Section 5.3 Separation of Variables

1. $\frac{dr}{ds} = 0.75\,r$

$\int \frac{dr}{r} = \int 0.75\,ds$

$\ln|r| = 0.75\,s + C_1$

$r = e^{0.75\,s + C_1}$

$r = Ce^{0.75\,s}$

2. $\frac{dr}{ds} = 0.75\,s$

$\int dr = \int 0.75\,s\,ds$

$r = 0.75\frac{s^2}{2} + C$

$r = 0.375\,s^2 + C$

3. $\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + C_1$

$y^2 - x^2 = C$

4. $\frac{dy}{dx} = \frac{3x^2}{y^2}$

$\int y^2\,dy = \int 3x^2\,dx$

$\frac{y^3}{3} = x^3 + C_1$

$y^3 - 3x^3 = C$

5. $\frac{dy}{dx} = \frac{x - 1}{y^3}$

$\int y^3\,dy = \int (x - 1)\,dx$

$\frac{1}{4}y^4 = \frac{1}{2}x^2 - x + C_1$

$y^4 - 2x^2 + 4x = C$

6. $\frac{dy}{dx} = \frac{6 - x^2}{2y^3}$

$\int 2y^3\,dy = \int (6 - x^2)\,dx$

$\frac{y^4}{2} = 6x - \frac{x^3}{3} + C_1$

$3y^4 + 2x^3 - 36x = C$

7. $(2 + x)y' = 3y$

$\int \frac{dy}{y} = \int \frac{3}{2 + x}\,dx$

$\ln|y| = 3\ln|2 + x| + \ln C = \ln\left|C(2 + x)^3\right|$

$y = C(x + 2)^3$

8. $xy' = y$

$\int \frac{dy}{y} = \int \frac{dx}{x}$

$\ln y = \ln x + \ln C = \ln Cx$

$y = Cx$

9. $yy' = 4\sin x$

$y\frac{dy}{dx} = 4\sin x$

$\int y\,dy = \int 4\sin x\,dx$

$\frac{y^2}{2} = -4\cos x + C_1$

$y^2 = C - 8\cos x$

10. $yy' = -8\cos(\pi x)$

$y\frac{dy}{dx} = -8\cos(\pi x)$

$\int y\,dy = \int -8\cos(\pi x)\,dx$

$\frac{y^2}{2} = \frac{-8\sin(\pi x)}{\pi} + C$

$y^2 = \frac{-16}{\pi}\sin(\pi x) + C$

11. $\sqrt{1 - 4x^2}\,y' = x$

$dy = \frac{x}{\sqrt{1 - 4x^2}}\,dx$

$\int dy = \int \frac{x}{\sqrt{1 - 4x^2}}\,dx$

$= -\frac{1}{8}\int (1 - 4x^2)^{-1/2}(-8x\,dx)$

$y = -\frac{1}{4}\sqrt{1 - 4x^2} + C$

12. $\sqrt{x^2 - 16}\,y' = 11x$

$\frac{dy}{dx} = \frac{11x}{\sqrt{x^2 - 16}}$

$\int dy = \int \frac{11x}{\sqrt{x^2 - 16}}\,dx$

$y = 11\sqrt{x^2 - 16} + C$

13. $y \ln x - xy' = 0$

$$\int \frac{dy}{y} = \int \frac{\ln x}{x}\,dx \quad \left(u = \ln x,\ du = \frac{dx}{x}\right)$$

$$\ln|y| = \frac{1}{2}(\ln x)^2 + C_1$$

$$y = e^{(1/2)(\ln x)^2 + C_1} = Ce^{(\ln x)^2/2}$$

14. $12yy' - 7e^x = 0$

$$12y\frac{dy}{dx} = 7e^x$$

$$\int 12y\,dy = \int 7e^x\,dx$$

$$6y^2 = 7e^x + C$$

15. $yy' - 2e^x = 0$

$$y\frac{dy}{dx} = 2e^x$$

$$\int y\,dy = \int 2e^x\,dx$$

$$\frac{y^2}{2} = 2e^x + C$$

Initial condition $(0, 6)$: $\frac{36}{2} = 2(1) + C \Rightarrow C = 16$

Particular solution: $\frac{y^2}{2} = 2e^x + 16$

$$y^2 = 4e^x + 32$$

16. $\sqrt{x} + \sqrt{y}\,y' = 0$

$$\int y^{1/2}dy = -\int x^{1/2}\,dx$$

$$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C_1$$

$$y^{3/2} + x^{3/2} = C$$

Initial condition $(1, 9)$:

