3.2 Rolle's Theorem and The Mean Value Theorem

The Mean Value Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there exists a number c in (a, b) such that:

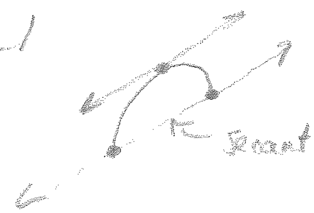
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) if $f(a) = f(b)$, the there is at least one number c in (a, b) such that $f'(c) = 0$.

Implications of MVT:

Geometric: There is a tangent line that is parallel to the secant line between the endpoints



Rate of Change: There is a point between the endpoints where the instantaneous rate of change is equal to the average rate of change between the endpoints.

Examples: Using MVT and Rolle's

For $f(x) = x^3 - 8x^2$, find all values of c in the open interval $(-2,5)$ such that $f'(c) = \frac{f(5)-f(-2)}{5-(-2)}$.

$f(x)$ is continuous $[-2,5]$

$f(x)$ is differentiable $(-2,5)$

$\therefore$ There must be a c

$f'(x) = 3x^2 - 16x$

$$\frac{f(5) - f(-2)}{5-(-2)} = \frac{(125 - 200) - (-8-32)}{7} = \frac{-75 + 40}{7} = \frac{-35}{7} = -5$$

$3x^2 - 16x = -5$

$3x^2 - 16x + 5 = 0$

$$x = \frac{16 \pm \sqrt{256 - 4(3)(5)}}{2(3)}$$

$$= \frac{16 \pm \sqrt{196}}{6} = \frac{16 \pm 14}{6} \nearrow 5, \searrow \frac{1}{3}$$

$\boxed{c = \frac{1}{3}}$

Find the two x-intercepts of $f(x) = x^2 + 2x - 8$ and show that $f'(x) = 0$ at some point between the two x-intercepts.

$x^2+2x-8=0$

$(x+4)(x-2)=0$

$x=-4 \quad x=2$

$f(-4)=16-8-8=0$

$f(2)=4+4-8=0$

$f(x)$ is continuous $[-4,2]$

$f(x)$ is differentiable $(-4,2)$

$f(-4)=f(2)$

$\therefore$ By Rolle's Theorem $f'(x)=0$ between $x=-4$ and $x=2$

Let $f(x) = x^4 - 8x^2 + 7$. Find all values of c in the interval $(-3,3)$ such that $f'(c) = 0$.

$f(x)$ is continuous $[-3,3]$

$f(x)$ is differentiable $(-3,3)$

$f(-3) = 81-72+7 = 16$

$f(3) = 81-72+7 = 16$

$\therefore f'(c)=0$ between -3 and 3

$f'(x) = 4x^3-16x$

$4x^3-16x=0$

$4x(x^2-4)=0$

$4x(x-2)(x+2)=0$

$\boxed{x=0 \quad x=2 \quad x=-2}$

A car traveling on the highway passes mile marker 147 traveling at a speed of 60 miles per hour. Five minutes later, the car passes mile marker 154 traveling at a speed of 65 miles per hour. Prove that the driver must have exceeded the speed limit of 65 mph at some time during the five minutes.

Position is continuous and differentiable

$\frac{7 \text{ miles}}{5 \text{ mins}} = \frac{7 \text{ mi}}{5/60} = 7 \cdot \frac{60}{5} = 84$ mph (average rate of change)

By MVT there had to be a point where the instantaneous rate of change was equal to the average rate of change.