The Inverted Pendulum

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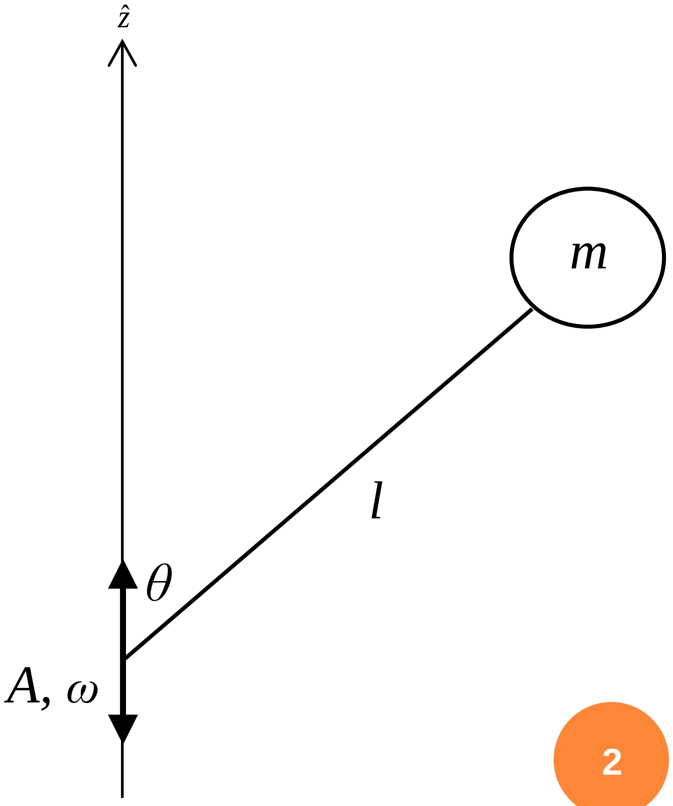
Formulating the problem

- Pendulum
  - Mass (m)
  - Length (l)

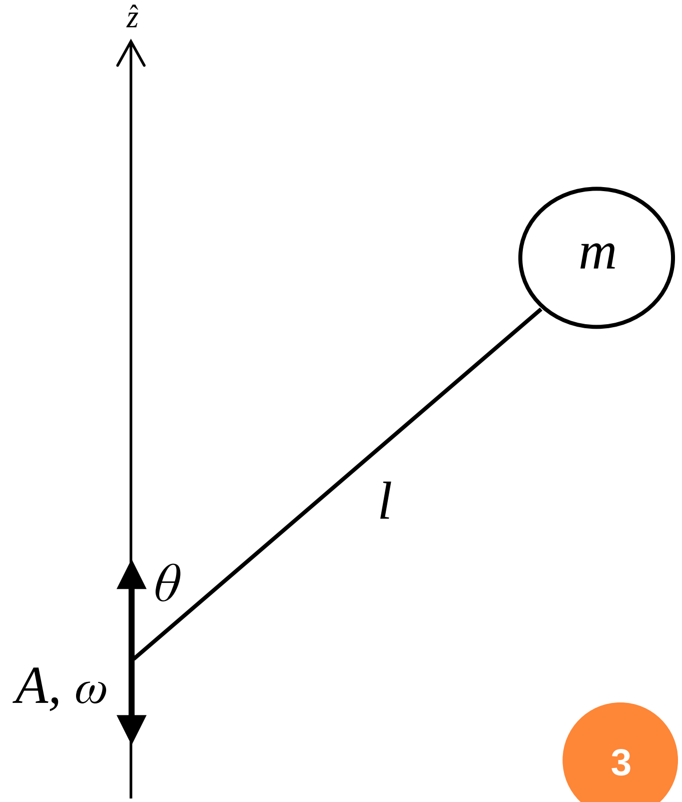
- Oscillating Pivot
  - Amplitude (A)
  - Frequency (ω)