$(9)^{3/2} + (1)^{3/2} = 27 + 1 = 28 = C$

Particular solution: $y^{3/2} + x^{3/2} = 28$

17. $y(x + 1) + y' = 0$

$$\int \frac{dy}{y} = -\int (x + 1)\,dx$$

$$\ln|y| = -\frac{(x + 1)^2}{2} + C_1$$

$$y = Ce^{-(x+1)^2/2}$$

Initial condition $(-2, 1)$: $1 = Ce^{-1/2}$, $C \overset{3}{=} e^{1/2}$

Particular solution: $y = e^{\left[1-(x+1)^2\right]/2} = e^{-\left(x^2+2x\right)/2}$

18. $2xy' - \ln x^2 = 0$

$$2x\frac{dy}{dx} = 2\ln x$$

$$\int dy = \int \frac{\ln x}{x}\,dx$$

$$y = \frac{(\ln x)^2}{2} + C$$

Initial condition $(1, 2)$: $2 = C$

Particular solution: $y = \frac{1}{2}(\ln x)^2 + 2$

19. $y(1 + x^2)y' = x(1 + y^2)$

$$\frac{y}{1 + y^2}\,dy = \frac{x}{1 + x^2}\,dx$$

$$\frac{1}{2}\ln(1 + y^2) = \frac{1}{2}\ln(1 + x^2) + C_1$$

$$\ln(1 + y^2) = \ln(1 + x^2) + \ln C = \ln\left[C(1 + x^2)\right]$$

$$1 + y^2 = C(1 + x^2)$$

Initial condition $(0, \sqrt{3})$: $1 + 3 = C \Rightarrow C = 4$

Particular solution: $1 + y^2 = 4(1 + x^2)$

$$y^2 = 3 + 4x^2$$

20. $y\sqrt{1 - x^2}\,\frac{dy}{dx} = x\sqrt{1 - y^2}$

$$\int (1 - y^2)^{-1/2}y\,dy = \int (1 - x^2)^{-1/2}x\,dx$$

$$-(1 - y^2)^{1/2} = -(1 - x^2)^{1/2} + C$$

Initial condition $(0, 1)$: $0 = -1 + C \Rightarrow C = 1$

Particular solution: $\sqrt{1 - y^2} = \sqrt{1 - x^2} - 1$

21. $\frac{du}{dv} = uv\sin v^2$

$$\int \frac{du}{u} = \int v\sin v^2\,dv$$

$$\ln|u| = -\frac{1}{2}\cos v^2 + C_1$$

$$u = Ce^{-\left(\cos v^2\right)/2}$$

Initial condition: $u(0) = 1$: $C = \frac{1}{e^{-1/2}} = e^{1/2}$

Particular solution: $u = e^{\left(1-\cos v^2\right)/2}$

22. $\frac{dr}{ds} = e^{r-2s}$

$\int e^{-r}\,dr = \int e^{-2s}\,ds$

$-e^{-r} = -\frac{1}{2}e^{-2s} + C$

Initial condition:

$r(0) = 0: -1 = -\frac{1}{2} + C \Rightarrow C = -\frac{1}{2}$

Particular solution:

$-e^{-r} = -\frac{1}{2}e^{-2s} - \frac{1}{2}$

$e^{-r} = \frac{1}{2}e^{-2s} + \frac{1}{2}$

$-r = \ln\left(\frac{1}{2}e^{-2s} + \frac{1}{2}\right) = \ln\left(\frac{1 + e^{-2s}}{2}\right)$

$r = \ln\left(\frac{2}{1 + e^{-2s}}\right)$

23. $dP - kP\,dt = 0$

$\int \frac{dP}{P} = k\int dt$

$\ln|P| = kt + C_1$

$P = Ce^{kt}$

Initial condition: $P(0) = P_0, P_0 = Ce^0 = C$

Particular solution: $P = P_0e^{kt}$

24. $dT + k(T - 70)\,dt = 0$

$\int \frac{dT}{T - 70} = -k\int dt$

$\ln(T - 70) = -kt + C_1$

$T - 70 = Ce^{-kt}$

Initial condition:

$T(0) = 140: 140 - 70 = 70 = Ce^0 = C$

Particular solution:

$T - 70 = 70e^{-kt}, T = 70(1 + e^{-kt})$

25. $y' = \frac{dy}{dx} = \frac{x}{4y}$

$\int 4y\,dy = \int x\,dx$

$2y^2 = \frac{x^2}{2} + C$

Initial condition $(0, 2)$: $2(2^2) = 0 + C \Rightarrow C = 8$

Particular solution: $2y^2 = \frac{x^2}{2} + 8$

$4y^2 - x^2 = 16$

26. $\frac{dy}{dx} = \frac{-9x}{16y}$

$\int 16y\,dy = -\int 9x\,dx$

$8y^2 = \frac{-9}{2}x^2 + C$

Initial condition $(1, 1)$: $8 = -\frac{9}{2} + C, C = \frac{25}{2}$

Particular solution: $8y^2 = \frac{-9}{2}x^2 + \frac{25}{2}$

$16y^2 + 9x^2 = 25$

27. $y' = \frac{dy}{dx} = \frac{y}{2x}$

$\int \frac{2}{y}\,dy = \int \frac{1}{x}\,dx$

$2\ln|y| = \ln|x| + C_1 = \ln|x| + \ln C$

$y^2 = Cx$

Initial condition $(9, 1)$: $1 = 9C \Rightarrow C = \frac{1}{9}$

Particular solution: $y^2 = \frac{1}{9}x$

$9y^2 - x = 0$

$y = \frac{1}{3}\sqrt{x}$

28. $\frac{dy}{dx} = \frac{2y}{3x}$

$\int \frac{3}{y}\,dy = \int \frac{2}{x}\,dx$

$\ln y^3 = \ln x^2 + \ln C$

$y^3 = Cx^2$

Initial condition $(8, 2)$: $2^3 = C(8^2), C = \frac{1}{8}$

Particular solution: $8y^3 = x^2, y = \frac{1}{2}x^{2/3}$

29. $m = \frac{dy}{dx} = \frac{0 - y}{(x + 2) - x} = -\frac{y}{2}$

$\int \frac{dy}{y} = \int -\frac{1}{2}\,dx$

$\ln|y| = -\frac{1}{2}x + C_1$

$y = Ce^{-x/2}$

30. $m = \dfrac{dy}{dx} = \dfrac{y-0}{x-0} = \dfrac{y}{x}$

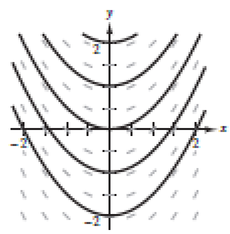
$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\ln y = \ln x + C_1 = \ln x + \ln C = \ln Cx$$

$$y = Cx$$

31. $\dfrac{dy}{dx} = x$

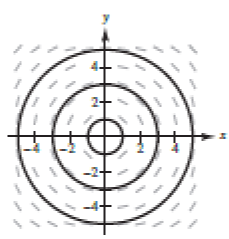
Graphs will vary.



$$y = \int x\,dx = \frac{1}{2}x^2 + C$$

32. $\dfrac{dy}{dx} = -\dfrac{x}{y}$

Graphs will vary.



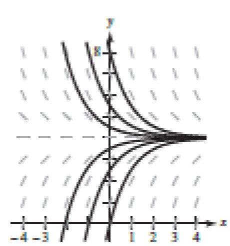
$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C_1$$

$$y^2 + x^2 = C$$

33. $\dfrac{dy}{dx} = 4 - y$

Graphs will vary.



$$\int \frac{dy}{4-y} = \int dx$$

$$\ln|4-y| = -x + C_1$$

$$4 - y = e^{-x+C_1}$$

$$y = 4 + Ce^{-x}$$

34. $\dfrac{dy}{dx} = 0.25x(4-y)$

$$\frac{dy}{4-y} = 0.25x\,dx$$

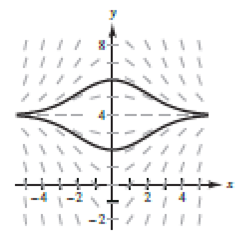
$$\int \frac{dy}{y-4} = \int -0.25x\,dx = -\frac{1}{4}\int x\,dx$$

$$\ln|y-4| = -\frac{1}{8}x^2 + C_1$$

$$y - 4 = e^{C_1 - (1/8)x^2} = Ce^{-(1/8)x^2}$$

$$y = 4 + Ce^{-(1/8)x^2}$$

Graphs will vary.



35. (a) Euler's Method gives $y \approx 0.1602$ when $x = 1$.

(b) $\dfrac{dy}{dx} = -6xy$

$$\int \frac{dy}{y} = \int -6x$$

$$\ln|y| = -3x^2 + C_1$$

$$y = Ce^{-3x^2}$$

$$y(0) = 5 \Rightarrow C = 5$$

$$y = 5e^{-3x^2}$$

(c) At $x = 1$, $y = 5e^{-3(1)} \approx 0.2489$.

Error: $0.2489 - 0.1602 \approx 0.0887$

36. (a) Euler's Method gives $y \approx 0.2622$ when $x = 1$.

(b) $\dfrac{dy}{dx} = -6xy^2$

$$\int \frac{dy}{y^2} = \int -6x\,dx$$

$$-\frac{1}{y} = -3x^2 + C_1$$

$$y = \frac{1}{3x^2 + C}$$

$$3 = \frac{1}{C} \Rightarrow C = \frac{1}{3}$$

$$y = \frac{1}{3x^2 + \frac{1}{3}} = \frac{3}{9x^2 + 1}$$

(c) At $x = 1$, $y = \dfrac{3}{9(1)+1} = \dfrac{3}{10} = 0.3$.

