Section 2.6 Derivatives of Inverse Functions

1. $f(x) = x^3 - 1, \quad a = 26$

$f'(x) = 3x^2$

f is monotonic (increasing) on $(-\infty, \infty)$ therefore f has an inverse.

$$f(3) = 26 \Rightarrow f^{-1}(26) = 3$$

$$\left(f^{-1}\right)'(26) = \frac{1}{f'\left(f^{-1}(26)\right)} = \frac{1}{f'(3)} = \frac{1}{3\left(3^2\right)} = \frac{1}{27}$$

2. $f(x) = 5 - 2x^3, \quad a = 7$

$f'(x) = -6x^2$

f is monotonic (decreasing) on $(-\infty, \infty)$ therefore f has an inverse.

$$f(-1) = 7 \Rightarrow f^{-1}(7) = -1$$

$$\left(f^{-1}\right)'(7) = \frac{1}{f'\left(f^{-1}(7)\right)} = \frac{1}{f'(-1)} = \frac{1}{-6(-1)^2} = \frac{-1}{6}$$

3. $f(x) = x^3 + 2x - 1, \quad a = 2$

$f'(x) = 3x^2 + 2 > 0$

f is monotonic (increasing) on $(-\infty, \infty)$ therefore f has an inverse.

$f(1) = 2 \Rightarrow f^{-1}(2) = 1$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{3(1^2) + 2} = \frac{1}{5}$$

4. $f(x) = \frac{1}{27}(x^5 + 2x^3), \quad a = -11$

$f'(x) = \frac{1}{27}(5x^4 + 6x^2)$

f is monotonic (increasing) on $(-\infty, \infty)$ therefore f has an inverse.

$f(-3) = \frac{1}{27}(-243 - 54) = -11 \Rightarrow f^{-1}(-11) = -3$

$$(f^{-1})'(-11) = \frac{1}{f'(f^{-1}(-11))} = \frac{1}{f'(-3)}$$

$$= \frac{1}{\frac{1}{27}\left(5(-3)^4 + 6(-3)^2\right)} = \frac{1}{\frac{1}{27}(459)} = \frac{1}{17}$$

5. $f(x) = \sin x, \quad a = 1/2, -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$f'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

f is monotonic (increasing) on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ therefore f has an inverse.

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} \Rightarrow f^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$$(f^{-1})'\left(\frac{1}{2}\right) = \frac{1}{f'\left(f^{-1}\left(\frac{1}{2}\right)\right)}$$

$$= \frac{1}{f'\left(\frac{\pi}{6}\right)} = \frac{1}{\cos\left(\frac{\pi}{6}\right)} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

6. $f(x) = \cos 2x, \quad a = 1, 0 \le x \le \pi/2$

$f'(x) = -2\sin 2x < 0$ on $(0, \pi/2)$

f is monotonic (decreasing) on $[0, \pi/2]$ therefore f has an inverse.

$f(0) = 1 \Rightarrow f^{-1}(1) = 0$

$$(f^{-1})'(1) = \frac{1}{f'(f^{-1}(1))} = \frac{1}{f'(0)} = \frac{1}{-2\sin 0} = \frac{1}{0}$$

So, $(f^{-1})'(1)$ is undefined.

7. $f(x) = \frac{x + 6}{x - 2}, \quad x > 0, a = 3$

$$f'(x) = \frac{(x - 2)(1) - (x + 6)(1)}{(x - 2)^2}$$

$$= \frac{-8}{(x - 2)^2} < 0 \text{ on } (2, \infty)$$

f is monotonic (decreasing) on $(2, \infty)$ therefore f has an inverse.

$f(6) = 3 \Rightarrow f^{-1}(3) = 6$

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(6)} = \frac{1}{-8/(6 - 2)^2} = -2$$

8. $f(x) = \frac{x + 3}{x + 1}, \quad x > -1, \ a = 2$

$$f'(x) = \frac{(x + 1)(1) - (x + 3)(1)}{(x + 1)^2}$$

$$= \frac{-2}{(x + 1)^2} < 0 \text{ on } (-1, \infty)$$

f is monotonic (decreasing) on $(-1, \infty)$ therefore f has an inverse.

$f(1) = 2 \Rightarrow f^{-1}(2) = 1$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(1)} = \frac{1}{(-2)/(1 + 1)^2} = -2$$

9. $f(x) = x^3 - \frac{4}{x}, \quad a = 6, x > 0$

$f'(x) = 3x^2 + \frac{4}{x^2} > 0$

f is monotonic (increasing) on $(0, \infty)$ therefore f has an inverse.

$f(2) = 6 \Rightarrow f^{-1}(6) = 2$

$$(f^{-1})'(6) = \frac{1}{f'(f^{-1}(6))} = \frac{1}{f'(2)} = \frac{1}{3(2^2) + 4/2^2} = \frac{1}{13}$$

10. $f(x) = \sqrt{x - 4}, \quad a = 2, \quad x \ge 4$

$$f'(x) = \frac{1}{2\sqrt{x - 4}} > 0 \text{ on } (4, \infty)$$

f is monotonic (increasing) on $[4, \infty)$ therefore f has an inverse.

