Section 3.5 A Summary of Curve Sketching

1. $y = \dfrac{1}{x - 2} - 3$

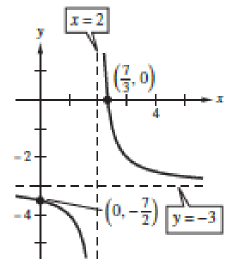
$y' = -\dfrac{1}{(x - 2)^2} \Rightarrow$ undefined when $x = 2$

$y'' = \dfrac{2}{(x - 2)^3} \Rightarrow$ undefined when $x = 2$

Intercepts: $\left(\dfrac{7}{3}, 0\right), \left(0, -\dfrac{7}{2}\right)$

Vertical asymptote: $x = 2$

Horizontal asymptote: $y = -3$



	y	y'	y''	Conclusion
$-\infty < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up

No relative extrema, no points of inflection

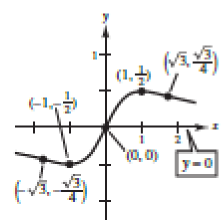
2. $y = \dfrac{x}{x^2 + 1}$

$$y' = \frac{1 - x^2}{\left(x^2 + 1\right)^2} = \frac{(1 - x)(x + 1)}{\left(x^2 + 1\right)^2} = 0 \text{ when } x = \pm 1.$$

$$y'' = -\frac{2x\left(3 - x^2\right)}{\left(x^2 + 1\right)^3} = 0 \text{ when } x = 0, \pm\sqrt{3}.$$

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -\sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = -\sqrt{3}$	$-\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$-\sqrt{3} < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\dfrac{1}{2}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	$\dfrac{1}{2}$	0	$-$	Relative maximum
$1 < x < \sqrt{3}$		$-$	$-$	Decreasing, concave down
$x = \sqrt{3}$	$\dfrac{\sqrt{3}}{4}$	$-$	0	Point of inflection
$\sqrt{3} < x < \infty$		$-$	$+$	Decreasing, concave up



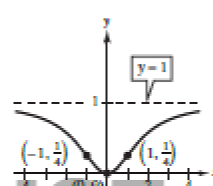
3. $y = \dfrac{x^2}{x^2 + 3}$

$$y' = \frac{6x}{\left(x^2 + 3\right)^2} = 0 \text{ when } x = 0.$$

$$y'' = \frac{18\left(1 - x^2\right)}{\left(x^2 + 3\right)^3} = 0 \text{ when } x = \pm 1.$$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$x = -1$	$\dfrac{1}{4}$	$-$	0	Point of inflection
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < 1$		$+$	$+$	Increasing, concave up
$x = 1$	$\dfrac{1}{4}$	$+$	0	2Point of inflection
$1 < x < \infty$		$+$	$-$	Increasing, concave down



4. $y = \dfrac{x^2 + 1}{x^2 - 4}$

$y' = \dfrac{-10x}{\left(x^2 - 4\right)^2} = 0$ when $x = 0$ and undefined when $x = \pm 2$.

$y'' = \dfrac{10\left(3x^2 + 4\right)}{\left(x^2 - 4\right)^3} < 0$ when $x = 0$.

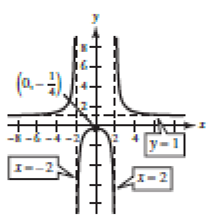
Intercept: $(0, -1/4)$

Symmetric about y-axis

Vertical asymptotes: $x = \pm 2$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$+$	$+$	Increasing, concave up
$-2 < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	$-\frac{1}{4}$			Relative maximum
$0 < x < 2$		$-$	$-$	Decreasing, concave down
$2 < x < \infty$		$-$	$+$	Decreasing, concave up



5. $y = \dfrac{3x}{x^2 - 1}$

$y' = \dfrac{-3\left(x^2 + 1\right)}{\left(x^2 - 1\right)^2}$ undefined when $x = \pm 1$

$y'' = \dfrac{6x\left(x^2 + 3\right)}{\left(x^2 - 1\right)^3}$

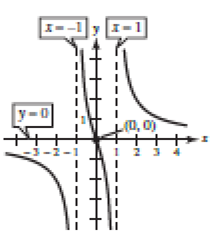
Intercept: $(0, 0)$

Symmetry with respect to origin

Vertical asymptotes: $x = \pm 1$

Horizontal asymptote: $y = 0$

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$-$	Decreasing, concave down
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	-3	0	Point of inflection
$0 < x < 1$		$-$	$-$	Decreasing, concave down
$1 < x < \infty$		$-$	$+$	Decreasing, concave up



6. $f(x) = \dfrac{x-3}{x} = 1 - \dfrac{3}{x}$

$f'(x) = \dfrac{3}{x^2}$ undefined when $x = 0$

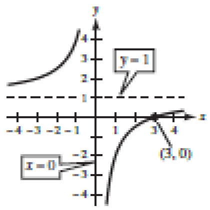
$f''(x) = -\dfrac{6}{x^3} \neq 0$

Vertical asymptote: $x = 0$

Intercept: $(3, 0)$

Horizontal asymptote: $y = 1$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$+$	Increasing, concave up
$0 < x < \infty$		$+$	$-$	Increasing, concave down



7. $f(x) = x + \dfrac{32}{x^2}$

$f'(x) = 1 - \dfrac{64}{x^3} = \dfrac{(x-4)(x^2+4x+16)}{x^3} = 0$ when $x = 4$ and undefined when $x = 0$.

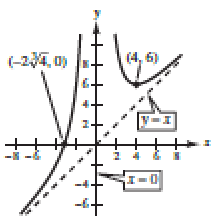
$f''(x) = \dfrac{192}{x^4}$

Intercept: $\left(-2\sqrt[3]{4}, 0\right)$

Vertical asymptote: $x = 0$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$+$	$+$	Increasing, concave up
$0 < x < 4$		$-$	$+$	Decreasing, concave up
$x = 4$	6	0	$+$	Relative minimum
$4 < x < \infty$		$+$	$+$	Increasing, concave up



8. $f(x) = \dfrac{x^3}{x^2 - 9} = x + \dfrac{9x}{x^2 - 9}$

$f'(x) = \dfrac{x^2(x^2 - 27)}{(x^2 - 9)^2} = 0$ when $x = 0, \ \pm 3\sqrt{3}$ and is undefined when $x = \pm 3$.

$f''(x) = \dfrac{18x(x^2 + 27)}{(x^2 - 9)^3} = 0$ when $x = 0$

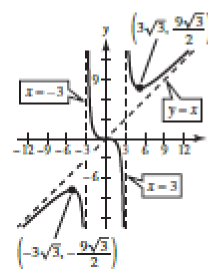
Intercept: $(0, 0)$

Symmetry: origin

Vertical asymptotes: $x = \pm 3$

Slant asymptote: $y = x$

	y	y'	y''	Conclusion
$-\infty < x < -3\sqrt{3}$		+	–	Increasing, concave down
$x = -3\sqrt{3}$	$-\frac{9\sqrt{3}}{2}$	0	–	Relative maximum
$-3\sqrt{3} < x < -3$		–	–	Decreasing, concave down
$-3 < x < 0$		–	+	Decreasing, concave up
$x = 0$	0	0	0	Point of inflection
$0 < x < 3$		–	–	Decreasing, concave down
$3 < x < 3\sqrt{3}$		–	+	Decreasing, concave up
$x = 3\sqrt{3}$	$\frac{9\sqrt{3}}{2}$	0	+	Relative minimum
$3\sqrt{3} < x < \infty$		+	+	Increasing, concave up



9. $y = \dfrac{x^2 - 6x + 12}{x - 4} = x - 2 + \dfrac{4}{x - 4}$

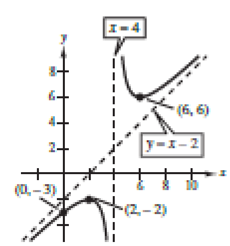
$y' = 1 - \dfrac{4}{(x - 4)^2} = \dfrac{(x - 2)(x - 6)}{(x - 4)^2} = 0$ when $x = 2, 6$ and is undefined when $x = 4$.

$y'' = \dfrac{8}{(x - 4)^3}$

Vertical asymptote: $x = 4$

Slant asymptote: $y = x - 2$

	y	y'	y''	Conclusion
$-\infty < x < 2$		+	–	Increasing, concave down
$x = 2$	-2	0	–	Relative maximum
$2 < x < 4$		–	–	Decreasing, concave down
$4 < x < 6$		–	+	Decreasing, concave up
$x = 6$	6	0	+	Relative minimum
$6 < x < \infty$		+	+	Increasing, concave up



10. $y = \dfrac{-x^2 - 4x - 7}{x + 3} = -x - 1 - \dfrac{4}{x + 3}$

$y' = -\dfrac{x^2 + 6x + 5}{(x + 3)^2} = -\dfrac{(x + 1)(x + 5)}{(x + 3)^2} = 0$ when $x = -1, -5$ and is undefined when $x = -3$.

