Section 2.8 Newton's Method

In the solutions for Exercises 1–4, the values in the tables have been rounded for convenience. Because a calculator and a computer program calculate internally using more digits than they display, you may produce slightly different values from those shown in the tables.

1. $f(x) = x^2 - 5$

$f'(x) = 2x$

$x_1 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	2.0000	−1.0000	4.0000	−0.2500	2.2500
2	2.2500	0.0625	4.5000	0.0139	2.2361

2. $f(x) = x^3 - 3$

$f'(x) = 3x^2$

$x_1 = 1.4$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.4000	−0.2560	5.8800	−0.0435	1.4435
2	1.4435	0.0080	6.2514	0.0013	1.4423

3. $f(x) = \cos x$

$f'(x) = -\sin x$

$x_1 = 1.6$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.6000	–0.0292	–0.9996	0.0292	1.5708
2	1.5708	0.0000	–1.0000	0.0000	1.5708

4. $f(x) = \tan x$

$f'(x) = \sec^2 x$

$x_1 = 0.1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	0.1000	0.1003	1.0101	0.0993	0.0007
2	0.0007	0.0007	1.0000	0.0007	0.0000

5. $f(x) = x^3 + 4$

$f'(x) = 3x^2$

$x_1 = -2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	–2.0000	–4.0000	12.0000	–0.3333	–1.6667
2	–1.6667	–0.6296	8.3333	–0.0756	–1.5911
3	–1.5911	–0.0281	7.5949	–0.0037	–1.5874
4	–1.5874	–0.0000	7.5596	0.0000	–1.5874

Approximation of the zero of f is –1.587.

6. $f(x) = 2 - x^3$

$f'(x) = -3x^2$

$x_1 = 1.0$

n	x_n	$f(x_n)$	$f'(x_n)$	$\frac{f(x_n)}{f'(x_n)}$	$x_n - \frac{f(x_n)}{f'(x_n)}$
1	1.0000	1.0000	–3.0000	–0.3333	1.3333
2	1.3333	–0.3704	–5.3333	0.0694	1.2639
3	1.2639	–0.0190	–4.7922	0.0040	1.2599
4	1.2599	0.0001	–4.7623	0.0000	1.2599

Approximation of the zero of f is 1.260.

7. $f(x) = x^3 + x - 1$

$f'(x) = 3x^2 + 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.3750	1.7500	–0.2143	0.7143
2	0.7143	0.0788	2.5307	0.0311	0.6832
3	0.6832	0.0021	2.4003	0.0009	0.6823

Approximation of the zero of f is 0.682.

8. $f(x) = x^5 + x - 1$

$f'(x) = 5x^4 + 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.4688	1.3125	–0.3571	0.8571
2	0.8571	0.3196	3.6983	0.0864	0.7707
3	0.7707	0.0426	2.7641	0.0154	0.7553
4	0.7553	0.0011	2.6272	0.0004	0.7549

Approximation of the zero of f is 0.755.

9. $f(x) = 5\sqrt{x - 1} - 2x$

$f'(x) = \dfrac{5}{2\sqrt{x - 1}} - 2$

From the graph you see that these are two zeros. Begin with $x = 1.2$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.2000	–0.1639	3.5902	–0.0457	1.2457
2	1.2457	–0.0131	3.0440	–0.0043	1.2500
3	1.2500	–0.0001	3.0003	–0.0003	1.2500

Approximation of the zero of f is 1.250.

Similarly, the other zero is approximately 5.000.

(**Note:** These answers are exact)

10. $f(x) = x - 2\sqrt{x + 1}$

$f'(x) = 1 - \dfrac{1}{\sqrt{x + 1}}$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	5.0000	0.1010	0.5918	0.1707	4.8293
2	4.8293	0.0005	0.5858	0.00085	4.8284

Approximation of the zero of f is 4.8284.

