3.1 Extrema on an Interval

Definition of Extrema

Let f be defined on an interval I containing c.

1. $f(c)$ is the minimum of f on I when $f(c) < f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I when $f(c) > f(x)$ for all x in I.

Other terms: Extreme values, Extrema, Absolute Minimum, Absolute Maximum, Global Minimum, Global Maximum

Extreme Value Theorem (EVT)

If f is continuous on a closed interval $[a, b]$ then f has both a maximum and a minimum on the interval.

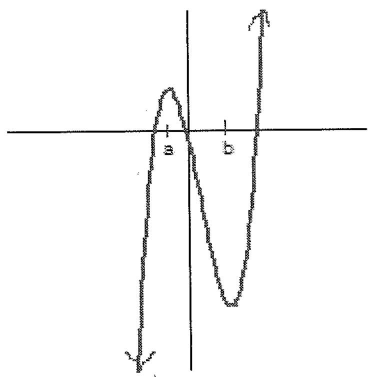
What do you think happens with a constant function?

If it is not continuous, does it not have extrema?

Definition of Relative Extrema

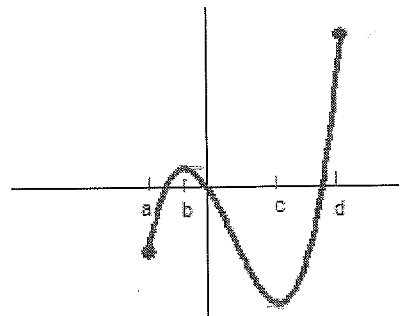
1. If there is an open interval containing c on which $f(c)$ is a maximum of f, then f has a Relative Maximum at $(c, f(c))$.
2. If there is an open interval containing c on which $f(c)$ is a minimum of f, then f has a Relative Minimum at $(c, f(c))$.

Other terms: Relative Maxima, Relative Minima, Local Maximum, Local Minimum



X= a Relative Max

X= b Relative Min



X= b Relative Max

X= c Absolute Min, Relative Min

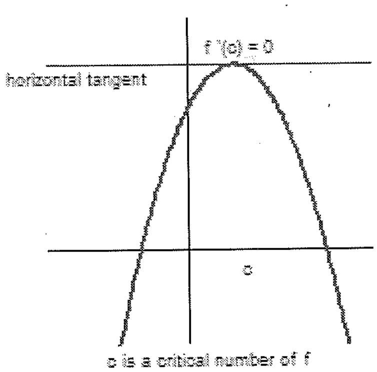
X= d Absolute Max

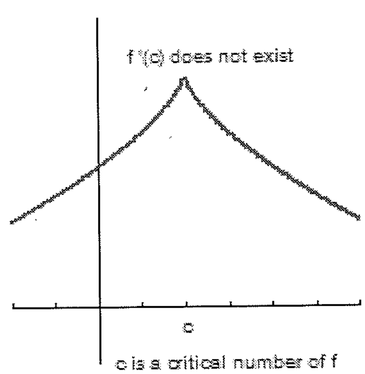
What do you notice about the behavior of the graph at a relative maximum or minimum?

Definition of a Critical Number

Let f be defined at c. If $f'(c) = 0$ or if f is not differentiable at c, then c is a critical number of f.

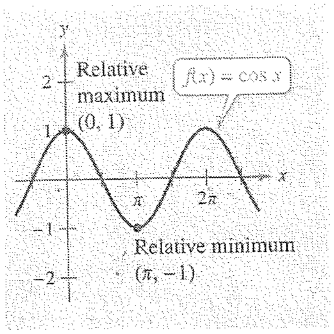
**f must be defined at c!

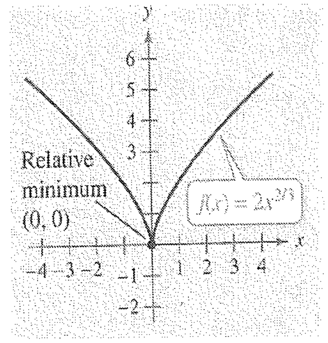




Examples – Value of Derivatives at Relative Extrema

Find the value of the derivative at each of the relative extrema shown on the graph.





