Section 3.7 Differentials

1. $f(x) = x^2$

$f'(x) = 2x$

Tangent line at (2, 4): $y - f(2) = f'(2)(x - 2)$

$y - 4 = 4(x - 2)$

$y = 4x - 4$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^2$	3.6100	3.9601	4	4.0401	4.4100
$T(x) = 4x - 4$	3.6000	3.9600	4	4.0400	4.4000

2. $f(x) = \frac{6}{x^2} = 6x^{-2}$

$f'(x) = -12x^{-3} = \frac{-12}{x^3}$

Tangent line at $\left(2, \frac{3}{2}\right)$:

$y - \frac{3}{2} = \frac{-12}{8}(x - 2) = \frac{-3}{2}(x - 2)$

$y = -\frac{3}{2}x + \frac{9}{2}$

x	1.9	1.99	2	2.01	2.1
$f(x) = \frac{6}{x^2}$	1.6620	1.5151	1.5	1.4851	1.3605
$T(x) = -\frac{3}{2}x + \frac{9}{2}$	1.65	1.515	1.5	1.485	1.35

3. $f(x) = x^5$

$f'(x) = 5x^4$

Tangent line at $(2, 32)$:

$y - f(2) = f'(2)(x - 2)$

$y - 32 = 80(x - 2)$

$y = 80x - 128$

x	1.9	1.99	2	2.01	2.1
$f(x) = x^5$	24.7610	31.2080	32	32.8080	40.8410
$T(x) = 80x - 128$	24.0000	31.2000	32	32.8000	40.0000

4. $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

Tangent line at $\left(2, \sqrt{2}\right)$:

$y - f(2) = f'(2)(x - 2)$

$y - \sqrt{2} = \dfrac{1}{2\sqrt{2}}(x - 2)$

$y = \dfrac{x}{2\sqrt{2}} + \dfrac{1}{\sqrt{2}}$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sqrt{x}$	1.3784	1.4107	1.4142	1.4177	1.4491
$T(x) = \dfrac{x}{2\sqrt{2}} + \dfrac{1}{\sqrt{2}}$	1.3789	1.4107	1.4142	1.4177	1.4496

5. $f(x) = \sin x$

$f'(x) = \cos x$

Tangent line at $(2, \sin 2)$:

$y - f(2) = f'(2)(x - 2)$

$y - \sin 2 = (\cos 2)(x - 2)$

$y = (\cos 2)(x - 2) + \sin 2$

x	1.9	1.99	2	2.01	2.1
$f(x) = \sin x$	0.9463	0.9134	0.9093	0.9051	0.8
$T(x) = (\cos 2)(x - 2) + \sin 2$	0.9509	0.9135	0.9093	0.9051	0.8

6. $f(x) = \log_2 x = \dfrac{\ln x}{\ln 2}, \quad (2, 1)$

$f'(x) = \dfrac{1}{x \ln 2}$

$f'(2) = \dfrac{1}{2 \ln 2}$

Tangent line at $(2, 1)$: $y - 1 = \dfrac{1}{2 \ln 2}(x - 2)$

$y = \dfrac{1}{2 \ln 2}x + 1 - \dfrac{1}{\ln 2}$

x	1.9	1.99	2	2.01	2.1
$f(x) = \log_2 x$	0.9260	0.9928	1	1.0072	1.0704
$T(x) = \dfrac{1}{2 \ln 2}x + 1 - \dfrac{1}{\ln 2}$	0.9279	0.9928	1	1.0072	1.0721

7. $y = f(x) = 0.5x^3, f'(x) = 1.5x^2, x = 1, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$
$= f(1.1) - f(1)$
$= 0.1655$

$dy = f'(x)\,dx$
$= 1.5x^2\,dx$
$= 1.5(1)^2(0.1)$
$= 0.15$

8. $y = f(x) = 6 - 2x^2, f'(x) = -4x, x = -2, \Delta x = dx = 0.1$

$\Delta y = f(x + \Delta x) - f(x)$ 2
$= f(-1.9) - f(-2)$
$= 6 - 2(-1.9)^2 - \left(6 - 2(-2)^2\right)$
$= -1.22 - (-2) = 0.78$