11. $f(x) = x - e^{-x}$

$f'(x) = 1 + e^{-x}$

$x_1 = 0.5$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.5	–0.1065	1.6065	–0.0663	0.5663
2	0.5663	0.0013	1.5676	0.0008	0.5671
3	0.5671	0.0001	1.5672	–0.0000	0.5671

Approximation of the zero of f is 0.567.

12. $f(x) = x - 3 + \ln x$

$f'(x) = 1 + \dfrac{1}{x}$

$x_1 = 2.0$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	2.0	–0.3069	1.5	–0.2046	2.2046
2	2.2046	–0.0049	1.4536	–0.0033	2.2079
3	2.2079	–0.0001	1.4529	–0.0000	2.2079

Approximation of the zero of f is 2.208.

13. $f(x) = x^3 - 3.9x^2 + 4.79x - 1.881$

$f'(x) = 3x^2 - 7.8x + 4.79$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.5000	–0.3360	1.6400	–0.2049	0.7049
2	0.7049	–0.0921	0.7824	–0.1177	0.8226
3	0.8226	–0.0231	0.4037	–0.0573	0.8799
4	0.8799	–0.0045	0.2495	–0.0181	0.8980
5	0.8980	–0.0004	0.2048	–0.0020	0.9000
6	0.9000	0.0000	0.2000	0.0000	0.9000

Approximation of the zero of f is 0.900.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.1	0.0000	–0.1600	–0.0000	1.1000

Approximation of the zero of f is 1.100.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.9	0.0000	0.8000	0.0000	1.9000

Approximation of the zero of f is 1.900.

14. $f(x) = x^4 + x^3 - 1$

$f'(x) = 4x^3 + 3x^2$

From the graph you see that these are two zeros. Begin with $x_1 = 1.0$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.0000	1.0000	7.0000	0.1429	0.8571
2	0.8571	0.1695	4.7230	0.0359	0.8213
3	0.8213	0.0088	4.2390	0.0021	0.8192
4	0.8192	0.0003	4.2120	0.0000	0.8192

Approximation of the zero of f is 0.819.
Similarly, the other zero is approximately –1.380.

15. $f(x) = 1 - x + \sin x$

$f'(x) = -1 + \cos x$

$x_1 = 2$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	2.0000	–0.0907	–1.4161	0.0640	1.9360
2	1.9360	–0.0019	–1.3571	0.0014	1.9346
3	1.9346	0.0000	–1.3558	0.0000	1.9346

Approximate zero: $x \approx 1.935$

16. $f(x) = x^3 - \cos x$

$f'(x) = 3x^2 + \sin x$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	0.9000	0.1074	3.2133	0.0334	0.8666
2	0.8666	0.0034	3.0151	0.0011	0.8655
3	0.8655	0.0000	3.0087	0.0000	0.8655

Approximation of the zero of f is 0.866.

17. $h(x) = f(x) - g(x) = 2x + 1 - \sqrt{x + 4}$

$h'(x) = 2 - \dfrac{1}{2\sqrt{x + 4}}$

n	x_n	$h(x_n)$	$h'(x_n)$	$\dfrac{h(x_n)}{h'(x_n)}$	$x_n - \dfrac{h(x_n)}{h'(x_n)}$
1	0.6000	0.0552	1.7669	0.0313	0.5687
2	0.5687	0.0000	1.7661	0.0000	0.5687

Point of intersection of the graphs of f and g occurs when $x \approx 0.569$.

18. $h(x) = e^{x/2} - 2 + x^2$

$h'(x) = \frac{1}{2}e^{x/2} + 2x$

Two points of intersection

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	−1	−0.3935	−1.6967	0.2319	−1.2319
2	−1.2319	0.0577	−2.1937	−0.0263	−1.2056
3	−1.2056	0.0007	−2.1376	−0.0004	−1.2052

One point of intersection of the graphs of f and g occurs when $x \approx -1.205$.

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	1	0.6487	2.8244	0.2297	0.7703
2	0.7703	0.0632	2.2755	0.0277	0.7425
3	0.7425	0.0009	2.2098	0.0004	0.7421

Another point of intersection of the graphs of f and g occurs when $x \approx 0.742$.

