1$

$$\frac{1}{2}(xy)^{-1/2}(xy' + y) = 2xy + x^2y'$$
$$\frac{x}{2\sqrt{xy}}y' + \frac{y}{2\sqrt{xy}} = 2xy + x^2y'$$
$$\left(\frac{x}{2\sqrt{xy}} - x^2\right)y' = 2xy - \frac{y}{2\sqrt{xy}}$$
$$y' = \frac{2xy - \frac{y}{2\sqrt{xy}}}{\frac{x}{2\sqrt{xy}} - x^2}$$
$$y' = \frac{4xy\sqrt{xy} - y}{x - 2x^2\sqrt{xy}}$$

9. $xe^y - 10x + 3y = 0$

$$xe^y\frac{dy}{dx} + e^y - 10 + 3\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(xe^y + 3) = 10 - e^y$$
$$\frac{dy}{dx} = \frac{10 - e^y}{xe^y + 3}$$

10. $e^{xy} + x^2 - y^2 = 10$

$$\left(x\frac{dy}{dx} + y\right)e^{xy} + 2x - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}(xe^{xy} - 2y) = -ye^{xy} - 2x$$
$$\frac{dy}{dx} = -\frac{ye^{xy} + 2x}{xe^{xy} - 2y}$$

11. $\sin x + 2\cos 2y = 1$

$$\cos x - 4(\sin 2y)y' = 0$$
$$y' = \frac{\cos x}{4\sin 2y}$$

12. $(\sin \pi x + \cos \pi y)^2 = 2$

$$2(\sin \pi x + \cos \pi y)[\pi \cos \pi x - \pi(\sin \pi y)y'] = 0$$
$$\pi \cos \pi x - \pi(\sin \pi y)y' = 0$$
$$y' = \frac{\cos \pi x}{\sin \pi y}$$

13. $\sin x = x(1 + \tan y)$

$$\cos x = x(\sec^2 y)y' + (1 + \tan y)(1)$$
$$y' = \frac{\cos x - \tan y - 1}{x \sec^2 y}$$

14. $\cot y = x - y$

$$(-\csc^2 y)y' = 1 - y'$$
$$y' = \frac{1}{1 - \csc^2 y} = \frac{1}{-\cot^2 y} = -\tan^2 y$$

15. $y = \sin xy$

$$y' = [xy' + y]\cos(xy)$$
$$y' - x\cos(xy)y' = y\cos(xy)$$
$$y' = \frac{y\cos(xy)}{1 - x\cos(xy)}$$

16. $x = \sec\frac{1}{y}$

$$1 = -\frac{y'}{y^2}\sec\frac{1}{y}\tan\frac{1}{y}$$
$$y' = \frac{-y^2}{\sec(1/y)\tan(1/y)} = -y^2\cos\left(\frac{1}{y}\right)\cot\left(\frac{1}{y}\right)$$

17. $x^2 - 3\ln y + y^2 = 10$

$$2x - \frac{3}{y}\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x = \frac{dy}{dx}\left(\frac{3}{y} - 2y\right)$$

$$\frac{dy}{dx} = \frac{2x}{(3/y) - 2y} = \frac{2xy}{3 - 2y^2}$$

18. $\ln(xy) + 5x = 30$

$$\ln x + \ln y + 5x = 30$$

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} + 5 = 0$$

$$\frac{1}{y}\frac{dy}{dx} = -\frac{1}{x} - 5$$

$$\frac{dy}{dx} = -\frac{y}{x} - 5y = -\left(\frac{y + 5xy}{x}\right)$$

19. $4x^3 + \ln y^2 + 2y = 2x$

$$12x^2 + \frac{2}{y}y' + 2y' = 2$$

$$\left(\frac{2}{y} + 2\right)y' = 2 - 12x^2$$

$$y' = \frac{2 - 12x^2}{2/y + 2}$$

$$y' = \frac{y - 6yx^2}{1 + y} = \frac{y(1 - 6x^2)}{1 + y}$$

20. $4xy + \ln x^2y = 7$

$$4xy + 2\ln x + \ln y = 7$$

$$4xy' + 4y + \frac{2}{x} + \frac{1}{y}y' = 0$$

$$\left(4x + \frac{1}{y}\right)y' = -4y - \frac{2}{x}$$

$$y' = \frac{-4y - \dfrac{2}{x}}{4x + \dfrac{1}{y}}$$

$$y' = \frac{-4xy^2 - 2y}{4x^2y + x}$$

21. The y-terms still need to be differentiated.

$$\frac{d}{dx}\left[4x^2 + 7x - 5y^2 + y\right] = 8x + 7 - 10y\frac{dy}{dx} + \frac{dy}{dx}$$

22. The derivative of e^y is $e^y\dfrac{dy}{dx}$.

$$\frac{d}{dx}\left[e^y + xy\right] = \frac{d}{dx}[4]$$

$$e^y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0$$

$$\left(e^y + x\right)\frac{dy}{dx} = -y$$

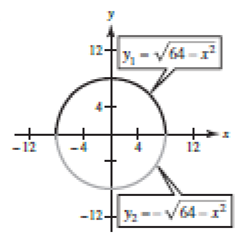
$$\frac{dy}{dx} = -\frac{y}{e^y + x}$$

23. (a) $x^2 + y^2 = 64$

$$y^2 = 64 - x^2$$

$$y = \pm\sqrt{64 - x^2}$$

(b)



