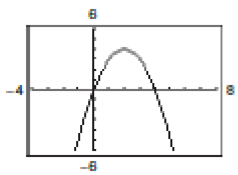
Section 1.3 Evaluating Limits Analytically

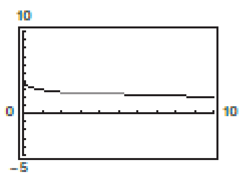
1.



(a) $\lim_{x \to 4} h(x) = 0$

(b) $\lim_{x \to -1} h(x) = -5$

2.

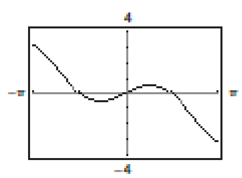


$$g(x) = \frac{12(\sqrt{x} - 3)}{x - 9}$$

(a) $\lim_{x \to 4} g(x) = 2.4$

(b) $\lim_{x \to 9} g(x) = 2$

3.

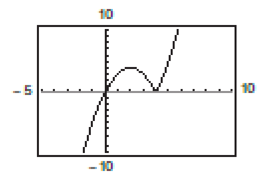


$f(x) = x \cos x$

(a) $\lim_{x \to 0} f(x) = 0$

(b) $\lim_{x \to \pi/3} f(x) \approx 0.524$ or $\dfrac{\pi}{6}$

4.



$f(t) = t|t - 4|$

(a) $\lim_{t \to 4} f(t) = 0$

(b) $\lim_{t \to -1} f(t) = -5$

5. $\lim_{x \to 2} x^3 = 2^3 = 8$

6. $\lim_{x \to -3} x^4 = (-3)^4 = 81$

7. $\lim_{x \to -3} (2x + 5) = 2(-3) + 5 = -1$

8. $\lim_{x \to 0} (2x - 1) = 2(0) - 1 = -1$

9. $\lim_{x \to -3} (x^2 + 3x) = (-3)^2 + 3(-3) = 9 - 9 = 0$

10. $\lim_{x \to 2} (-x^3 + 1) = (-2)^3 + 1 = -8 + 1 = -7$

11. $\lim_{x \to -3} (2x^2 + 4x + 1) = 2(-3)^2 + 4(-3) + 1$
$= 18 - 12 + 1 = 7$

12. $\lim_{x \to 1} (2x^3 - 6x + 5) = 2(1)^3 - 6(1) + 5$
$= 2 - 6 + 5 = 1$

13. $\lim_{x \to 3} \sqrt{x + 1} = \sqrt{3 + 1} = 2$

14. $\lim_{x \to 2} \sqrt[3]{12x + 3} = \sqrt[3]{12(2) + 3}$
$= \sqrt[3]{24 + 3} = \sqrt[3]{27} = 3$

15. $\lim_{x \to -4} (x + 3)^2 = (-4 + 3)^2 = 1$

16. $\lim_{x \to 0} (3x - 2)^4 = [3(0) - 2]^4 = (-2)^4 = 16$

17. $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$

18. $\lim_{x \to -5} \frac{5}{x + 3} = \frac{5}{-5 + 3} = -\frac{5}{2}$

19. $\lim_{x \to 1} \frac{x}{x^2 + 4} = \frac{1}{1^2 + 4} = \frac{1}{5}$

20. $\lim_{x \to 1} \frac{3x + 5}{x + 1} = \frac{3(1) + 5}{1 + 1} = \frac{3 + 5}{2} = \frac{8}{2} = 4$

21. $\lim_{x \to 7} \frac{3x}{\sqrt{x + 2}} = \frac{3(7)}{\sqrt{7 + 2}} = \frac{21}{3} = 7$

22. $\lim_{x \to 3} \frac{\sqrt{x + 6}}{x + 2} = \frac{\sqrt{3 + 6}}{3 + 2} = \frac{\sqrt{9}}{5} = \frac{3}{5}$

23. $\lim_{x \to \pi/2} \sin x = \sin \frac{\pi}{2} = 1$

24. $\lim_{x \to \pi} \tan x = \tan \pi = 0$

25. $\lim_{x \to 1} \cos \frac{\pi x}{3} = \cos \frac{\pi}{3} = \frac{1}{2}$

26. $\lim_{x \to 2} \sin \frac{\pi x}{2} = \sin \frac{\pi(2)}{2} = 0$

27. $\lim_{x \to 0} \sec 2x = \sec 0 = 1$

28. $\lim_{x \to \pi} \cos 3x = \cos 3\pi = -1$

29. $\lim_{x \to 5\pi/6} \sin x = \sin \frac{5\pi}{6} = \frac{1}{2}$

30. $\lim_{x \to 5\pi/3} \cos x = \cos \frac{5\pi}{3} = \frac{1}{2}$

31. $\lim_{x \to 3} \tan \frac{\pi x}{4} = \tan \frac{3\pi}{4} = -1$

32. $\lim_{x \to 7} \sec \frac{\pi x}{6} = \sec \frac{7\pi}{6} = -\frac{2\sqrt{3}}{3}$

33. $\lim_{x \to 0} e^x \cos 2x = e^0 \cos 0 = 1$

34. $\lim_{x \to 0} e^{-x} \sin \pi x = e^0 \sin 0 = 0$

35. $\lim_{x \to 1} (\ln 3x + e^x) = \ln 3 + e$

36. $\lim_{x \to 1} \ln\left(\frac{x}{e^x}\right) = \ln\left(\frac{1}{e}\right) = \ln e^{-1} = -1$