19. $h(x) = f(x) - g(x) = x - \tan x$

$h'(x) = 1 - \sec^2 x$

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	4.5000	−0.1373	−21.5048	0.0064	4.4936
2	4.4936	−0.0039	−20.2271	0.0002	4.4934

Point of intersection of the graphs of f and g occurs when $x \approx 4.493$.

Note: $f(x) = x$ and $g(x) = \tan x$ intersect infinitely often.

20. $h(x) = \arctan x - \arccos x$

$h'(x) = \frac{1}{1 + x^2} + \frac{1}{\sqrt{1 - x^2}}$

n	x_n	$h(x_n)$	$h'(x_n)$	$\frac{h(x_n)}{h'(x_n)}$	$x_n - \frac{h(x_n)}{h'(x_n)}$
1	0.5	−0.5835	1.9547	−0.2985	0.7985
2	0.7985	0.0278	2.2718	0.0122	0.7863
3	0.7863	0.0003	2.2365	0.0001	0.7862

Point of intersection of the graphs of f and g occurs when $x \approx 0.786$.

21. (a) $f(x) = x^2 - a, a > 0$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

(b) $\sqrt{5}$: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right), x_1 = 2$

n	1	2	3	4
x_n	2	2.25	2.2361	2.2361

For example, given $x_1 = 2$,

$$x_2 = \frac{1}{2}\left(2 + \frac{5}{2}\right) = \frac{9}{4} = 2.25.$$

$\sqrt{5} \approx 2.236$

$\sqrt{7}$: $x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right), x_1 = 2$

n	1	2	3	4	5
x_n	2	2.75	2.6477	2.6458	2.6458

$\sqrt{7} \approx 2.646$

22. (a) $f(x) = x^n - a, a > 0$

$f'(x) = nx^{n-1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{x_i^n - a}{nx_i^{n-1}} = \frac{(n-1)x_i^n + a}{nx_i^{n-1}}$$

(b) $\sqrt[4]{6}$: $x_{i+1} = \frac{3x_i^4 + 6}{4x_i^3}, x_1 = 1.5$

i	1	2	3	4
x_i	1.5	1.5694	1.5651	1.5651

$\sqrt[4]{6} \approx 1.565$

$\sqrt[3]{15}$: $x_{i+1} = \frac{2x_i^3 + 15}{3x_i^2}, x_1 = 2.5$

i	1	2	3	4
x_i	2.5	2.4667	2.4662	2.4662

$\sqrt[3]{15} \approx 2.466$

23. $y = 2x^3 - 6x^2 + 6x - 1 = f(x)$

$y' = 6x^2 - 12x + 6 = f'(x)$

$x_1 = 1$

$f'(x) = 0$; therefore, the method fails.

n	x_n	$f(x_n)$	$f'(x_n)$
1	1	1	0

24. $y = x^3 - 2x - 2, x_1 = 0$

$y' = 3x^2 - 2$

$x_1 = 0$

$x_2 = -1$

$x_3 = 0$

$x_4 = -1$ and so on.

Fails to converge

25. Let $g(x) = f(x) - x = \cos x - x$

$g'(x) = -\sin x - 1.$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.4597	−1.8415	0.2496	0.7504
2	0.7504	−0.0190	−1.6819	0.0113	0.7391
3	0.7391	0.0000	−1.6736	0.0000	0.7391

The fixed point is approximately 0.74.

26. Let $g(x) = f(x) - x = \cot x - x$

$g'(x) = -\csc^2 x - 1.$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0000	−0.3579	−2.4123	0.1484	0.8516
2	0.8516	0.0240	−2.7668	−0.0087	0.8603
3	0.8603	0.0001	−2.7403	0.0000	0.8603

The fixed point is approximately 0.86.

27. Let $g(x) = e^{x/10} - x$

$g'(x) = \frac{1}{10}e^{x/10} - 1$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	1.0	0.1052	−0.8895	−0.1182	1.1182
2	1.1182	0.0001	−0.8882	−0.0001	1.1183

The fixed point is approximately 1.12.

