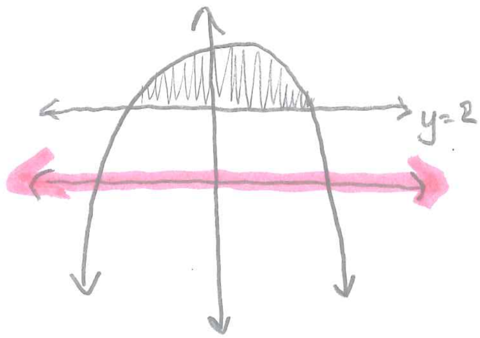
6.2 Volume: The Disk and Washer Method (Day 2)

Consider the following problem:

Find the volume of the solid formed by revolving the region bound by $y = 4 - \frac{x^2}{4}$ and $y = 2$ about the $x - axis$.

How is this problem different than the last problem that we did in our notes on Day 1?



There is a gap between our figure and our axis of revolution.

The Washer Method

$$\pi \int_a^b [\text{Big Radius}]^2 - [\text{Small Radius}]^2 \, dx$$

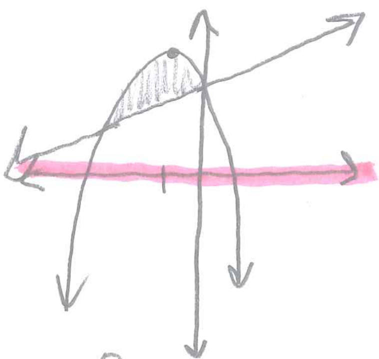
$$\pi \int_a^b [f(x) - \text{axis}]^2 - [g(x) - \text{axis}]^2 \, dx \quad \text{OR} \quad \pi \int_a^b [\text{axis} - f(x)]^2 - [\text{axis} - g(x)]^2 \, dx$$

Hint:

- Remember π
- Remember to square the radius functions
- Remember it's always big - small

Examples – Volumes of solids of revolution

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 6 - 2x - x^2$ and $y = x + 6$ about the $x - axis$.



$6 - 2x - x^2 = x + 6$

$0 = x^2 + 3x$

$0 = x(x+3)$

$\frac{2}{2(-1)} = -1$

$y = 6 + 2 - 1 = 7$

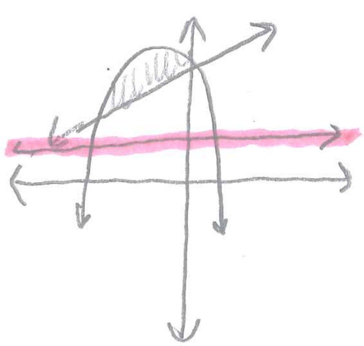
$\pi \int_{-3}^{0} [6 - 2x - x^2 - 0]^2 - [x + 6 - 0]^2\, dx$

$= \pi \int_{-3}^{0} 36 - 12x - 6x^2 - 12x + 4x^2 + 2x^3 - 6x^2 + 2x^3 + x^4 - (x^2 + 12x + 36)\, dx$

$= \pi \int_{-3}^{0} -36x - 9x^2 + 4x^3 + x^4\, dx = \pi \left[-36\left(\frac{x^2}{2}\right) - 9\left(\frac{x^3}{3}\right) + 4\left(\frac{x^4}{4}\right) + \frac{1}{5}x^5\right]_{-3}^{0}$

$= 0 - [-18(9) - 3(-27) + 81 + \frac{1}{5}(-243)]\pi = \frac{243}{5}\pi$

Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 6 - 2x - x^2$ and $y = x + 6$ about the line $y = 3$.



$\pi \int_{-3}^{0} (6 - 2x - x^2 - 3)^2 - (x + 6 - 3)^2\, dx$

$\pi \int_{-3}^{0} (3 - 2x - x^2)^2 - (x+3)^2\, dx =$

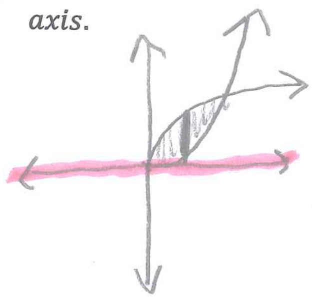
$\pi \int_{-3}^{0} 9 - 6x - 3x^2 - 6x + 4x^2 + 2x^3 - 3x^2 + 2x^3 + x^4 - (x^2 + 6x + 9)\, dx$

$\pi \int_{-3}^{0} -18x - 3x^2 + 4x^3 + x^4\, dx = \pi \left[-18\left(\frac{x^2}{2}\right) - 3\left(\frac{x^3}{3}\right) + 4\left(\frac{x^4}{4}\right) + \frac{1}{5}x^5\right]_{-3}^{0}$

$= \pi \left[-9x^2 - x^3 + x^4 + \frac{1}{5}x^5\right]_{-3}^{0} = 0 - [-9(9) + 27 + 81 + \frac{1}{5}(-243)]\pi$

$= \frac{108}{5}\pi$

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \sqrt{x}$ and $y = x^2$ about the $x-axis$.



$\sqrt{x} = x^2$

$x^2 = x^4$

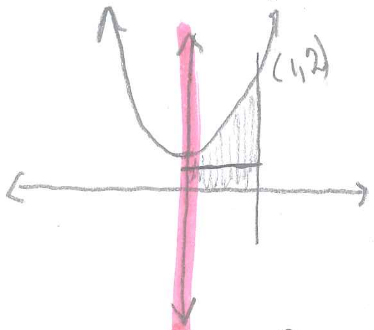
$0 = x^4 - x^2$

$0 = x^2(x^2 - 1)$

$\pi \int_0^1 (\sqrt{x}-0)^2 - (x^2-0)^2 dx = \pi \int_0^1 x - x^4 dx = \pi \left[\frac{1}{2}x^2 - \frac{1}{5}x^5\right]\Big|_0^1$

$\pi\left[\frac{1}{2} - \frac{1}{5}\right] = \left[\frac{5}{10} - \frac{2}{10}\right]\pi = \frac{3}{10}\pi$

Find the volume of the solid formed by revolving the region bounded by the graphs of $y = x^2 + 1$, $y = 0$, $x = 0$ and $x = 1$ about the $y - axis$.



$y - 1 = x^2$

$x = \pm\sqrt{y-1}$

$\pi \int_0^1 (1-0)^2 dy + \pi \int_1^2 (1-0)^2 - (\sqrt{y-1} - 0)^2 dy$

$\pi \int_0^1 1 dy + \pi \int_1^2 1 - (y-1) dy$

$\pi (y)\Big|_0^1 + \pi \int_1^2 -y + 2 dy = \pi (y)\Big|_0^1 + -\frac{1}{2}y^2 + 2y \Big|_1^2 (\pi)$

$\pi(1-0) + \pi\left[\left(-\frac{1}{2}(4) + 4\right) - \left(-\frac{1}{2} + 2\right)\right] = \pi + \pi\left[-2 + 4 - \frac{3}{2}\right]$

$= \pi + \pi\left[\frac{1}{2}\right] = \frac{3}{2}\pi$