(c) Explicitly: $\dfrac{dy}{dx} = \pm\dfrac{1}{2}\left(64 - x^2\right)^{-1/2}(-2x)$

$$= \frac{\mp x}{\sqrt{64 - x^2}}$$

$$= \frac{-x}{\pm\sqrt{64 - x^2}}$$

$$= -\frac{x}{y}$$

(d) Implicitly: $2x + 2yy' = 0$

$$y' = -\frac{x}{y}$$

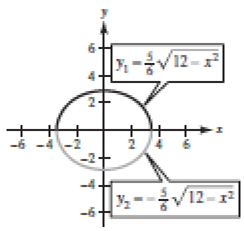
24. (a) $25x^2 + 36y^2 = 300$

$$36y^2 = 300 - 25x^2 = 25(12 - x^2)$$

$$y^2 = \frac{25}{36}(12 - x^2)$$

$$y = \pm\frac{5}{6}\sqrt{12 - x^2}$$

(b)



(c) Explicitly: $\dfrac{dy}{dx} = \pm\dfrac{5}{6}\left(\dfrac{1}{2}\right)(12 - x^2)^{-1/2}(-2x)$

$$= \mp\frac{5x}{6\sqrt{12 - x^2}}$$

$$= -\frac{25x}{36y}$$

(d) Implicitly: $50x + 72y \cdot y' = 0$

$$y' = \frac{-50x}{72y} = -\frac{25x}{36y}$$

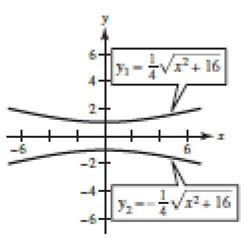
25. (a) $16y^2 - x^2 = 16$

$$16y^2 = x^2 + 16$$

$$y^2 = \frac{x^2}{16} + 1 = \frac{x^2 + 16}{16}$$

$$y = \frac{\pm\sqrt{x^2 + 16}}{4}$$

(b)



(c) Explicitly: $\dfrac{dy}{dx} = \dfrac{\pm\frac{1}{2}(x^2 + 16)^{-1/2}(-2x)}{4}$

$$= \frac{\pm x}{4\sqrt{x^2 + 16}} = \frac{\pm x}{4(\pm 4y)} = \frac{x}{16y}$$

(d) Implicitly: $16y^2 - x^2 = 16$

$$32yy' - 2x = 0$$

$$32yy' = 2x$$

$$y' = \frac{2x}{32y} = \frac{x}{16y}$$

26. (a)

$$x^2 + y^2 - 4x + 6y + 9 = 0$$

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) = -9 + 4 + 9$$

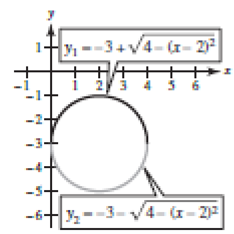
$$(x - 2)^2 + (y + 3)^2 = 4$$

$$(y + 3)^2 = 4 - (x - 2)^2$$

$$y + 3 = \pm\sqrt{4 - (x - 2)^2}$$

$$y = -3 \pm \sqrt{4 - (x - 2)^2}$$

(b)



(c) Explicitly:

$$\frac{dy}{dx} = \pm\frac{1}{2}\left[4 - (x - 2)^2\right]^{-1/2}\left[-2(x - 2)\right]$$

$$= \mp\frac{x - 2}{\sqrt{4 - (x - 2)^2}}$$

$$= -\frac{x - 2}{y + 3}$$

(d) Implicitly:

$$2x + 2yy' - 4 + 6y' = 0$$

$$2yy' + 6y' = -2x + 4$$

$$y'(2y + 6) = -2(x - 2)$$

$$y' = \frac{-2(x - 2)}{2(y + 3)} = -\frac{x - 2}{y + 3}$$

27.

$$xy = 6$$

$$xy' + y(1) = 0$$

$$xy' = -y$$

$$y' = -\frac{y}{x}$$

At $(-6, -1)$: $y' = -\dfrac{1}{6}$

28.

$$y^3 - x^2 = 4$$

$$3y^2y' - 2x = 0$$

$$y' = \frac{2x}{3y^2}$$

At $(2, 2)$: $y' = \dfrac{2(2)}{3(2^2)} = \dfrac{1}{3}$

29. $y^2 = \dfrac{x^2 - 49}{x^2 + 49}$

$$2yy' = \frac{(x^2 + 49)(2x) - (x^2 - 49)(2x)}{(x^2 + 49)^2}$$

$$2yy' = \frac{196x}{(x^2 + 49)^2}$$

$$y' = \frac{98x}{y(x^2 + 49)^2}$$

At $(7, 0)$: y' is undefined.