28. Let $g(x) = x + \ln x$

$g'(x) = \frac{1}{x} + 1$

n	x_n	$g(x_n)$	$g'(x_n)$	$\frac{g(x_n)}{g'(x_n)}$	$x_n - \frac{g(x_n)}{g'(x_n)}$
1	0.5	−0.1931	3	−0.0644	0.5644
2	0.5644	−0.0076	2.7718	−0.0027	0.5671

The fixed point is approximately 0.57.

29. $f(x) = \frac{1}{x} - a = 0$

$f'(x) = -\frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{(1/x_n) - a}{-1/x_n^2} = x_n + x_n^2\left(\frac{1}{x_n} - a\right) = x_n + x_n - x_n^2 a = 2x_n - x_n^2 a = x_n(2 - ax_n)$$

30. (a) $x_{n+1} = x_n(2 - 3x_n)$

i	1	2	3	4
x_i	0.3000	0.3300	0.3333	0.3333

$\frac{1}{3} \approx 0.333$

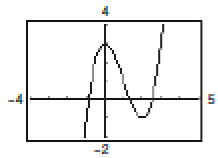
(b) $x_{n+1} = x_n(2 - 11x_n)$

i	1	2	3	4
x_i	0.1000	0.0900	0.0909	0.0909

$\frac{1}{11} \approx 0.091$

31. $f(x) = x^3 - 3x^2 + 3, f'(x) = 3x^2 - 6x$

(a)



(b) $x_1 = 1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 1.333$$

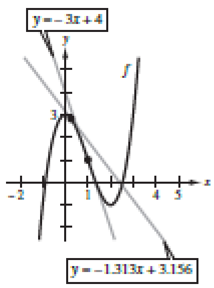
Continuing, the zero is 1.347.

(c) $x_1 = \frac{1}{4}$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 2.405$$

Continuing, the zero is 2.532.

(d)



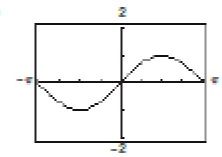
The x-intercept of $y = -3x + 4$ is $\frac{4}{3}$. The x-intercept of $y = 1.313x + 3.156$ is approximately 2.405.

The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

32. $f(x) = \sin x, f'(x) = \cos x$

(a)



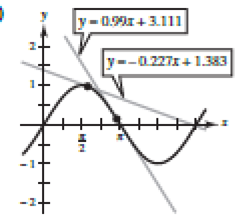
(b) $x_1 = 1.8$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 6.086$$

(c) $x_1 = 3$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 3.143$$

(d)



The x-intercept of $y = -0.227x + 1.383$ is approximately 6.086. The x-intercept of $y = 0.99x + 3.111$ is approximately 3.143.

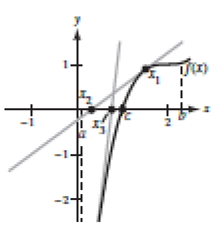
The x-intercepts correspond to the values resulting from the first iteration of Newton's Method.

(e) If the initial guess x_1 is not "close to" the desired zero of the function, the x-intercept of the tangent line may approximate another zero of the function.

33. Answers will vary. See page 229.

If f is a function continuous on $[a, b]$ and differentiable on (a, b) where $c \in [a, b]$ and $f(c) = 0$, Newton's Method uses tangent lines to approximate c such that $f(c) = 0$.

First, estimate an initial x_1 close to c (see graph).



Then determine x_2 by $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$.

Calculate a third estimate by $x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$.

Continue this process until $|x_n - x_{n+1}|$ is within the desired accuracy.

Let x_{n+1} be the final approximation of c.

34. At $x = -3$ and $x = 2$, the tangent lines to the curve are horizontal. Hence, Newton's Method will not converge for these initial approximations.

