ALGEBRA 1

LEAVING CERT HIGHER LEVEL MATHS

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These notes are not endorsed by any teachers, and I have modified them and added my own details to these after the classes. All errors are almost surely mine.

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1 POLYNOMIAL EXPRESSIONS

Lecture 1
28^{th} August, 2018

Definition 1.1 (Polynomial). *A **polynomial** is an expression of the form:*

$$a_x x^n + a_{n-1}x^{n-1} + \cdots + a^2x^2 + a_1x + a_0$$

Where $a_i \in \mathbb{R}$; $i = 0, 1, 2, \ldots, n$, $n \in \mathbf{N}$.

Definition 1.2 (Degree). *The **degree** of a polynomial is the highest power of its monomials with non-zero coefficients.*

There are a few types of polynomial which are useful to know:

a) A **linear** polynomial is of degree 1 ($ax + b$)

b) A **quadratic** polynomial is of degree 2 ($ax^2 + bx + c$)

c) A **cubic** polynomial is of degree 3 ($ax^3 + bx^2 + cx + d$)

Definition 1.3 (Coefficent). *A quantity that multiplies the variable in an algebraic expression (for example 4 in $4x^3$).*

Definition 1.4 (Constant Term). *A term in an algebraic expression that does not contain any modifiable variables.*

Definition 1.5 (Binomial). *An polynomial containing two terms.*

For more visit `github.com/adamisntdead/notes`.

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Definition 1.6 (Trinomial). *An polynomial containing three terms.*

1.1 *Addition And Subtraction Of Polynomial Expressions*

Adding and subtracting polynomials involves the combination of like terms.

Definition 1.7 (Like terms). *Monomials that contain the same variable raised to the same power.*

Example 1.1 (Simplify). $(6x^2–7x + 4) + (7x^2–9x + 8)$

$$
\begin{aligned}
&(6x^2–7x + 4) + (7x^2–9x + 8) \\
&= (6x^2 + 7x^2) + (-7x - 9x) + (4 + 8) \\
&= 13x^2 - 16x^2 + 12
\end{aligned}
$$

Example 1.2 (Subtract). $7x^3–6x^5 + 2x^2$ *from* $9x^5–6x^3 + 7x^2$

$$
\begin{aligned}
&(9x^5–6x^3 + 7x^2) - (7x^3–6x^5 + 2x^2) \\
&= (9x^5 - (-6x^5)) + (-6x^3 - 7x^3) + (7x^2 - 2x^2) \\
&= 15x^5 - 13x^3 + 5x^2
\end{aligned}
$$

1.2 *Multiplying Polynomial Expressions*

Polynomials are multiplied using the distributive property of addition:

$$a(b + c) = ab + ac$$

Example 1.3 (Simplify). $(x + 4)(2x + 5)$

$$
\begin{aligned}
&(x + 4)(2x + 5) \\
&= 2x^2 + 8x + 5x + 20 \\
&= 2x^2 + 13x + 20
\end{aligned}
$$

1.3 *Dividing Polynomial Expressions*

Lecture 2
30th August, 2018

There are a few cases for the division of polynomials. Some of these are detailed below

Remember, the following equalities **do not hold** when the quotient's denominator is zero. For example, if you have the expression $(2x + 2) \div (x + 1)$, and you simplify it to 2, you must remember that this expression is **not equal** to two if $x = -1$, as that would be division by zero and thus undefined.

1.3.1 Denominator Is A Factor Of Each Term

If the denominator is a factor of each term of the numerator, the quotient can be simplified by dividing each term of the numerator by the denominator.

Example 1.4 (Simplify). $(-16x^4 + 8x^2 + 2x) \div 2x$

$$\begin{aligned} &\frac{-16x^4 + 8x^2 + 2x}{2x} \\ &= \frac{-16x^4}{2x} + \frac{8x^2}{2x} + \frac{2x}{2x} \\ &= -7x^3 + 4x + 1 \end{aligned}$$

1.3.2 Denominator Is A Factor Of The Numerator

If the denominator is a factor of the numerator, the quotient can be simplified by factoring the numerator and then canceling out the factor.

For more on factoring, see section 2

Example 1.5 (Simplify). $(15x^2 + 22x + 8) \div (5x + 4)$

$$\begin{aligned} &\frac{15x^2 + 22x + 8}{5x + 4} \\ &= \frac{(3x + 2)\cancel{(5x + 4)}}{\cancel{5x + 4}} \\ &= 3x + 2 \end{aligned}$$

1.3.3 Polynomial Long Division

If the other two methods don't work, you can use polynomial long division. This is a generalized version of the arithmetic technique called long division.