Examples – Finding Critical Numbers

Find any critical numbers of the function $f(x) = 3x^4 - 4x^3$

$$f'(x) = 12x^3 - 12x^2$$
$$0 = 12x^2(x-1)$$
$$\{x=0 \quad x=1\}$$

Find any critical numbers of the function $f(x) = 3x\sqrt{x-5}$

$[5, \infty)$

$$f'(x) = (3x)\left(\tfrac{1}{2}\right)(x-5)^{-\frac{1}{2}} + 3(x-5)^{\frac{1}{2}}$$
$$f'(x) = 3(x-5)^{-\frac{1}{2}}\left[\tfrac{1}{2}x + x - 5\right]$$
$$f'(x) = \frac{3\left(\frac{3}{2}x - 5\right)}{(x-5)^{1/2}}$$

$\cancel{x = 10/3}$

$x = 5$

Finding extrema on a closed interval

To find the extrema of a continuous function f on a closed interval $[a, b]$.

1. Find critical numbers
2. Evaluate the function at the critical numbers
3. Evalute the function at the endpoints
4. Highest → Max
   Lowest → Min

Examples – Finding extrema

Find the extrema of $f(x) = 3x^5 - 7x^4$ on the interval $[-1,2]$.

$f'(x) = 15x^4 - 28x^3$

$0 = x^3(15x - 28)$

$x = 0 \qquad x = \frac{28}{15}$

$f(-1) = -3 - 7 = -10$

$f(0) = 0$

$f\left(\frac{28}{15}\right) = 3\left(\frac{28}{15}\right)^5 - 7\left(\frac{28}{15}\right)^4 \approx -17$

$f(2) = 3(32) - 7(16) = -16$

Abs. Min $\left(28/15, -17\right)$ Abs. Max $(0,0)$

Find the extrema of $f(x) = |x^2 - 4|$ on $[-3,4]$

$(x-2)(x+2)$

$+ \quad - \quad +$

$-2 \quad 2$

$$f(x) = \begin{cases} x^2-4 & x \leq -2 \\ -x^2+4 & -2 \leq x \leq 2 \\ x^2-4 & x > 2 \end{cases}$$

$$f'(x) \begin{cases} 2x & x < -2 \\ -2x & -2 \leq x \leq 2 \\ 2x & x > 2 \end{cases}$$

$x = 0$

$x = 2$

$x = -2$

$f(-3) = 5$

$f(-2) = 0$

$f(0) = 4$

$f(2) = 0$

$f(4) = 12$

Abs Max $(4,12)$

Abs. Min $(-2,0)$ $(2,0)$

Find the extrema of $f(x) = sin2x + 2cosx$ on $[0,2\pi]$

$f'(x) = \cos(2x)\cdot 2 + 2(-\sin x)$

$0 = 2(1 - 2\sin^2 x) - 2\sin x$

$0 = -4\sin^2 x - 2\sin x + 2$

$0 = -2(2\sin^2 x + \sin x - 1)$

$0 = -2(2\sin x - 1)(\sin x + 1)$

$\sin x = \frac{1}{2} \qquad \sin x = -1$

$x = \frac{\pi}{6}, \frac{5\pi}{6} \qquad x = \frac{3\pi}{2}$

$f(0) = \sin 0 + 2\cos 0 = 2$

$f\left(\frac{\pi}{6}\right) = \sin\frac{\pi}{3} + 2\cos\frac{\pi}{6} = \frac{3\sqrt{3}}{2}$

$f\left(\frac{5\pi}{6}\right) = \sin\frac{5\pi}{3} + 2\cos\frac{5\pi}{6} = -\frac{3\sqrt{3}}{2}$

$f\left(\frac{3\pi}{2}\right) = 0$

$f(2\pi) = 2$

Abs. Min $\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$

Abs. Max $\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$