37. (a) $\lim_{x \to 1} f(x) = 5 - 1 = 4$

(b) $\lim_{x \to 4} g(x) = 4^3 = 64$

(c) $\lim_{x \to 1} g(f(x)) = g(f(1)) = g(4) = 64$

38. (a) $\lim_{x \to -3} f(x) = (-3) + 7 = 4$

(b) $\lim_{x \to 4} g(x) = 4^2 = 16$

(c) $\lim_{x \to -3} g(f(x)) = g(4) = 16$

39. (a) $\lim_{x \to 1} f(x) = 4 - 1 = 3$

(b) $\lim_{x \to 3} g(x) = \sqrt{3 + 1} = 2$

(c) $\lim_{x \to 1} g(f(x)) = g(3) = 2$

40. (a) $\lim_{x \to 4} f(x) = 2(4^2) - 3(4) + 1 = 21$

(b) $\lim_{x \to 21} g(x) = \sqrt[3]{21 + 6} = 3$

(c) $\lim_{x \to 4} g(f(x)) = g(21) = 3$

41. (a) $\lim_{x \to c} [5g(x)] = 5 \lim_{x \to c} g(x) = 5(2) = 10$

(b) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 3 + 2 = 5$

(c) $\lim_{x \to c} [f(x)g(x)] = \left[\lim_{x \to c} f(x)\right]\left[\lim_{x \to c} g(x)\right] = (3)(2) = 6$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{3}{2}$

42. (a) $\lim_{x \to c} [4f(x)] = 4 \lim_{x \to c} f(x) = 4(2) = 8$

(b) $\lim_{x \to c} [f(x) + g(x)] = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = 2 + \frac{3}{4} = \frac{11}{4}$

(c) $\lim_{x \to c} [f(x)g(x)] = \left[\lim_{x \to c} f(x)\right]\left[\lim_{x \to c} g(x)\right] = 2\left(\frac{3}{4}\right) = \frac{3}{2}$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{2}{(3/4)} = \frac{8}{3}$

43. (a) $\lim_{x \to c} [f(x)]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (16)^2 = 256$

(b) $\lim_{x \to c} \sqrt{f(x)} = \sqrt{\lim_{x \to c} f(x)} = \sqrt{16} = 4$

(c) $\lim_{x \to c} [3f(x)] = 3\left[\lim_{x \to c} f(x)\right] = 3(16) = 48$

(d) $\lim_{x \to c} [f(x)]^{3/2} = \left[\lim_{x \to c} f(x)\right]^{3/2} = (16)^{3/2} = 64$

44. (a) $\lim_{x \to c} \sqrt[3]{f(x)} = \sqrt[3]{\lim_{x \to c} f(x)} = \sqrt[3]{27} = 3$

(b) $\lim_{x \to c} \frac{f(x)}{18} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} 18} = \frac{27}{18} = \frac{3}{2}$

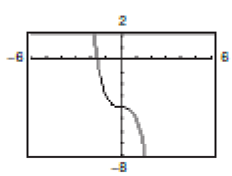
(c) $\lim_{x \to c} [f(x)]^2 = \left[\lim_{x \to c} f(x)\right]^2 = (27)^2 = 729$

(d) $\lim_{x \to c} [f(x)]^{2/3} = \left[\lim_{x \to c} f(x)\right]^{2/3} = (27)^{2/3} = 9$

45. $f(x) = \begin{cases} -x^3 - 4, & x \neq -2 \\ -2, & x = -2 \end{cases}$ and $g(x) = -x^3 - 4$

agree except at $x = -2$.

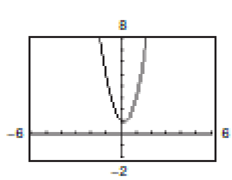
$\lim_{x \to -2} f(x) = \lim_{x \to -2} g(x) = 4$



46. $g(x) = \begin{cases} 3x^2 - x + 1, & x \neq 3 \\ 3, & x = 3 \end{cases}$ and

$h(x) = 3x^2 - x + 1$ agree except at $x = 3$.

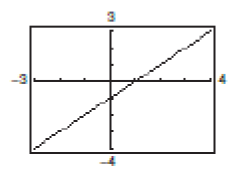
$\lim_{x \to 3} g(x) = \lim_{x \to 3} h(x) = 25$



47. $f(x) = \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1}$ and $g(x) = x - 1$

agree except at $x = -1$.

$\lim_{x \to -1} f(x) = \lim_{x \to -1} g(x) = \lim_{x \to -1} (x - 1) = -1 - 1 = -2$

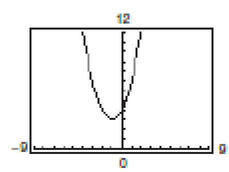


48. $f(x) = \frac{x^3 - 8}{x - 2}$ and $g(x) = x^2 + 2x + 4$ agree except

at $x = 2$.

$\lim_{x \to 2} f(x) = \lim_{x \to 2} g(x) = \lim_{x \to 2} (x^2 + 2x + 4)$

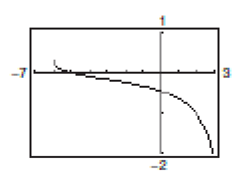
$= 2^2 + 2(2) + 4 = 12$



49. $f(x) = \frac{(x + 4)\ln(x + 6)}{x^2 - 16}$ and $g(x) = \frac{\ln(x + 6)}{x - 4}$

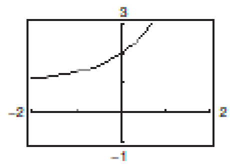
agree except at $x = -4$.