30. $$(x + y)^3 = x^3 + y^3$$

$$x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3$$

$$3x^2y + 3xy^2 = 0$$

$$x^2y + xy^2 = 0$$

$$x^2y' + 2xy + 2xyy' + y^2 = 0$$

$$(x^2 + 2xy)y' = -(y^2 + 2xy)$$

$$y' = -\frac{y(y + 2x)}{x(x + 2y)}$$

At $(-1, 1)$: $y' = -1$

31. $$\tan(x + y) = x$$

$$(1 + y')\sec^2(x + y) = 1$$

$$y' = \frac{1 - \sec^2(x + y)}{\sec^2(x + y)}$$

$$= \frac{-\tan^2(x + y)}{\tan^2(x + y) + 1}$$

$$= -\sin^2(x + y)$$

$$= -\frac{x^2}{x^2 + 1}$$

At $(0, 0)$: $y' = 0$

32. $$x \cos y = 1$$

$$x[-y' \sin y] + \cos y = 0$$

$$y' = \frac{\cos y}{x \sin y}$$

$$= \frac{1}{x}\cot y$$

$$= \frac{\cot y}{x}$$

At $\left(2, \dfrac{\pi}{3}\right)$: $y' = \dfrac{1}{2\sqrt{3}}$

33. $$3e^{xy} - x = 0$$

$$3e^{xy}[xy' + y] - 1 = 0$$

$$3e^{xy}xy' = 1 - 3ye^{xy}$$

$$y' = \frac{1 - 3ye^{xy}}{3xe^{xy}}$$

At $(3, 0)$: $y' = \dfrac{1}{9}$

34. $y^2 = \ln x$

$$2yy' = \frac{1}{x}$$

$$y' = \frac{1}{2xy}$$

At $(e, 1)$: $y' = \dfrac{1}{2e}$

35. $$(x^2 + 4)y = 8$$

$$(x^2 + 4)y' + y(2x) = 0$$

$$y' = \frac{-2xy}{x^2 + 4}$$

$$= \frac{-2x\left[8/(x^2 + 4)\right]}{x^2 + 4}$$

$$= \frac{-16x}{(x^2 + 4)^2}$$

At $(2, 1)$: $y' = \dfrac{-32}{64} = -\dfrac{1}{2}$

$\left(\text{Or, you could just solve for } y\text{: } y = \dfrac{8}{x^2 + 4}\right)$

36. $$(4 - x)y^2 = x^3$$

$$(4 - x)(2yy') + y^2(-1) = 3x^2$$

$$y' = \frac{3x^2 + y^2}{2y(4 - x)}$$

At $(2, 2)$: $y' = 2$

37. $(y - 3)^2 = 4(x - 5)$, $(6, 1)$

$$2(y - 3)y' = 4$$

$$y' = \frac{2}{y - 3}$$

At $(6, 1)$: $y' = \dfrac{2}{1 - 3} = -1$

Tangent line: $y - 1 = -1(x - 6)$

$$y = -x + 7$$

38. $7x^2 - 6\sqrt{3}xy + 13y^2 - 16 = 0, \quad (\sqrt{3}, 1)$

$$14x - 6\sqrt{3}xy' - 6\sqrt{3}y + 26yy' = 0$$

$$y' = \frac{6\sqrt{3}y - 14x}{26y - 6\sqrt{3}x}$$

At $(\sqrt{3}, 1)$: $y' = \dfrac{6\sqrt{3} - 14\sqrt{3}}{26 - 6\sqrt{3}\sqrt{3}} = \dfrac{-8\sqrt{3}}{8} = -\sqrt{3}$

Tangent line: $y - 1 = -\sqrt{3}(x - \sqrt{3})$

$$y = -\sqrt{3}x + 4$$

39. $x^2y^2 - 9x^2 - 4y^2 = 0, \quad (-4, 2\sqrt{3})$

$$x^2 2yy' + 2xy^2 - 18x - 8yy' = 0$$

$$y' = \frac{18x - 2xy^2}{2x^2y - 8y}$$

At $(-4, 2\sqrt{3})$: $y' = \dfrac{18(-4) - 2(-4)(12)}{2(16)(2\sqrt{3}) - 16\sqrt{3}}$

$$= \frac{24}{48\sqrt{3}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$$

Tangent line: $y - 2\sqrt{3} = \dfrac{\sqrt{3}}{6}(x + 4)$

$$y = \frac{\sqrt{3}}{6}x + \frac{8}{3}\sqrt{3}$$

40. $y^2(x^2 + y^2) = 2x^2, \quad (1, 1)$

$$y^2x^2 + y^4 = 2x^2$$

$$2yy'x^2 + 2xy^2 + 4y^3y' = 4x$$

At $(1, 1)$:

$$2y' + 2 + 4y' = 4$$

$$6y' = 2$$

$$y' = \tfrac{1}{3}$$

Tangent line: $y - 1 = \tfrac{1}{3}(x - 1)$

$$y = \tfrac{1}{3}x + \tfrac{2}{3}$$

41. $4xy = 9, \quad \left(1, \dfrac{9}{4}\right)$

$$4xy' + 4y = 0$$

$$xy' = -y$$

$$y' = \frac{-y}{x}$$

At $\left(1, \dfrac{9}{4}\right)$, $y' = \dfrac{-9/4}{1} = \dfrac{-9}{4}$

Tangent line: $y - \dfrac{9}{4} = \dfrac{-9}{4}(x - 1)$

$$4y - 9 = -9x + 9$$

$$4y + 9x = 18$$

$$y = \frac{-9}{4}x + \frac{9}{2}$$

42. $x^2 + xy + y^2 = 4, \quad (2, 0)$

$$2x + xy' + y + 2yy' = 0$$

$$(x + 2y)y' = -2x - y$$

$$y' = \frac{-2x - y}{x + 2y}$$

At $(2, 0)$, $y' = \dfrac{-4}{2} = -2$

Tangent line: $y - 0 = -2(x - 2)$

$$y = -2x + 4$$

43. $x + y - 1 = \ln(x^2 + y^2), \quad (1, 0)$

$$1 + y' = \frac{2x + 2yy'}{x^2 + y^2}$$

$$x^2 + y^2 + (x^2 + y^2)y' = 2x + 2yy'$$

At $(1, 0)$: $1 + y' = 2$

$$y' = 1$$

Tangent line: $y = x - 1$

44. $y^2 + \ln(xy) = 2, \quad (e, 1)$

$$2yy' + \frac{xy' + y}{xy} = 0$$

$$2xy^2y' + xy' + y = 0$$

At $(e, 1)$: $2ey' + ey' + 1 = 0$

$$y' = \frac{-1}{3e}$$

Tangent line: $y - 1 = \dfrac{-1}{3e}(x - e)$

$$y = \frac{-1}{3e}x + \frac{4}{3}$$

45. (a) $\dfrac{x^2}{2} + \dfrac{y^2}{8} = 1, \quad (1, 2)$

$$x + \frac{yy'}{4} = 0$$

$$y' = -\frac{4x}{y}$$

At $(1, 2)$: $y' = -2$

Tangent line: $y - 2 = -2(x - 1)$

$$y = -2x + 4$$

(b) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{-b^2x}{a^2y}$

$y - y_0 = \dfrac{-b^2x_0}{a^2y_0}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{y_0y}{b^2} - \frac{y_0^2}{b^2} = \frac{-x_0x}{a^2} + \frac{x_0^2}{a^2}$$