$f(8) = 2 \Rightarrow f^{-1}(2) = 8$

$$f'(8) = \frac{1}{2\sqrt{8 - 4}} = \frac{1}{4}$$

$$(f^{-1})'(2) = \frac{1}{f'(f^{-1}(2))} = \frac{1}{f'(8)} = \frac{1}{1/4} = 4$$

11. $f(x) = x^3, \quad \left(\frac{1}{2}, \frac{1}{8}\right)$

$f'(x) = 3x^2$

$f'\left(\frac{1}{2}\right) = \frac{3}{4}$

$f^{-1}(x) = \sqrt[3]{x}, \quad \left(\frac{1}{8}, \frac{1}{2}\right)$

$\left(f^{-1}\right)'(x) = \frac{1}{3\sqrt[3]{x}}$

$\left(f^{-1}\right)'\left(\frac{1}{8}\right) = \frac{4}{3}$

12. $f(x) = 3 - 4x, \quad (1, -1)$

$f'(x) = -4$

$f'(1) = -4$

$f^{-1}(x) = \frac{3 - x}{4}, \quad (-1, 1)$

$\left(f^{-1}\right)'(x) = -\frac{1}{4}$

$\left(f^{-1}\right)'(-1) = -\frac{1}{4}$

13. $f(x) = \sqrt{x - 4}, \quad (5, 1)$

$f'(x) = \frac{1}{2\sqrt{x - 4}}$

$f'(5) = \frac{1}{2}$

$f^{-1}(x) = x^2 + 4, \quad (1, 5)$

$\left(f^{-1}\right)'(x) = 2x$

$\left(f^{-1}\right)'(1) = 2$

14. $f(x) = \frac{4}{1 + x^2}$

$f'(x) = \frac{-8x}{\left(x^2 + 1\right)^2}$

$f'(1) = -2$

$f^{-1}(x) = \sqrt{\frac{4 - x}{x}}$

$\left(f^{-1}\right)'(x) = \frac{-2}{x^2\sqrt{(4 - x)/x}}$

$\left(f^{-1}\right)'(2) = -\frac{1}{2}$

15. (a) $f(x) = \arccos\left(x^2\right)$

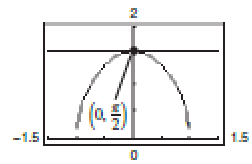
$f'(x) = \frac{-1}{\sqrt{1 - x^4}}(2x) = \frac{-2x}{\sqrt{1 - x^4}}$

$f'(0) = 0$

$y - \frac{\pi}{2} = 0(x - 0)$

$y = \frac{\pi}{2}, \quad$ tangent line

(b)



16. (a) $f(x) = \arctan x$

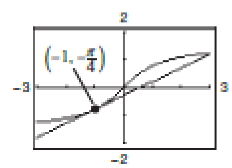
$f'(x) = \frac{1}{1 + x^2}$

$f'(-1) = \frac{1}{2}$

$y + \frac{\pi}{4} = \frac{1}{2}(x + 1)$

$y = \frac{1}{2}x + \frac{1}{2} - \frac{\pi}{4}, \quad$ tangent line

(b)



17. (a) $f(x) = \arcsin 3x$

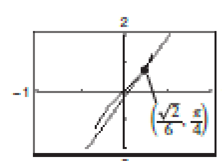
$f'(x) = \frac{1}{\sqrt{1 - (3x)^2}}(3) = \frac{3}{\sqrt{1 - 9x^2}}$

$f'\left(\sqrt{2}/6\right) = \frac{3}{\sqrt{1 - 9(1/18)}} = \frac{3}{\sqrt{1/2}} = 3\sqrt{2}$

$y - \frac{\pi}{4} = 3\sqrt{2}\left(x - \sqrt{2}/6\right)$

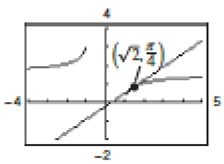
$y = 3\sqrt{2}x + \frac{\pi}{4} - 1, \quad$ Tangent line

(b)



18. (a)
$$f(x) = \operatorname{arcsec} x$$
$$f'(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$
$$f'(\sqrt{2}) = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$y - \frac{\pi}{4} = \frac{\sqrt{2}}{2}\left(x - \sqrt{2}\right)$$
$$y = \frac{\sqrt{2}}{2}x + \frac{\pi}{4} - 1, \quad \text{tangent line}$$

(b)



19. $x = y^3 - 7y^2 + 2$
$$1 = 3y^2\frac{dy}{dx} - 14y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{3y^2 - 14y}$$
At $(-4, 1)$: $\dfrac{dy}{dx} = \dfrac{1}{3 - 14} = \dfrac{-1}{11}.$

Alternate Solution:

Let $f(x) = x^3 - 7x^2 + 2$. Then $f'(x) = 3x^2 - 14x$ and $f'(1) = -11$. So,
$$\frac{dy}{dx} = \frac{1}{-11} = \frac{-1}{11}.$$

20. $x = 2\ln\left(y^2 - 3\right)$
$$1 = 2\frac{1}{y^2 - 3}2y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{y^2 - 3}{4y}$$
At $(0, 2)$: $\dfrac{dy}{dx} = \dfrac{4 - 3}{8} = \dfrac{1}{8}$

21.
$$x \arctan x = e^y$$
$$x\frac{1}{1 + x^2} + \arctan x = e^y \cdot \frac{dy}{dx}$$
At $\left(1, \ln\frac{\pi}{4}\right)$: $\dfrac{1}{2} + \dfrac{\pi}{4} = \dfrac{\pi}{4}\dfrac{dy}{dx}$
$$\frac{dy}{dx} = \frac{\pi + 2}{\pi}$$