$\lim_{x \to -4} f(x) = \lim_{x \to -4} g(x) = -\frac{\ln 2}{8} \approx -0.0866$



50. $f(x) = \dfrac{e^{2x} - 1}{e^x - 1}$ and $g(x) = e^x + 1$ agree except at $x = 0$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = e^0 + 1 = 2$$



51. $$\lim_{x\to 0}\frac{x}{x^2 - x} = \lim_{x\to 0}\frac{x}{x(x-1)} = \lim_{x\to 0}\frac{1}{x-1} = \frac{1}{0-1} = -1$$

52. $$\lim_{x\to 0}\frac{2x}{x^2 + 4x} = \lim_{x\to 0}\frac{2x}{x(x+4)} = \lim_{x\to 0}\frac{2}{x+4}$$
$$= \frac{2}{0+4} = \frac{2}{4} = \frac{1}{2}$$

53. $$\lim_{x\to -3}\frac{x^2 - 9}{x + 3} = \lim_{x\to -3}\frac{(x+3)(x-3)}{x+3}$$
$$= \lim_{x\to -3}(x - 3) = (-3) - 3 = -6$$

54. $$\lim_{x\to 5}\frac{5 - x}{x^2 - 25} = \lim_{x\to 5}\frac{-(x-5)}{(x-5)(x+5)}$$
$$= \lim_{x\to 5}\frac{-1}{x+5} = \frac{-1}{5+5} = -\frac{1}{10}$$

55. $$\lim_{x\to 4}\frac{x^2 - 5x + 4}{x^2 - 16} = \lim_{x\to 4}\frac{(x-4)(x-1)}{(x+4)(x-4)}$$
$$= \lim_{x\to 4}\frac{x-1}{x+4} = \frac{4-1}{4+4} = \frac{3}{8}$$

56. $$\lim_{x\to 2}\frac{x^2 + 2x - 8}{x^2 - x - 2} = \lim_{x\to 2}\frac{(x-2)(x+4)}{(x-2)(x+1)}$$
$$= \lim_{x\to 2}\frac{x+4}{x+1} = \frac{2+4}{2+1} = \frac{6}{3} = 2$$

57. $$\lim_{x\to 4}\frac{\sqrt{x+5} - 3}{x - 4} = \lim_{x\to 4}\frac{\sqrt{x+5} - 3}{x - 4}\cdot\frac{\sqrt{x+5} + 3}{\sqrt{x+5} + 3}$$
$$= \lim_{x\to 4}\frac{(x+5) - 9}{(x-4)\left(\sqrt{x+5} + 3\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x+5} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}$$

58. $$\lim_{x\to 3}\frac{\sqrt{x+1} - 2}{x - 3} = \lim_{x\to 3}\frac{\sqrt{x+1} - 2}{x - 3}\cdot\frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2} = \lim_{x\to 3}\frac{x - 3}{(x-3)\left[\sqrt{x+1} + 2\right]}$$
$$= \lim_{x\to 3}\frac{1}{\sqrt{x+1} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}$$

59. $$\lim_{x\to 0}\frac{\sqrt{x+5} - \sqrt{5}}{x} = \lim_{x\to 0}\frac{\sqrt{x+5} - \sqrt{5}}{x}\cdot\frac{\sqrt{x+5} + \sqrt{5}}{\sqrt{x+5} + \sqrt{5}}$$
$$= \lim_{x\to 0}\frac{(x+5) - 5}{x\left(\sqrt{x+5} + \sqrt{5}\right)} = \lim_{x\to 0}\frac{1}{\sqrt{x+5} + \sqrt{5}} = \frac{1}{\sqrt{5} + \sqrt{5}} = \frac{1}{2\sqrt{5}} = \frac{\sqrt{5}}{10}$$

60. $$\lim_{x\to 0}\frac{\sqrt{2+x} - \sqrt{2}}{x} = \lim_{x\to 0}\frac{\sqrt{2+x} - \sqrt{2}}{x}\cdot\frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}}$$
$$= \lim_{x\to 0}\frac{2 + x - 2}{\left(\sqrt{2+x} + \sqrt{2}\right)x} = \lim_{x\to 0}\frac{1}{\sqrt{2+x} + \sqrt{2}} = \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$$

61. $$\lim_{x\to 0}\frac{\dfrac{1}{3+x} - \dfrac{1}{3}}{x} = \lim_{x\to 0}\frac{3 - (3+x)}{(3+x)3(x)} = \lim_{x\to 0}\frac{-x}{(3+x)(3)(x)} = \lim_{x\to 0}\frac{-1}{(3+x)3} = \frac{-1}{(3)3} = -\frac{1}{9}$$

62. $$\lim_{x\to 0}\frac{\dfrac{1}{x+4} - \dfrac{1}{4}}{x} = \lim_{x\to 0}\frac{\dfrac{4 - (x+4)}{4(x+4)}}{x} = \overset{4}{\lim_{x\to 0}}\frac{-1}{4(x+4)} = \frac{-1}{4(4)} = -\frac{1}{16}$$

63. $$\lim_{\Delta x\to 0}\frac{2(x + \Delta x) - 2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2x + 2\Delta x - 2x}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2 = 2$$

64. $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - x^2}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - x^2}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta x(2x + \Delta x)}{\Delta x} = \lim\limits_{\Delta x \to 0} (2x + \Delta x) = 2x$

