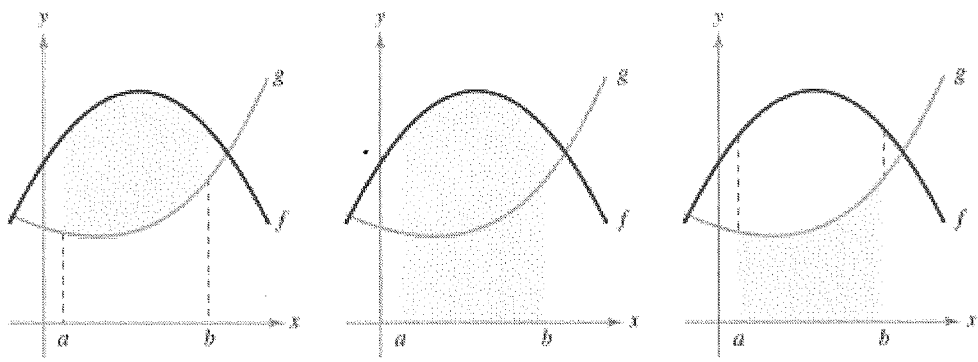
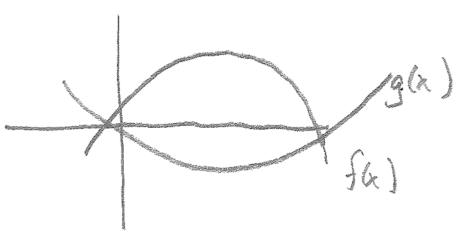
6.1 Area of a Region Between Two Curves

If f and g are continuous on $[a, b]$ and $g(x) \le f(x)$ for all x in $[a, b]$, then the area of the region bounded by the graphs of f and g and the vertical lines $x = a$ and $x = b$ is



Area Between $= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

But what happens if it goes below the $x -$ axis?



The area below the x-axis would be evaluated as negative, when it is subtracted it would become positive, so the two areas would be added.

Finding the area using vertical rectangles:

$$\int_a^b [f(x) - g(x)]\,dx$$

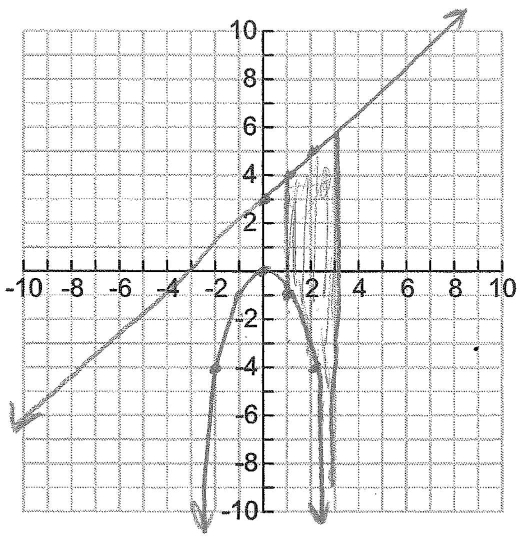
* Upper - lower

Finding the area using horizontal rectangles:

$$\int_a^b [f(y) - g(y)]\,dy$$

* Left - Right

Find the area of the region bounded by the graphs of $y = x + 3$, $y = -x^2$, $x = 1$, and $x = 3$.



$$\int_1^3 x+3-(-x^2)\,dx$$

$$\int_1^3 x^2+x+3\,dx$$

$$\frac{1}{3}x^3+\frac{1}{2}x^2+3x\Big|_1^3$$

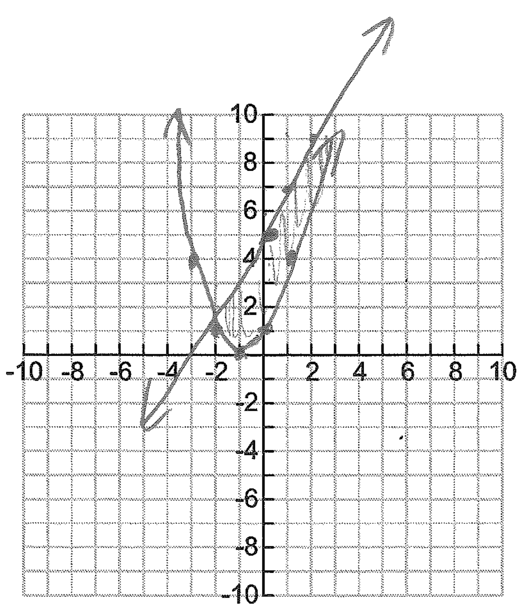
$$\left(\frac{1}{3}(27)+\frac{1}{2}(9)+9\right)-\left(\frac{1}{3}+\frac{1}{2}+3\right)$$