22.
$$\arctan(xy) = \frac{2}{3}\arctan(2x)$$
$$\frac{1}{\sqrt{1 - (xy)^2}}\left(x\frac{dy}{dx} + y\right) = \frac{2}{3}\frac{1}{1 + 4x^2}(2)$$
At $\left(\frac{1}{2}, 1\right)$: $\dfrac{1}{\sqrt{3/4}}\left(\dfrac{1}{2}y' + 1\right) = \dfrac{2}{3}$
$$\frac{2}{\sqrt{3}}\left(\frac{1}{2}y' + 1\right) = \frac{2}{3}$$
$$y' = \left(\frac{\sqrt{3}}{3} - 1\right)2 = \frac{2\sqrt{3} - 6}{3}$$

23. $f(x) = \arcsin(x + 1)$
$$f'(x) = \frac{1}{\sqrt{1 - (x + 1)^2}} = \frac{1}{\sqrt{-x^2 - 2x}}$$

24. $f(t) = \arcsin t^2$
$$f'(t) = \frac{2t}{\sqrt{1 - t^4}}$$

25. $g(x) = 3\arccos\dfrac{x}{2}$
$$g'(x) = \frac{-3(1/2)}{\sqrt{1 - (x^2/4)}} = \frac{-3}{\sqrt{4 - x^2}}$$

26. $f(x) = \operatorname{arcsec} 2x$
$$f'(x) = \frac{2}{|2x|\sqrt{4x^2 - 1}} = \frac{1}{|x|\sqrt{4x^2 - 1}}$$

27. $f(x) = \arctan\left(e^x\right)$
$$f'(x) = \frac{1}{1 + \left(e^x\right)^2}e^x = \frac{e^x}{1 + e^{2x}}$$

28. $f(x) = \arctan\sqrt{x}$
$$f'(x) = \left(\frac{1}{1 + x}\right)\left(\frac{1}{2\sqrt{x}}\right) = \frac{1}{2\sqrt{x}(1 + x)}$$

29. $g(x) = \dfrac{\arcsin 3x}{x}$
$$g'(x) = \frac{x\left(3/\sqrt{1 - 9x^2}\right) - \arcsin 3x}{x^2}$$
$$= \frac{3x - \sqrt{1 - 9x^2}\arcsin 3x}{x^2\sqrt{1 - 9x^2}}$$

30. $g(x) = \dfrac{\arccos x}{x+1}$

$$g'(x) = \frac{(x+1)\dfrac{-1}{\sqrt{1-x^2}} - \arccos x}{(x+1)^2}$$

$$= -\frac{x+1+\sqrt{1-x^2}\arccos x}{(x+1)^2\sqrt{1-x^2}}$$

31. $g(x) = e^{2x}\arcsin x$

$$g'(x) = e^{2x}\frac{1}{\sqrt{1-x^2}} + 2e^{2x}\arcsin x$$

$$= e^{2x}\left[2\arcsin x + \frac{1}{\sqrt{1-x^2}}\right]$$

32. $h(x) = x^2\arctan(5x)$

$$h'(x) = 2x\arctan(5x) + x^2\frac{1}{1+(5x)^2}(5)$$

$$= 2x\arctan(5x) + \frac{5x^2}{1+25x^2}$$

33. $h(x) = \operatorname{arccot} 6x$

$$h'(x) = \frac{-6}{1+36x^2}$$

34. $f(x) = \operatorname{arccsc} 3x$

$$f'(x) = \frac{-3}{|3x|\sqrt{9x^2-1}}$$

$$= \frac{-1}{|x|\sqrt{9x^2-1}}$$

35. $h(t) = \sin(\arccos t) = \sqrt{1-t^2}$

$$h'(t) = \frac{1}{2}(1-t^2)^{-1/2}(-2t)$$

$$= \frac{-t}{\sqrt{1-t^2}}$$

36. $f(x) = \arcsin x + \arccos x = \dfrac{\pi}{2}$

$$f'(x) = 0$$

37. $y = 2x\arccos x - 2\sqrt{1-x^2}$

$$y' = 2\arccos x - 2x\frac{1}{\sqrt{1-x^2}} - 2\left(\frac{1}{2}\right)(1-x^2)^{-1/2}(-2x)$$

$$= 2\arccos x - \frac{2x}{\sqrt{1-x^2}} + \frac{2x}{\sqrt{1-x^2}} = 2\arccos x$$

38. $y = \ln(t^2+4) - \dfrac{1}{2}\arctan\dfrac{t}{2}$

$$y' = \frac{2t}{t^2+4} - \frac{1}{2}\cdot\frac{1}{1+(t/2)^2}\left(\frac{1}{2}\right)$$

$$= \frac{2t}{t^2+4} - \frac{1}{t^2+4} = \frac{2t-1}{t^2+4}$$

39. $y = \dfrac{1}{2}\left(\dfrac{1}{2}\ln\dfrac{x+1}{x-1} + \arctan x\right) = \dfrac{1}{4}[\ln(x+1) - \ln(x-1)] + \dfrac{1}{2}\arctan x$

$$\frac{dy}{dx} = \frac{1}{4}\left(\frac{1}{x+1} - \frac{1}{x-1}\right) + \frac{1/2}{1+x^2} = \frac{1}{1-x^4}$$

40. $y = \dfrac{1}{2}\left[x\sqrt{4-x^2} + 4\arcsin\left(\dfrac{x}{2}\right)\right]$

$$y' = \frac{1}{2}\left[x\frac{1}{2}(4-x^2)^{-1/2}(-2x) + \sqrt{4-x^2} + 2\frac{1}{\sqrt{1-(x/2)^2}}\right] = \frac{1}{2}\left[\frac{-x^2}{\sqrt{4-x^2}} + \sqrt{4-x^2} + \frac{4}{\sqrt{4-x^2}}\right] = \sqrt{4-x^2}$$