$dy = f'(x)\,dx$
$= -4(-2)(0.1)$
$= 0.8$

9. $y = f(x) = x^4 + 1,\ f'(x) = 4x^3,\ x = -1,\ \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) \qquad dy = f'(x)\,dx$$

$$= f(-0.99) - f(-1) \qquad = f'(-1)(0.01)$$

$$= \left[(-0.99)^4 + 1\right] - \left[(-1)^4 + 1\right] \approx -0.0394 \qquad = (-4)(0.01) = -0.04$$

10. $y = f(x) = 2 - x^4,\ f'(x) = -4x^3,\ x = 2,\ \Delta x = dx = 0.01$

$$\Delta y = f(x + \Delta x) - f(x) \qquad dy = f'(x)\,dx$$

$$= f(2.01) - f(2) \qquad = \left(-4x^3\right) dx$$

$$\approx -14.3224 - (-14) = -0.3224 \qquad = -4(2)^3(0.01)$$

$$= -0.32$$

11. $y = 3x^2 - 4$

$dy = 6x\,dx$

12. $y = 3x^{2/3}$

$dy = 2x^{-1/3}\,dx = \dfrac{2}{x^{1/3}}\,dx$

13. $y = x\tan x$

$dy = \left(x\sec^2 x + \tan x\right)dx$

14. $y = \csc 2x$

$dy = (-2\csc 2x\cot 2x)dx$

15. $y = \dfrac{x+1}{2x-1}$

$dy = -\dfrac{3}{(2x-1)^2}\,dx$

16. $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

$dy = \left(\dfrac{1}{2\sqrt{x}} - \dfrac{1}{2x\sqrt{x}}\right)dx = \dfrac{x-1}{2x\sqrt{x}}\,dx$

17. $y = \sqrt{9 - x^2}$

$dy = \dfrac{1}{2}\left(9 - x^2\right)^{-1/2}(-2x)\,dx = \dfrac{-x}{\sqrt{9 - x^2}}\,dx$

18. $y = x\sqrt{1 - x^2}$

$dy = \left(x\dfrac{-x}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right)dx = \dfrac{1 - 2x^2}{\sqrt{1-x^2}}\,dx$

19. $y = 3x - \sin^2 x$

$dy = (3 - 2\sin x\cos x)\,dx = (3 - \sin 2x)\,dx$

20. $y = \dfrac{\sec^2 x}{x^2 + 1}$

$$dy = \left[\frac{\left(x^2+1\right)2\sec^2 x\tan x - \sec^2 x(2x)}{\left(x^2+1\right)^2}\right]dx$$

$$= \left[\frac{2\sec^2 x\left(x^2\tan x + \tan x - x\right)}{\left(x^2+1\right)^2}\right]dx$$

21. $y = \ln\sqrt{4 - x^2} = \dfrac{1}{2}\ln\left(4 - x^2\right)$

$dy = \dfrac{1}{2}\left(\dfrac{-2x}{4 - x^2}\right)dx = \dfrac{-x}{4 - x^2}\,dx$

22. $y = e^{-0.5x}\cos 4x$

$dy = \left[e^{-0.5x}(-4\sin 4x) + (-0.5)e^{-0.5x}\cos 4x\right]dx$

$= e^{-0.5x}\left[-4\sin 4x - 0.5\cos 4x\right]dx$

23. $y = x\arcsin x$

$dy = \left(\dfrac{x}{\sqrt{1 - x^2}} + \arcsin x\right)dx$

24. $y = \arctan(x - 2)$

$dy = \dfrac{1}{1 + (x-2)^2}\,dx$

25. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$\approx 1 + (1)(-0.1) = 0.9$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04$