Because $\dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2} = 1$, you have $\dfrac{y_0y}{b^2} + \dfrac{x_0x}{a^2} = 1$.

Note: From part (a),

$\dfrac{1(x)}{2} + \dfrac{2(y)}{8} = 1 \Rightarrow \dfrac{1}{4}y = -\dfrac{1}{2}x + 1 \Rightarrow y = -2x + 4$,

Tangent line.

46. (a) $\dfrac{x^2}{6} - \dfrac{y^2}{8} = 1, \quad (3, -2)$

$$\frac{x}{3} - \frac{y}{4}y' = 0$$

$$\frac{y}{4}y' = \frac{x}{3}$$

$$y' = \frac{4x}{3y}$$

At $(3, -2)$: $y' = \dfrac{4(3)}{3(-2)} = -2$

Tangent line: $y + 2 = -2(x - 3)$

$$y = -2x + 4$$

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \Rightarrow \dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \Rightarrow y' = \dfrac{xb^2}{ya^2}$

$y - y_0 = \dfrac{x_0b^2}{y_0a^2}(x - x_0)$, Tangent line at (x_0, y_0)

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{x_0x}{a^2} - \frac{x_0^2}{a^2}$$

Because $\dfrac{x_0^2}{a^2} - \dfrac{y_0^2}{b^2} = 1$, you have $\dfrac{x_0x}{a^2} - \dfrac{yy_0}{b^2} = 1$.

Note: From part (a),

$\dfrac{3x}{6} - \dfrac{(-2)y}{8} = 1 \Rightarrow \dfrac{1}{2}x + \dfrac{y}{4} = 1 \Rightarrow y = -2x + 4$,

Tangent line.

47. $\tan y = x$

$$y' \sec^2 y = 1$$

$$y' = \frac{1}{\sec^2 y} = \cos^2 y, -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\sec^2 y = 1 + \tan^2 y = 1 + x^2$$

$$y' = \frac{1}{1 + x^2}$$

48. $\cos y = x$

$$-\sin y \cdot y' = 1$$

$$y' = \frac{-1}{\sin y}, \quad 0 < y < \pi$$

$$\sin^2 y + \cos^2 y = 1$$

$$\sin^2 y = 1 - \cos^2 y$$

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$$

$$y' = \frac{-1}{\sqrt{1 - x^2}}, \quad -1 < x < 1$$

49. $x^2 + y^2 = 4$

$$2x + 2yy' = 0$$

$$y' = \frac{-x}{y}$$

$$y'' = \frac{y(-1) + xy'}{y^2}$$

$$= \frac{-y + x(-x/y)}{y^2}$$

$$= \frac{-y^2 - x^2}{y^3}$$

$$= -\frac{4}{y^3}$$

50. $x^2y - 4x = 5$

$$x^2y' + 2xy - 4 = 0$$

$$y' = \frac{4 - 2xy}{x^2}$$

$$x^2y'' + 2xy' + 2xy' + 2y = 0$$

$$x^2y'' + 4x\left[\frac{4 - 2xy}{x^2}\right] + 2y = 0$$

$$x^4y'' + 4x(4 - 2xy) + 2x^2y = 0$$

$$x^4y'' + 16x - 8x^2y + 2x^2y = 0$$

$$x^4y'' = 6x^2y - 16x$$

$$y'' = \frac{6xy - 16}{x^3}$$

51.

$$
\begin{aligned}
x^2 - y^2 &= 36 \\
2x - 2yy' &= 0 \\
y' &= \frac{x}{y} \\
x - yy' &= 0 \\
1 - yy'' - (y')^2 &= 0 \\
1 - yy'' - \left(\frac{x}{y}\right)^2 &= 0 \\
y^2 - y^3y'' &= x^2 \\
y'' &= \frac{y^2 - x^2}{y^3} = -\frac{36}{y^3}
\end{aligned}
$$

52.

$$
\begin{aligned}
xy - 1 &= 2x + y^2 \\
xy' + y &= 2 + 2yy' \\
xy' - 2yy' &= 2 - y \\
(x - 2y)y' &= 2 - y \\
y' &= \frac{2 - y}{x - 2y} \\
xy'' + y' + y' &= 2yy'' + 2(y')^2 \\
xy'' - 2yy'' &= 2(y')^2 - 2y' \\
(x - 2y)y'' &= 2(y')^2 - 2y' = 2\left(\frac{2 - y}{x - 2y}\right)^2 - 2\left(\frac{2 - y}{x - 2y}\right) \\
y'' &= \frac{2(2 - y)[(2 - y) - (x - 2y)]}{(x - 2y)^3} = \frac{2(2 - y)(2 - x + y)}{(x - 2y)^3} \\
&= \frac{2(4 - 2x + 2y - 2y + xy - y^2)}{(x - 2y)^3} = \frac{2(y^2 - xy + 2x - 4)}{(2y - x)^3} \\
&= \frac{2(-5)}{(2y - x)^3} = \frac{10}{(x - 2y)^3}
\end{aligned}
$$