41. $g(t) = \tan(\arcsin t) = \dfrac{t}{\sqrt{1-t^2}}$

$$g'(t) = \frac{\sqrt{1-t^2} - t\left(-t/\sqrt{1-t^2}\right)}{1-t^2} = \frac{1}{(1-t^2)^{3/2}}$$

42. $f(x) = \operatorname{arcsec} x + \operatorname{arccsc} x = \dfrac{\pi}{2}$

$$f'(x) = 0$$

43. $y = x\arcsin x + \sqrt{1-x^2}$

$$\frac{dy}{dx} = x\left(\frac{1}{\sqrt{1-x^2}}\right) + \arcsin x - \frac{x}{\sqrt{1-x^2}} = \arcsin x$$

44. $y = x\arctan 2x - \dfrac{1}{4}\ln(1+4x^2)$

$$\frac{dy}{dx} = \frac{2x}{1+4x^2} + \arctan(2x) - \frac{1}{4}\left(\frac{8x}{1+4x^2}\right) = \arctan(2x)$$

45. $y = 8\arcsin\frac{x}{4} - \frac{x\sqrt{16-x^2}}{2}$

$$y' = 2\frac{1}{\sqrt{1-(x/4)^2}} - \frac{\sqrt{16-x^2}}{2} - \frac{x}{4}(16-x^2)^{-1/2}(-2x)$$

$$= \frac{8}{\sqrt{16-x^2}} - \frac{\sqrt{16-x^2}}{2} + \frac{x^2}{2\sqrt{16-x^2}} = \frac{16-(16-x^2)+x^2}{2\sqrt{16-x^2}} = \frac{x^2}{\sqrt{16-x^2}}$$

46. $y = 25\arcsin\frac{x}{5} - x\sqrt{25-x^2}$

$$y' = 5\frac{1}{\sqrt{1-(x/2)^2}} - \sqrt{25-x^2} - x\frac{1}{2}(25-x^2)^{-1/2}(-2x) = \frac{25}{\sqrt{25-x^2}} - \frac{(25-x^2)}{\sqrt{25-x^2}} + \frac{x^2}{\sqrt{25-x^2}} = \frac{2x^2}{\sqrt{25-x^2}}$$

47. $y = \arctan x + \frac{x}{1+x^2}$

$$y' = \frac{1}{1+x^2} + \frac{(1+x^2)-x(2x)}{(1+x^2)^2} = \frac{(1+x^2)+(1-x^2)}{(1+x^2)^2} = \frac{2}{(1+x^2)^2}$$

48. $y = \arctan\frac{x}{2} - \frac{1}{2(x^2+4)}$

$$y' = \frac{1}{2}\frac{1}{1+(x/2)^2} + \frac{1}{2}(x^2+4)^{-2}(2x) = \frac{2}{x^2+4} + \frac{x}{(x^2+4)^2} = \frac{2x^2+8+x}{(x^2+4)^2}$$

49. The Chain Rule should have been used for $4x$ in e^{4x}.

$$\frac{d}{dx}\left[\arcsin e^{4x}\right] = \frac{e^{4x}(4)}{\sqrt{1-(e^{4x})^2}} = \frac{4e^{4x}}{\sqrt{1-e^{8x}}}$$

50. Under the radical, x^2 should be $(x^2)^2 = x^4$.

$$\frac{d}{dx}\left[\operatorname{arcsec} x^2\right] = \frac{2x}{x^2\sqrt{(x^2)^2-1}} = \frac{2}{x\sqrt{x^4-1}}$$

51. $y = 2\arcsin x$, $\left(\frac{1}{2}, \frac{\pi}{3}\right)$

$$y' = \frac{2}{\sqrt{1-x^2}}$$

At $\left(\frac{1}{2}, \frac{\pi}{3}\right)$, $y' = \frac{2}{\sqrt{1-(1/4)}} = \frac{4}{\sqrt{3}}$.

Tangent line: $y - \frac{\pi}{3} = \frac{4}{\sqrt{3}}\left(x - \frac{1}{2}\right)$

$$y = \frac{4}{\sqrt{3}}x + \frac{\pi}{3} - \frac{2}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}x + \frac{\pi}{3} - \frac{2\sqrt{3}}{3}$$

52. $y = \frac{1}{2}\arccos x$, $\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$

$$y' = \frac{-1}{2\sqrt{1-x^2}}$$

At $\left(-\frac{\sqrt{2}}{2}, \frac{3\pi}{8}\right)$, $y' = \frac{-1}{2\sqrt{1/2}} = -\frac{\sqrt{2}}{2}$.

Tangent line: $y - \frac{3\pi}{8} = -\frac{\sqrt{2}}{2}\left(x + \frac{\sqrt{2}}{2}\right)$

$$y = -\frac{\sqrt{2}}{2}x + \frac{3\pi}{8} - \frac{1}{2}$$

53. $y = \arcsin\left(\frac{x}{2}\right)$, $\left(2, \frac{\pi}{4}\right)$

$$y' = \frac{1}{1+(x^2/4)}\left(\frac{1}{2}\right) = \frac{2}{4+x^2}$$

At $\left(2, \frac{\pi}{4}\right)$, $y' = \frac{2}{4+4} = \frac{1}{4}$.

