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Linearization problem

Consider a non-linear time-invariant system of the form

$$\begin{cases} \dot{x}(t) = f\big(x(t), u(t)\big), \\ y(t) = g\big(x(t), u(t)\big) \end{cases} \tag{1}$$

such that $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and

A: $\frac{\partial f(x,u)}{\partial x}$, $\frac{\partial f(x,u)}{\partial u}$, $\frac{\partial g(x,u)}{\partial x}$, and $\frac{\partial g(x,u)}{\partial u}$ exist and are continuous.

The question is:

✗ How to approximate system (1) by a linear model?

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The simplest example

Consider the static time-invariant system

$$y(t) = ku(t) + \eta. \tag{2}$$

This system is nonlinear, but intuitively it seems to be "almost linear." Indeed, this system can easily be made linear. To this end notice that for any *constant* $u(t) = u_e$ the output $y(t)$ is also constant and given by $y_e \doteq ku_e + \eta$. Let's call the pair (y_e, u_e) the *equilibrium* point.

Define now new signals:

$$y_\delta(t) \doteq y(t) - y_e \quad \text{and} \quad u_\delta(t) \doteq u(t) - u_e,$$

which are deviations of y and u from the equilibrium point. Then

$$y_\delta(t) = ku_\delta(t), \tag{2_δ}$$

which is *linear*. Thus, nonlinear system (2) can be made linear by an appropriate (cf. equilibrium) shift of its input and output signals.

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A little bit more complicated example

Consider now the system

$$\begin{cases} \dot{x}(t) = -ax(t) + bu(t) + \eta_x, \\ y(t) = x(t) + \eta_y. \end{cases} \tag{3}$$

Assume again that the input $u(t) = u_e$ is constant. Then

$$x(t) = \int_0^t e^{-a(t-s)}(bu_e + \eta_x)dt = \tfrac{1}{a}(1 - e^{-at})(bu_e + \eta_x).$$

As $\lim_{t\to\infty} e^{-at} = 0$, system (3) approaches it equilibrium point[a] (x_e, u_e), where

$$x_e = \tfrac{1}{a}(bu_e + \eta_x).$$

[a] This is called the *steady-state*.

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A little bit more complicated example (contd)

Let's write now system (3) in terms of the deviations

$$x_\delta(t) \doteq x(t) - x_e \quad \text{and} \quad u_\delta(t) \doteq u(t) - u_e.$$

We get:

$$\begin{cases} \dot{x}_\delta(t) = -ax_\delta(t) + bu_\delta(t), \\ y_\delta(t) = x_\delta(t), \end{cases} \tag{3_δ}$$

where $y_\delta(t) \doteq y(t) - x_e - \eta_y$. Equation (3_δ) is thus a linear equivalent of the nonlinear system (3).

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Equilibrium

The pair (x_e, u_e) is called *equilibrium* if

$$f\big(x_e, u_e\big) = 0.$$

Let (x_e, u_e) be an equilibrium of (1). Then $x(t)$, $y(t)$, and $u(t)$ can be presented as follows:

$$x(t) = x_e + x_\delta(t), \quad y(t) = y_e + y_\delta(t), \quad \text{and} \quad u(t) = u_e + u_\delta(t),$$

where $y_e \doteq g\big(x_e, u_e\big)$.

Eqn. (1) can then be rewritten as follows:

$$\begin{cases} \dot{x}_\delta(t) = f\big(x_e + x_\delta(t), u_e + u_\delta(t)\big), \\ y_\delta(t) = g\big(x_e + x_\delta(t), u_e + u_\delta(t)\big) - y_e \end{cases} \tag{1_δ}$$

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Tailor series expansion

One can expand $f(\cdot)$ and $g(\cdot)$ as follows[a]:

$$f\big(x, u\big) = f(x_e, u_e) + \tfrac{\partial f}{\partial x}\big|_{x_e, u_e} x_\delta + \tfrac{\partial f}{\partial u}\big|_{x_e, u_e} u_\delta + o\big(x_\delta, u_\delta\big),$$
$$g\big(x, u\big) = g(x_e, u_e) + \tfrac{\partial g}{\partial x}\big|_{x_e, u_e} x_\delta + \tfrac{\partial g}{\partial u}\big|_{x_e, u_e} u_\delta + o\big(x_\delta, u_\delta\big),$$

where $o(\eta)$ is any function of a variable η such that $\lim_{\|\eta\|\to 0} \frac{\|o(\eta)\|}{\|\eta\|} = 0$.

[a] Given an n-dimensional function $\phi(\eta)$ of an m-dimensional argument η, the partial derivative of ϕ with respect to η (the *Jacobian matrix*) is defined as the following $n \times m$ matrix:

$$\frac{\partial \phi(\eta)}{\partial \eta} \doteq \begin{bmatrix} \frac{\partial \phi_1(\eta)}{\partial \eta_1} & \cdots & \frac{\partial \phi_1(\eta)}{\partial \eta_m} \\ \vdots & & \vdots \\ \frac{\partial \phi_n(\eta)}{\partial \eta_1} & \cdots & \frac{\partial \phi_n(\eta)}{\partial \eta_m} \end{bmatrix}.$$

Linearization around (x_e, u_e)

If the deviations x_δ and u_δ are "small enough," then Eqn. (1_δ) can be approximated by the following *linear* model:

$$\begin{cases} \dot{x}_\delta(t) = A_\delta x_\delta(t) + B_\delta u_\delta(t), \\ y_\delta(t) = C_\delta x_\delta(t) + D_\delta u_\delta(t), \end{cases} \qquad (1_{\text{linearized}})$$

