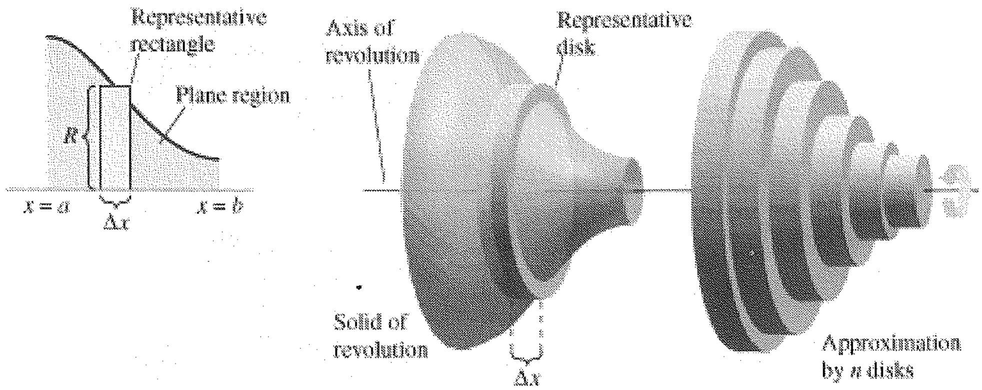
6.2 Volume: The Disk and Washer Method (Day 1)

Solid of Revolution: Obtained when a plan region is revolved about a line (axis of revolution)



The resulting solid is a solid of revolution. The most common is a right circular cylinder, a disk, that results from rotating a rectangle about the axis of revolution.

Recall: The volume of a disk $V = \pi r^2 h$

Therefore:

Horizontal Axis of Revolution: $\pi \int_a^b [f(x)]^2 \, dx$ (radius)

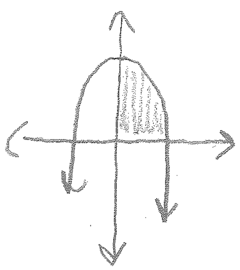
(Vertical Rotation)

Vertical Axis of Revolution: $\pi \int_a^b [f(y)]^2 \, dy$ (radius)

(Horizontal Rotation)

**** The representative rectangle is always perpendicular to the axis of revolution**

Find the volume of the solid formed when the region defined by $y = 4 - x^2, x = 0$ and $y = 0$ is revolved about the $x - axis$.

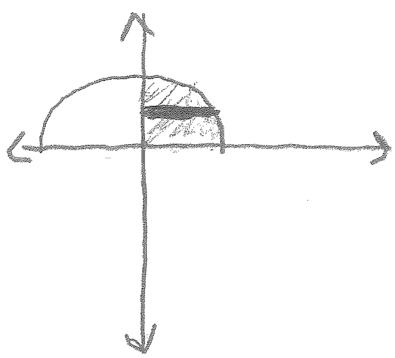


$$\pi \int_0^2 [4-x^2]^2 dx = \pi \int_0^2 16 - 8x^2 + x^4 dx$$

$$= \pi \left[16x - 8\left(\frac{x^3}{3}\right) + \frac{1}{5}x^5\right]\Big|_0^2 = \pi \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]\Big|_0^2$$

$$= \pi \left[32 - \frac{64}{3} + \frac{32}{5}\right] = \frac{256}{15}\pi$$

Determine the volume of the solid formed when the region defined by $y = \sqrt{16 - x^2}$ in the first quadrant is revolved about the y-axis.



$$\pi \int_0^4 \left(\sqrt{16-y^2}\right)^2 dy$$

$$y^2 = 16 - x^2$$
$$x^2 = 16 - y^2$$
$$x = \pm\sqrt{16-y^2}$$

$$\pi \int_0^4 16 - y^2 \, dy$$

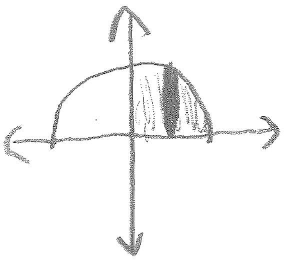
$$\pi \left[16y - \frac{1}{3}y^3\right]\Big|_0^4$$

$$\pi \left[64 - \frac{64}{3}\right] = \frac{128}{3}\pi$$

Examples – Finding Volumes

Set up and evaluate the integral that gives the volume of the solid formed by revolving the region defined by

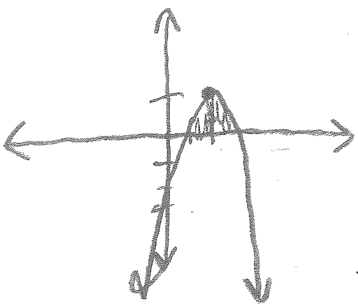
$y = \sqrt{4 - x^2}, x = 0$ *and* $y = 0$ about the $x - axis$.



$$\pi\int_0^2 \left(\sqrt{4-x^2}\right)^2 dx = \pi\int_0^2 4 - x^2\, dx = \pi\left[4x - \tfrac{1}{3}x^3\right]\Big|_0^2$$

$$= \pi\left[8 - \tfrac{8}{3}\right] = \frac{16}{3}\pi$$

Find the volume of the solid formed by revolving the region bounded by the graph of $f(x) = -x^2 + 4x - 3$ and the $x -$ $axis$ about the $x - axis$.