Formulating the Problem

Forces Acting on the Pendulum

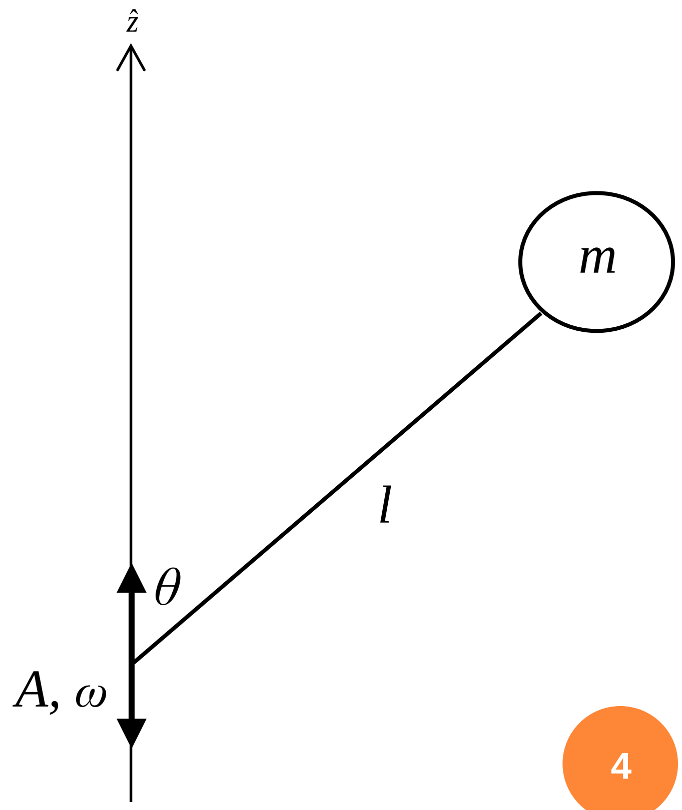
1. Gravity
2. Force of the pivot
3. [Friction]



Equation of Motion

- Newton's Second Law
  - Linear motion: $$F = ma = m\frac{d^2x}{dt^2}$$
  - Circular motion: $$\tau = I\alpha = I\frac{d^2\theta}{dt^2}$$
- Equation of Motion

$$\boxed{\tau_{\text{total}} = I\frac{d^2\theta}{dt^2}}$$



TORQUES

- **Gravity:**

$$\tau_{\text{grav}} = r\,F\sin\theta$$

$$\boxed{\tau_{\text{grav}} = mgl\sin\theta}$$

- **Oscillating pivot:**
  - **Force:** $y(t) = A\cos(\omega t)$

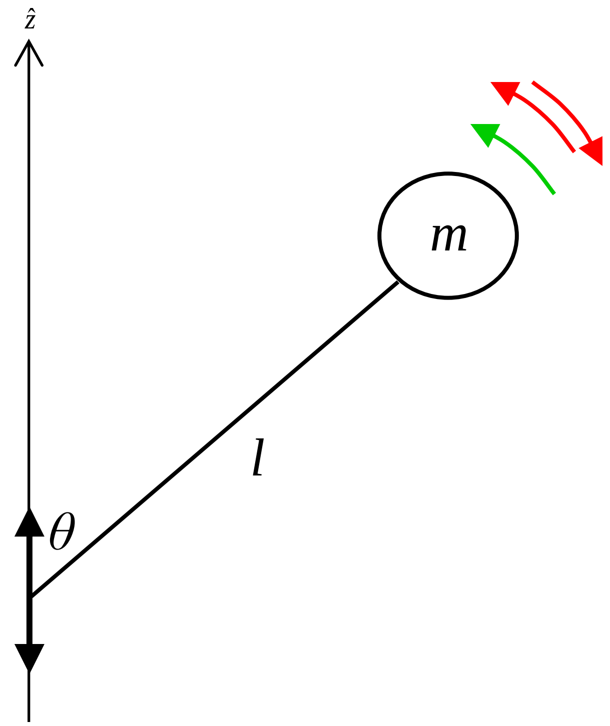
$$a = \frac{d^2 y(t)}{dt^2} = -\omega^2 A\cos(\omega t)$$

$$F = ma = m\frac{d^2 y(t)}{dt^2} = -m\omega^2 A\cos(\omega t)$$

  - **Torque:**

$$\tau_{\text{pivot}} = r\,F\sin\theta$$

$$\boxed{\tau_{\text{pivot}} = -ml\omega^2 A\cos(\omega t)\sin\theta}$$



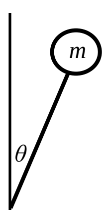
Equation of Motion

$$\tau_{\text{total}} = \tau_{\text{grav}} + \tau_{\text{pivot}} = I\frac{d^2\theta}{dt^2} = ml^2\frac{d^2\theta}{dt^2}$$

$$ml^2\frac{d^2\theta}{dt^2} = mgl\sin\theta - mg\omega^2 A\cos(\omega t)\sin\theta$$