Tangent line: $y - \frac{\pi}{4} = \frac{1}{4}(x-2)$

$$y = \frac{1}{4}x + \frac{\pi}{4} - \frac{1}{2}$$

54. $y = \operatorname{arcsec}(4x), \left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right)$

$$y' = \frac{4}{|4x|\sqrt{16x^2 - 1}} = \frac{1}{x\sqrt{16x^2 - 1}} \text{ for } x > 0$$

At $\left(\frac{\sqrt{2}}{4}, \frac{\pi}{4}\right), y' = \frac{1}{\left(\sqrt{2}/4\right)\sqrt{2 - 1}} = 2\sqrt{2}.$

Tangent line: $y - \frac{\pi}{4} = 2\sqrt{2}\left(x - \frac{\sqrt{2}}{4}\right)$

$$y = 2\sqrt{2}x + \frac{\pi}{4} - 1$$

55. $y = 4x \arccos(x - 1), (1, 2\pi)$

$$y' = 4x\frac{-1}{\sqrt{1 - (x - 1)^2}} + 4 \arccos(x - 1)$$

At $(1, 2\pi)$, $y' = -4 + 2\pi$.

Tangent line: $y - 2\pi = (2\pi - 4)(x - 1)$

$$y = (2\pi - 4)x + 4$$

56. $y = 3x \arcsin x, \left(\frac{1}{2}, \frac{\pi}{4}\right)$

$$y' = 3x\frac{1}{\sqrt{1 - x^2}} + 3 \arcsin x$$

At $\left(\frac{1}{2}, \frac{\pi}{4}\right), y' = \frac{3}{2}\frac{1}{\sqrt{3/4}} + 3\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{2}.$

Tangent line: $y - \frac{\pi}{4} = \left(\sqrt{3} + \frac{\pi}{2}\right)\left(x - \frac{1}{2}\right)$

$$y = \left(\sqrt{3} + \frac{\pi}{2}\right)x - \frac{\sqrt{3}}{2}$$

57. $f(x) = \arccos x$

$$f'(x) = \frac{-1}{\sqrt{1 - x^2}} = -2 \text{ when } x = \pm\frac{\sqrt{3}}{2}.$$

When $x = \sqrt{3}/2, f\left(\sqrt{3}/2\right) = \pi/6.$

When $x = -\sqrt{3}/2, f\left(-\sqrt{3}/2\right) = 5\pi/6.$

Tangent lines:

$$y - \frac{\pi}{6} = -2\left(x - \frac{\sqrt{3}}{2}\right) \Rightarrow y = -2x + \left(\frac{\pi}{6} + \sqrt{3}\right)$$

$$y - \frac{5\pi}{6} = -2\left(x + \frac{\sqrt{3}}{2}\right) \Rightarrow y = -2x \mp \left(\frac{5\pi}{6} - \sqrt{3}\right)$$

58. $g(x) = \arctan x, g'(x) = \frac{1}{1 + x^2}, g'(1) = \frac{1}{2}$

Tangent line: $y - \frac{\pi}{4} = \frac{1}{2}(x - 1)$

$$y = \frac{1}{2}x + \frac{\pi}{4} - \frac{1}{2}$$

59. $f(x) = \arctan x, a = 0$

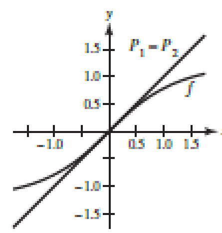
$f(0) = 0$

$f'(x) = \frac{1}{1 + x^2}, \qquad f'(0) = 1$

$f''(x) = \frac{-2x}{\left(1 + x^2\right)^2}, \quad f''(0) = 0$

$P_1(x) = f(0) + f'(0)x = x$

$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = x$



60. $f(x) = \arccos x, a = 0$

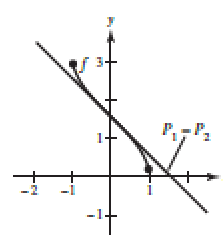
$f(0) = \frac{\pi}{2}$

$f'(x) = \frac{-1}{\sqrt{1 - x^2}}, \qquad f'(0) = -1$

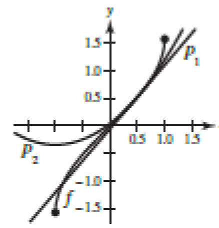
$f''(x) = \frac{-x}{\left(1 - x^2\right)^{3/2}}, \quad f''(0) = 0$

$P_1(x) = f(0) + f'(0)x = \frac{\pi}{2} - x$

$P_2(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 = \frac{\pi}{2} - x$



61. $f(x) = \arcsin x,\ a = \frac{1}{2}$



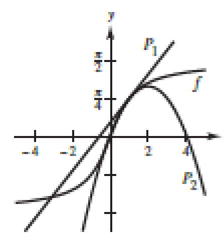
$$f'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$f''(x) = \frac{x}{(1 - x^2)^{3/2}}$$

$$P_1(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right)$$

$$P_2(x) = f\left(\frac{1}{2}\right) + f'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) + \frac{1}{2}f''\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right)^2 = \frac{\pi}{6} + \frac{2\sqrt{3}}{3}\left(x - \frac{1}{2}\right) + \frac{2\sqrt{3}}{9}\left(x - \frac{1}{2}\right)^2$$

62. $f(x) = \arcsin x,\ a = 1$



$$f'(x) = \frac{1}{1 + x^2}$$

$$f''(x) = \frac{-2x}{(1 + x^2)^2}$$

$$P_1(x) = f(1) + f'(1)(x - 1) = \frac{\pi}{4} + \frac{1}{2}(x - 1)$$

$$P_2(x) = f(1) + f'(1)(x - 1) + \frac{1}{2}f''(1)(x - 1)^2 = \frac{\pi}{4} + \frac{1}{2}(x - 1) - \frac{1}{4}(x - 1)^2$$