65. $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - 2(x + \Delta x) + 1 - (x^2 - 2x + 1)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} (2x + \Delta x - 2) = 2x - 2$

66. $\lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$

$= \lim\limits_{\Delta x \to 0} \dfrac{\Delta x\left(3x^2 + 3x\Delta x + (\Delta x)^2\right)}{\Delta x} = \lim\limits_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$

67. Because $(-2)^3 = -8, 10(-2)^3 = -80.$

$\lim\limits_{x \to -2} \dfrac{10x^3 + 12x^2 + 2x}{x^2 - 8x + 11} = \dfrac{10(-2)^3 + 12(-2)^2 + 2(-2)}{(-2)^2 - 8(-2) + 11}$

$= \dfrac{-80 + 48 - 4}{4 + 16 + 11}$

$= -\dfrac{36}{31}$

68. $x^3 - 125$ was factored incorrectly.

$\lim\limits_{x \to 5} \dfrac{x^3 - 125}{x - 5} = \lim\limits_{x \to 5} \dfrac{(x - 5)(x^2 + 5x + 25)}{(x - 5)}$

$= \lim\limits_{x \to 5} \dfrac{\cancel{(x - 5)}(x^2 + 5x + 25)}{\cancel{(x - 5)}}$

$= \lim\limits_{x \to 5} (x^2 + 5x + 25)$

$= 25 + 25 + 27$

$= 75$

69. $\lim\limits_{x \to 0} \dfrac{\sin x}{5x} = \lim\limits_{x \to 0} \left[\left(\dfrac{\sin x}{x}\right)\left(\dfrac{1}{5}\right)\right] = (1)\left(\dfrac{1}{5}\right) = \dfrac{1}{5}$

70. $\lim\limits_{x \to 0} \dfrac{3(1 - \cos x)}{x} = \lim\limits_{x \to 0} \left[3\left(\dfrac{(1 - \cos x)}{x}\right)\right] = (3)(0) = 0$

71. $\lim\limits_{x \to 0} \dfrac{\sin x(1 - \cos x)}{x^2} = \lim\limits_{x \to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{1 - \cos x}{x}\right]$

$= (1)(0) = 0$

72. $\lim\limits_{\theta \to 0} \dfrac{\cos \theta \tan \theta}{\theta} = \lim\limits_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$

73. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{x} = \lim\limits_{x \to 0} \left[\dfrac{\sin x}{x} \sin x\right] = (1) \sin 0 = 0$

74. $\lim\limits_{x \to 0} \dfrac{\tan^2 x}{x} = \lim\limits_{x \to 0} \dfrac{\sin^2 x}{x \cos^2 x} = \lim\limits_{x \to 0} \left[\dfrac{\sin x}{x} \cdot \dfrac{\sin x}{\cos^2 x}\right]$

$= (1)(0) = 0$

75. $\lim\limits_{h \to 0} \dfrac{(1 - \cos h)^2}{h} = \lim\limits_{h \to 0} \left[\dfrac{1 - \cos h}{h}(1 - \cos h)\right]$

$= (0)(0) = 0$

76. $\lim\limits_{\phi \to \pi} \phi \sec \phi = \pi(-1) = -\pi$

77. $\lim\limits_{x \to \pi/2} \dfrac{\cos x}{\cot x} = \lim\limits_{x \to \pi/2} \sin x = 1$

78. $\lim\limits_{x \to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x} = \lim\limits_{x \to \pi/4} \dfrac{\cos x - \sin x}{\sin x \cos x - \cos^2 x}$

$= \lim\limits_{x \to \pi/4} \dfrac{-(\sin x - \cos x)}{\cos x(\sin x - \cos x)}$

$= \lim\limits_{x \to \pi/4} \dfrac{-1}{\cos x}$

$= \lim\limits_{x \to \pi/4} (-\sec x)$

$= -\sqrt{2}$

79. $\lim\limits_{x \to 0} \dfrac{1 - e^{-x}}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{1 - e^{-x}}{e^x - 1} \cdot \dfrac{e^{-x}}{e^{-x}} = \lim\limits_{x \to 0} \dfrac{(1 - e^{-x})e^{-x}}{1 - e^{-x}}$

$= \lim\limits_{x \to 0} e^{-x} = 1$

80. $\lim\limits_{x \to 0} \dfrac{4(e^{2x} - 1)}{e^x - 1} = \lim\limits_{x \to 0} \dfrac{4(e^x - 1)(e^x + 1)}{e^x - 1}$

$= \lim\limits_{x \to 0} 4(e^x + 1) = 4(2) = 8$

81. $\lim\limits_{t \to 0} \dfrac{\sin 3t}{2t} = \lim\limits_{t \to 0} \left(\dfrac{\sin 3t}{3t}\right)\left(\dfrac{3}{2}\right) = (1)\left(\dfrac{3}{2}\right) = \dfrac{3}{2}$

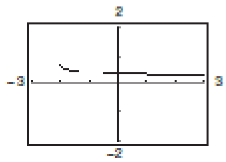
82. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{\sin 3x} = \lim\limits_{x \to 0} \left[2\left(\dfrac{\sin 2x}{2x}\right)\left(\dfrac{1}{3}\right)\left(\dfrac{3x}{\sin 3x}\right)\right]$

$= 2(1)\left(\dfrac{1}{3}\right)(1) = \dfrac{2}{3}$

83. $f(x) = \dfrac{\sqrt{x+2} - \sqrt{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	0.358	0.354	0.354	?	0.354	0.353	0.349

It appears that the limit is 0.354.