$y'' = \dfrac{-8}{(x + 3)^3}$

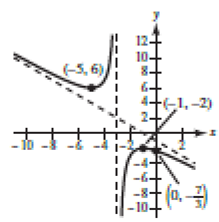
Intercept: $\left(0, -\frac{7}{3}\right)$

No symmetry

Vertical asymptote: $x = -3$

Slant asymptote: $y = -x - 1$

	y	y'	y''	Conclusion
$-\infty < x < -5$		$-$	$+$	Decreasing, concave up
$x = -5$	6	0	$+$	Relative minimum
$-5 < x < -3$		$+$	$+$	Increasing, concave up
$-3 < x < -1$		$+$	$-$	Increasing, concave down
$x = -1$	-2	0	$-$	Relative maximum
$-1 < x < \infty$		$-$	$-$	Decreasing, concave down



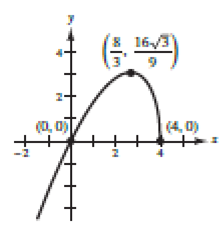
11. $y = x\sqrt{4 - x}$, Domain: $(-\infty, 4]$

$y' = \dfrac{8 - 3x}{2\sqrt{4 - x}} = 0$ when $x = \dfrac{8}{3}$ and undefined when $x = 4$.

$y'' = \dfrac{3x - 16}{4(4 - x)^{3/2}} = 0$ when $x = \dfrac{16}{3}$ and undefined when $x = 4$.

Note: $x = \dfrac{16}{3}$ is not in the domain.

	y	y'	y''	Conclusion
$-\infty < x < \frac{8}{3}$		$+$	$-$	Increasing, concave down
$x = \frac{8}{3}$	$\frac{16}{3\sqrt{3}}$	0	$-$	Relative maximum
$\frac{8}{3} < x < 4$		$-$	$-$	Decreasing, concave down
$x = 4$	0	Undefined	Undefined	Endpoint



12. $h(x) = x\sqrt{9 - x^2}$, Domain: $-3 \le x \le 3$

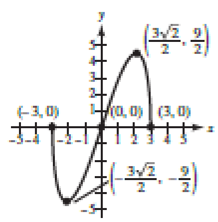
$$h'(x) = \frac{9 - 2x^2}{\sqrt{9 - x^2}} = 0 \text{ when } x = \pm\frac{3}{\sqrt{2}} = \pm\frac{3\sqrt{2}}{2} \text{ and undefined when } x = \pm 3.$$

$$h''(x) = \frac{x(2x^2 - 27)}{(9 - x^2)^{3/2}} = 0 \text{ when } x = 0 \text{ and undefined when } x = \pm 3.$$

Intercepts: $(0, 0), (\pm 3, 0)$

Symmetric with respect to the origin

	y	y'	y''	Conclusion
$x = -3$	0	Undefined	Undefined	Endpoint
$-3 < x < -\frac{3}{\sqrt{2}}$		$-$	$+$	Decreasing, concave up
$x = -\frac{3}{\sqrt{2}}$	$-\frac{9}{2}$	0	$+$	Relative minimum
$-\frac{3}{\sqrt{2}} < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	0	3	0	Point of inflection
$0 < x < \frac{3}{\sqrt{2}}$		$+$	$-$	Increasing, concave down
$x = \frac{3}{\sqrt{2}}$	$\frac{9}{2}$	0	$-$	Relative maximum
$\frac{3}{\sqrt{2}} < x < 3$		$-$	$-$	Decreasing, concave down
$x = 3$	0	Undefined	Undefined	Endpoint

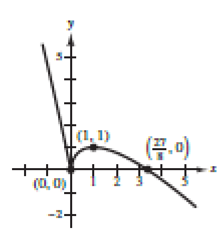


13. $y = 3x^{2/3} - 2x$

$$y' = 2x^{-1/3} - 2 = \frac{2(1 - x^{1/3})}{x^{1/3}} = 0 \text{ when } x = 1 \text{ and undefined when } x = 0.$$

$$y'' = \frac{-2}{3x^{4/3}} < 0 \text{ when } x \ne 0.$$

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	0	Undefined	Undefined	Relative minimum
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	1	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down



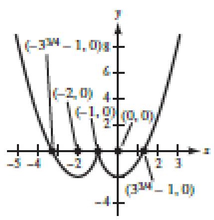
14. $y = (x+1)^2 - 3(x+1)^{2/3}$

$y' = 2(x+1) - 2(x+1)^{-1/3} = \dfrac{2(x+1)^{4/3} - 2}{(x+1)^{1/3}} = 0$ when $x = 0, -2$ and undefined when $x = -1$.

$y'' = 2 + \dfrac{2}{3}(x+1)^{-4/3} = \dfrac{6(x+1)^{4/3} + 2}{3(x+1)^{4/3}}$

Intercepts: $(-1, 0), \left(\pm 3^{3/4} - 1, 0\right)$

	y	y'	y''	Conclusion
$-\infty < x < -2$		$-$	$+$	Decreasing, concave up
$x = -2$	-2	0	$+$	Relative minimum
$-2 < x < -1$		$+$	$+$	Increasing, concave up
$x = -1$	0	Undefined	$+$	Relative maximum
$-1 < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	0	0	$+$	Relative minimum
$0 < x < \infty$		$+$	$+$	Increasing, concave up



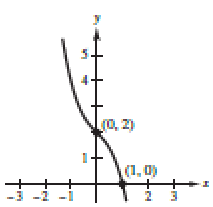
15. $y = 2 - x - x^3$

$y' = -1 - 3x^2$

No critical numbers

$y'' = -6x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < 0$		$-$	$+$	Decreasing, concave up
$x = 0$	2	$-$	0	Point of inflection
$0 < x < \infty$		$-$	$-$	Decreasing, concave down

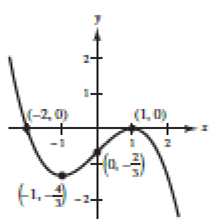


16. $y = -\frac{1}{3}(x^3 - 3x + 2)$

$y' = -x^2 + 1 = 0$ when $x = \pm 1$.

$y'' = -2x = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	$-\frac{4}{3}$	0	$+$	Relative minimum
$-1 < x < 0$		$+$	$+$	Increasing, concave up
$x = 0$	$-\frac{2}{3}$	$+$	0	Point of inflection
$0 < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	$-$	Relative maximum
$1 < x < \infty$		$-$	$-$	Decreasing, concave down

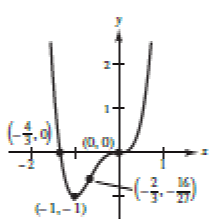


17. $y = 3x^4 + 4x^3$

$y' = 12x^3 + 12x^2 = 12x^2(x + 1) = 0$ when $x = 0, x = -1$.

$y'' = 36x^2 + 24x = 12x(3x + 2) = 0$ when $x = 0, x = -\frac{2}{3}$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		$-$	$+$	Decreasing, concave up
$x = -1$	-1	0	$+$	Relative minimum
$-1 < x < -\frac{2}{3}$		$+$	$+$	Increasing, concave up
$x = -\frac{2}{3}$	$-\frac{16}{27}$	$+$	0	Point of inflection
$-\frac{2}{3} < x < 0$		$+$	$-$	Increasing, concave down
$x = 0$	0	0	0	Point of inflection
$0 < x < \infty$		$+$	$+$	Increasing, concave up



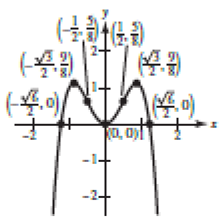
18. $y = -2x^4 + 3x^2$

$y' = -8x^3 + 6x = 0$ when $x = 0, \pm\dfrac{\sqrt{3}}{2}$.

$y'' = -24x^2 + 6 = 0$ when $x = \pm\dfrac{1}{2}$.