63. $x^2 + x\arctan y = y - 1, \quad \left(-\frac{\pi}{4}, 1\right)$

$$2x + \arctan y + \frac{x}{1 + y^2}y' = y'$$

$$\left(1 - \frac{x}{1 + y^2}\right)y' = 2x + \arctan y$$

$$y' = \frac{2x + \arctan y}{1 - \dfrac{x}{1 + y^2}}$$

At $\left(-\frac{\pi}{4}, 1\right)$: $y' = \dfrac{-\frac{\pi}{2} + \frac{\pi}{4}}{1 - \frac{-\pi/4}{2}} = \dfrac{-\frac{\pi}{2}}{2 + \frac{\pi}{4}} = \dfrac{-2\pi}{8 + \pi}$

Tangent line: $y - 1 = \dfrac{-2\pi}{8 + \pi}\left(x + \dfrac{\pi}{4}\right)$

$$y = \frac{-2\pi}{8 + \pi}x + 1 - \frac{\pi^2}{16 + 2\pi}$$

64. $\arctan(xy) = \arcsin(x + y), \quad (0, 0)$

$$\frac{1}{1 + (xy)^2}[y + xy'] = \frac{1}{\sqrt{1 - (x + y)^2}}[1 + y']$$

At $(0, 0)$: $0 = 1 + y' \Rightarrow y' = -1$

Tangent line: $y = -x$

65. $\arcsin x + \arcsin y = \frac{\pi}{2}, \quad \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

$$\frac{1}{\sqrt{1 - x^2}} + \frac{1}{\sqrt{1 - y^2}}y' = 0$$

$$\frac{1}{\sqrt{1 - y^2}}y' = \frac{-1}{\sqrt{1 - x^2}}$$

At $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$: $y' = -1$

Tangent line: $y - \dfrac{\sqrt{2}}{2} = -1\left(x - \dfrac{\sqrt{2}}{2}\right)$

$$y = -x + \sqrt{2}$$

66. $\arctan(x + y) = y^2 + \frac{\pi}{4}, \quad (1, 0)$

$$\frac{1}{1 + (x + y)^2}[1 + y'] = 2yy'$$

At $(1, 0)$: $\frac{1}{2}[1 + y'] = 0 \Rightarrow y' = -1$

Tangent line: $y - 0 = -1(x - 1)$

$$y = -x + 1$$

67. f is not one-to-one because many different x-values yield the same y-value.

Example: $f(0) = f(\pi) = 0$

Not continuous at $\dfrac{(2n - 1)\pi}{2}$, where n is an integer.

68. f is not one-to-one because different x-values yield the same y-value.

Example: $f(3) = f\left(-\frac{4}{3}\right) = \frac{3}{5}$

Not continuous at ± 2.

69. Because you know that f^{-1} exists and that $y_1 = f(x_1)$ by Theorem 3.17, then $\left(f^{-1}\right)'(y_1) = \dfrac{1}{f'(x_1)}$, provided that $f'(x_1) \neq 0$.

70. Theorem 3.17: Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$.

Moreover, $g'(x) = \dfrac{1}{f'(g(x))}$, $f'(g(x)) \neq 0$.

71. The derivatives are algebraic. See Theorem 3.18.

72. (a) Since the slope of the tangent line to f at $\left(-1, -\frac{1}{2}\right)$ is $\frac{1}{2}$, the slope of the tangent line to f^{-1} at $\left(-\frac{1}{2}, 1\right)$ is $m = \dfrac{1}{(1/2)} = 2$.

(b) Since the slope of the tangent line to f at $(2, 1)$ is 2, the slope of the tangent line to f^{-1} at $(1, 2)$ is $m = \frac{1}{2}$.

73. (a) $\cot\theta = \dfrac{x}{5}$

$$\theta = \operatorname{arccot}\left(\frac{x}{5}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-1/5}{1 + (x/5)^2}\dfrac{dx}{dt} = \dfrac{-5}{x^2 + 25}\dfrac{dx}{dt}$

If $\dfrac{dx}{dt} = -400$ and $x = 10$, $\dfrac{d\theta}{dt} = 16$ rad/h.

If $\dfrac{dx}{dt} = -400$ and $x = 3$, $\dfrac{d\theta}{dt} \approx 58.824$ rad/h.

74. (a) $\cot\theta = \dfrac{x}{3}$

$$\theta = \operatorname{arccot}\left(\frac{x}{3}\right)$$

(b) $\dfrac{d\theta}{dt} = \dfrac{-3}{x^2 + 9}\dfrac{dx}{dt}$

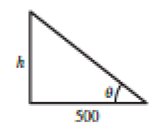
If $x = 10$, $\dfrac{d\theta}{dt} \approx 11.001$ rad/h.

If $x = 3$, $\dfrac{d\theta}{dt} \approx 66.667$ rad/h.

A lower altitude results in a greater rate of change of θ.

75. (a) $h(t) = -16t^2 + 256$

$-16t^2 + 256 = 0$ when $t = 4$ sec



(b) $\tan\theta = \dfrac{h}{500} = \dfrac{-16t^2 + 256}{500}$

$$\theta = \arctan\left[\frac{16}{500}\left(-t^2 + 16\right)\right]$$

$$\frac{d\theta}{dt} = \frac{-8t/125}{1 + \left[(4/125)\left(-t^2 + 16\right)\right]^2}$$

$$= \frac{-1000t}{15{,}625 + 16\left(16 - t^2\right)^2}$$

When $t = 1$, $d\theta/dt \approx -0.0520$ rad/sec.

When $t = 2$, $d\theta/dt \approx -0.1116$ rad/sec.