The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} = \lim_{x\to 0} \frac{\sqrt{x+2} - \sqrt{2}}{x} \cdot \frac{\sqrt{x+2} + \sqrt{2}}{\sqrt{x+2} + \sqrt{2}}$

$$= \lim_{x\to 0} \frac{x+2-2}{x\left(\sqrt{x+2} + \sqrt{2}\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x+2} + \sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4} \approx 0.354.$$

84. $f(x) = \dfrac{4 - \sqrt{x}}{x - 16}$

x	15.9	15.99	15.999	16	16.001	16.01	16.1
$f(x)$	–0.1252	–0.125	–0.125	?	–0.125	–0.125	–0.1248

It appears that the limit is –0.125.

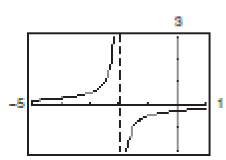
The graph has a hole at $x = 16$.

Analytically, $\displaystyle\lim_{x\to 16} \frac{4 - \sqrt{x}}{x - 16} = \lim_{x\to 16} \frac{\left(4 - \sqrt{x}\right)}{\left(\sqrt{x} + 4\right)\left(\sqrt{x} - 4\right)} = \lim_{x\to 16} \frac{-1}{\sqrt{x} + 4} = -\frac{1}{8}.$

85. $f(x) = \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

x	–0.1	–0.01	–0.001	0	0.001	0.01	0.1
$f(x)$	–0.263	–0.251	–0.250	?	–0.250	–0.249	–0.238

It appears that the limit is –0.250.



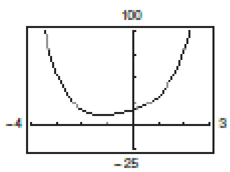
The graph has a hole at $x = 0$.

Analytically, $\displaystyle\lim_{x\to 0} \frac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x} = \lim_{x\to 0} \frac{2 - (2+x)}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-x}{2(2+x)} \cdot \frac{1}{x} = \lim_{x\to 0} \frac{-1}{2(2+x)} = -\frac{1}{4}.$

86. $f(x) = \dfrac{x^5 - 32}{x - 2}$

x	1.9	1.99	1.999	1.9999	2	2.0001	2.001	2.01	2.1
$f(x)$	72.39	79.20	79.92	79.99	?	80.01	80.08	80.80	88.41

It appears that the limit is 80.



The graph has a hole at $x = 2$.

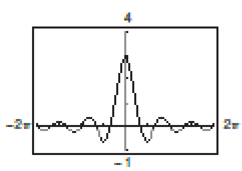
Analytically, $\lim_{x\to 2} \dfrac{x^5 - 32}{x - 2} = \lim_{x\to 2} \dfrac{(x-2)(x^4 + 2x^3 + 4x^2 + 8x + 16)}{x - 2} = \lim_{x\to 2}(x^4 + 2x^3 + 4x^2 + 8x + 16) = 80.$

(*Hint:* Use long division to factor $x^5 - 32$.)

87. $f(t) = \dfrac{\sin 3t}{t}$

t	−0.1	−0.01	−0.001	0	0.001	0.01	0.1
$f(t)$	2.96	2.9996	3	?	3	2.9996	2.96

It appears that the limit is 3.



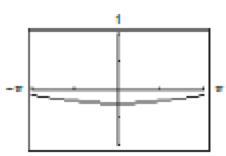
The graph has a hole at $t = 0$.

Analytically, $\lim_{t\to 0} \dfrac{\sin 3t}{t} = \lim_{t\to 0} 3\left(\dfrac{\sin 3t}{3t}\right) = 3(1) = 3.$

88. $f(x) = \dfrac{\cos x - 1}{2x^2}$

x	−1	−0.1	−0.01	0	0.01	0.1	1
$f(x)$	−0.2298	−0.2498	−0.25	?	−0.25	−0.2498	−0.2298

It appears that the limit is −0.25.



The graph has a hole at $x = 0$.

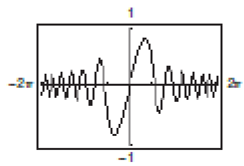
Analytically, $\dfrac{\cos x - 1}{2x^2} \cdot \dfrac{\cos x + 1}{\cos x + 1} = \dfrac{\cos^2 x - 1}{\cancel{2}x^2(\cos x + 1)} = \dfrac{-\sin^2 x}{2x^2(\cos x + 1)} = \dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}$

$\lim_{x\to 0}\left[\dfrac{\sin^2 x}{x^2} \cdot \dfrac{-1}{2(\cos x + 1)}\right] = 1\left(\dfrac{-1}{4}\right) = -\dfrac{1}{4} = -0.25$

89. $f(x) = \dfrac{\sin x^2}{x}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	-0.099998	-0.01	-0.001	?	0.001	0.01	0.099998

It appears that the limit is 0.



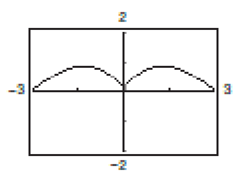
The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x} = \lim\limits_{x\to 0} x\left(\dfrac{\sin x^2}{x^2}\right) = 0(1) = 0.$

90. $f(x) = \dfrac{\sin x}{\sqrt[3]{x}}$

x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0.215	0.0464	0.01	?	0.01	0.0464	0.215

It appears that the limit is 0.

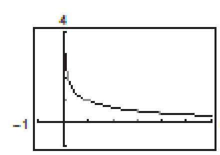


The graph has a hole at $x = 0$.