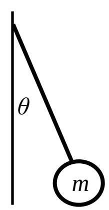
$$\boxed{\frac{d^2\theta}{dt^2} - \left[\frac{g}{l} - \frac{\omega^2 A}{l}\cos(\omega t)\right]\sin\theta = 0}$$

Gravity term *Pivot term*



$$\frac{d^2\theta}{dt^2} - \frac{g}{l}\theta = 0$$

$$\theta(t) = \exp(\omega_0 t)$$



$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

$$\theta(t) = \sin\left(\omega_0 t\right)$$

$$\boxed{\omega_0^2 = \frac{g}{l}}$$

For example:

$g = 9.81\,\text{m.s}^{-2}$

$l = 19\,\text{cm}$

$$\boxed{\omega_0 = 7.19\ \text{rad.s}^{-1}}$$

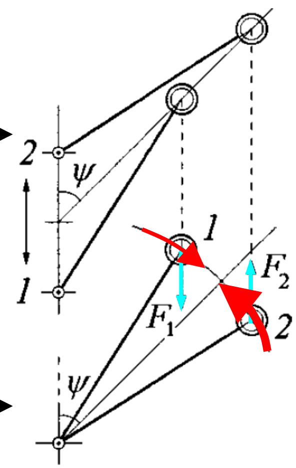
Physical Intuition

$$\boxed{\frac{d^2\theta}{dt^2} - \left[\omega_0^2 - \frac{\omega^2 A}{l}\cos(\omega t)\right]\sin\theta = 0}$$

Pivot term
(dominates for
large values of ω)

Inertial (lab) frame

Non-inertial frame (of pendulum)



$$F = -m\omega^2 A\cos(\omega t)$$

$$F_{\max} = \pm m\omega^2 A$$

$$\boxed{|F_1| = |F_2|}$$

$$\tau = rF\sin\theta$$

$$\theta_2 > \theta_1$$

$$\therefore \boxed{\tau_2 > \tau_1}$$

Butikov, Eugene. *On the dynamic stabilization of an inverted pendulum.* Am. J. Phys., Vol. 69, No. 7, July 2001

Reparameterization

Reparameterize: $\boxed{\tau = \omega t}$ $\left(t = \frac{\tau}{\omega} \right)$

$$\frac{d^2\theta}{d\tau^2} - \left[\frac{\omega_0^2}{\omega^2} - \frac{A}{l}\cos(\tau) \right] \sin\theta = 0$$

Let: $\boxed{\delta = -\left(\frac{\omega_0}{\omega} \right)^2}$ $\boxed{\varepsilon = \frac{A}{l}}$

$$\boxed{\frac{d^2\theta}{d\tau^2} + [\delta + \varepsilon \cos(\tau)] \sin\theta = 0}$$

For small values of θ*:* $\sin\theta \approx \theta$

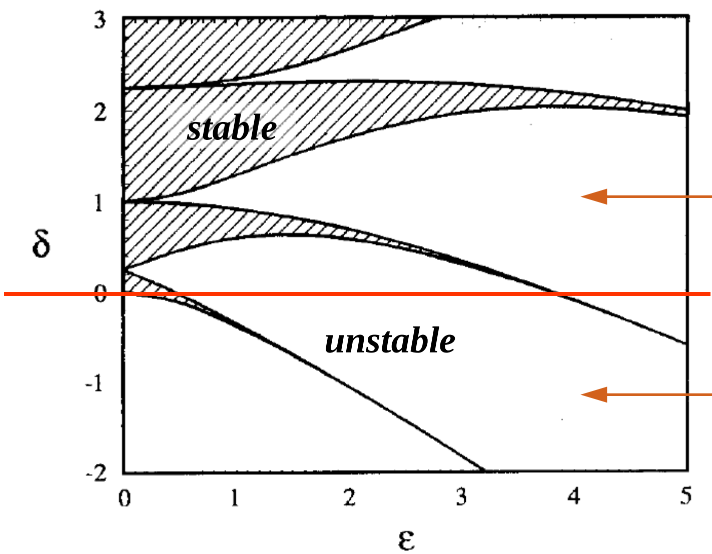
$$\boxed{\frac{d^2\theta}{d\tau^2} + [\delta + \varepsilon \cos(\tau)] \theta = 0}$$

$\uparrow$

Mathieu's Equation

Stability

$$\boxed{\frac{d^2\theta}{d\tau^2} + [\delta + \varepsilon\cos(\tau)]\theta = 0}$$