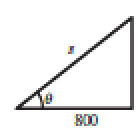
76. $\cos\theta = \dfrac{800}{s}$

$$\theta = \arccos\left(\frac{800}{s}\right)$$

$$\frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt}$$

$$= \frac{-1}{\sqrt{1 - (800/s)^2}}\left(\frac{-800}{s^2}\right)\frac{ds}{dt}$$

$$= \frac{800}{s\sqrt{s^2 - 800^2}}\frac{ds}{dt}, \quad s > 800$$



77. $\tan\theta = \dfrac{h}{300}$

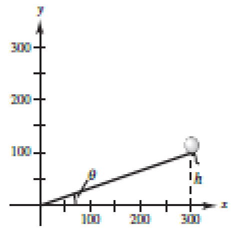
$\dfrac{dh}{dt} = 5$ ft/sec

$\theta = \arctan\left(\dfrac{h}{300}\right)$

$\dfrac{d\theta}{dt} = \dfrac{1/300}{1 + \left(h^2/300^2\right)}\left(\dfrac{dh}{dt}\right)$

$= \dfrac{300}{300^2 + h^2}(5)$

$= \dfrac{1500}{300^2 + h^2} = \dfrac{3}{200}$ rad/sec when $h = 100$



78. $\dfrac{d\theta}{dt} = 30(2\pi) = 60\pi$ rad/min

$\tan\theta = \dfrac{x}{50}$

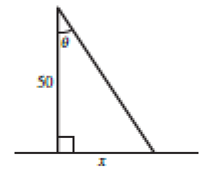
$\theta = \arctan\left(\dfrac{x}{50}\right)$

$\dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{50}{x^2 + 2500}\dfrac{dx}{dt}$

$\dfrac{dx}{dt} = \dfrac{x^2 + 2500}{50}\dfrac{d\theta}{dt}$

When $\theta = 45° = \dfrac{\pi}{4}, x = 50$:

$\dfrac{dx}{dt} = \dfrac{(50)^2 + 2500}{50}(60\pi) = 6000\pi$ ft/min



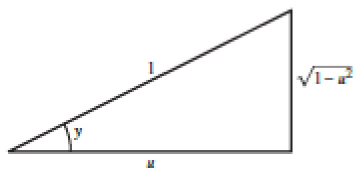
79. Prove $\dfrac{d}{dx}[\arcsin u] = \dfrac{u'}{\sqrt{1 - u^2}}$.

Let $y = \arccos u$. Then

$\cos y = u$

$-\sin y\dfrac{dy}{dx} = u'$

$\dfrac{dy}{dx} = -\dfrac{u'}{\sin y} = -\dfrac{u'}{\sqrt{1 - u^2}}$.



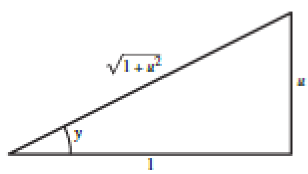
Prove $\dfrac{d}{dx}[\arctan u] = \dfrac{u'}{1 + u^2}$.

Let $y = \arctan u$. Then

$\tan y = u$

$\sec^2 y\dfrac{dy}{dx} = u'$

$\dfrac{dy}{dx} = \dfrac{u'}{\sec^2 y} = \dfrac{u'}{1 - u^2}$.



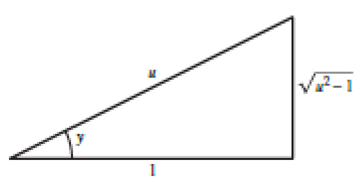
Prove $\dfrac{d}{dx}[\operatorname{arcsec} u] = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$.

Let $y = \operatorname{arcsec} u$. Then

$\sec y = u$

$\sec y\tan y\dfrac{dy}{dx} = u'$

$\dfrac{dy}{dx} = \dfrac{u'}{\sec y\tan y} = \dfrac{u'}{|u|\sqrt{u^2 - 1}}$.



Note: The absolute value sign in the formula for the derivative of arcsec u is necessary because the inverse secant function has a positive slope at every value in its domain.

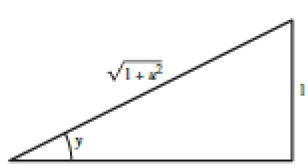
Prove $\dfrac{d}{dx}[\operatorname{arccot} u] = \dfrac{-u'}{1 + u^2}$.

Let $y = \operatorname{arccot} u$. Then

$\cot y = u$

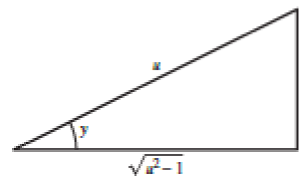
$-\csc^2 y\dfrac{dy}{dx} = u'$

$\dfrac{dy}{dx} = \dfrac{u'}{-\csc^2 y} = -\dfrac{u'}{1 + u^2}$.



Prove $\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$.

$$\begin{aligned}
\text{Let } y &= \operatorname{arccsc} u. \text{ Then} \\
\csc y &= u \\
-\csc y \cot y \frac{dy}{dx} &= u' \\
\frac{dy}{dx} &= \frac{u'}{-\csc y \cot y} = -\frac{u'}{|u|\sqrt{u^2 - 1}}.
\end{aligned}$$



Note: The absolute value sign in the formula for the derivative of arccsc u is necessary because the inverse cosecant function has a negative slope at every value in its domain.

80. $f(x) = kx + \sin x$

For $k \geq 1$, f is one-to-one, and for $k \leq -1$, f is one-to-one. Therefore, f has an inverse for $k \geq 1$ and $k \leq -1$.