Analytically, $\lim\limits_{x\to 0}\dfrac{\sin x}{\sqrt[3]{x}} = \lim\limits_{x\to 0}\sqrt[3]{x^2}\left(\dfrac{\sin x}{x}\right) = (0)(1) = 0.$

91. $f(x) = \dfrac{\ln x}{x - 1}$

x	0.5	0.9	0.99	1	1.01	1.1	1.5
$f(x)$	1.3863	1.0536	1.0050	?	0.9950	0.9531	0.8109

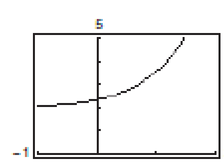


It appears that the limit is 1.

Analytically, $\lim\limits_{x\to 1}\dfrac{\ln x}{x - 1} = 1.$

92. $f(x) = \dfrac{e^{3x} - 8}{e^{2x} - 4}$

x	0.5	0.6	0.69	ln 2	0.7	0.8	0.9
$f(x)$	2.7450	2.8687	2.9953	0	3.0103	3.1722	3.3565



It appears that the limit is 3.

Analytically, $\lim\limits_{x\to \ln 2}\dfrac{e^{3x} - 8}{e^{2x} - 4} = \lim\limits_{x\to \ln 2}\dfrac{(e^x - 2)(e^{2x} + 2e^x + 4)}{(e^x - 2)(e^x + 2)} = \lim\limits_{x\to \ln 2}\dfrac{e^{2x} + 2e^x + 4}{e^x + 2} = \dfrac{4 + 4 + 4}{2 + 2} = 3.$

93. $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{3(x + \Delta x) - 2 - (3x - 2)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{3x + 3\Delta x - 2 - 3x + 2}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{3\Delta x}{\Delta x} = 3$

94. $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{(x + \Delta x)^2 - 4(x + \Delta x) - (x^2 - 4x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{x^2 + 2x\Delta x + \Delta x^2 - 4x - 4\Delta x - x^2 + 4x}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 4)}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x - 4) = 2x - 4$$

95. $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\dfrac{1}{x + \Delta x + 3} - \dfrac{1}{x + 3}}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{x + 3 - (x + \Delta x + 3)}{(x + \Delta x + 3)(x + 3)} \cdot \frac{1}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x + 3)(x + 3)\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x + 3)(x + 3)} = -\frac{1}{(x + 3)^2}$$

96. $\lim\limits_{\Delta x \to 0} \dfrac{f(x + \Delta x) - f(x)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \dfrac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

97. $\lim\limits_{x \to 0} \left(4 - x^2\right) \le \lim\limits_{x \to 0} f(x) \le \lim\limits_{x \to 0} \left(4 + x^2\right)$

$$4 \le \lim_{x \to 0} f(x) \le 4$$

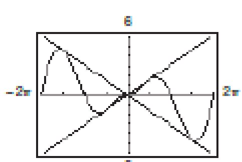
Therefore, $\lim\limits_{x \to 0} f(x) = 4$.

98. $\lim\limits_{x \to a} \left[b - |x - a|\right] \le \lim\limits_{x \to a} f(x) \le \lim\limits_{x \to a} \left[b + |x - a|\right]$

$$b \le \lim_{x \to a} f(x) \le b$$

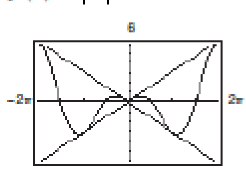
Therefore, $\lim\limits_{x \to a} f(x) = b$.

99. $f(x) = |x| \sin x$



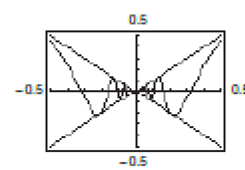
$\lim\limits_{x \to 0} |x| \sin x = 0$

100. $f(x) = |x| \cos x$



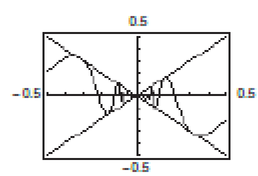
$\lim\limits_{x \to 0} |x| \cos x = 0$

101. $f(x) = x \sin \dfrac{1}{x}$



$\lim\limits_{x \to 0} \left(x \sin \dfrac{1}{x}\right) = 0$

102. $h(x) = x \cos \dfrac{1}{x}$



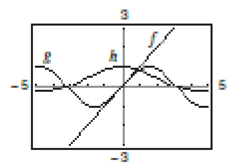
$\lim\limits_{x \to 0} \left(x \cos \dfrac{1}{x}\right) = 0$

103. (a) Two functions f and g agree at all but one point (on an open interval) if $f(x) = g(x)$ for all x in the interval except for $x = c$, where c is in the interval.

(b) Answers will vary. Sample answer:

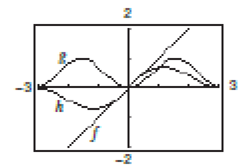
$f(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x + 1)(x - 1)}{x - 1}$ and $g(x) = x + 1$ agree at all points except $x = 1$.

104. $f(x) = x,\ g(x) = \sin x,\ h(x) = \dfrac{\sin x}{x}$



When the x-values are "close to" 0 the magnitude of f is approximately equal to the magnitude of g. So, $|g|/|f| \approx 1$ when x is "close to" 0.

105. $f(x) = x,\ g(x) = \sin^2 x,\ h(x) = \dfrac{\sin^2 x}{x}$



When the x-values are "close to" 0 the magnitude of g is "smaller" than the magnitude of f and the magnitude of g is approaching zero "faster" than the magnitude of f. So, $|g|/|f| \approx 0$ when x is "close to" 0.