Symmetry: y-axis

Intercepts: $\left(\pm\dfrac{\sqrt{6}}{2}, 0\right)$



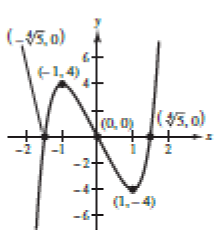
	y	y'	y''	Conclusion
$-\infty < x < -\frac{\sqrt{3}}{2}$		+	−	Increasing, concave down
$x = -\frac{\sqrt{3}}{2}$	$\frac{9}{8}$	0	−	Relative maximum
$-\frac{\sqrt{3}}{2} < x < -\frac{1}{2}$		−	−	Decreasing, concave down
$x = -\frac{1}{2}$	$\frac{5}{8}$	-2	0	Point of inflection
$-\frac{1}{2} < x < 0$		−	+	Decreasing, concave up
$x = 0$	0	0	+	Relative minimum
$0 < x < \frac{1}{2}$		+	+	Increasing, concave up
$x = \frac{1}{2}$	$\frac{5}{8}$	2	0	Point of inflection
$\frac{1}{2} < x < \frac{\sqrt{3}}{2}$		+	−	Increasing, concave down
$x = \frac{\sqrt{3}}{2}$	$\frac{9}{8}$	0	−	Relative maximum
$\frac{\sqrt{3}}{2} < x < \infty$		−	−	Decreasing, concave down

19. $y = x^5 - 5x$

$y' = 5x^4 - 5 = 5(x^4 - 1) = 0$ when $x = \pm 1$.

$y'' = 20x^3 = 0$ when $x = 0$.

	y	y'	y''	Conclusion
$-\infty < x < -1$		+	−	Increasing, concave down
$x = -1$	4	0	−	Relative maximum
$-1 < x < 0$		−	−	Decreasing, concave down
$x = 0$	0	−	0	Point of inflection
$0 < x < 1$		−	+	Decreasing, concave up
$x = 1$	-4	0	+	Relative minimum
$1 < x < \infty$		+	+	Increasing, concave up

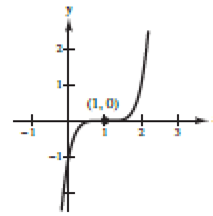


20. $y = (x-1)^5$

$y' = 5(x-1)^4 = 0$ when $x = 1$.

$y'' = 20(x-1)^3 = 0$ when $x = 1$.

	y	y'	y''	Conclusion
$-\infty < x < 1$		$+$	$-$	Increasing, concave down
$x = 1$	0	0	0	Point of inflection
$1 < x < \infty$		$+$	$+$	Increasing, concave up

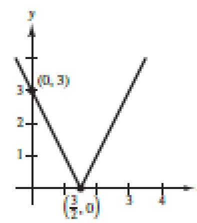


21. $y = |2x-3|$

$y' = \dfrac{2(2x-3)}{|2x-3|}$ undefined at $x = \dfrac{3}{2}$.

$y'' = 0$

	y	y'	Conclusion
$-\infty < x < \frac{3}{2}$		$-$	Decreasing
$x = \frac{3}{2}$	0	Undefined	Relative minimum
$\frac{3}{2} < x < \infty$		$+$	Increasing



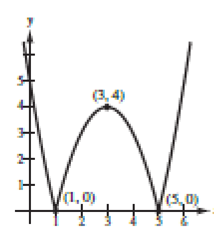
22. $y = |x^2 - 6x + 5|$

$y' = \dfrac{2(x-3)(x^2-6x+5)}{|x^2-6x+5|} = \dfrac{2(x-3)(x-5)(x-1)}{|(x-5)(x-1)|}$

$= 0$ when $x = 3$ and undefined when $x = 1, x = 5$.

$y'' = \dfrac{2(x^2-6x+5)}{|x^2-6x+5|} = \dfrac{2(x-5)(x-1)}{|(x-5)(x-1)|}$ undefined when $x = 1, x = 5$.

	y	y'	y''	Conclusion
$-\infty < x < 1$		$-$	$+$	Decreasing, concave up
$x = 1$	0	Undefined	Undefined	Relative minimum, point of inflection
$1 < x < 3$		$+$	$-$	Increasing, concave down
$x = 3$	4	0	$-$	Relative maximum
$3 < x < 5$		$-$	$-$	Decreasing, concave down
$x = 5$	0	Undefined	Undefined	Relative minimum, point of inflection
$5 < x < \infty$		$+$	$+$	Increasing, concave up

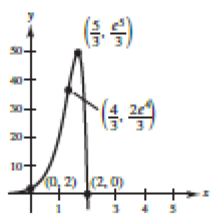


23. $f(x) = e^{3x}(2 - x)$

$f'(x) = -e^{3x} + 2(2 - x)e^{3x} = e^{3x}(5 - 3x) = 0$ when $x = \frac{5}{3}$.

$f''(x) = -3e^{3x}(-4 + 3x) = 0$ when $x = \frac{4}{3}$.

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < \frac{4}{3}$		+	+	Increasing, concave up
$x = \frac{4}{3}$	$\frac{2e^4}{3}$	54.6	0	Point of inflection
$\frac{4}{3} < x < \frac{5}{3}$		+	−	Increasing, concave down
$x = \frac{5}{3}$	$\frac{e^5}{3}$	0	−445.2	Relative maximum
$\frac{5}{3} < x < \infty$		−	−	Decreasing, concave down



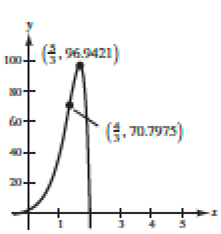
24. $f(x) = -2 + e^{3x}(4 - 2x)$

$f'(x) = -2e^{3x}(3x - 5) = 0$ when $x = \frac{5}{3}$.

$f''(x) = -6e^{3x}(3x - 4) = 0$ when $x = \frac{4}{3}$.

Horizontal asymptote (to left): $y = -2$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < \frac{4}{3}$		+	+	Increasing, concave up
$x = \frac{4}{3}$	70.7975	109.1963	0	Point of inflection
$\frac{4}{3} < x < \frac{5}{3}$		+	−	Increasing, concave down
$x = \frac{5}{3}$	96.9421	0	−890.4790	Relative maximum
$\frac{5}{3} < x < \infty$		−	−	Decreasing, concave down



25. $g(t) = \dfrac{10}{1 + 4e^{-t}}$

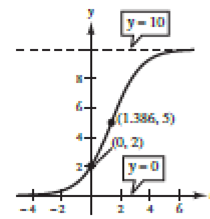
$g'(t) = \dfrac{40e^{-t}}{\left(1 + 4e^{-t}\right)^2} > 0$ for all t.

$g''(t) = \dfrac{40e^{-t}\left(4e^{-t} - 1\right)}{\left(1 + 4e^{-t}\right)^3} = 0$ at $t \approx 1.386$.

$\lim\limits_{t \to \infty} g(t) = 10 \Rightarrow t = 10$ is a horizontal asymptote.

$\lim\limits_{t \to -\infty} g(t) = 0 \Rightarrow t = 0$ is a horizontal asymptote.

	$g(t)$	$g'(t)$	$g''(t)$	Conclusion
$-\infty < t < 1.386$		$+$	$+$	Increasing, concave up
$t = 1.386$	5	2.5	0	Point of inflection
$1.386 < t < \infty$		$+$	$-$	Increasing, concave down



26. $h(x) = \dfrac{8}{2 + 3e^{-x/2}}$

$h'(x) = \dfrac{12e^{x/2}}{\left(2e^{x/2} + 3\right)^2}$

$h''(x) = \dfrac{6e^{x/2}\left(3 - 2e^{x/2}\right)}{\left(2e^{x/2} + 3\right)^3}$

No critical numbers, no relative extrema

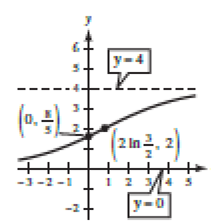
$\lim\limits_{x \to \infty} h(x) = \dfrac{8}{2} = 4 \Rightarrow x = 4$ is a horizontal asymptote.

$\lim\limits_{x \to -\infty} h(x) = 0 \Rightarrow x = 0$ is a horizontal asymptote.