$$9+\frac{9}{2}+9-\frac{1}{3}-\frac{1}{2}-3$$

$$15+4-\frac{1}{3} = 19-\frac{1}{3} = \frac{57}{3}-\frac{1}{3}$$

$$=\frac{56}{3}$$

Find the area of the region bounded by the graphs of $f(x) = x^2 + 2x + 1$ and $g(x) = 2x + 5$.



$$f(x) = (x+1)(x+1)$$

$$x^2+2x+1 = 2x+5$$

$$x^2-4=0$$

$$x=\pm 2$$

$$\int_{-2}^{2} (2x+5)-(x^2+2x+1)\,dx$$

$$\int_{-2}^{2} 2x+5-x^2-2x-1\,dx$$

$$\int_{-2}^{2} -x^2+4\,dx = -\frac{1}{3}x^3+4x\Big|_{-2}^{2}$$

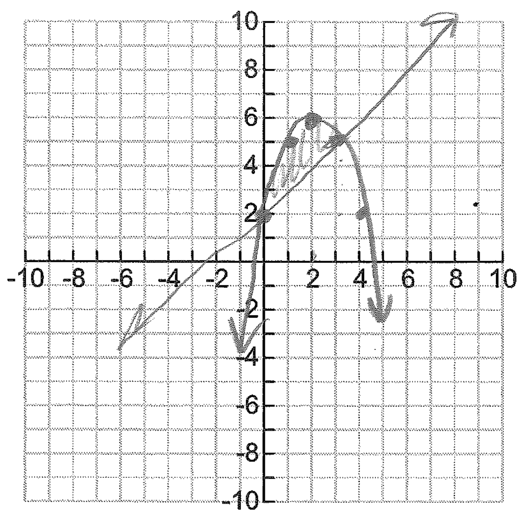
$$\left(-\frac{1}{3}(8)+8\right)-\left(-\frac{1}{3}(-8)-8\right)$$

$$-\frac{8}{3}+8-\frac{8}{3}+8 = 16-\frac{16}{3} = \frac{48-16}{3} = \frac{32}{3}$$