35. Maximize: $C = \dfrac{3t^2 + t}{50 + t^3}$

$$C' = \frac{-3t^4 - 2t^3 + 300t + 50}{\left(50 + t^3\right)^2} = 0$$

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	4.5000	12.4375	915.0000	0.0136	4.4864
2	4.4864	0.0658	904.3822	0.0001	4.4863

Let $f(x) = 3t^4 + 2t^3 - 300t - 50$

$f'(x) = 12t^3 + 6t^2 - 300.$

Because $f(4) = -354$ and $f(5) = 575$, the solution is in the interval $(4, 5)$.

Approximation: $t \approx 4.486$ hours

36. Minimize: $T = \dfrac{\text{Distance rowed}}{\text{Rate rowed}} + \dfrac{\text{Distance walked}}{\text{Rate walked}}$

$$T = \frac{\sqrt{x^2 + 4}}{3} + \frac{\sqrt{x^2 - 6x + 10}}{4}$$

$$T' = \frac{x}{3\sqrt{x^2 + 4}} + \frac{x - 3}{4\sqrt{x^2 - 6x + 10}} = 0$$

$$4x\sqrt{x^2 - 6x + 10} = -3(x - 3)\sqrt{x^2 + 4}$$

$$16x^2\left(x^2 - 6x + 10\right) = 9(x - 3)^2\left(x^2 + 4\right)$$

$$7x^4 - 42x^3 + 43x^2 + 216x - 324 = 0$$

Let $f(x) = 7x^4 - 42x^3 + 43x^2 + 216x - 324$ and $f'(x) = 28x^3 - 126x^2 + 86x + 216.$

Because $f(1) = -100$ and $f(2) = 56$, the solution is in the interval $(1, 2)$.

n	x_n	$f(x_n)$	$f'(x_n)$	$\dfrac{f(x_n)}{f'(x_n)}$	$x_n - \dfrac{f(x_n)}{f'(x_n)}$
1	1.7000	19.5887	135.6240	0.1444	1.5556
2	1.5556	−1.0480	150.2780	−0.0070	1.5626
3	1.5626	0.0014	49.5591	0.0000	1.5626

Approximation: $x \approx 1.563$ mi

37. True

38. True

39. $f(x) = -\sin x$

$f'(x) = -\cos x$

Let $(x_0, y_1) = (x_0, -\sin(x_0))$ be a point on the graph of f. If (x_0, y_0) is a point of tangency, then

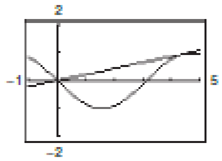
$$-\cos(x_0) = \frac{y_0 - 0}{x_0 - 0} = \frac{y_0}{x_0} = \frac{-\sin(x_0)}{x_0}.$$

So, $x_0 = \tan(x_0)$.

$x_0 \approx 4.4934$

Slope $= -\cos(x_0) \approx 0.217$

You can verify this answer by graphing $y_1 = -\sin x$ and the tangent line $y_2 = 0.217x$.



40. Let (x_1, y_1) be the point of tangency.

$f(x) = \cos x,\ f'(x) = -\sin x,\ f'(x_1) = -\sin(x_1).$

At the point of tangency,

$$f'(x_1) = \frac{y_1 - 0}{x_1 - 0}$$

$-\sin(x_1) = \cos(x_1)/x_1$

$\cos(x_1) + x_1 \sin(x_1) = 0$

Using Newton's method with initial guess 3, you obtai
$x_1 \approx 2.798$ and $y_1 \approx -0.942$.

41. $\lim_{h\to 0} \frac{\sin(\pi + h) - \sin \pi}{h} = \lim_{h\to 0} \frac{\sin(\pi + h)}{h} - \lim_{h\to 0} \frac{\sin}{h}$

$= -1 - 0 = -1$

So, the answer is A.

42. $y = -5\sqrt[3]{15x^2 - 1}$

$$y' = -5\left[\frac{1}{3}(15x^2 - 1)^{-2/3} \cdot 30x\right] = -\frac{50}{(15x^2 - 1)^{2/3}}$$

So, the answer is D.