81. True

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2 - 1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2 - 1}}$$

82. True

$$\frac{d}{dx}[\arctan x] = \frac{1}{1 + x^2} > 0 \text{ for all } x.$$

83. True

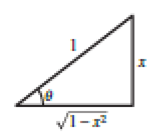
$$\frac{d}{dx}[\arctan(\tan x)] = \frac{\sec^2 x}{1 + \tan^2 x} = \frac{\sec^2 x}{\sec^2 x} = 1$$

84. False. The derivative $\frac{dy}{dx}$ is undefined when $x = \pm 1$.

85. Let $\theta = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right)$, $\quad -1 < x < 1$

$$\begin{aligned}
\tan\theta &= \frac{x}{\sqrt{1 - x^2}} \\
\sin\theta &= \frac{x}{1} = x \\
\arcsin x &= \theta.
\end{aligned}$$

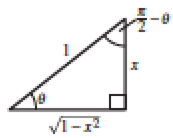
So, $\arcsin x = \arctan\left(\frac{x}{\sqrt{1 - x^2}}\right)$ for $-1 < x < 1$.



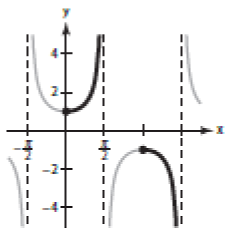
86. Let $\theta = \arctan\frac{x}{\sqrt{1 - x^2}}$, $\quad |x| < 1$.

Then $\tan\theta = \frac{x}{\sqrt{1 - x^2}}$, as indicated in the figure.

So, $\cos\left(\frac{\pi}{2} - \theta\right) = x$ and $\frac{\pi}{2} - \theta = \arccos x$ which gives $\arccos x = \frac{\pi}{2} - \arctan\frac{x}{\sqrt{1 - x^2}}$.

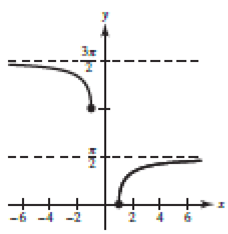


87. $f(x) = \sec x, \quad 0 \le x < \frac{\pi}{2}, \pi \le x < \frac{3\pi}{2}$



(a) $y = \operatorname{arcsec} x, \quad x \le -1 \quad \text{or} \quad x \ge 1$

$0 \le y < \frac{\pi}{2} \quad \text{or} \quad \pi \le y < \frac{3\pi}{2}$



(b)
$$\begin{aligned} y &= \operatorname{arcsec} x \\ x &= \sec y \\ 1 &= \sec y \tan y \cdot y' \\ y' &= \frac{1}{\sec y \tan y} \\ &= \frac{1}{x\sqrt{x^2 - 1}} \end{aligned}$$

$$\tan^2 y + 1 = \sec^2 y$$
$$\tan y = \pm\sqrt{\sec^2 y - 1}$$

On $0 \le y < \pi/2$ and $\pi \le y < 3\pi/2$, $\tan y \ge 0$.

88. $f(x) = \arcsin\left(\frac{x-2}{2}\right) - 2\arcsin\frac{\sqrt{x}}{2}, \quad 0 \le x \le 4$

$$\begin{aligned} f'(x) &= \frac{1/2}{\sqrt{1 - [(x-2)/2]^2}} - 2\left[\frac{1/(4\sqrt{x})}{1 - (\sqrt{x}/2)^2}\right] \\ &= \frac{1}{2\sqrt{1 - (1/4)(x^2 - 4x + 4)}} - \frac{1}{2\sqrt{x}\sqrt{1 - (x/4)}} \\ &= \frac{1}{2\sqrt{x - (x^2/4)}} - \frac{1}{2\sqrt{x - (x^2/4)}} \\ &= 0 \end{aligned}$$

Because the derivative is zero, you can conclude that the function is constant. (By letting $x = 0$ in $f(x)$, you can see that the constant is $-\pi/2$.)

89. $f(x) = 2x\sqrt{x - 6}$

Because $f(x) = 2x\sqrt{x-6} = 40$ when $x = 10$, $f(10) = 40$ and $f^{-1}(40) = 10$.

$$\begin{aligned} f^{-1}(40) &= \frac{1}{f'(f^{-1}(40))} \\ &= \frac{1}{f'(10)}. \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x\left(\frac{1}{2\sqrt{x-6}}\right) + 2\sqrt{x-6} \\ &= \frac{x}{\sqrt{x-6}} + 2\sqrt{x-6} \\ &= \frac{3x - 12}{\sqrt{x-6}} \end{aligned}$$

$$\begin{aligned} (f^{-1})'(40) &= \frac{1}{f'(10)} = \frac{1}{(3(10) - 12)/\sqrt{(10) - 6}} \\ &= \frac{1}{18/2} = \frac{1}{9}. \end{aligned}$$

So, the answer is A.

90. $f(x) = \frac{1}{3}\arctan\frac{x}{3}$

$$\begin{aligned} f'(x) &= \frac{1}{3} \cdot \frac{\frac{1}{3}}{1 + \left(\frac{x}{3}\right)^2} \\ &= \frac{1}{9}\left(\frac{1}{1 + \frac{x^2}{9}}\right) \\ &= \frac{1}{9 + x^2} \end{aligned}$$

So, the answer is D.

91. $f(x) = \ln(x^3 + 1) + \arctan 4x$

$$f'(x) = \frac{3x^2}{x^3 + 1} + \frac{4}{1 + 16x^2}$$

When $x = 1$,

$$\begin{aligned} f'(1) &= \frac{3(1)^2}{(1)^3 + 1} + \frac{4}{1 + 16(1)^2} \\ &= \frac{3}{2} + \frac{4}{17} \\ &= \frac{59}{34}. \end{aligned}$$

So, the answer is D.