Error: $0.3 - 0.2622 = 0.0378$

37. (a) Euler's Method gives $y \approx 3.0318$ when $x = 2$.

(b) $\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$

$$\int (3y^2 - 4)\,dy = \int (2x + 12)\,dx$$

$$y^3 - 4y = x^2 + 12x + C$$

$$y(1) = 2: 2^3 - 4(2) = 1 + 12 + C \Rightarrow C = -13$$

$$y^3 - 4y = x^2 + 12x - 13$$

(c) At $x = 2$,

$$y^3 - 4y = 2^2 + 12(2) - 13 = 15$$

$$y^3 - 4y - 15 = 0$$

$$(y - 3)(y^2 + 3y + 5) = 0 \Rightarrow y = 3.$$

Error: $3.0318 - 3 = 0.0318$

38. (a) Euler's Method gives $y \approx 1.7270$ when $x = 1.5$.

(b) $\dfrac{dy}{dx} = 2x(1 + y^2)$

$$\int \frac{dy}{1 + y^2} = \int 2x\,dx$$

$$\arctan y = x^2 + C$$

$$\arctan(0) = 1^2 + C \Rightarrow C = -1$$

$$\arctan(y) = x^2 - 1$$

$$y = \tan(x^2 - 1)$$

(c) At $x = 1.5$, $y = \tan(1.5^2 - 1) \approx 3.0096$.

Error: $1.7270 - 3.0096 = -1.2826$

39. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial amount: $y(0) = y_0 = C$

Half-life: $\dfrac{y_0}{2} = y_0 e^{k(1599)}$

$$k = \frac{1}{1599}\ln\left(\frac{1}{2}\right)$$

$$y = Ce^{[\ln(1/2)/1599]t}$$

When $t = 50$, $y = 0.9786C$ or 97.86%.

40. $\dfrac{dy}{dt} = ky, \quad y = Ce^{kt}$

Initial conditions: $y(0) = 40$, $y(1) = 35$

$$40 = Ce^0 = C$$

$$35 = 40e^k$$

$$k = \ln\frac{7}{8}$$

Particular solution: $y = 40e^{t\ln(7/8)}$

When 75% has been changed:

$$10 = 40e^{t\ln(7/8)}$$

$$\frac{1}{4} = e^{t\ln(7/8)}$$

$$t = \frac{\ln(1/4)}{\ln(7/8)} \approx 10.38 \text{ hours}$$

41. (a) $\dfrac{dy}{dx} = k(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 4$; but not along $y = 0$. Matches (a).

42. (a) $\dfrac{dy}{dx} = k(x - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $x = 4$. Matches (b).

43. (a) $\dfrac{dy}{dx} = ky(y - 4)$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$ and $y = 4$. Matches (c).

44. (a) $\dfrac{dy}{dx} = ky^2$

(b) The direction field satisfies $(dy/dx) = 0$ along $y = 0$, and grows more positive as y increases. Matches (d).

45. (a) $$\frac{dw}{dt} = k(1200 - w)$$

$$\int \frac{dw}{1200 - w} = \int k\,dt$$

$$\ln|1200 - w| = -kt + C_1$$

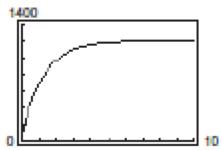
$$1200 - w = e^{-kt + C_1} = Ce^{-kt}$$

$$w = 1200 - Ce^{-kt}$$

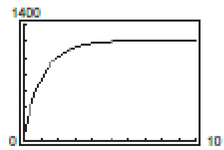
$$w(0) = 60 = 1200 - C \Rightarrow C = 1200 - 60 = 1140$$

$$w = 1200 - 1140e^{-kt}$$

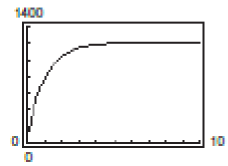
(b)



$k = 0.8$



$k = 0.9$



$k = 1$

(c) $k = 0.8$: $t = 1.31$ years

$k = 0.9$: $t = 1.16$ years

$k = 1.0$: $t = 1.05$ years

(d) Maximum weight: 1200 pounds

$$\lim_{x \to \infty} w = 1200$$

46. From Exercise 45:

$$w = 1200 - Ce^{-kt},\ k = 1$$

$$w = 1200 - Ce^{-t}$$

$$w(0) = w_0 = 1200 - C \Rightarrow C = 1200 - w_0$$

$$w = 1200 - (1200 - w_0)e^{-t}$$

47. $$\frac{dN}{dt} = kN(500 - N)$$

$$\int \frac{dN}{N(500 - N)} = \int k\,dt$$

$$\frac{1}{500}\int\left[\frac{1}{N} + \frac{1}{500 - N}\right]dN = \int k\,dt$$

$$\ln|N| - \ln|500 - N| = 500(kt + C_1)$$

$$\frac{N}{500 - N} = e^{500kt + C_2} = Ce^{500kt}$$

$$N = \frac{500Ce^{500kt}}{1 + Ce^{500kt}}$$

When $t = 0, N = 100$. So, $100 = \dfrac{500C}{1 + C} \Rightarrow C = 0.25$. Therefore, $N = \dfrac{125e^{500kt}}{1 + 0.25e^{500kt}}$.