$\approx 1 + (1)(0.04) = 1.0$

26. (a) $f(1.9) = f(2 - 0.1) \approx f(2) + f'(2)(-0.1)$

$$\approx 1 + \left(-\tfrac{1}{2}\right)(-0.1) = 1.05$$

(b) $f(2.04) = f(2 + 0.04) \approx f(2) + f'(2)(0.04)$

$$\approx 1 + \left(-\tfrac{1}{2}\right)(0.04) = 0.98$$

27. The denominator in Step 2 should be $\left(12x^2\right)^{2/3} = \sqrt[3]{144x^4}$.

$$dy = \frac{1}{3}\left(12x^2\right)^{-2/3}(24x)\,dx$$

$$= \frac{8x}{\sqrt[3]{144x^4}}\,dx$$

$$= \frac{4}{\sqrt[3]{18x}}\,dx$$

28. The Chain Rule should have been used for $\cos 2x$.

If $y = x^2 \cos 2x$, then

$dy = \left(2x \cos 2x - 2x^2 \sin 2x\right) dx.$

29. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + \left(-\tfrac{1}{2}\right)(-0.07) = 8.035$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + \left(-\tfrac{1}{2}\right)(0.1) = 7.95$$

30. (a) $g(2.93) = g(3 - 0.07) \approx g(3) + g'(3)(-0.07)$

$$\approx 8 + (3)(-0.07) = 7.79$$

(b) $g(3.1) = g(3 + 0.1) \approx g(3) + g'(3)(0.1)$

$$\approx 8 + (3)(0.1) = 8.3$$

31. $x = 10$ in., $\Delta x = dx = \pm\dfrac{1}{32}$ in.

(a) $A = x^2$

$dA = 2x dx$

$$\Delta A \approx dA = 2(10)\left(\pm\frac{1}{32}\right) = \pm\frac{5}{8}\text{ in}^2$$

(b) Percent error:

$$\frac{dA}{A} = \frac{5/8}{100} = \frac{5}{800} = \frac{1}{100} = 0.00625 = 0.625\%$$

32. (a) $C = 64$ cm

$\Delta C = dC = \pm 0.9$ cm

$$C = 2\pi r \Rightarrow r = \frac{C}{2\pi}$$

$$A = \pi r^2 = \pi\left(\frac{C}{2\pi}\right)^2 = \frac{1}{4\pi}C^2$$

$$dA = \frac{1}{2\pi}C\,dC = \frac{1}{2\pi}(64)(\pm 0.9) = \frac{\pm 28.8}{\pi}$$

$$\frac{dA}{A} = \frac{28.8/\pi}{\left[1/(4\pi)\right](64)^2} \approx 0.028125 = 2.8\%$$

(b) $$\frac{dA}{A} = \frac{\left[1/(2\pi)\right]C\,dC}{\left[1/(4\pi)\right]C^2} = \frac{2\,dC}{C} \le 0.03$$

$$\frac{dC}{C} \le \frac{0.03}{2} = 0.015 = 1.5\%$$

33. $x = 15$ in., $\Delta x = dx = \pm 0.03$ in.

(a) $V = x^3$

$dV = 3x^2 dx$

$\Delta V \approx dV = 3(15)^2(\pm 0.03) = \pm 20.25\text{ in}^3$

(b) $S = 6x^2$

$dS = 12x\,dx$

$\Delta S \approx dS = 12(15)(\pm 0.03) = \pm 5.4\text{ in}^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{20.25}{15^3} = 0.006 \text{ or } 0.6\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{5.4}{6(15)^2} = 0.004 \text{ or } 0.4\%$$

34. $r = 8$ in., $dr = \Delta r = \pm 0.02$ in.

(a) $V = \dfrac{4}{3}\pi r^3$

$dV = 4\pi r^2 dr$

$\Delta V \approx dV = 4\pi(8)^2(\pm 0.02) = \pm 5.12\pi\text{ in}^3$

(b) $S = 4\pi r^2$

$dS = 8\pi r\,dr$

$\Delta S \approx dS = 8\pi(8)(\pm 0.02) = \pm 1.28\pi\text{ in}^2$

(c) Percent error of volume:

$$\frac{dV}{V} = \frac{5.12\pi}{\frac{4}{3}\pi(8)^3} = 0.0075 \text{ or } 0.75\%$$