$h''(x) = 0$: $3 = 2e^{x/2} \Rightarrow e^{x/2} = \dfrac{3}{2} \Rightarrow x = 2\ln\left(\dfrac{3}{2}\right)$

Intercept: $\left(0, \dfrac{8}{5}\right)$

	$h(x)$	$h'(x)$	$h''(x)$	Conclusion
$-\infty < x < 2\ln\frac{3}{2}$		$+$	$+$	Increasing, concave up
$x = 2\ln\frac{3}{2}$	2	$\frac{1}{2}$	0	Point of inflection
$2\ln\frac{3}{2} < x < \infty$		$+$	$-$	Increasing, concave down

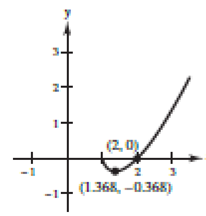


27. $y = (x-1)\ln(x-1)$, Domain: $x > 1$

$y' = 1 + \ln(x-1) = 0$ when $\ln(x-1) = -1 \Rightarrow (x-1) = e^{-1} \Rightarrow x = 1 + e^{-1}$

$$y'' = \frac{1}{x-1}$$

	y	y'	y''	Conclusion
$1 < x < 1 + e^{-1}$		$-$	$+$	Decreasing, concave up
$x = 1 + e^{-1}$	$-e^{-1}$	0	e	Relative minimum
$1 + e^{-1} < x < \infty$		$+$	$+$	Increasing, concave up

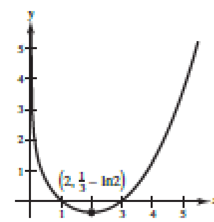


28. $y = \frac{1}{24}x^3 - \ln x$, Domain: $x > 0$

$y' = \dfrac{(x-2)(x^2+2x+4)}{8x} = 0$ when $x = 2$.

$$y'' = \frac{x^3+4}{4x^2}$$

	y	y'	y''	Conclusion
$0 < x < 2$		$-$	$+$	Decreasing, concave up
$x = 2$	-0.3598	0	3	Relative minimum
$2 < x < \infty$		$+$	$+$	Increasing, concave down

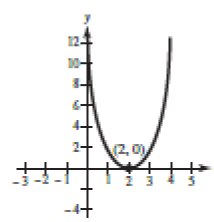


29. $g(x) = 6\arcsin\left(\frac{x-2}{2}\right)^2$, Domain: $[0, 4]$

$g'(x) = \dfrac{12(x-2)}{\sqrt{(4x-x^2)(x^2-4x+8)}} = 0$ when $x = 2$.

$$g''(x) = \frac{12(x^4 - 8x^3 + 24x^2 - 32x + 32)}{\left[(4x-x^2)(x^2-4x+8)\right]^{3/2}}$$

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$0 < x < 2$		$-$	$+$	Decreasing, concave up
$x = 2$	0	0	$+$	Relative minimum
$2 < x < 4$		$+$	$+$	Increasing, concave down

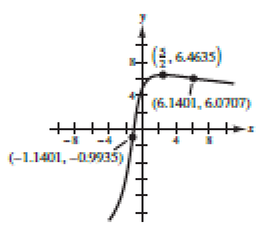


30. $h(x) = 7\arctan(x+1) - \ln(x^2 + 2x + 2)$

$$h'(x) = \frac{5 - 2x}{x^2 + 2x + 2} = 0 \text{ when } x = \frac{5}{2}.$$

$$h''(x) = \frac{2(x^2 - 5x - 7)}{(x^2 + 2x + 2)^2} = 0 \text{ when } x = \frac{5 \pm \sqrt{53}}{2}$$

	$h(x)$	$h'(x)$	$h''(x)$	Conclusion
$-\infty < x < -1.1401$		+	+	Increasing, concave up
$x = -1.1401$	-0.9935	+	0	Point of inflection
$-1.1401 < x < \frac{5}{2}$		+	–	Increasing, concave down
$x = \frac{5}{2}$	6.4635	0	–	Relative maximum
$\frac{5}{2} < x < 6.1401$		–	–	Decreasing, concave down
$x = 6.1401$	6.0707	–	0	Point of inflection
$6.1401 < x < \infty$		–	+	Decreasing, concave up



31. $f(x) = \frac{x}{3^{x-3}} = \frac{27x}{3^x}$

$$f'(x) = \frac{27(1 - x\ln 3)}{3^x} = 0 \Rightarrow x = \frac{1}{\ln 3} \approx 0.910$$

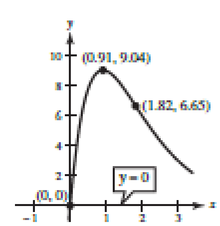
$$f''(x) = \frac{27\ln 3(x\ln 3 - 2)}{3^x} = 0 \Rightarrow x = \frac{2}{\ln 3} \approx 1.820$$

$$\lim_{x\to\infty} f(x) = 0,\ \lim_{x\to-\infty} f(x) = -\infty$$

Horizontal asymptote: $y = 0$

Intercept: $(0, 0)$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 0.910$		+	–	Increasing, concave down
$x = 0.910$	9.041	0	–	Relative maximum
$0.910 < x < 1.820$		–	–	Decreasing, concave down
$x = 1.820$		–	0	Point of inflection
$1.820 < x < \infty$	6.652	–	+	Decreasing, concave up



32. $g(t) = (5 - t)5^t$

$$g'(t) = 5^t(5 \ln 5 - 1 - t \ln 5) = 0 \Rightarrow t \ln 5 = 5 \ln 5 - 1 \Rightarrow t = \frac{5 \ln 5 - 1}{\ln 5} = 5 - \frac{1}{\ln 5} \approx 4.379$$

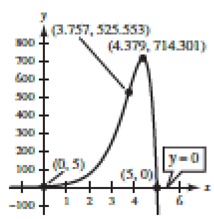
$$g''(t) = 5^t \ln 5(5 \ln 5 - 2 - t \ln 5) = 0 \Rightarrow t = \frac{5 \ln 5 - 2}{\ln 5} = 5 - \frac{2}{\ln 5} \approx 3.757$$

$\lim\limits_{t \to \infty} g(t) = -\infty$ and $\lim\limits_{t \to -\infty} g(t) = 0$

Horizontal asymptote: $y = 0$

Intercepts: $(5, 0)$, $(0, 5)$

	$g(t)$	$g'(t)$	$g''(t)$	Conclusion
$-\infty < t < 3.757$		+	+	Increasing, concave up
$t = 3.757$	525.553	+	0	Point of inflection
$3.757 < t < 4.379$		+	–	Increasing, concave down
$t = 4.379$	714.301	0	–	Relative maximum
$4.379 < t < \infty$		–	–	Decreasing, concave down

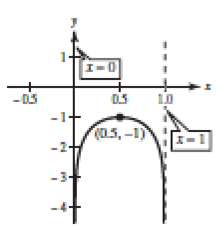


33. $g(x) = \log_4(x - x^2) = \dfrac{\ln(x - x^2)}{\ln 4}$, Domain: $0 < x < 1$

$$g'(x) = \frac{2x - 1}{\ln 4 \cdot x(x - 1)} = 0 \text{ when } x = \frac{1}{2}.$$

$$g''(x) = \frac{-2x^2 + 2x - 1}{\ln 4 \cdot x^2(x - 1)^2}$$

	$g(x)$	$g'(x)$	$g''(x)$	Conclusion
$0 < x < \frac{1}{2}$		+	–	Increasing, concave down
$x = \frac{1}{2}$	–1	0	–	Relative maximum
$\frac{1}{2} < x < 1$		–	–	Decreasing, concave down

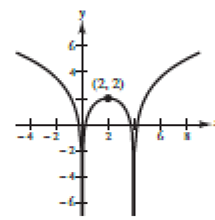


34. $f(x) = \log_2\left|x^2 - 4x\right| = \dfrac{\ln\left|x^2 - 4x\right|}{\ln 2}$

$f'(x) = \dfrac{2(x - 2)}{x(x - 4)\ln 2} = 0$ when $x = 2$ and undefined when $x = 0$ and $x = 4$.

$f''(x) = \dfrac{-2\left(x^2 - 4x + 8\right)}{x^2(x - 4)^2 \ln 2}$

	$f(x)$	$f'(x)$	$f''(x)$	Conclusion
$-\infty < x < 0$		$-$	$-$	Decreasing, concave down
$x = 0$	Undefined	Undefined	Undefined	Undefined
$0 < x < 2$		$+$	$-$	Increasing, concave down
$x = 2$	2	0	$-$	Relative maximum
$2 < x < 4$		$-$	$-$	Decreasing, concave down
$x = 4$	Undefined	Undefined	Undefined	Undefined
$4 < x < \infty$		$+$	$-$	Increasing, concave down



35. Because $1 + x^2 \neq 0$ for any real number x and $f(0) = 15$, the graph of f does not have a vertical asymptote. So, the graph of f has a horizontal asymptote at $y = 0$.

36. $f(x) = \dfrac{15}{1 + x^2} = 15\left(1 + x^2\right)^{-1}$

$$f'(x) = -15\left(1 + x^2\right)^{-2}(2x)$$

$$= \frac{-30x}{\left(1 + x^2\right)^2}$$

$$0 = \frac{-30x}{\left(1 + x^2\right)^2}$$

$$x = 0$$

$$f(0) = \frac{15}{1 + (0)^2} = 15$$

So, the graph of f has a relative maximum at $(0, 15)$.

37. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x} = \dfrac{19x^2 - 1}{x\left(x^2 + 1\right)}$

$f'(x) = \dfrac{-\left(19x^4 - 22x^2 - 1\right)}{x^2\left(x^2 + 1\right)^2} = 0$ for $x \approx \pm 1.10$

$f''(x) = \dfrac{2\left(19x^6 - 63x^9 - 3x^2 - 1\right)}{x^3\left(x^2 + 1\right)^3} = 0$ for $x \approx \pm 1.84$

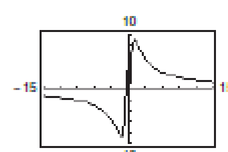
Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$

Minimum: $(-1.10, -9.05)$

Maximum: $(1.10, 9.05)$

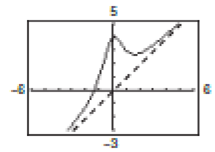
Points of inflection: $(-1.84, -7.86), (1.84, 7.86)$



38. $f(x) = x + \dfrac{4}{x^2+1} = \dfrac{x^3+x+4}{x^2+1} = 0$ for $x \approx -1.379$

$f'(x) = \dfrac{x^4+2x^2-8x+1}{\left(x^2+1\right)^2} = 0$ for $x \approx 1.608, x \approx 0.129$

$f''(x) = \dfrac{8\left(3x^2-1\right)}{\left(x^2+1\right)^3} = 0$ for $x = \pm\dfrac{1}{\sqrt{3}} \approx \pm 0.577$



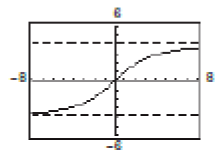
Slant asymptote: $y = x$

Points of inflection: $(-0.577, 2.423), (0.577, 3.577)$

Relative maximum: $(0.129, 4.064)$

Relative minimum: $(1.608, 2.724)$

39. $f(x) = \dfrac{4x}{\sqrt{x^2+15}}$



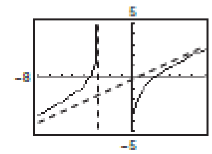
$f'(x) = \dfrac{60}{\left(x^2+15\right)^{3/2}} > 0$

$f''(x) = \dfrac{-180x}{\left(x^2+15\right)^{5/2}} = 0$ at $x = 0$

Horizontal asymptotes: $y = \pm 4$

Point of inflection: $(0, 0)$

40. $y = \dfrac{x}{2} + \ln\left(\dfrac{x}{x+3}\right)$



$y' = \dfrac{1}{2} + \dfrac{3}{x(x+3)}$

$y'' = \dfrac{-3(2x+3)}{x^2(x+3)^2}$

Vertical asymptotes: $x = -3, x = 0$

Slant asymptote: $y = \dfrac{x}{2}$

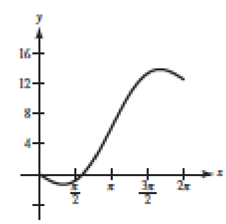
41. $f(x) = 2x - 4\sin x, 0 \le x \le 2\pi$

$f'(x) = 2 - 4\cos x$

$f''(x) = 4\sin x$

$f'(x) = 0 \Rightarrow \cos x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

$f''(x) = 0 \Rightarrow x = 0, \pi, 2\pi$



Relative minimum: $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3} - 2\sqrt{3}\right)$

Relative maximum: $\left(\dfrac{5\pi}{3}, \dfrac{10\pi}{3} + 2\sqrt{3}\right)$

Points of inflection: $(0, 0), (\pi, 2\pi), (2\pi, 4\pi)$

42. $f(x) = -x + 2\cos x, 0 \le x \le 2\pi$

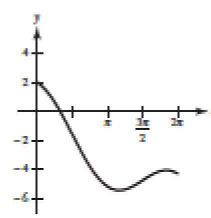
$f'(x) = -1 - 2\sin x$

$f''(x) = -2\cos x$

$f(x) = 0$ at $x \approx 1.030$

$f'(x) = 0 \Rightarrow \sin x = -\dfrac{1}{2} \Rightarrow x = \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

$f''(x) = 0 \Rightarrow x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$



Relative minimum: $\left(\dfrac{7\pi}{6}, -\sqrt{3} - \dfrac{7\pi}{6}\right) \approx (3.665, -5.397)$

Relative maximum: $\left(\dfrac{11\pi}{6}, \sqrt{3} - \dfrac{11\pi}{6}\right) \approx (5.760, -4.028)$

43. $y = \sin x - \dfrac{1}{18}\sin 3x, 0 \le x \le 2\pi$

$$y' = \cos x - \frac{1}{6}\cos 3x$$
$$= \cos x - \frac{1}{6}\left[\cos 2x \cos x - \sin 2x \sin x\right]$$
$$= \cos x - \frac{1}{6}\left[\left(1 - 2\sin^2 x\right)\cos x - 2\sin^2 x \cos x\right]$$
$$= \cos x\left[1 - \frac{1}{6}\left(1 - 2\sin^2 x - 2\sin^2 x\right)\right] = \cos x\left[\frac{5}{6} + \frac{2}{3}\sin^2 x\right]$$

$y' = 0$: $\quad \cos x = 0 \Rightarrow x = \pi/2, 3\pi/2$

$\dfrac{5}{6} + \dfrac{2}{3}\sin^2 x = 0 \Rightarrow \sin^2 x = -5/4$, impossible

$$y'' = -\sin x + \frac{1}{2}\sin 3x = 0 \Rightarrow 2\sin x = \sin 3x$$
$$= \sin 2x \cos x + \cos 2x \sin x$$
$$= 2\sin x \cos^2 x + \left(2\cos^2 x - 1\right)\sin x$$
$$= \sin x\left(2\cos^2 x + 2\cos^2 x - 1\right)$$
$$= \sin x\left(4\cos^2 x - 1\right)$$

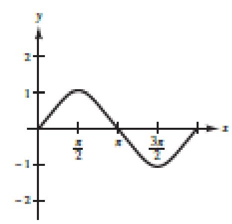
$\sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$2 = 4\cos^2 x - 1 \Rightarrow \cos x = \pm\sqrt{3}/2 \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

Relative maximum: $\left(\dfrac{\pi}{2}, \dfrac{19}{18}\right)$

Relative minimum: $\left(\dfrac{3\pi}{2}, -\dfrac{19}{18}\right)$

Points of inflection: $\left(\dfrac{\pi}{6}, \dfrac{4}{9}\right), \left(\dfrac{5\pi}{6}, \dfrac{4}{9}\right), (\pi, 0), \left(\dfrac{7\pi}{6}, -\dfrac{4}{9}\right), \left(\dfrac{11\pi}{6}, -\dfrac{4}{9}\right)$



44. $y = \cos x - \dfrac{1}{4}\cos 2x, 0 \le x \le 2\pi$

$$y' = -\sin x + \frac{1}{2}\sin 2x = -\sin x + \sin x \cos x$$
$$= \sin x(\cos x - 1)$$

$y' = 0$: $\quad \sin x = 0 \Rightarrow x = 0, \pi, 2\pi$

$\cos x - 1 = 0 \Rightarrow x = 0, 2\pi$

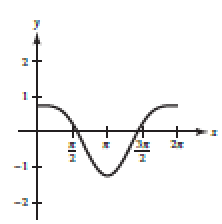
$$y'' = -\cos x + \cos 2x$$
$$= -\cos x + 2\cos^2 x - 1$$
$$= (2\cos x + 1)(\cos x - 1)$$

$y'' = 0$: $2\cos x + 1 = 0 \Rightarrow x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$\cos x - 1 = 0 \Rightarrow x = 0, 2\pi$

Relative minimum: $\left(\pi, -\dfrac{5}{4}\right)$

Points of inflection: $\left(\dfrac{2\pi}{3}, -\dfrac{3}{8}\right), \left(\dfrac{4\pi}{3}, -\dfrac{3}{8}\right)$



45. $y = 2x - \tan x, -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

$y' = 2 - \sec^2 x = 0$ when $x = \pm\dfrac{\pi}{4}$.

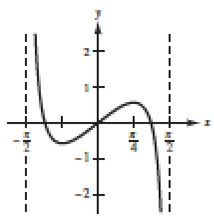
$y'' = -2\sec^2 x \tan x = 0$ when $x = 0$.