When $t = 4, N = 200$. So, $200 = \dfrac{125e^{2000k}}{1 + 0.25e^{2000k}} \Rightarrow k = \dfrac{\ln(8/3)}{2000} \approx 0.00049$.

Therefore, $N = \dfrac{125e^{0.2452t}}{1 + 0.25e^{0.2452t}} = \dfrac{500}{1 + 4e^{-0.2452t}}$.

48. The differential equation is given by the following.

$$\frac{dS}{dt} = kS(L - S)$$

$$\int \frac{dS}{S(L - S)} = \int k\,dt$$

$$\frac{1}{L}\left[\ln|S| - \ln|L - S|\right] = kt + C_1$$

$$\frac{S}{L - S} = Ce^{Lkt}$$

$$S = \frac{CLe^{Lkt}}{1 + Ce^{Lkt}} = \frac{CL}{C + e^{-Lkt}}$$

When $t = 0, S = 10$. So, $C = \dfrac{10}{L - 10}$.

Therefore, $S = \dfrac{CL}{C + e^{-Lkt}} = \dfrac{\left[10/(L - 10)\right]L}{\left[10/(L - 10)\right] + e^{-Lkt}} = \dfrac{10L}{10 + (L - 10)e^{-Lkt}}$.

49. The general solution is $y = 1 - Ce^{-kt}$. Because $y = 0$ when $t = 0$, it follows that $C = 1$.

Because $y = 0.75$ when $t = 1$, you have

$$0.75 = 1 - e^{-k(1)}$$
$$-0.25 = -e^{-k}$$
$$0.25 = e^{-k}$$
$$\ln 0.25 = -k$$
$$k = \ln 0.25 = \ln 4 \approx 1.386.$$

So, $y \approx 1 - e^{-1.386t}$.

Note: This can be written as $y = 1 - 4^{-x}$.

50. The general solution is $y = 1 - Ce^{-kt}$. Because $y = 0$ when $t = 0$, it follows that $C = 1$.

Because $y = 0.9$ when $t = 2$, you have

$$0.9 = 1 - e^{-2k}$$
$$-0.1 = -e^{-2k}$$
$$0.1 = e^{-2k}$$
$$\ln 0.1 = -2k$$
$$k = -\frac{1}{2}\ln 0.1 = \frac{1}{2}\ln 10 \approx 1.151.$$

So, $y \approx 1 - e^{-1.151t}$.

Note: This can be written as $y = 1 - 10^{-x/2}$.

51. The general solution is $y = -\dfrac{1}{kt + C}$.

Because $y = 45$ when $t = 0$, it follows that

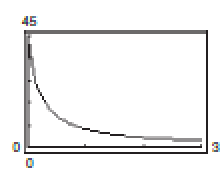
$45 = -\dfrac{1}{C}$ and $C = -\dfrac{1}{45}$.

Therefore, $y = -\dfrac{1}{kt - (1/45)} = \dfrac{45}{1 - 45kt}$.

Because $y = 4$ when $t = 2$, you have

$4 = \dfrac{45}{1 - 45k(2)} \Rightarrow k = -\dfrac{41}{360}$.

So, $y = \dfrac{45}{1 + (41/8)t} = \dfrac{360}{8 + 41t}$.



52. The general solution is $y = -1/(kt + C)$.

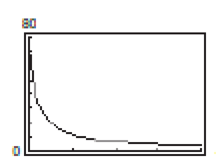
Because $y = 75$ when $t = 0$, you have $C = -1/75$.

So, $y = -\dfrac{1}{kt - (1/75)} = \dfrac{75}{1 - 75kt}$.

Because $y = 12$ when $t = 1$, you have

$12 = \dfrac{75}{1 - 75k} \Rightarrow k = -\dfrac{7}{100}$.

So, you have $y = \dfrac{75}{1 + 5.25t} = \dfrac{300}{4 + 21t}$.