Examples – Calculating Areas Between Two Curves

Find the area of the region bound by $f(x) = -x^2 + 4x + 2$ and $g(x) = x + 2$

$\frac{-4}{2(-1)} = \frac{-4}{-2} = 2 \qquad f(2) = -4 + 8 + 2 = 6$



$-x^2 + 4x + 2 = x + 2$

$0 = x^2 - 3x$

$0 = x(x-3) \qquad x = 0 \qquad x = 3$

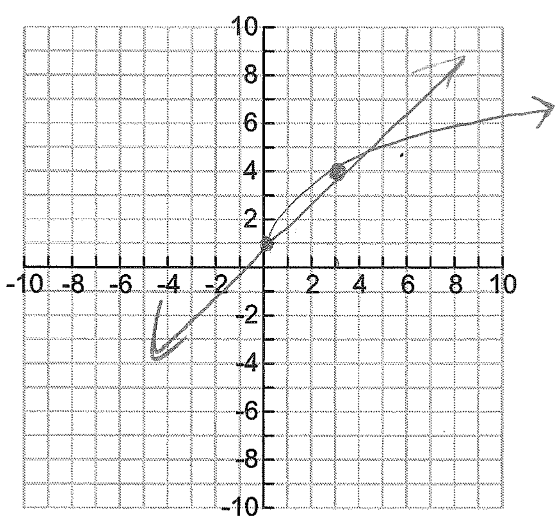
$\int_0^3 (-x^2+4x+2) - (x+2)\, dx$

$\int_0^3 -x^2 + 3x\, dx = -\frac{1}{3}x^3 + \frac{3}{2}x^2 \Big|_0^3$

$-\frac{1}{3}(27) + \frac{3}{2}(9) - 0 = -9 + \frac{27}{2}$

$= \frac{-18}{2} + \frac{27}{2} = \frac{9}{2}$

Find the area of the region bound by $f(x) = \sqrt{3x} + 1$ and $g(x) = x + 1$



$\sqrt{3x} + 1 = x + 1$

$\sqrt{3x} = x$

$3x = x^2$

$x^2 - 3x = 0$

$x(x-3) = 0$

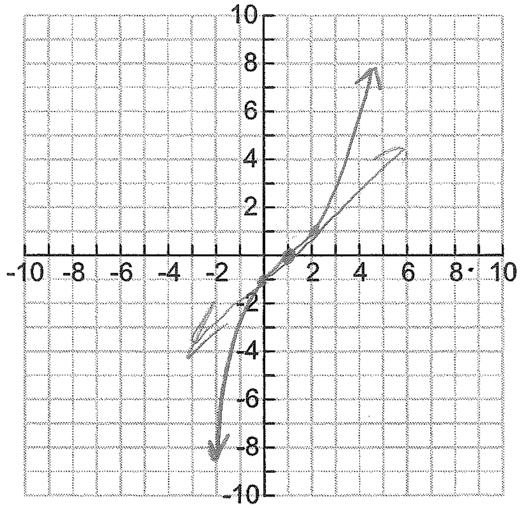
$\int_0^3 \sqrt{3x} + 1 - (x+1)\, dx$

$\int_0^3 \sqrt{3x} - x\, dx = \frac{1}{3}\frac{(3x)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{1}{2}x^2 \Big|_0^3$

$\frac{2}{9}(3x)^{\frac{3}{2}} - \frac{1}{2}x^2 = \frac{2}{3}(9)^{\frac{3}{2}} - \frac{1}{2}(9) = \frac{54}{3} - \frac{9}{2}$

$= \frac{36}{2} - \frac{9}{2} = \frac{3}{2}$

Find the area of the region bound by $f(x) = (x-1)^3$ and $g(x) = x - 1$



$$\int_0^1 (x-1)^3 - (x-1)\,dx + \int_1^2 (x-1) - (x-1)^3\,dx$$

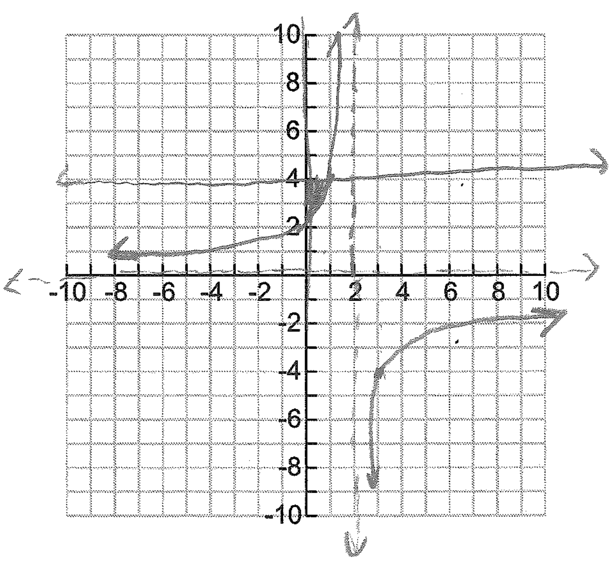
$$\frac{1}{4}(x-1)^4 - \frac{1}{2}x^2 + x \Big|_0^1 + \frac{1}{2}x^2 - x - \frac{1}{4}(x-1)^4 \Big|_1^2$$

$$\left(0 - \frac{1}{2} + 1\right) - \left(\frac{1}{4}\right) + \left(2 - 2 - \frac{1}{4}\right) - \left(\frac{1}{2} - 1 - 0\right)$$

$$-\frac{1}{2} + 1 - \frac{1}{4} - \frac{1}{4} - \frac{1}{2} + 1$$

$$-1 + 1 - \frac{1}{2} + 1 = \frac{1}{2}$$

Find the area of the region bound by $f(x) = \frac{4}{2-x}$, $g(x) = 4$ and $x = 0$.