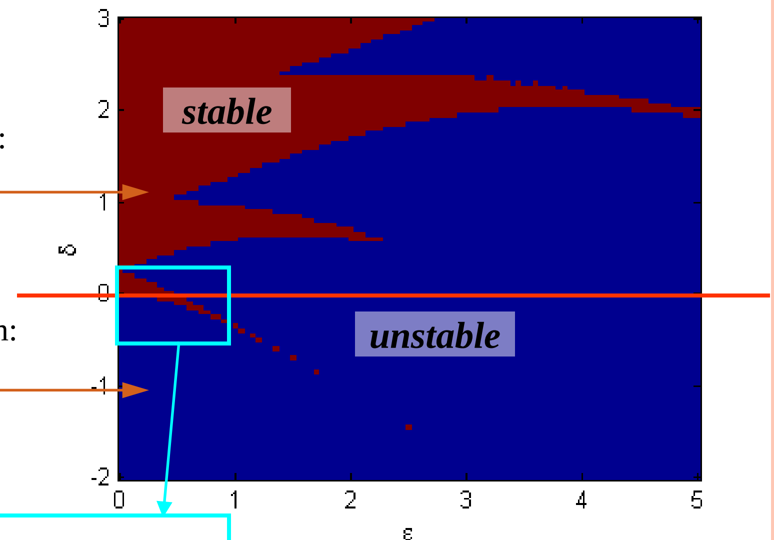
Smith, Blackburn, et. al.
Stability and Hopf bifurcations in an inverted pendulum. Am J. Phys. 60 (10), Oct 1992

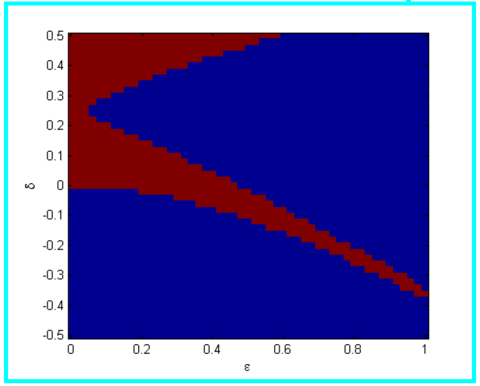
Regular Pendulum:

$\delta > 0$

Inverted Pendulum:

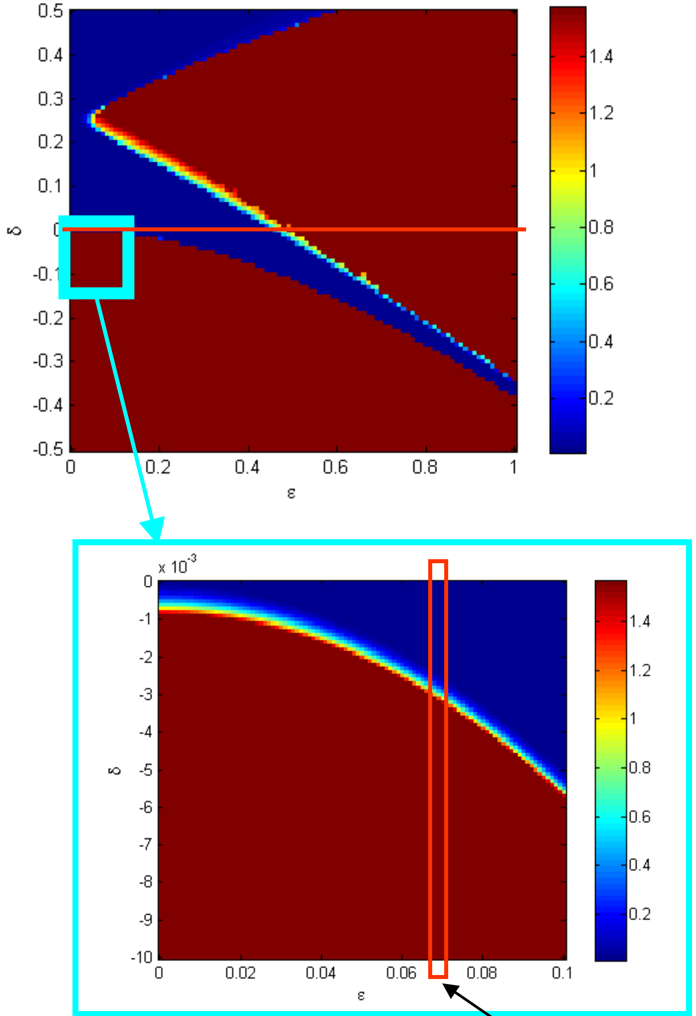
$\delta < 0$





Matlab Plot

Physical Region



$\varepsilon = 0.067$

Our values of ε and δ

$$\varepsilon = \frac{A}{l} = \frac{\frac{1}{2}\,\text{inch}}{19\,\text{cm}} = \frac{0.013\text{m}}{0.19\text{m}} = 0.067$$ **(fixed)**