53. Because $y = 100$ when $t = 0$, it follows that $100 = 500e^{-C}$, which implies that $C = \ln 5$.

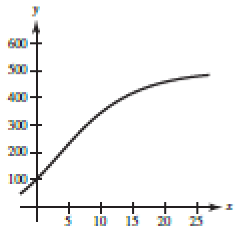
So, you have $y = 500e^{(-\ln 5)e^{-kt}}$. Because $y = 150$ when $t = 2$, it follows that

$$150 = 500e^{(-\ln 5)e^{-2k}}$$

$$e^{-2k} = \frac{\ln 0.3}{\ln 0.2}$$

$$k = -\frac{1}{2}\ln\frac{\ln 0.3}{\ln 0.2}$$

$$\approx 0.1452.$$



So, y is given by

$$y = 500e^{-1.6904e^{-0.1451t}}.$$

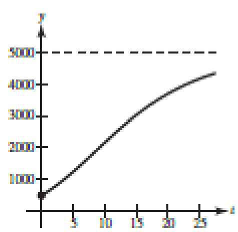
54. The general solution is $y = 5000e^{-Ce^{-kt}}$. Because $y = 500$ when $t = 0$, it follows that $500 = 5000e^{-C}$ which implies that $C = -\ln\frac{1}{10} = \ln 10$. So, you have $y = 5000e^{(-\ln 10)e^{-kt}}$. Because $y = 625$ when $t = 1$, it follows that

$$625 = 5000e^{(-\ln 10)e^{-k}}$$

$$e^{-k} = \frac{\ln(1/8)}{\ln(1/10)}$$

$$k = -\ln\left(\frac{\ln(1/8)}{\ln(1/10)}\right)$$

$$\approx 0.1019.$$



So, you have $y = 5000e^{(-2.3026)e^{(-0.1019)t}}$.

55. From Example 7, the general solution is $y = 60e^{-Ce^{-kt}}$.

Because $y = 8$ when $t = 0$,

$$8 = 60e^{-C} \Rightarrow C = \ln\frac{15}{2} \approx 2.0149.$$

Because $y = 15$ when $t = 3$,

$$15 = 60e^{-2.0149e^{-3k}}$$

$$\frac{1}{4} = e^{-2.0149e^{-3k}}$$

$$\ln\frac{1}{4} = -2.0149e^{-3k}$$

$$k = -\frac{1}{3}\ln\left(\frac{\ln(1/4)}{-2.0149}\right) \approx 0.1246.$$

So, $y = 60e^{-2.0149e^{-0.1246t}}$.

When $t = 10$, $y \approx 34$ beavers.

56. From Example 7, the general solution is $y = 400e^{-Ce^{-kt}}$.

Because $y = 30$ when $t = 0$,

$$30 = 400e^{-C} \Rightarrow C = \ln\left(\frac{40}{3}\right) \approx 2.5903.$$

Because $y = 90$ when $t = 1$,

$$90 = 400e^{-2.5903e^{-k}}$$

$$\frac{9}{40} = e^{-2.5903e^{-k}}$$

$$\ln\left(\frac{9}{40}\right) = -2.5903e^{-k}$$

$$k = -\ln\left(\frac{\ln(9/40)}{-2.5903}\right) \approx 0.5519.$$

So, $y = 400e^{-2.5903e^{-0.5519t}}$.

Finally, when $t = 3$, $y \approx 244$ rabbits.

57. (a)

$$\frac{dQ}{dt} = -\frac{Q}{20}$$

$$\int\frac{dQ}{Q} = \int -\frac{1}{20}dt$$

$$\ln|Q| = -\frac{1}{20}t + C_1$$

$$Q = e^{-(1/20)t + C_1} = Ce^{-(1/20)t}$$

Because $Q = 25$ when $t = 0$, you have $25 = C$.

So, the particular solution is $Q = 25e^{-(1/20)t}$.

(b) When $Q = 15$, you have $15 = 25e^{-(1/20)t}$.

$$\frac{3}{5} = e^{-(1/20)t}$$

$$\ln\left(\frac{3}{5}\right) = -\frac{1}{20}t$$

$$-20\ln\left(\frac{3}{5}\right) = t$$

$$t \approx 10.217 \text{ minutes}$$

58. Because $Q' + \frac{1}{20}Q = \frac{5}{2}$ is a first-order linear differential equation with $P(x) = \frac{1}{20}$ and $R(x) = \frac{5}{2}$, you have the integrating factor $u(t) = e^{\int(1/20)dt} = e^{(1/20)t}$, and the general solution is

$Q = e^{-0.05t}\int\frac{5}{2}e^{0.05t}\,dt = e^{-0.05t}\left(50e^{0.05t} + C\right) = 50 + Ce^{-0.05t}.$

Because $Q = 0$ when $t = 0$, you have $C = -50$ and $Q = 50\left(1 - e^{-0.05t}\right)$.

Finally, when $t = 30$, you have $Q \approx 38.843$ lb/gal.