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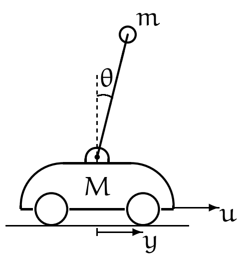
where

$$A_\delta \doteq \frac{\partial f}{\partial x}\Big|_{x_e,u_e}, \qquad B_\delta \doteq \frac{\partial f}{\partial u}\Big|_{x_e,u_e},$$

$$C_\delta \doteq \frac{\partial g}{\partial x}\Big|_{x_e,u_e}, \qquad D_\delta \doteq \frac{\partial g}{\partial u}\Big|_{x_e,u_e}.$$

Inverted pendulum



M : mass of the cart
m : mass of the pendulum
l : length of the pendulum

y : position of the cart
θ : angular rotation
u : force on the cart

μ : friction coefficient
g : acceleration of gravity

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Nonlinear model:

$$\begin{cases} (M+m)\ddot{y} + ml\,\ddot{\theta}\cos\theta - ml(\dot{\theta})^2\sin\theta + \mu\dot{y} = u \\ l\ddot{\theta} - g\sin\theta + \ddot{y}\cos\theta = 0 \end{cases}$$

or, equivalently,

$$\begin{bmatrix} M+m & ml\cos\theta \\ \cos\theta & l \end{bmatrix} \begin{bmatrix} \ddot{y} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} ml(\dot{\theta})^2\sin\theta - \mu\dot{y} \\ g\sin\theta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u.$$

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Inverted pendulum: the state-space realization

The state vector:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y \\ \theta \\ \dot{y} \\ \dot{\theta} \end{bmatrix}$$

The state equation:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \underbrace{\begin{bmatrix} x_3 \\ x_4 \\ \begin{bmatrix} M+m & ml\cos x_2 \\ \cos x_2 & l \end{bmatrix}^{-1}\left(\begin{bmatrix} mlx_4^2 \sin x_2 - \mu x_3 \\ g\sin x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u\right) \end{bmatrix}}_{f(x,u)}. \tag{4}$$

where $\Lambda(x_2) \doteq \begin{bmatrix} M+m & ml\cos x_2 \\ \cos x_2 & l \end{bmatrix}$ is nonsingular ($\det \Lambda(x_2) = l(M + m\sin^2 x_2)$).

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Inverted pendulum: equilibriums

Equilibrium points should correspond to $\dot{x} \equiv 0$. Then, from Eq. (4) one can get that $x_3 \equiv 0$, $x_4 \equiv 0$, and

$$\Lambda(x_2)^{-1}\begin{bmatrix} u \\ g\sin x_2 \end{bmatrix} = 0.$$

Therefore, the equilibrium points are:

$$(x_e, u_e) = \left(\begin{bmatrix} x_1 \\ k\pi \\ 0 \\ 0 \end{bmatrix}, 0\right), \qquad \text{for every } x_1 \in \mathbb{R} \text{ and } k = 0, \pm 1, \pm 2, \ldots$$

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Inverted pendulum: Jacobian matrices

Jacobian matrices[a]:

$$\frac{\partial f_{1,2}}{\partial x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\frac{\partial f_{3,4}}{\partial x} = \Lambda(x_2)^{-1} \begin{bmatrix} 0 & mlx_4^2 \cos x_2 + \phi_1(x_2, x_3, x_4) \sin x_2 & -\mu & 2mlx_4 \sin x_2 \\ 0 & g \cos x_2 + \phi_2(x_2, x_3, x_4) \sin x_2 & 0 & 0 \end{bmatrix},$$

where ϕ_1 and ϕ_2 are some algebraic functions of x_2, x_3, and x_4,

$$\frac{\partial f}{\partial u} = \frac{1}{l(M + m \sin^2 x_2)} \begin{bmatrix} 0 \\ 0 \\ l \\ -\cos x_2 \end{bmatrix}.$$

[a] Given two matrix functions $M_1(\eta)$ and $M_2(\eta)$, then

$$\frac{\partial}{\partial \eta}\left(M_1(\eta)^{-1} M_2(\eta)\right) = M_1(\eta)^{-1}\left(\frac{\partial M_2(\eta)}{\partial \eta} - \frac{\partial M_1(\eta)}{\partial \eta} M_1(\eta)^{-1} M_2(\eta)\right).$$

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Inverted pendulum: linearization around $(0, 0)$

It's readily verified that

$$\left.\frac{\partial f}{\partial x}\right|_0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{m}{M} g & -\frac{\mu}{M} & 0 \\ 0 & \frac{m+M}{Ml} g & \frac{\mu}{Ml} & 0 \end{bmatrix}$$

and

$$\left.\frac{\partial f}{\partial u}\right|_0 = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M} \\ -\frac{1}{Ml} \end{bmatrix},$$

which yield the "A" and "B" matrices of the linearized realization.

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Linearization of discrete-time systems

The linearization problem for discrete-time system

$$\begin{cases} x_{k+1} = f(x_k, u_k), \\ \quad y_k = g(x_k, u_k) \end{cases}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, has exactly the same solution as in the continuous time, except that the *equilibrium* is obtained from

$$f(x_e, u_e) = x_e.$$