Relative maximum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 1\right)$

Relative minimum: $\left(-\dfrac{\pi}{4}, 1 - \dfrac{\pi}{2}\right)$

Point of inflection: $(0, 0)$

Vertical asymptotes: $x = \pm\dfrac{\pi}{2}$



46. $y = 2(x - 2) + \cot x, 0 < x < \pi$

$y' = 2 - \csc^2 x = 0$ when $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

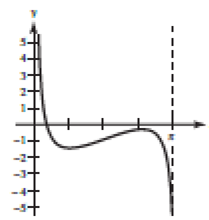
$y'' = 2\csc^2 x \cot x = 0$ when $x = \dfrac{\pi}{2}$.

Relative maximum: $\left(\dfrac{3\pi}{4}, \dfrac{3\pi}{2} - 5\right)$

Relative minimum: $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2} - 3\right)$

Point of inflection: $\left(\dfrac{\pi}{2}, \pi - 4\right)$

Vertical asymptotes: $x = 0, \pi$

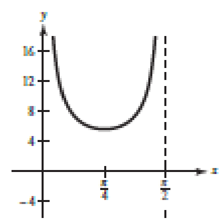


47. $y = 2(\csc x + \sec x), 0 < x < \dfrac{\pi}{2}$

$y' = 2(\sec x \tan x - \csc x \cot x) = 0 \Rightarrow x = \dfrac{\pi}{4}$

Relative minimum: $\left(\dfrac{\pi}{4}, 4\sqrt{2}\right)$

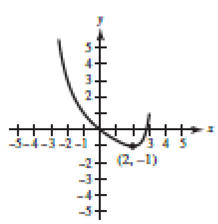
Vertical asymptotes: $x = 0, \dfrac{\pi}{2}$



48. $y = \sec^2\left(\dfrac{\pi x}{8}\right) - 2\tan\left(\dfrac{\pi x}{8}\right) - 1, -3 < x < 3$

$y' = 2\sec^2\left(\dfrac{\pi x}{8}\right)\tan\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) - 2\sec^2\left(\dfrac{\pi x}{8}\right)\left(\dfrac{\pi}{8}\right) = 0 \Rightarrow x = 2$

Relative minimum: $(2, -1)$



49. $g(x) = x \tan x, -\frac{3\pi}{2} < x < \frac{3\pi}{2}$

$g'(x) = \dfrac{x + \sin x \cos x}{\cos^2 x} = 0$ when $x = 0$.

$g''(x) = \dfrac{2(\cos x + x \sin x)}{\cos^3 x}$

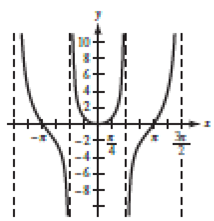
Vertical asymptotes: $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$

Intercepts: $(-\pi, 0), (0, 0), (\pi, 0)$

Symmetric with respect to y-axis.

Increasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

Points of inflection: $(\pm 2.80, -1)$



50. $g(x) = x \cot x, -2\pi < x < 2\pi$

$g'(x) = \dfrac{\sin x \cos x - x}{\sin^2 x}$

$g'(0)$ does not exist. But $\lim_{x \to 0} x \cot x = \lim_{x \to 0} \dfrac{x}{\tan x} = 1.$

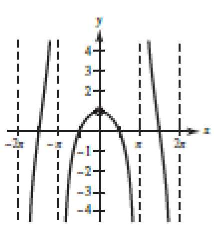
Vertical asymptotes: $x = \pm 2\pi, \pm\pi$

Intercepts: $\left(-\frac{3\pi}{2}, 0\right), \left(-\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$

Symmetric with respect to y-axis.

Decreasing on $(0, \pi)$ and $(\pi, 2\pi)$

Points of inflection: $(\pm 4.49, 1)$



51. Because the slope is negative, the function is decreasing on $(2, 8)$, and so $f(3) > f(5)$.

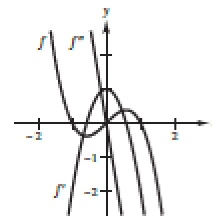
52. If $f'(x) = 2$ in $[-5, 5]$, then $f(x) = 2x + 3$ and $f(2) = 7$ is the least possible value of $f(2)$. If $f'(x) = 4$ in $[-5, 5]$, then $f(x) = 4x + 3$ and $f(2) = 11$ is the greatest possible value of $f(2)$.

53. f is cubic.

f' is quadratic.

f'' is linear.

The zeros of f' correspond to the points where the graph of f has horizontal tangents. The zero of f'' corresponds to the point where the graph of f' has a horizontal tangent.

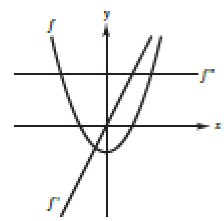


54. f'' is constant.

f' is linear.

f is quadratic.

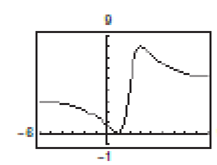
The zero of f' corresponds to the points where the graph of f has a horizontal tangent. There are no zeros on of f'', which means the graph of f' has no horizontal tangent.



55. $f(x) = \dfrac{4(x-1)^2}{x^2 - 4x + 5}$

Vertical asymptote: none

Horizontal asymptote: $y = 4$

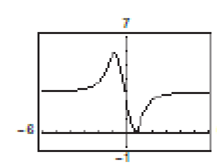


The graph crosses the horizontal asymptote $y = 4$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

56. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

Vertical asymptote: none

Horizontal asymptote: $y = 3$

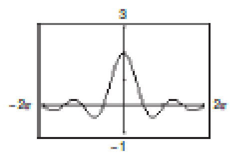


The graph crosses the horizontal asymptote $y = 3$. If a function has a vertical asymptote at $x = c$, the graph would not cross it because $f(c)$ is undefined.

57. $h(x) = \dfrac{\sin 2x}{x}$

Vertical asymptote: none

Horizontal asymptote: $y = 0$



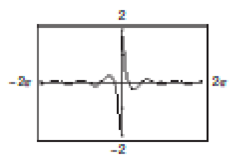
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

58. $f(x) = \dfrac{\cos 3x}{4x}$

Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 0$



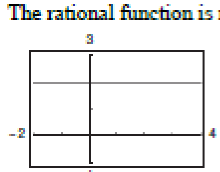
Yes, it is possible for a graph to cross its horizontal asymptote.

It is not possible to cross a vertical asymptote because the function is not continuous there.

59. $h(x) = \dfrac{6 - 2x}{3 - x}$

$$= \frac{2(3 - x)}{3 - x} = \begin{cases} 2, & \text{if } x \neq 3 \\ \text{Undefined}, & \text{if } x = 3 \end{cases}$$

The rational function is not reduced to lowest terms.

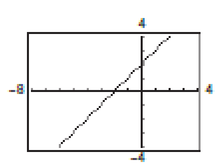


There is a hole at $(3, 2)$.

60. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

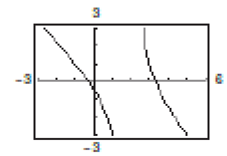
$$= \frac{(x + 2)(x - 1)}{x - 1} = \begin{cases} x + 2, & \text{if } x \neq 1 \\ \text{Undefined}, & \text{if } x = 1 \end{cases}$$

The rational function is not reduced to lowest terms.



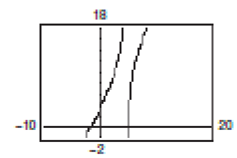
There is a hole at $(1, 3)$.

61. $f(x) = -\dfrac{x^2 - 3x - 1}{x - 2} = -x + 1 + \dfrac{3}{x - 2}$



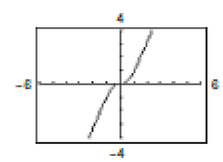
The graph appears to approach the slant asymptote $y = -x + 1$.

62. $g(x) = \dfrac{2x^2 - 8x - 15}{x - 5} = 2x + 2 - \dfrac{5}{x - 5}$



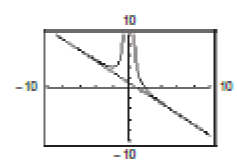
The graph appears to approach the slant asymptote $y = 2x + 2$.

63. $f(x) = \dfrac{2x^3}{x^2 + 1} = 2x - \dfrac{2x}{x^2 + 1}$



The graph appears to approach the slant asymptote $y = 2x$.

64. $h(x) = \dfrac{-x^3 + x^2 + 4}{x^2} = -x + 1 + \dfrac{4}{x^2}$



The graph appears to approach the slant asymptote $y = -x + 1$.

65. (a) $f'(x) < 0$ when $-3 < x < 1$.

So, f is decreasing on the interval $(-3, 1)$.

(b) $f'(x) = x^2 + 2x$

$f''(x) = 2x + 2 < 0$ when $x < -1$.

So, the graph of f is concave downward on the interval $(-7, -1)$.

(c) $f'(x) = 0$ when $x = -3$ and $x - 1$.