59. (a) $\frac{dy}{dt} = ky$

$\int\frac{dy}{y} = \int k\,dt$

$\ln y = kt + C_1$

$y = e^{kt+C_1} = Ce^{kt}$

(b) $y(0) = 20 \Rightarrow C = 20$

$y(1) = 16 = 20e^{k} \Rightarrow k = \ln\frac{16}{20} = \ln\left(\frac{4}{5}\right)$

$y = 20e^{t\,\ln(4/5)}$

When 75% has changed:

$5 = 20e^{t\,\ln(4/5)}$

$\frac{1}{4} = e^{t\,\ln(4/5)}$

$t = \frac{\ln(1/4)}{\ln(4/5)} \approx 6.2$ hours

60. $\frac{ds}{dh} = \frac{k}{h}$

$\int ds = \int\frac{k}{h}\,dh$

$s = k\ln h + C_1 = k\ln Ch$

Because $s = 25$ when $h = 2$ and $s = 12$ when $h = 6$, it follows that $25 = k\ln(2C)$ and $12 = k\ln(6C)$, which implies

$C = \frac{1}{2}e^{-(25/13)\ln 3} \approx 0.0605$

and

$k = \frac{25}{\ln(2C)} = \frac{-13}{\ln 3} \approx -11.8331.$

Therefore, s is given by the following.

$$\begin{aligned} s &= -\frac{13}{\ln 3}\ln\left[\frac{h}{2}e^{-(25/13)\ln 3}\right] \\ &= -\frac{13}{\ln 3}\left[\ln\frac{h}{2} - \frac{25}{13}\ln 3\right] \\ &= -\frac{1}{\ln 3}\left[13\ln\frac{h}{2} - 25\ln 3\right] \\ &= 25 - \frac{13\ln(h/2)}{\ln 3},\quad 2 \le h \le 15 \end{aligned}$$

61. The general solution is $y = Ce^{kt}$. Because $y = 0.60C$ when $t = 1$, you have

$0.60C = Ce^{k} \Rightarrow k = \ln 0.60 \approx -0.5108.$

So, $y = Ce^{-0.5108t}$. When $y = 0.20C$, you have

$0.20C = Ce^{-0.5108t}$

$\ln 0.20 = -0.5108t$

$t \approx 3.15$ hours.

62. $\int\left(\frac{1}{y}\frac{dy}{dt}\right)dt = \int\left(\frac{1}{x}\frac{dx}{dt}\right)dt$

$\int\frac{1}{y}dy = \int\frac{1}{x}dx$

$\ln|y| = \ln|x| + C_1 = \ln|Cx|$

$y = Cx$

63. $\frac{dA}{dt} = rA + P$

$\frac{dA}{rA + P} = dt$

$\int\frac{dA}{rA + P} = \int dt$

$\frac{1}{r}\ln(rA + P) = t + C_1$

$\ln(rA + P) = rt + C_2$

$rA + P = e^{rt+C_2}$

$A = \frac{C_3e^{rt} - P}{r}$

$A = Ce^{rt} - \frac{P}{r}$

When $t = 0$: $A = 0$

$0 = C - \frac{P}{r} \Rightarrow C = \frac{P}{r}$

$A = \frac{P}{r}\left(e^{rt} - 1\right)$

64. $A = \frac{P}{r}\left(e^{rt} - 1\right)$

$A = \frac{275{,}000}{0.06}\left(e^{0.08(10)} - 1\right) \approx \$4{,}212{,}796.94$

65. From Exercise 63,

$$A = \frac{P}{r}\left(e^{rt} - 1\right).$$

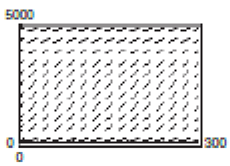
Because $A = 260{,}000{,}000$ when $t = 8$ and $r = 0.0725$, you have

$$P = \frac{Ar}{e^{rt} - 1} = \frac{(260{,}000{,}000)(0.0725)}{e^{(0.0725)(8)} - 1} \approx \$23{,}981{,}015.77.$$

66. $$1{,}000{,}000 = \frac{125{,}000}{0.08}\left(e^{0.08t} - 1\right)$$
$$1.64 = e^{0.08t}$$
$$t = \frac{\ln(1.64)}{0.08} \approx 6.18 \text{ years}$$

67. $$\frac{dy}{dt} = 0.02y\ln\left(\frac{5000}{y}\right)$$

(a)



(b) As $t \to \infty$, $y \to L = 5000$.

(c) Using a computer algebra system or separation of variables, the general solution is

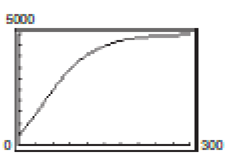
$$y = 5000e^{-Ce^{-kt}} = 5000e^{-Ce^{-0.02t}}.$$

Using the initial condition $y(0) = 500$, you obtain

$$500 = 5000e^{-C} \Rightarrow C = \ln 10 \approx 2.3026.$$

So, $y = 5000e^{-2.3026e^{-0.02t}}$.