106. (a) Use the dividing out technique because the numerator and denominator have a common factor.

$$\begin{aligned}\lim_{x\to-2}\frac{x^2 + x - 2}{x + 2} &= \lim_{x\to-2}\frac{(x + 2)(x - 1)}{x + 2}\\ &= \lim_{x\to-2}(x - 1) = -2 - 1 = -3\end{aligned}$$

(b) Use the rationalizing technique because the numerator involves a radical expression.

$$\begin{aligned}\lim_{x\to0}\frac{\sqrt{x + 4} - 2}{x} &= \lim_{x\to0}\frac{\sqrt{x + 4} - 2}{x}\cdot\frac{\sqrt{x + 4} + 2}{\sqrt{x + 4} + 2}\\ &= \lim_{x\to0}\frac{(x + 4) - 4}{x\left(\sqrt{x + 4} + 2\right)}\\ &= \lim_{x\to0}\frac{1}{\sqrt{x + 4} + 2} = \frac{1}{\sqrt{4} + 2} = \frac{1}{4}\end{aligned}$$

107. $s(t) = -16t^2 + 500$

$$\begin{aligned}\lim_{t\to2}\frac{s(2) - s(t)}{2 - t} &= \lim_{t\to2}\frac{-16(2)^2 + 500 - \left(-16t^2 + 500\right)}{2 - t}\\ &= \lim_{t\to2}\frac{436 + 16t^2 - 500}{2 - t}\\ &= \lim_{t\to2}\frac{16\left(t^2 - 4\right)}{2 - t}\\ &= \lim_{t\to2}\frac{16(t - 2)(t + 2)}{2 - t}\\ &= \lim_{t\to2} -16(t + 2) = -64 \text{ ft/sec}\end{aligned}$$

The paint can is falling at about 64 feet/second.

108. $s(t) = -16t^2 + 500 = 0$ when $t = \sqrt{\dfrac{500}{16}} = \dfrac{5\sqrt{5}}{2}$ sec. The velocity at time $a = \dfrac{5\sqrt{5}}{2}$ is

$$\lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{s\left(\frac{5\sqrt{5}}{2}\right) - s(t)}{\frac{5\sqrt{5}}{2} - t} = \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{0 - \left(-16t^2 + 500\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t^2 - \frac{125}{4}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\left(\frac{5\sqrt{5}}{2}\right)} \frac{16\left(t + \frac{5\sqrt{5}}{2}\right)\left(t - \frac{5\sqrt{5}}{2}\right)}{\frac{5\sqrt{5}}{2} - t}$$

$$= \lim_{t\to\frac{5\sqrt{5}}{2}} \left[-16\left(t + \frac{5\sqrt{5}}{2}\right)\right] = -80\sqrt{5} \text{ ft/sec}$$

$$\approx -178.9 \text{ ft/sec.}$$

The velocity of the paint can when it hits the ground is about 178.9 ft/sec.

109. $s(t) = -4.9t^2 + 200$

$$\lim_{t\to3} \frac{s(3) - s(t)}{3 - t} = \lim_{t\to3} \frac{-4.9(3)^2 + 200 - \left(-4.9t^2 + 200\right)}{3 - t}$$

$$= \lim_{t\to3} \frac{4.9\left(t^2 - 9\right)}{3 - t}$$

$$= \lim_{t\to3} \frac{4.9(t - 3)(t + 3)}{3 - t}$$

$$= \lim_{t\to3} \left[-4.9(t + 3)\right]$$

$$= -29.4 \text{ m/sec}$$

The object is falling about 29.4 m/sec.

110. $-4.9t^2 + 200 = 0$ when $t = \sqrt{\dfrac{200}{4.9}} = \dfrac{20\sqrt{5}}{7}$ sec. The velocity at time $a = \dfrac{20\sqrt{5}}{7}$ is

$$\lim_{t\to a} \frac{s(a) - s(t)}{a - t} = \lim_{t\to a} \frac{0 - \left[-4.9t^2 + 200\right]}{a - t}$$

$$= \lim_{t\to a} \frac{4.9(t + a)(t - a)}{a - t}$$

$$= \lim_{t\to\frac{20\sqrt{5}}{7}} \left[-4.9\left(t + \frac{20\sqrt{5}}{7}\right)\right] = -28\sqrt{5} \text{ m/sec}$$

$$\approx -62.6 \text{ m/sec.}$$

The velocity of the object when it hits the ground is about 62.6 m/sec.

111. Let $f(x) = 1/x$ and $g(x) = -1/x$. $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x)$ do not exist. However,

$$\lim_{x\to0} \left[f(x) + g(x)\right] = \lim_{x\to0} \left[\frac{1}{x} + \left(-\frac{1}{x}\right)\right] = \lim_{x\to0} [0] = 0 \text{ and therefore does not exist.}$$

112. Suppose, on the contrary, that $\lim_{x\to c} g(x)$ exists. Then, because $\lim_{x\to c} f(x)$ exists, so would $\lim_{x\to c}\left[f(x) + g(x)\right]$, which is a contradiction. So, $\lim_{x\to c} g(x)$ does not exist.

113. Given $f(x) = b$, show that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\left|f(x) - b\right| < \varepsilon$ whenever $\left|x - c\right| < \delta$. Because $\left|f(x) - b\right| = \left|b - b\right| = 0 < \varepsilon$ for every $\varepsilon > 0$, any value of $\delta > 0$ will work.