53.

$$
\begin{aligned}
y^2 &= x^3 \\
2yy' &= 3x^2 \\
y' &= \frac{3x^2}{2y} = \frac{3x^2}{2y} \cdot \frac{xy}{xy} = \frac{3y}{2x} \cdot \frac{x^3}{y^2} = \frac{3y}{2x} \\
y'' &= \frac{2x(3y') - 3y(2)}{4x^2} \\
&= \frac{2x[3 \cdot (3y/2x)] - 6y}{4x^2} = \frac{3y}{4x^2} = \frac{3x}{4y}
\end{aligned}
$$

54.

$$
\begin{aligned}
y^3 &= 4x \\
3y^2y' &= 4 \\
y' &= \frac{4}{3y^2} \\
3y^2y'' + 6y(y')^2 &= 0 \\
yy'' + 2(y')^2 &= 0 \\
y'' &= \frac{-2(y')^2}{y} = \frac{-2}{y}\left(\frac{4}{3y^2}\right)^2 \\
y'' &= -\frac{32}{9y^5}
\end{aligned}
$$

Note:

$$
\begin{aligned}
y &= (4x)^{1/3} \\
y' &= \frac{4}{3}(4x)^{-2/3} \\
y'' &= -\frac{8}{9}(4)(4x)^{-5/3} = -\frac{32}{9(4x)^{5/3}} = -\frac{32}{9y^5}
\end{aligned}
$$

55. $x^2 + y^2 = 25$

$2x + 2yy' = 0$

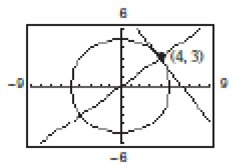
$$y' = \frac{-x}{y}$$

At $(4, 3)$:

Tangent line:

$y - 3 = \frac{-4}{3}(x - 4) \Rightarrow 4x + 3y - 25 = 0$

Normal line: $y - 3 = \frac{3}{4}(x - 4) \Rightarrow 3x - 4y = 0$

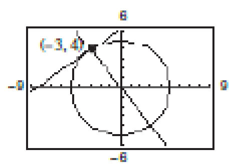


At $(-3, 4)$:

Tangent line:

$y - 4 = \frac{3}{4}(x + 3) \Rightarrow 3x - 4y + 25 = 0$

Normal line: $y - 4 = \frac{-4}{3}(x + 3) \Rightarrow 4x + 3y = 0$



56. $x^2 + y^2 = 36$

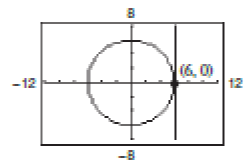
$2x + 2yy' = 0$

$$y' = -\frac{x}{y}$$

At $(6, 0)$; slope is undefined.

Tangent line: $x = 6$

Normal line: $y = 0$



At $\left(5, \sqrt{11}\right)$, slope is $\frac{-5}{\sqrt{11}}$

Tangent line: $y - \sqrt{11} = \frac{-5}{\sqrt{11}}(x - 5)$

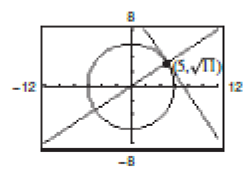
$$\sqrt{11}y - 11 = -5x + 25$$

$$5x + \sqrt{11}y - 36 = 0$$

Normal line: $y - \sqrt{11} = \frac{\sqrt{11}}{5}(x - 5)$

$$5y - 5\sqrt{11} = \sqrt{11}x - 5\sqrt{11}$$

$$5y - \sqrt{11}x = 0$$



57. $x^2 + y^2 = r^2$

$2x + 2yy' = 0$

$y' = \frac{-x}{y}$ = slope of tangent line

$\frac{y}{x}$ = slope of normal line

Let (x_0, y_0) be a point on the circle. If $x_0 = 0$, then the tangent line is horizontal, the normal line is vertical and, hence, passes through the origin. If $x_0 \neq 0$, then the equation of the normal line is

$$y - y_0 = \frac{y_0}{x_0}(x - x_0)$$

$$y = \frac{y_0}{x_0}x$$

which passes through the origin.

58. $y^2 = 4x$

$2yy' = 4$

$y' = \frac{2}{y} = 1$ at $(1, 2)$

Equation of normal line at $(1, 2)$ is

$y - 2 = -1(x - 1), y = 3 - x.$

The centers of the circles must be on the normal line and at a distance of 4 units from $(1, 2)$.

Therefore,

$$(x - 1)^2 + [(3 - x) - 2]^2 = 16$$
$$2(x - 1)^2 = 16$$
$$x = 1 \pm 2\sqrt{2}.$$

Centers of the circles: $\left(1 + 2\sqrt{2}, 2 - 2\sqrt{2}\right)$ and

$\left(1 - 2\sqrt{2}, 2 + 2\sqrt{2}\right)$

Equations:

$$\left(x - 1 - 2\sqrt{2}\right)^2 + \left(y - 2 + 2\sqrt{2}\right)^2 = 16$$
$$\left(x - 1 + 2\sqrt{2}\right)^2 + \left(y - 2 - 2\sqrt{2}\right)^2 = 16$$

59. $25x^2 + 16y^2 + 200x - 160y + 400 = 0$

$50x + 32yy' + 200 - 160y' = 0$

$$y' = \frac{200 + 50x}{160 - 32y}$$

Horizontal tangents occur when $x = -4$:

$25(16) + 16y^2 + 200(-4) - 160y + 400 = 0$

$y(y - 10) = 0 \Rightarrow y = 0, 10$

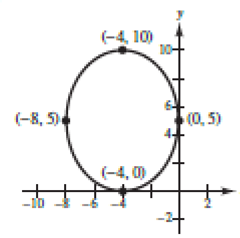
Horizontal tangents: $(-4, 0), (-4, 10)$

Vertical tangents occur when $y = 5$:

$25x^2 + 400 + 200x - 800 + 400 = 0$

$25x(x + 8) = 0 \Rightarrow x = 0, -8$

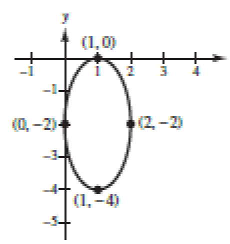
Vertical tangents: $(0, 5), (-8, 5)$



60. $4x^2 + y^2 - 8x + 4y + 4 = 0$

$8x + 2yy' - 8 + 4y' = 0$

$$y' = \frac{8 - 8x}{2y + 4} = \frac{4 - 4x}{y + 2}$$



Horizontal tangents occur when $x = 1$:

$4(1)^2 + y^2 - 8(1) + 4y + 4 = 0$

$y^2 + 4y = y(y + 4) = 0 \Rightarrow y = 0, -4$

Horizontal tangents: $(1, 0), (1, -4)$

Vertical tangents occur when $y = -2$:

$4x^2 + (-2)^2 - 8x + 4(-2) + 4 = 0$

$4x^2 - 8x = 4x(x - 2) = 0 \Rightarrow x = 0, 2$

Vertical tangents: $(0, -2), (2, -2)$

61. $y = x\sqrt{x^2 + 1}$

$\ln y = \ln x + \frac{1}{2}\ln(x^2 + 1)$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{x}{x^2 + 1}$$

$$\frac{dy}{dx} = y\left[\frac{2x^2 + 1}{x(x^2 + 1)}\right] = \frac{2x^2 + 1}{\sqrt{x^2 + 1}}$$

62. $$y = \sqrt{x^2(x+1)(x+2)},\ x > 0$$
$$y^2 = x^2(x+1)(x+2)$$
$$2\ln y = 2\ln x + \ln(x+1) + \ln(x+2)$$
$$\frac{2}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2}$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{1}{x+1} + \frac{1}{x+2}\right]$$
$$\frac{dy}{dx} = \frac{\sqrt{x^2(x+1)(x+2)}}{2}\left[\frac{2(x+1)(x+2) + x(x+2) + x(x+1)}{x(x+1)(x+2)}\right] = \frac{4x^2 + 9x + 4}{2\sqrt{(x+1)(x+2)}}$$

63. $$y = \frac{x^2\sqrt{3x-2}}{(x+1)^2}$$
$$\ln y = 2\ln x + \frac{1}{2}\ln(3x-2) - 2\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x} + \frac{3}{2(3x-2)} - \frac{2}{x+1}$$
$$\frac{dy}{dx} = y\left[\frac{3x^2 + 15x - 8}{2x(3x-2)(x+1)}\right]$$
$$= \frac{3x^3 + 15x^2 - 8x}{2(x+1)^3\sqrt{3x-2}}$$

64. $$y = \sqrt{\frac{x^2-1}{x^2+1}}$$
$$\ln y = \frac{1}{2}\left[\ln(x^2-1) - \ln(x^2+1)\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{2}\left[\frac{2x}{x^2-1} - \frac{2x}{x^2+1}\right]$$
$$\frac{dy}{dx} = \sqrt{\frac{x^2-1}{x^2+1}}\left[\frac{2x}{x^4-1}\right]$$
$$= \frac{(x^2-1)^{1/2}2x}{(x^2+1)^{1/2}(x^2-1)(x^2+1)}$$
$$= \frac{2x}{(x^2+1)^{3/2}(x^2-1)^{1/2}}$$

65. $$y = \frac{x(x-1)^{3/2}}{\sqrt{x+1}}$$
$$\ln y = \ln x + \frac{3}{2}\ln(x-1) - \frac{1}{2}\ln(x+1)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x} + \frac{3}{2}\left(\frac{1}{x-1}\right) - \frac{1}{2}\left(\frac{1}{x+1}\right)$$
$$\frac{dy}{dx} = \frac{y}{2}\left[\frac{2}{x} + \frac{3}{x-1} - \frac{1}{x+1}\right]$$
$$= \frac{y}{2}\left[\frac{4x^2 + 4x - 2}{x(x^2-1)}\right] = \frac{(2x^2 + 2x - 1)\sqrt{x-1}}{(x+1)^{3/2}}$$

66. $$y = \frac{(x+1)(x-2)}{(x-1)(x+2)}$$
$$\ln y = \ln(x+1) + \ln(x-2) - \ln(x-1) - \ln(x+2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x+1} + \frac{1}{x-2} - \frac{1}{x-1} - \frac{1}{x+2}$$
$$\frac{dy}{dx} = y\left[\frac{-2}{x^2-1} + \frac{4}{x^2-4}\right] = y\left[\frac{2x^2+4}{(x^2-1)(x^2-4)}\right]$$
$$= \frac{(x+1)(x+2)}{(x-1)(x-2)} \cdot \frac{2x^2+4}{(x+1)(x-1)(x+2)(x-2)}$$
$$= \frac{2(x^2+2)}{(x-1)^2(x-2)^2}$$