Because f is decreasing on the interval $(-3, 1)$, f has a relative maximum at $x = -3$ and a relative minimum at $x = 1$.

(d) $f''(x) = 2x + 2 = 0$ when $x = -1$.

So, the graph of f has a point of inflection at $x = -1$.

66. (a) $f'(x) > 0$ when $x < -2$ and $-2 < x < 1$.

So, f is increasing on the intervals $(-4, 2)$ and $(-2, 1)$.

(b) Because $f'(x)$ is increasing when $-2 < x < 0$, the graph of f is concave upward on the interval $(-2, 0)$.

(c) $f'(x) = 0$ when $x = -2$ and $x = 1$.

Because f is increasing on the intervals $(-4, -2)$ and $(-2, 1)$ and decreasing on the interval $(1, 2)$, the relative maximum is at $x = 1$ and there is no relative minimum.

(d) Because $f''(x) = 0$ when $x = -2$ and $x = 0$, the graph of f has points of inflection at $x = -2$ and $x = 0$.

67. (a) Because $f'' < 0$ when $x < -2$ and $x > 6$, the graph of f is concave downward on $(-\infty, 2)$ and $(6, \infty)$. Because $f'' > 0$ when $-2 < x < 6$, the graph of f is concave upward on $(-2, 6)$.

(b) f' is decreasing on $(-\infty, -2)$ and $(6, \infty)$ and increasing on $(-2, 6)$.

(c) Because $f'' = 0$ when $x = -2$ and $x = 6$, the graph of f has points of inflection at $x = -2$ and $x = 6$.

68. (a) Because $f'' < 0$ when $x < -2$ and $x > 2$, the graph of f is concave downward on $(-\infty, -2)$ and $(2, \infty)$. Because $f'' > 0$ when $-2 < x < 0$ and $0 < x < 2$, the graph of f is concave upward on $(-2, 2)$.

(b) f' is decreasing on $(-\infty, -2)$ and $(2, \infty)$ and increasing on $(-2, 2)$.

(c) Because $f'' = 0$ when $x = -2$ and $x = 2$, the graph has points of inflection at $x = -2$ and $x = 2$.

69. (a) Because $f'' < 0$ when $0 < x < \pi$, the graph of f is concave downward on $(0, \pi)$. Because $f'' > 0$ when $\pi < x < 2\pi$, the graph of f is concave upward on $(\pi, 2\pi)$.

(b) f' is decreasing on $(0, \pi)$ and increasing on $(\pi, 2\pi)$.

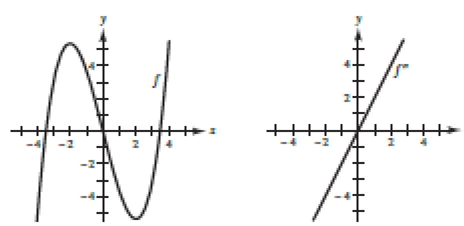
(c) Because $f'' = 0$ when $x = \pi$, the graph of f has a point of inflection at $x = \pi$.

70. (a) Because $f'' < 0$ when $x < -1$ and $x > 1$, the graph of f is concave downward on $(-\infty, -1)$ and $(1, \infty)$. Because $f'' > 0$ when $-1 < x < 1$, the graph of f is concave upward on $(-1, 1)$.

(b) f' is decreasing on $(-\infty, -1)$ and $(1, \infty)$ and increasing on $(-1, 1)$.

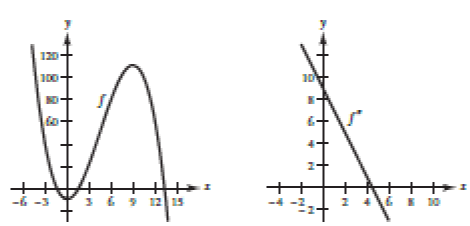
(c) Because $f'' = 0$ when $x = -1$ and $x = 1$, the graph has points of inflection at $x = -1$ and $x = 1$.

71.



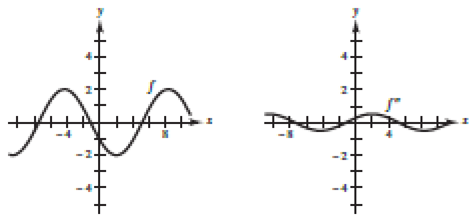
(or any vertical translation of f)

72.



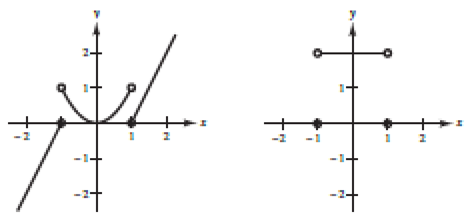
(or any vertical translation of f)

73.



(or any vertical translation of f)

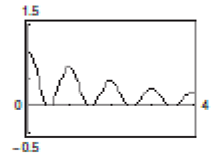
74.



(or any vertical translation of the 3 segments of f)

75. $f(x) = \dfrac{\cos^2 \pi x}{\sqrt{x^2+1}}, (0, 4)$

(a)



On $(0, 4)$ there seem to be 7 critical numbers: 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5

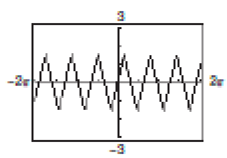
(b) $$f'(x) = \frac{-\cos \pi x\left(x \cos \pi x + 2\pi\left(x^2+1\right)\sin \pi x\right)}{\left(x^2+1\right)^{3/2}} = 0$$

Critical numbers $\approx \frac{1}{2}, 0.97, \frac{3}{2}, 1.98, \frac{5}{2}, 2.98, \frac{7}{2}$.

The critical numbers where maxima occur appear to be integers in part (a), but approximating them using f' shows that they are not integers.

76. $f(x) = \tan(\sin \pi x)$

(a)



(b) $f(-x) = \tan(\sin(-\pi x)) = \tan(-\sin \pi x) = -\tan(\sin \pi x) = -f(x)$

Symmetry with respect to the origin

(c) Periodic with period 2

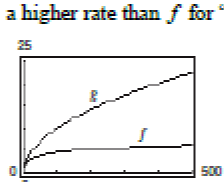
(d) On $(-1, 1)$, there is a relative maximum at $\left(\frac{1}{2}, \tan 1\right)$ and a relative minimum at $\left(-\frac{1}{2}, -\tan 1\right)$.

(e) On $(0, 1)$, the graph of f is concave downward.

77. (a) $f(x) = \ln x$, $g(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{x}, g'(x) = \dfrac{1}{2\sqrt{x}}$

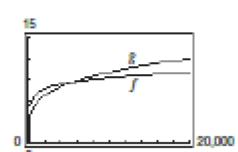
For $x > 4$, $g'(x) > f'(x)$. g is increasing at a higher rate than f for "large" values of x.



(b) $f(x) = \ln x$, $g(x) = \sqrt[4]{x}$

$f'(x) = \dfrac{1}{x}, g'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

For $x > 256$, $g'(x) > f'(x)$. g is increasing at a higher rate than f for "large" values of x. $f(x) = \ln x$ increases very slowly for "large" values of x.



78. $g(x) = \ln f(x), f(x) > 0$

$$g'(x) = \frac{f'(x)}{f(x)}$$

(a) Yes. If the graph of g is increasing, then $g'(x) > 0$. Because $f(x) > 0$, you know that $f'(x) = g'(x)f(x)$ and $f'(x) > 0$. So, the graph of f is increasing.

(b) No. Let $f(x) = x^2 + 1$ (positive and concave up). $g(x) = \ln(x^2 + 1)$ is not concave up.

79. (a) $f'(x) = 0$ at x_0, x_2 and x_4 (horizontal tangent).

(b) $f''(x) = 0$ at x_2 and x_3 (point of inflection).

(c) $f'(x)$ does not exist at x_1 (sharp corner).

(d) f has a relative maximum at x_1.

(e) f has a point of inflection at x_2 and x_3 (change in concavity).

80. (a) $f'(x) = 0$ for $x = -2$ (relative maximum) and $x = 2$ (relative minimum).

f' is negative for $-2 < x < 2$ (decreasing).

f' is positive for $x > 2$ and $x < -2$ (increasing).

(b) $f''(x) = 0$ at $x = 0$ (point of inflection).

f'' is positive for $x > 0$ (concave upward).

f'' is negative for $x < 0$ (concave downward).

(c) f' is increasing on $(0, \infty)$. $(f'' > 0)$

(d) $f'(x)$ is minimum at $x = 0$. The rate of change of f at $x = 0$ is less than the rate of change of f for all other values of x.