Percent error of surface area:

$$\frac{dS}{S} = \frac{1.28\pi}{4\pi(8)^2} = 0.005 \text{ or } 0.5\%$$

35. $T = 2.5x + 0.5x^2, \Delta x = dx = 26 - 25 = 1, x = 25$

$dT = (2.5 + x)dx = (2.5 + 25)(1) = 27.5$ mi

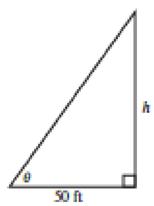
Percentage change $= \dfrac{dT}{T} = \dfrac{27.5}{375} \approx 7.3\%$

36. Because the slope of the tangent line is greater at $x = 900$ than at $x = 400$, the change in profit is greater at $x = 900$ units.

37. $dH = -\dfrac{401{,}493{,}267}{2{,}000{,}000} \dfrac{e^{369{,}444/(50t+19{,}793)}}{(50t + 19{,}793)^2} dt$

At $t = 72$ and $dt = 1$, $dH \approx -2.65$.

38. $h = 50 \tan \theta$



$\theta = 71.5° = 1.2479$ radians

$dh = 50 \sec^2\theta \cdot d\theta$

$$\left|\frac{dh}{h}\right| = \left|\frac{50 \sec^2(1.2479)}{50 \tan(1.2479)} d\theta\right| \le 0.06$$

$$\left|\frac{9.9316}{2.9886} d\theta\right| \le 0.06$$

$|d\theta| \le 0.018$

39. Let $f(x) = \sqrt{x}, x = 100, dx = -0.6$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt{x} + \frac{1}{2\sqrt{x}}dx$$

$$f(x + \Delta x) = \sqrt{99.4} \approx \sqrt{100} + \frac{1}{2\sqrt{100}}(-0.6) = 9.97$$

Using a calculator: $\sqrt{99.4} \approx 9.96995$

40. Let $f(x) = \sqrt[3]{x}, x = 27, dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[3]{x} + \frac{1}{3\sqrt[3]{x^2}}dx$$

$$\sqrt[3]{26} \approx \sqrt[3]{27} + \frac{1}{3\sqrt[3]{27^2}}(-1) = 3 - \frac{1}{27} \approx 2.9630$$

Using a calculator, $\sqrt[3]{26} \approx 2.9625$

41. Let $f(x) = \sqrt[4]{x}, x = 625, dx = -1$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = \sqrt[4]{x} + \frac{1}{4\sqrt[4]{x^3}}dx$$

$$f(x + \Delta x) = \sqrt[4]{624} \approx \sqrt[4]{625} + \frac{1}{4\left(\sqrt[4]{625}\right)^3}(-1) = 5 - \frac{1}{500} = 4.998$$

Using a calculator, $\sqrt[4]{624} \approx 4.9980$.

42. Let $f(x) = x^3, x = 3, dx = -0.01$.

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx = x^3 + 3x^2dx$$

$$f(x + \Delta x) = (2.99)^3 \approx 3^3 + 3(3)^2(-0.01) = 27 - 0.27 = 26.73$$

Using a calculator: $(2.99)^3 \approx 26.7309$

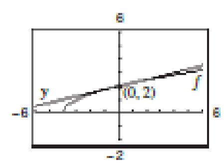
43. $f(x) = \sqrt{x + 4}$

$f'(x) = \dfrac{1}{2\sqrt{x + 4}}$

At $(0, 2)$, $f(0) = 2$, $f'(0) = \dfrac{1}{4}$

Tangent line: $y - 2 = \dfrac{1}{4}(x - 0)$

$y = \dfrac{1}{4}x + 2$



44. $f(x) = \tan x$

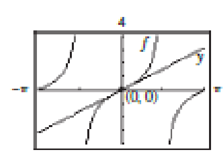
$f'(x) = \sec^2 x$

$f(0) = 0$

$f'(0) = 1$

Tangent line at $(0, 0)$: $y - 0 = (x - 0)$

$y = x$



45. (a) Let $f(x) = \sqrt{x}, x = 4, dx = 0.02,$

$$f'(x) = 1/(2\sqrt{x}).$$

Then

$$f(4.02) \approx f(4) + f'(4)\,dx$$

$$\sqrt{4.02} \approx \sqrt{4} + \frac{1}{2\sqrt{4}}(0.02) = 2 + \frac{1}{4}(0.02).$$