67. $$y = x^{2/x}$$
$$\ln y = \frac{2}{x}\ln x$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{2}{x}\left(\frac{1}{x}\right) + \ln x\left(-\frac{2}{x^2}\right) = \frac{2}{x^2}(1 - \ln x)$$
$$\frac{dy}{dx} = \frac{2y}{x^2}(1 - \ln x) = 2x^{(2/x)-2}(1 - \ln x)$$

68. $$y = x^{x-1}$$
$$\ln y = (x-1)(\ln x)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x-1)\left(\frac{1}{x}\right) + \ln x$$
$$\frac{dy}{dx} = y\left[\frac{x-1}{x} + \ln x\right]$$
$$= x^{x-2}(x - 1 + x\ln x)$$

69. $$y = (x-2)^{x+1}$$
$$\ln y = (x+1)\ln(x-2)$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right) = (x+1)\left(\frac{1}{x-2}\right) + \ln(x-2)$$
$$\frac{dy}{dx} = y\left[\frac{x+1}{x-2} + \ln(x-2)\right]$$
$$= (x-2)^{x+1}\left[\frac{x+1}{x-2} + \ln(x-2)\right]$$

70. $$y = (1 + x)^{1/x}$$

$$\ln y = \frac{1}{x}\ln(1 + x)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \frac{1}{x}\left(\frac{1}{1 + x}\right) + \ln(1 + x)\left(-\frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$

$$= \frac{(1 + x)^{1/x}}{x}\left[\frac{1}{x + 1} - \frac{\ln(x + 1)}{x}\right]$$

71. $y = x^{\ln x}, \quad x > 0$

$$\ln y = \ln x^{\ln x} = (\ln x)(\ln x) = (\ln x)^2$$

$$\frac{y'}{y} = 2\ln x(1/x)$$

$$y' = \frac{2y\ln x}{x} = \frac{2x^{\ln x}\cdot\ln x}{x}$$

72. $y = (\ln x)^{\ln x}, \qquad x > 1$

$$\ln y = \ln\left[(\ln x)^{\ln x}\right] = (\ln x)\ln(\ln x)$$

$$\frac{y'}{y} = (\ln x)\frac{1}{\ln x}\cdot\frac{1}{x} + \frac{1}{x}\ln(\ln x)$$

$$= \frac{1}{x}(1 + \ln(\ln x))$$

$$y' = \frac{y}{x}(1 + \ln(\ln x))$$

$$= (\ln x)^{\ln x}\left[1 + \ln(\ln x)\right]/x$$

73. $y = -x$ and $x = \sin y$

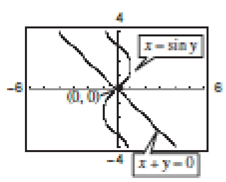
Point of intersection: $(0, 0)$

$y = -x$:	$x = \sin y$:
$y' = -1$	$1 = y'\cos y$
	$y' = \sec y$

At $(0, 0)$, the slopes are:

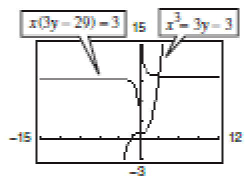
$y' = -1 \qquad\qquad y' = 1$

Tangents are perpendicular.



74. Rewriting each equation and differentiating:

$x^3 = 3(y - 1)$	$x(3y - 29) = 3$
$y = \dfrac{x^3}{3} + 1$	$y = \dfrac{1}{3}\left(\dfrac{3}{x} + 29\right)$
$y' = x^2$	$y' = -\dfrac{1}{x^2}$

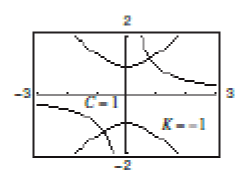


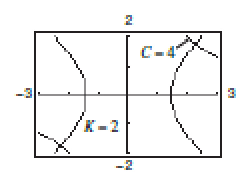
For each value of x, the derivatives are negative reciprocals of each other. So, the tangent lines are orthogonal at both points of intersection.

75.

$xy = C$	$x^2 - y^2 = K$
$xy' + y = 0$	$2x - 2yy' = 0$
$y' = -\dfrac{y}{x}$	$y' = \dfrac{x}{y}$

At any point of intersection (x, y) the product of the slopes is $(-y/x)(x/y) = -1$. The curves are orthogonal.

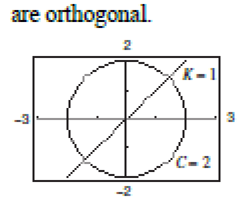


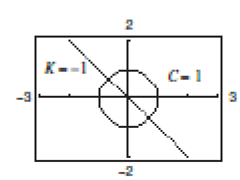


76.

$x^2 + y^2 = C^2$	$y = Kx$
$2x + 2yy' = 0$	$y' = K$
$y' = -\dfrac{x}{y}$	

At the point of intersection (x, y), the product of the slopes is $(-x/y)(K) = (-x/Kx)(K) = -1$. The curves are orthogonal.





77. Answers will vary. *Sample answer:* In the explicit form of a function, the variable is explicitly written as a function of x. In an implicit equation, the function is only implied by an equation. An example of an implicit function is $x^2 + xy = 5$. In explicit form it would be $y = (5 - x^2)/x$.

78. Answers will vary. *Sample answer:* Given an implicit equation, first differentiate both sides with respect to x. Collect all terms involving y' on the left side of the equation, and move all other terms to the right side of the equation. Factor out y' on the left side. Finally, divide both sides by the left-hand factor that does not contain y'.

79. (a) True

(b) False. $\frac{d}{dy}\cos(y^2) = -2y\sin(y^2)$.