81. Tangent line at P: $y - y_0 = f'(x_0)(x - x_0)$

(a) Let $y = 0$: $-y_0 = f'(x_0)(x - x_0)$

$$f'(x_0)x = x_0 f'(x_0) - y_0$$

$$x = x_0 - \frac{y_0}{f'(x_0)} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x-intercept: $\left(x_0 - \dfrac{f(x_0)}{f'(x_0)}, 0\right)$

(b) Let $x = 0$: $y - y_0 = f'(x_0)(-x_0)$

$$y = y_0 - x_0 f'(x_0)$$

$$y = f(x_0) - x_0 f'(x_0)$$

y-intercept: $(0, f(x_0) - x_0 f'(x_0))$

(c) Normal line: $y - y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

Let $y = 0$: $-y_0 = -\dfrac{1}{f'(x_0)}(x - x_0)$

$$-y_0 f'(x_0) = -x + x_0$$

$$x = x_0 + y_0 f'(x_0) = x_0 + f(x_0)f'(x_0)$$

x-intercept: $(x_0 + f(x_0)f'(x_0), 0)$

(d) Let $x = 0$: $y - y_0 = \dfrac{-1}{f'(x_0)}(-x_0)$

$$y = y_0 + \frac{x_0}{f'(x_0)}$$

y-intercept: $\left(0, y_0 + \dfrac{x_0}{f'(x_0)}\right)$

(e) $|BC| = \left|x_0 - \dfrac{f(x_0)}{f'(x_0)} - x_0\right| = \left|\dfrac{f(x_0)}{f'(x_0)}\right|$

(f) $|PC|^2 = y_0^2 + \left(\dfrac{f(x_0)}{f'(x_0)}\right) = \dfrac{f(x_0)^2 f'(x_0)^2 + f(x_0)^2}{f'(x_0)^2}$

$$|PC| = \left|\frac{f(x_0)\sqrt{1 + [f'(x_0)]^2}}{f'(x_0)}\right|$$

(g) $|AB| = |x_0 - (x_0 + f(x_0)f'(x_0))| = |f(x_0)f'(x_0)|$

(h) $|AP|^2 = f(x_0)^2 f'(x_0)^2 + y_0^2$

$$|AP| = |f(x_0)|\sqrt{1 + [f'(x_0)]^2}$$

82. $f(x) = \dfrac{2x^n}{x^4 + 1}$

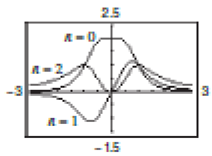
(a) For n even, f is symmetric about the y-axis. For n odd, f is symmetric about the origin.

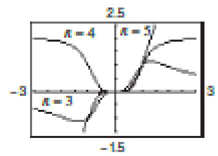
(b) The x-axis will be the horizontal asymptote if the degree of the numerator is less than 4. That is, $n = 0, 1, 2, 3$.

(c) $n = 4$ gives $y = 2$ as the horizontal asymptote.

(d) There is a slant asymptote $y = 2x$ if $n = 5$: $\dfrac{2x^5}{x^4 + 1} = 2x - \dfrac{2x}{x^4 + 1}$.

(e)





n	0	1	2	3	4	5
M	1	2	3	2	1	0
N	2	3	4	5	2	3

83. $f(x) = \dfrac{ax}{(x - b)^2}$

Answers will vary. *Sample answer*: The graph has a vertical asymptote at $x = b$. If a and b are both positive, or both negative, then the graph of f approaches ∞ as x approaches b, and the graph has a minimum at $x = -b$. If a and b have opposite signs, then the graph of f approaches $-\infty$ as x approaches b, and the graph has a maximum at $x = -b$.

84. $f(x) = \frac{1}{2}(ax)^2 - (ax) = \frac{1}{2}(ax)(ax - 2), a \neq 0$

$f'(x) = a^2x - a = a(ax - 1) = 0$ when $x = \dfrac{1}{a}$.

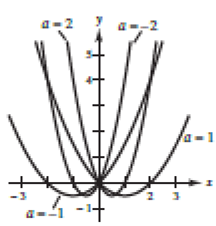
$f''(x) = a^2 > 0$ for all x.

(a) Intercepts: $(0, 0), \left(\dfrac{2}{a}, 0\right)$

Relative minimum: $\left(\dfrac{1}{a}, -\dfrac{1}{2}\right)$

Points of inflection: none

(b)



85. Vertical asymptote: $x = 3$

Horizontal asymptote: $y = 0$

$y = \dfrac{1}{x - 3}$

86. Vertical asymptote: $x = 2$

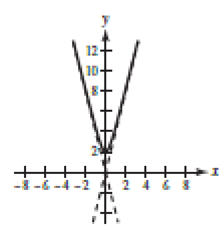
Slant asymptote: $y = -x$

$y = -x + \dfrac{1}{x - 2} = \dfrac{-x^2 + 2x + 1}{x - 2}$

87. $y = \sqrt{4 + 16x^2}$

As $x \to \infty, y \to 4x$. As $x \to -\infty, y \to -4x$.

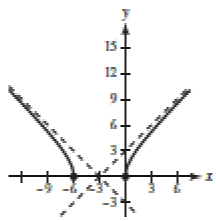
Slant asymptotes: $y = \pm 4x$



88. $y = \sqrt{x^2 + 6x} = \sqrt{(x+3)^2 - 9}$

$y \to x + 3$ as $x \to \infty$, and $y \to -x - 3$ as $x \to -\infty$.

Slant asymptotes: $y = x + 3$, $y = -x - 3$



89. $f(x) = \dfrac{x^3 - 1}{x^2 - x} = \dfrac{(x-1)(x^2 + x + 1)}{x(x-1)} = \dfrac{x^2 + x + 1}{x}$

$\lim\limits_{x \to -\infty} \dfrac{x^2 + x + 1}{x} = \lim\limits_{x \to -\infty} \left(x + 1 + \dfrac{1}{x}\right) = -\infty$

$\lim\limits_{x \to -\infty} \dfrac{x^2 + x + 1}{x} = \lim\limits_{x \to -\infty} \left(x + 1 + \dfrac{1}{x}\right) = \infty$

So, the graph of $y = x + 1$ is a slant asymptote of f.

$f(x) = \dfrac{x^3 - 1}{x^2 - x} = \dfrac{x^2 + x + 1}{x}$ has a vertical asymptote at $x = 0$, and $x = 1$ is a nonremovable discontinuity.

Because the asymptotes of f are $y = x + 1$ and $x = 0$, the answer is C.

90. Evaluate each statement.

A: Because $f'' < 0$ when $x < -8$ and $8 < x < 16$, the graph is concave downward on $(-\infty, -8)$ and $(8, 16)$.
The statement is false.

B: Because $f' > 0$ when $x < -16$ and $0 < x < 16$, the graph of f is increasing on $(-\infty, -16)$ and $(0, 16)$.
The statement is false.

C: $f' = 0$ when $x = -16$, $x = 0$, and $x = 16$.
The statement is false.

D: Because $f'' = 0$ when $x = -16$, $x = 0$, and $x = 16$, the graph of f has points of inflection at $x = -16$, $x = 0$, and $x = 16$.
The statement is true.

So, the answer is D.

91. Evaluate each statement.

I: Because $g'' > 0$ when $x > 2$, the graph of g is concave upward on $(2, \infty)$.
The statement is true.

II: Because $g'' < 0$ when $x < 2$, the graph of g' is decreasing on the interval $(-\infty, 2)$.
The statement is true.

III: Because $g'' = 0$ when $x = 2$, the graph has a point of inflection at $x = 2$.
The statement is true.

So, the answer is D.

92. (a) $f'(x) = -12(x-2)^2(x-4) = 0$
$x = 2, 4$

Because $f'(x) > 0$ on $(-\infty, 2)$ and $(2, 4)$ and $f'(x) < 0$ on $(4, \infty)$, the graph of f has a relative maximum at $x = 4$.

(b) Because $f'(x) < 0$ on $(4, \infty)$, the graph of f is decreasing on $(4, \infty)$.

(c) $f'(x) = -12(x-2)^2(x-4)$

$f'(x)$ has relative extrema at $x = 2$ and $x = \frac{10}{3}$, so $f''(x) = 0$ at $x = 2$ and $x = \frac{10}{3}$. Because $f''(x) = 0$ when $x < 2$ and $x > \frac{10}{3}$, the graph of f is concave downward on $(-\infty, 2)$ and $\left(\frac{10}{3}, \infty\right)$.

(d) Because $f''(x) = 0$ at $x = 2$ and $x = \frac{10}{3}$, the graph of f has two points of inflection at $x = 2$ and $x = \frac{10}{3}$.