$$\frac{-4}{2(-1)} = 2$$

$$y = -4 + 8 - 3 = 1$$

$$0 = -x^2 + 4x - 3$$

$$0 = -1(x^2 - 4x + 3)$$

$$0 = -1(x-1)(x-3)$$

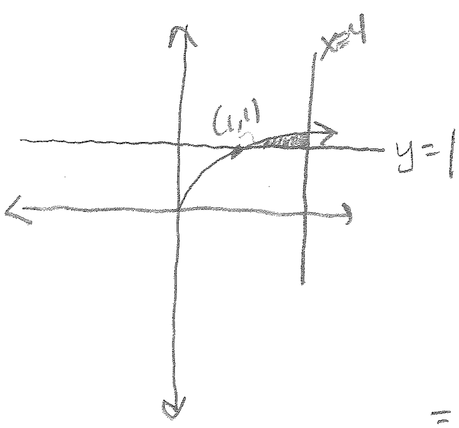
$$\pi\int_0^1 \left(-x^2 + 4x - 3\right)^2 dx$$

$$\pi\int_0^1 x^4 - 4x^3 + 3x^2 - 4x^3 + 16x^2 - 12x + 3x^2 - 12x + 9\, dx$$

$$\pi\int_0^1 x^4 - 8x^3 + 22x^2 - 24x + 9\, dx = \pi\left[\tfrac{1}{5}x^5 - 8\left(\tfrac{x^4}{4}\right) + 22\left(\tfrac{x^3}{3}\right) + 24\left(\tfrac{x^2}{2}\right) + 9x\right]\Big|_0^1$$

$$= \pi\left[\tfrac{1}{5} - 2 + \tfrac{22}{3} + 12 + 9\right] = \frac{398}{15}\pi$$

Find the volume of the solid formed by revolving the graphs of $y = \sqrt{x}$, $y = 1$ and $x = 4$ about the line $y = 1$.



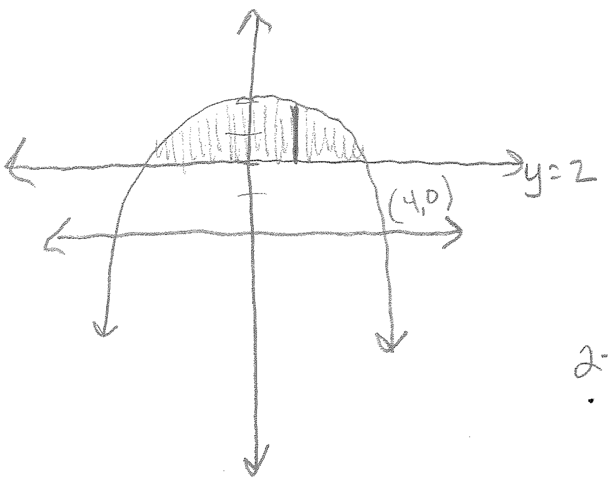
$$\pi\int_1^4 \left(\sqrt{x} - 1\right)^2 dx = \pi\int_1^4 x - 2\sqrt{x} + 1\, dx$$

$$= \pi\left[\tfrac{1}{2}x^2 - 2\left(\frac{x^{3/2}}{3/2}\right) + x\right]\Big|_1^4 = \pi\left[\tfrac{1}{2}x^2 - \tfrac{4}{3}x^{3/2} + x\right]\Big|_1^4$$

$$= \pi\left[\left(\tfrac{1}{2}(16) - \tfrac{4}{3}(4)^{3/2} + 4\right) - \left(\tfrac{1}{2} - \tfrac{4}{3} + 1\right)\right]$$

$$= \pi\left[8 - \tfrac{32}{3} + 4 - \tfrac{1}{2} + \tfrac{4}{3} - 1\right] = \frac{7}{6}\pi$$

Find the volume of the solid formed by revolving the region bound by $y = 4 - \frac{x^2}{4}$ and $y = 2$ about the line $y = 2$.



$$\frac{16-x^2}{4} = 2$$
$$16-x^2 = 8$$
$$8 = x^2$$
$$x = \pm 2\sqrt{2}$$

$$2\pi \int_0^{2\sqrt{2}} \left(4 - \frac{x^2}{4} - 2\right)^2 dx \quad \text{(Top) (Bottom)}$$

$$2\pi \int_0^{2\sqrt{2}} \left(2 - \frac{x^2}{4}\right)^2 dx = 2\pi \int_0^{2\sqrt{2}} 4 - x^2 + \frac{x^4}{16}\, dx = 2\pi \left[4x - \frac{1}{3}x^3 + \frac{1}{16}\left(\frac{x^5}{5}\right)\right]\Bigg|_0^{2\sqrt{2}}$$

$$2\pi\left[4(2\sqrt{2}) - \frac{1}{3}(2\sqrt{2})^3 + \frac{1}{80}(2\sqrt{2})^5\right] = 2\pi\left[8\sqrt{2} - \frac{1}{3}(16\sqrt{2}) + \frac{1}{80}(128\sqrt{2})\right]$$

$$= 2\pi\left[\frac{64}{15}\sqrt{2}\right] = \frac{128}{15}\pi\sqrt{2}$$