114. Given $f(x) = x^n$, n is a positive integer, then

$$\begin{aligned}\lim_{x\to c} x^n &= \lim_{x\to c}\left(xx^{n-1}\right)\\ &= \left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-1}\right] = c\left[\lim_{x\to c}\left(xx^{n-2}\right)\right]\\ &= c\left[\lim_{x\to c} x\right]\left[\lim_{x\to c} x^{n-2}\right] = c(c)\lim_{x\to c}\left(xx^{n-3}\right)\\ &= \cdots = c^n.\end{aligned}$$

115. If $b = 0$, the property is true because both sides are equal to 0. If $b \neq 0$, let $\varepsilon > 0$ be given. Because $\lim_{x\to c} f(x) = L$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon/\left|b\right|$ whenever $0 < \left|x - c\right| < \delta$. So, whenever $0 < \left|x - c\right| < \delta$, we have

$$\left|b\right|\left|f(x) - L\right| < \varepsilon \quad \text{or} \quad \left|bf(x) - bL\right| < \varepsilon$$

which implies that $\lim_{x\to c}\left[bf(x)\right] = bL$.

116. Given $\lim_{x\to c} f(x) = 0$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - 0\right| < \varepsilon$ whenever $0 < \left|x - c\right| < \delta$.

Now $\left|f(x) - 0\right| = \left|f(x)\right| = \left|\left|f(x)\right| - 0\right| < \varepsilon$ for $\left|x - c\right| < \delta$. Therefore, $\lim_{x\to c}\left|f(x)\right| = 0$.

117. (a) If $\lim_{x\to c}\left|f(x)\right| = 0$, then $\lim_{x\to c}\left[-\left|f(x)\right|\right] = 0$.

$$-\left|f(x)\right| \le f(x) \le \left|f(x)\right|$$

$$\lim_{x\to c}\left[-\left|f(x)\right|\right] \le \lim_{x\to c} f(x) \le \lim_{x\to c}\left|f(x)\right|$$

$$0 \le \lim_{x\to c} f(x) \le 0$$

Therefore, $\lim_{x\to c} f(x) = 0$.

(b) Given $\lim_{x\to c} f(x) = L$:

For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\left|f(x) - L\right| < \varepsilon$ whenever $0 < \left|x - c\right| < \delta$. Because $\left|\left|f(x)\right| - \left|L\right|\right| \le \left|f(x) - L\right| < \varepsilon$ for $\left|x - c\right| < \delta$, then $\lim_{x\to c}\left|f(x)\right| = \left|L\right|$.

118. Let

$$f(x) = \begin{cases} 4, & \text{if } x \ge 0\\ -4, & \text{if } x < 0\end{cases}$$

$\lim_{x\to 0}\left|f(x)\right| = \lim_{x\to 0} 4 = 4.$

$\lim_{x\to 0} f(x)$ does not exist because for $x < 0$, $f(x) = -4$ and for $x \ge 0$, $f(x) = 4$.

119.
$$\begin{aligned}\lim_{x\to 0}\frac{1 - \cos x}{x} &= \lim_{x\to 0}\frac{1 - \cos x}{x}\cdot\frac{1 + \cos x}{1 + \cos x}\\ &= \lim_{x\to 0}\frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x\to 0}\frac{\sin^2 x}{x(1 + \cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1 + \cos x}\\ &= \left[\lim_{x\to 0}\frac{\sin x}{x}\right]\left[\lim_{x\to 0}\frac{\sin x}{1 + \cos x}\right]\\ &= (1)(0) = 0\end{aligned}$$

120. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational}\\ 1, & \text{if } x \text{ is irrational}\end{cases}$

$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational}\\ x, & \text{if } x \text{ is irrational}\end{cases}$

$\lim_{x\to 0} f(x)$ does not exist.

No matter how "close to" 0 x is, there are still an infinite number of rational and irrational numbers so that $\lim_{x\to 0} f(x)$ does not exist.

$\lim_{x\to 0} g(x) = 0$

When x is "close to" 0, both parts of the function are "close to" 0.

121. $\lim_{x\to 2}\dfrac{2x^2 - 3x + 1}{2x^3 - 25} = \dfrac{2(2)^2 - 3(2) + 1}{2(2)^3 - 25} = \dfrac{3}{-9} = -\dfrac{1}{3}$

So, the answer is A.

122. Evaluate each limit.

I: Using a graphing utility, $\lim_{x\to 1}\dfrac{x^3 + 1}{x - 1}$ does not exist.

II: $\lim_{x\to 0}\dfrac{\left|x\right|}{x} = \lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} -1, & x < 0\\ 1, & x \ge 0\end{cases}$ does not exist because the limits on each side of $x = 0$ do not agree.

III: $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} 3, & x \le 2\\ 0, & x > 2\end{cases}$ does not exist because the limits on each side of $x = 2$ do not agree.

Because the limits of I, II, and III do not exist, the answer is D.

123. $\lim_{x \to 3} \frac{x^4 - 81}{x - 3} = \lim_{x \to 3} \frac{(x^2 + 9)(x^2 - 9)}{x - 3}$

$= \lim_{x \to 3} \frac{(x^2 + 9)(x + 3)(x - 3)}{x - 3}$

$= \lim_{x \to 3} (x^2 + 9)(x + 3)$

$= (3^2 + 9)(3 + 3)$

$= 108$

So, the answer is C.