(b) Let

$$f(x) = \tan x, x = 0, dx = 0.05, f'(x) = \sec^2 x.$$

Then

$$f(0.05) \approx f(0) + f'(0)\,dx$$

$$\tan 0.05 \approx \tan 0 + \sec^2 0(0.05) = 0 + 1(0.05).$$

46. Yes. $y = x$ is the tangent line approximation to $f(x) = \sin x$ at $(0, 0)$.

$$f'(x) = \cos x$$

$$f'(0) = 1$$

Tangent line: $y - 0 = 1(x - 0)$

$$y = x$$

47. True, $\dfrac{\Delta y}{\Delta x} = \dfrac{dy}{dx} = a$

48. True

49. False

Let $f(x) = \sqrt{x}, x = 1$, and $\Delta x = dx = 3$. Then

$$\Delta y = f(x + \Delta x) - f(x) = f(4) - f(1) = 1$$

and $dy = f'(x)\,dx = \dfrac{1}{2\sqrt{1}}(3) = \dfrac{3}{2}.$

So, $dy > \Delta y$ in this example.

50. $f(x) = \cos^{-1} x \Rightarrow x = \cos y$

$$\frac{d}{dx}\cos y = \frac{d}{dx}x$$

$$-\sin y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = -\frac{1}{\sin y}$$

$$= -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

Tangent line at $\left(\frac{1}{2}, \frac{\pi}{3}\right)$:

$$y - f\left(\frac{1}{2}\right) = f'\left(\frac{1}{2}\right)(x - 2)$$

$$y - \frac{\pi}{3} = \frac{1}{\sqrt{1 - \frac{1}{4}}}(x - 2)$$

$$y = -\frac{2\sqrt{3}}{3}x + \frac{\sqrt{3} + \pi}{3}$$

At $x = 0.52, y = \dfrac{-2\sqrt{3}}{3}(0.52) + \dfrac{\sqrt{3} + \pi}{3} \approx 1.$

So, the answer is C.

51. $y = x^2 \ln x$

$$dy = \left(x^2 \cdot \frac{1}{x} + 2x\ln x\right)dx$$

$$= (x + 2x\ln x)\,dx$$

So, the answer is A.

52. $y = f(c) = f'(c)(x - c)$

$$y = f(3) + f'(3)(x - 3)$$

$$y = 8 + 22(x - 3)$$

$$y = 22x - 58$$

$$f(2.9) = 22(2.9) - 58 = 5.8$$

So, the answer is B.

53. (a) $P = 100xe^{-x/400}$

$$P' = 100x\left(-\frac{1}{400}e^{-x/400}\right) + 100e^{-x/400} = -\frac{1}{4}xe^{-x/400} + 100e^{-x/400}$$

(b)

$$P'(x) = 0$$

$$-\frac{1}{4}xe^{-x/400} + 100e^{-x/400} = 0$$

$$100e^{-x/400} = \frac{1}{4}xe^{-x/400}$$

$$400 = x$$

The profit is maximum when $x = 400$ units.

(c) Change in profit:

$$f(130) - f(120) = 100(130)e^{-130/400} - 100(120)e^{-120/400} \approx \$503$$

Percent change:

$$P = 100xe^{-x/400}, \Delta x = dx = 130 - 120 = 10, x = 120$$

$$dP = \left[100x\left(-\frac{1}{400}e^{-x/400}\right) + 100e^{-x/400}\right]dx$$

$$= \left[100(120)\left(-\frac{1}{400}e^{-120/400}\right) + 100e^{-120/400}\right](10)$$

$$= 700e^{-0.3}$$

$$\text{Percent change} = \frac{dp}{p} = \frac{700e^{-0.3}}{12{,}000e^{-0.3}} \approx 5.8\%$$