(c) False. $\frac{d}{dx}\cos(y^2) = -2yy'\sin(y^2)$.

80. (a) The slope is greater at $x = -3$.

(b) The graph has vertical tangent lines at about $(-2, 3)$ and $(2, 3)$.

(c) The graph has a horizontal tangent line at about $(0, 6)$.

81.
$$x^2 + y^2 = 100, \text{ slope } = \frac{3}{4}$$
$$2x + 2yy' = 0$$
$$y' = -\frac{x}{y} = \frac{3}{4} \Rightarrow y = -\frac{4}{3}x$$
$$x^2 + \left(\frac{16}{9}x^2\right) = 100$$
$$\frac{25}{9}x^2 = 100$$
$$x = \pm 6$$

Points: $(6, -8)$ and $(-6, 8)$

82. (a)
$$y = x^{p/q};\ p, q \text{ integers and } q > 0$$
$$y^q = x^p$$
$$qy^{q-1}y' = px^{p-1}$$
$$y' = \frac{p}{q}\cdot\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\cdot\frac{x^{p-1}y}{y^q}$$
$$= \frac{p}{q}\cdot\frac{x^{p-1}}{x^p}x^{p/q} = \frac{p}{q}x^{p/q-1}$$

So, if $y = x^n$, $n = p/q$, then $y' = nx^{n-1}$.

(b)
$$y = x^r, \quad r \text{ real}$$
$$\ln y = \ln(x^r) = r\ln x$$
$$\frac{y'}{y} = \frac{r}{x}$$
$$y' = \frac{yr}{x} = \frac{x^r \cdot r}{x} = rx^{r-1}.$$

83.
$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \quad (4, 0)$$
$$\frac{2x}{4} + \frac{2yy'}{9} = 0$$
$$y' = \frac{-9x}{4y}$$
$$\frac{-9x}{4y} = \frac{y - 0}{x - 4}$$
$$-9x(x - 4) = 4y^2$$

But, $9x^2 + 4y^2 = 36 \Rightarrow 4y^2 = 36 - 9x^2$.

So, $-9x^2 + 36x = 4y^2 = 36 - 9x^2 \Rightarrow x = 1$.

Points on ellipse: $\left(1, \pm\frac{3}{2}\sqrt{3}\right)$

At $\left(1, \frac{3}{2}\sqrt{3}\right)$: $y' = \frac{-9x}{4y} = \frac{-9}{4\left[(3/2)\sqrt{3}\right]} = -\frac{\sqrt{3}}{2}$

At $\left(1, -\frac{3}{2}\sqrt{3}\right)$: $y' = \frac{\sqrt{3}}{2}$

Tangent lines: $y = -\frac{\sqrt{3}}{2}(x - 4) = -\frac{\sqrt{3}}{2}x + 2\sqrt{3}$

$y = \frac{\sqrt{3}}{2}(x - 4) = \frac{\sqrt{3}}{2}x - 2\sqrt{3}$

84.
$$x\ln y = 2$$
$$x\left(\frac{1}{y}\frac{dy}{dx}\right) + (1)\ln y = 0$$
$$\frac{x}{y}\frac{dy}{dx} = -\ln y$$
$$\frac{dy}{dx} = -\frac{y\ln y}{x}$$

So, the answer is B.

85.
$$x^4 - x^2y + y^4 = 1$$
$$4x^3 - \left[x^2y' + 2xy\right] + 4y^3y' = 0$$
$$4x^3 - x^2y' - 2xy + 4y^3y' = 0$$
$$-x^2y' + 4y^3y' = -4x^3 + 2xy$$
$$y'(-x^2 + 4y^3) = -4x^3 + 2xy$$
$$y' = \frac{-4x^3 + 2xy}{-x^2 + 4y^3}$$

At $(1, 1)$, $y' = \frac{-4(1)^3 + 2(1)(1)}{-(1)^2 + 4(1)^3} = -\frac{2}{3}$.

So, the answer is B.

86. $x^2 + y^2 = 100$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{x\frac{dy}{dx} - y}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{x\left(-\frac{x}{4}\right) - y}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{x^2}{y^3} - \frac{1}{y}$$

At $(6, 8)$, $-\frac{(6)^2}{(8)^3} - \frac{1}{8} = -\frac{100}{512} = -\frac{25}{128}$.

So, the answer is A.

87. (a) $x^2 + 4y^2 + 6x - 8y + 9 = 0$

$$2x + 8yy' + 6 - 8y' + 0 = 0$$

$$8yy' - 8y' = -2x - 6$$

$$y' = \frac{-2x - 6}{8y - 8} = \frac{-2(x + 3)}{2(4y - 4)}$$

So, $\frac{dy}{dx} = -\frac{x + 3}{4y - 4}$.

(b) Vertical tangents occur at $y = 1$.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$

$$x^2 + 4(1)^2 + 6x - 8(1) + 9 = 0$$

$$x^2 + 6x + 5 = 0$$

$$(x + 5)(x + 1) = 0$$

$$x = -5, -1$$

So, the vertical tangents occur at $(-5, -1)$ and $(-1, 1)$.

(c) Horizontal tangents occur at $x = -3$.

$$x^2 + 4y^2 + 6x - 8y + 9 = 0$$

$$(-3)^2 + 4y^2 + 6(-3) - 8y + 9 = 0$$

$$9 + 4y^2 - 18 - 8y + 9 = 0$$

$$4y^2 - 8y = 0$$

$$4y(y - 2) = 0$$

$$y = 0, 2$$

So, the horizontal tangents occur at $(-3, 0)$ and $(-3, 2)$.