(d)



The graph is concave upward on $(0, 41.7)$ and concave downward on $(41.7, \infty)$.

68. A differential equation can be solved by separation of variables if it can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

To solve a separable equation, rewrite as,

$$M(x)\,dx = -N(y)\,dy$$

69. $$y(1 + x)\,dx + x\,dy = 0$$
$$x\,dy = -y(1 + x)\,dx$$
$$\frac{1}{y}\,dy = -\frac{1 + x}{x}\,dx$$

Separable

70. $$y' = \frac{dy}{dx} = y^{1/2}$$
$$\frac{dy}{y^{1/2}} = dx$$

Separable

71. $$\frac{dy}{dx} + xy = 5$$

Not separable

72. $$\frac{dy}{dx} = x - xy - y + 1$$
$$\frac{dy}{dx} = x(1 - y) + (1 - y)$$
$$\frac{dy}{dx} = (x + 1)(1 - y)$$
$$\frac{dy}{1 - y} = (x + 1)\,dx$$

Separable

73. (a) $$\frac{dv}{dt} = k(W - v)$$
$$\int \frac{dv}{W - v} = \int k\,dt$$
$$-\ln|W - v| = kt + C_1$$
$$v = W - Ce^{-kt}$$

Initial conditions:

$W = 20, v = 0$ when $t = 0$ and $v = 10$ when $t = 0.5$ so, $C = 20, k = \ln 4$.

Particular solution:

$$v = 20\left(1 - e^{-(\ln 4)t}\right) = 20\left(1 - \left(\frac{1}{4}\right)^t\right)$$

or

$$v = 20\left(1 - e^{-1.386t}\right)$$

(b) $s = \int 20\left(1 - e^{-1.386t}\right)dt \approx 20\left(t + 0.7215e^{-1.386t}\right) + C$

Because $s(0) = 0, C \approx -14.43$ and you have

$$s \approx 20t + 14.43\left(e^{-1.386t} - 1\right).$$

74. Use the y-intercepts to match the graphs with the appropriate value of C.

For graph (a), the y-intercept is $(0, 6)$, so $C = 3$.

For graph (b), the y-intercept is $(0, 4)$, so $C = 2$.

For graph (c), the y-intercept is $(0, 2)$, so $C = 1$.

75. False. $\frac{dy}{dx} = \frac{x}{y}$ is separable, but $y = 0$ is not a solution.

76. True

$$\frac{dy}{dx} = (x - 2)(y + 1)$$

77.
$$\frac{dy}{dx} = \frac{y^2 + 1}{x + 2}$$
$$\frac{dy}{y^2 + 1} = \frac{dx}{x + 2}$$
$$\int \frac{dy}{y^2 + 1} = \int \frac{dx}{x + 2}$$
$$\arctan y = \ln|x + 2| + C$$
$$y = \tan[\ln|x + 2| + C]$$
$$y = \tan[\ln(x + 2) + C]$$

So, the answer is A.

78.
$$\frac{dy}{dt} = y \sec^2 t$$
$$\frac{dy}{y} = \sec^2 t\, dt$$
$$\int \frac{dy}{y} = \int \sec^2 t\, dt$$
$$\ln|y| = \tan t + C_1$$
$$y = e^{\tan t + C_1}$$
$$y = Ce^{\tan t}$$

Use $t = 0, y = 4$, and $y = Ce^{\tan t}$ to find C.

$$4 = Ce^{\tan 0}$$
$$4 = C$$

The equation is $y = 4e^{\tan t}$.

So, the answer is B.

79. (a)
$$\frac{dy}{dx} = \frac{y - 4}{x^2}$$
$$\frac{dy}{y - 4} = \frac{dx}{x^2}$$
$$\int \frac{dy}{y - 4} = \int \frac{dx}{x^2}$$
$$\ln|y - 4| = -\frac{1}{x} + C_1$$
$$y - 4 = e^{(-1/x) + C_1}$$
$$y = 4 + Ce^{-1/x}$$

Use $y = 4 + Ce^{-1/x}$ to find C when $f(3) = 0$.

$$0 = 4 + Ce^{-1/3}$$
$$C = -4e^{1/3}$$

So, $y = 4 + \left(-4e^{1/3}\right)\left(e^{-1/x}\right)$

$$= -4e^{(-1/x) + (1/3)} + 4.$$

(b)
$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(-4e^{(-1/x) + (1/3)} + 4\right)$$
$$= \lim_{x \to \infty} -4e^{(-1/x) + (1/3)} + \lim_{x \to \infty} 4$$
$$= -4e^{1/3} + 4$$