$$\frac{4}{2-x} = 4$$

$$4 = 8 - 4x$$

$$-4 = -4x$$

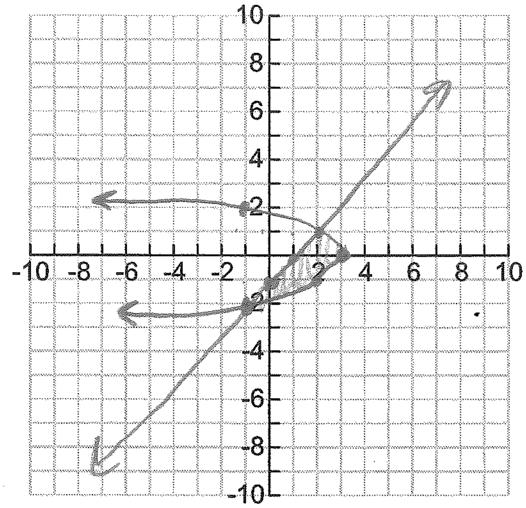
$$1 = x$$

$$\int_0^1 \left(4 - \frac{4}{2-x}\right)dx = 4x + 4\ln|2-x| \Big|_0^1$$

$$(4 + 4\ln 1) - (0 + 4\ln 2)$$

$$4 - 4\ln 2$$

Find the area of the region bound by the graphs of $x = 3 - y^2$ and $x = y + 1$



$y^2 = 3 - x$ $\quad$ $y = x - 1$ $\quad$ $3 - y^2 = y + 1$

$y^2 = -1(x-3)$ $\quad$ $y^2 + y - 2 = 0$

$(y+2)(y-1) = 0$

$y = -2 \quad y = 1$

$$\int_{-2}^{1} (3-y^2) - (y+1)\,dy$$

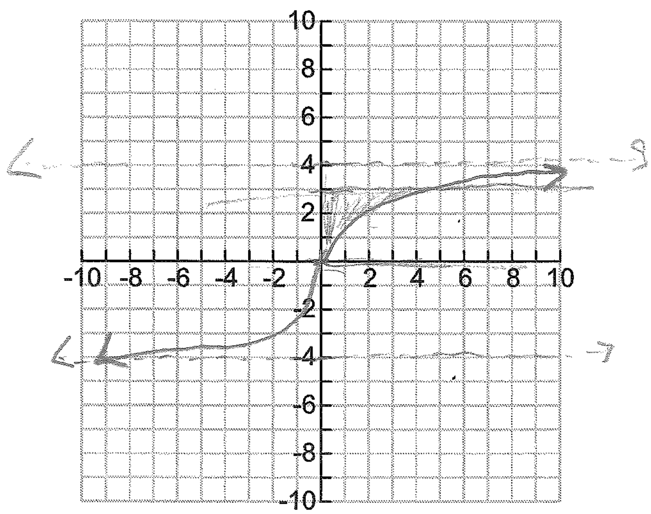
$$\int_{-2}^{1} -y^2 - y + 2\,dy = -\frac{1}{3}y^3 - \frac{1}{2}y^2 + 2y \Big|_{-2}^{1}$$

$$\left[-\frac{1}{3}(1) - \frac{1}{2}(1) + 2\right] - \left[-\frac{1}{3}(-8) - \frac{1}{2}(4) + 2(-2)\right]$$

$$-\frac{1}{3} - \frac{1}{2} + 2 - \frac{8}{3} + 2 + 4 = -\frac{2}{6} - \frac{3}{6} - \frac{16}{6} + \frac{48}{6} = \frac{27}{6}$$

Find the area of the region bound by the graphs $x = \frac{y}{\sqrt{16-y^2}}$, $x = 0$ and $y = 3$.

x	y



$$\int_{0}^{3} \frac{y}{\sqrt{16-y^2}} - 0\,dy$$

$u = 16 - y^2$

$du = -2y\,dy$

$dy = \frac{du}{-2y}$

$$-\frac{1}{2}\left(\frac{(16-y^2)^{\frac{1}{2}}}{\frac{1}{2}}\right)\Big|_{0}^{3}$$

$$-1(16-y^2)^{\frac{1}{2}}\Big|_{0}^{3}$$

$$-1(16-9)^{\frac{1}{2}} - (-1)(16-0)^{\frac{1}{2}} = -(7)^{\frac{1}{2}} + 4$$

Accumulation

Given $F(y) = \int_{-1}^{y} 4e^{\frac{x}{2}}\,dx$ find $F(-1)$ and $F(4)$

$$F(-1) = \int_{-1}^{-1} 4e^{\frac{x}{2}}\,dx = 0$$

$$F(4) = \int_{-1}^{4} 4e^{\frac{x}{2}}\,dx$$

Using your calculator