Deriving Newtons Laws from the Principle of Least Action

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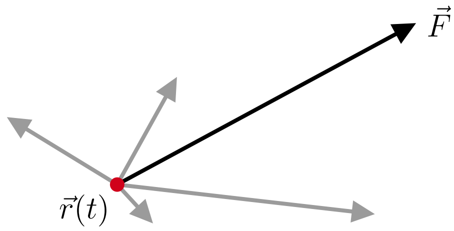
1 Newtonian Mechanics

In the last 250 years, classical mechanics has been completely described from a single principle. This principle is known as *the principle of least action*, and was developed by William Rowan Hamilton. In this section, this principle will be examined, starting by considering Newtons laws.
Consider a single particle at position $\vec{r}(t)$, acted on by a force $\vec{F}$. This force can be thought of as the sum of many different forces.



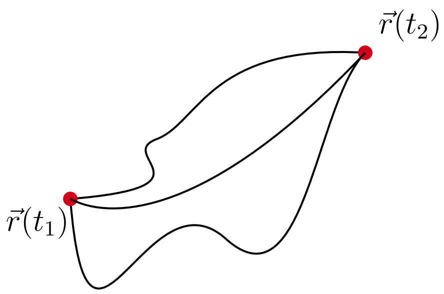
Newtonian mechanics states that

$$\vec{F} = m\vec{a} = m\ddot{\vec{r}}.$$

It is the aim of classical mechanics to solve this equation for forces such as gravity, friction and so on. This equation is a second-order differential equation, thus it's solution has two constants of integration. These constants correspond with the initial position $\vec{r}(t_1)$ and initial velocity $\dot{\vec{r}}(t_1)$ of the particle being considered.

2 Principle Of Least Action

Instead of specifying the initial position and velocity, consider instead the initial and final positions, $\vec{r}(t_1)$ and $\vec{r}(t_2)$. Consider also all of the possible paths that could connect them (here a few are shown):



What path does the particle take? That is the question that the *principle of least action* addresses. First, let me introduce the *action*

DEFINITION

Action.

For each path $\vec{r}(t)$*, the* action *is*

$$S_{\vec{r}(t)} = \int_{t_1}^{t_2} T - V \, dt,$$

Where T *is the kinetic energy, and* V *is the potential energy.*

This may look somewhat arbitrary, but it comes together with the following theorem:

THEOREM 2.1

Principle of Least Action.

The true path taken by the particle is an extremum of the action S*.*

The finding of an extremum of a function is well known, and comes from elementary differential calculus. However, if you examine the action you will see that it is an integral, and thus a function of a function. This is known as a *functional*, and is solved using a branch of maths called the 'calculus of variations'. The proof of this theorem is above the level of this document.

3 The Lagrangian

If you look at the definition of the action, there is a special name for the integrand.

DEFINITION

Lagrangian.

The integrand of the action is called the Lagrangian L*, where*

$$L(\vec{r}, \dot{\vec{r}}, t) = T - V$$

Thus $S = \int_{t_1}^{t_2} L \, dt$*.*

In a simple example, the kinetic energy is $T = \frac{1}{2}m\dot{\vec{r}}^2$ and potential energy is $V(\vec{r})$. From this we can define a generalized momentum $\vec{P}$ and a generalized force $\vec{F}$,

$$\vec{P} = \frac{\partial L}{\partial \dot{\vec{r}}}$$
$$\vec{F} = \frac{\partial L}{\partial \vec{r}}.$$

Following from this, we can rewrite Newton's equations as

$$\frac{d\vec{P}}{dt} = \vec{F} \implies \frac{d}{dt}(\frac{\partial L}{\partial \dot{\vec{r}}}) = \frac{\partial L}{\partial \vec{r}}.$$

This is known as the *Euler-Lagrange equation.*