Example 1.6 (Simplify). $(x^3 + x^2 - 2x) \div (x - 1)$

$$\begin{array}{r|rrr} & & x^2 & +2x \\ \hline x - 1 & x^3 & +x^2 & -2x \\ & -x^3 & +x^2 & \\ \hline & & 2x^2 & -2x \\ & & -2x^2 & +2x \\ \hline & & & 0 \end{array}$$

$$\therefore \frac{x^3 + x^2 - 2x}{x - 1} = x^2 + 2x$$

You should note that it is possible that it doesn't divide in evenly, in which case the division will leave a *remainder*. This can just be added at the end. Here is an example of a remainder

Example 1.7 (Simplify). $(x^3 + x^2 - 1) \div (x - 1)$

$$\begin{array}{r l}
 & \ x^2 + 2x + 2 \\ \cline{2-2}
x - 1 & \big)\ x^3 + x^2 \qquad\ \ - 1 \\
 & \ \ - x^3 + x^2 \\ \cline{2-2}
 & \qquad 2x^2 \\
 & \quad - 2x^2 + 2x \\ \cline{2-2}
 & \qquad\qquad 2x - 1 \\
 & \qquad\quad - 2x + 2 \\ \cline{2-2}
 & \qquad\qquad\qquad 1
\end{array}$$

$$\therefore \frac{x^3 + x^2 - 1}{x - 1} = x^2 + 2x + 2 + \frac{1}{x - 1}$$

1.3.4 Synthetic Division

If the denominator of the expression is $x - a$, where $a \in \mathbb{R}$, then you can use a division technique called synthetic division.

This technique is generally used in the process of finding the roots of a polynomial.

Example 1.8 (Complete). *example 1.6 using synthetic division.*

$$\begin{array}{c|cccc}
 & 1 & 1 & -2 & 0 \\
1 & & 1 & 2 & 0 \\ \hline
 & 1 & 2 & 0 & 0
\end{array}$$

$$\therefore \frac{x^3 + x^2 - 2x}{x - 1} = x^2 + 2x$$

2 FACTORISING POLYNOMIAL EXPRESSIONS

Lecture 3
31st July, 18

Definition 2.1 (Factor). *An algebraic factor is an expression that divides a polynomial leaving no remainder.*

Factorising is a very important technique in algebra. A few factoring techniques are shown below.

2.1 *Highest Common Factor*

This method can be done by looking for the highest common factor of each of the expressions and then working backwards from there.

Example 2.1 (Factor).

$$2x^2 + 6x = 2x(x+3)$$
$$15axy + 2xyb + 4xy = 2xy(7a + b + 2)$$

2.2 *Grouping*

An expression can sometimes be factored by breaking the expression up into groups and factoring the individual parts.

Example 2.2 (Factor).

$$\begin{aligned} 6x^2y + 3xy^2 - 12x - 6y &= 3xy(2x+y) - 6(2x+y) \\ &= (2x+y)(2xy-5) \end{aligned}$$

2.3 *Difference Of Two Squares*

Theorem 2.1 (Difference of two squares). *An expression of the form $a^2 - b^2$ can be factored into $(a+b)(a-b)$*

Proof.

$$\begin{aligned} (a+b)(a-b) &= a(a-b) + b(a-b) \\ &= a^2 - ab + ab - b^2 \\ &= a^2 - b^2 \end{aligned}$$ □

Example 2.3 (Factor).

$$\begin{aligned} x^2 - 36 &= (x+6)(x-6) \\ 4x^2 - 81 &= (2x+9)(2x-9) \\ 9x^3 - 81x &= 9x(x^2-9) = 9x(x+3)(x-3) \\ 16x^4 - 1 &= (4x^2+1)(4x^2-1) = (4x^2+1)(2x+1)(2x-1) \end{aligned}$$

2.4 *Difference Of Two Cubes*

Theorem 2.2 (Difference of two cubes). *An expression of the form $a^3 - b^3$ can be factored into $(a-b)(a^2 + ab + b^2)$*

Proof.

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2) - b(a^2+ab+b^2)\\ &= (a^3+a^2b+ab^2)-(a^2b+ab^2+b^3)\\ &= (a^3+\cancel{a^2b+ab^2})-(\cancel{a^2b+ab^2}+b^3)\\ &= a^3-b^3\end{aligned}$$ □

Example 2.4 (Factor).

$$\begin{aligned}x^3-8 &= (x-2)(x^2+2x+4)\\ 125x^3-27y^3 &= (5x-3y)(25x^2+15xy+9y^2)\end{aligned}$$

2.5 *Sum Of Two Cubes*

Theorem 2.3 (Sum of two cubes). *An expression of the form a^3+b^3 can be factored into $(a+b)(a^2-ab+b^2)$*

Proof.

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a(a^2-ab+b^2)+b(a^2-ab+b^2)\\ &= a^3-a^2b+ab^2+ba^2-ab^2+b^3\\ &= a^3-\cancel{a^2b+ab^2+ba^2-ab^2}+b^3\\ &= a^3+b^3\end{aligned}$$ □

Example 2.5 (Factor).

$$\begin{aligned}27x^3+1 &= (3x+1)(9x^2-3x+1)\\ 81x^3+192b^3 &= (5x+6b)(25x^2-30xb+36b^2)\end{aligned}$$

2.6 *Factoring With The Quadratic Formula*

If the previous methods don't work and you with to factor a quadratic, you can use ths following method. The best way to show this is with an example.

Example 2.6 (Factor). $3x^2-17x+20$
Answer:
First, you must let the expression equal 0, and then solve for x:

$$
\begin{aligned}
&3x^2 - 17x + 20 = 0 \\
&\implies x = \frac{17 \pm \sqrt{(-17)^2 - 4(3)(20)}}{2(3)} = \frac{17 \pm 7}{6} \\
&\implies x = 4 \textit{ or } x = \frac{5}{3}
\end{aligned}
$$

From this you can find that if $x = 4$, then $x - 4$ is a factor. If $x = \frac{5}{3} \implies 3x = 5$, then $3x - 5$ is a factor

$$\therefore 3x^2 - 17x + 20 = (3x - 5)(x - 4)$$