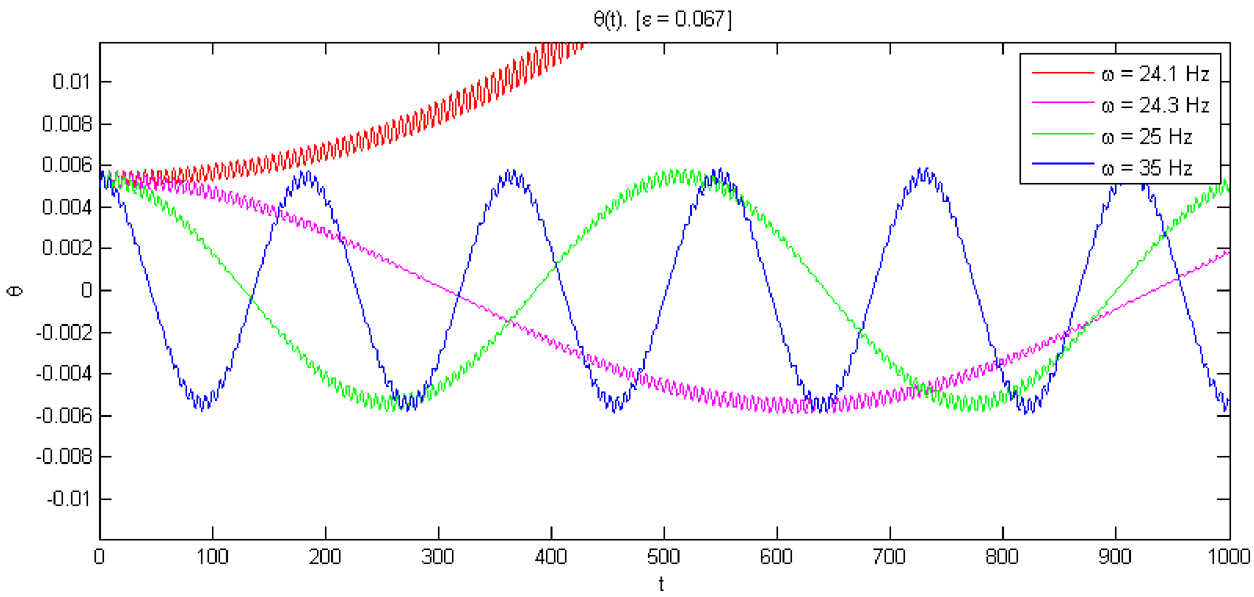
$$\delta = \frac{\omega_0^2}{\omega^2} = \frac{7.19^2}{[5\text{ Hz};\ 200\,\text{Hz}]^2} = [2;\ 0.001]$$ **(depends on *ω*)**

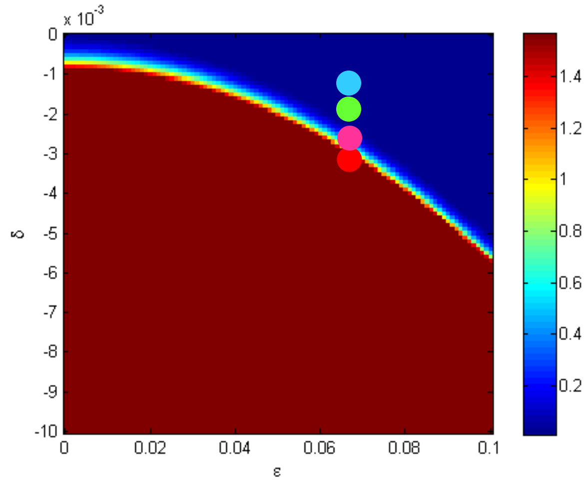
Stability condition:
(approximating *A* << *l*)

$$\frac{A}{l}\frac{\omega}{\omega_0} = \frac{\varepsilon}{\sqrt{\delta}} > \sqrt{2}$$

$$\boxed{\begin{aligned}\delta_c &\approx 2.2\times10^{-3}\\ \omega_c &\approx 24.2\ \text{Hz}\end{aligned}}$$

Sample Plots





HOW WE BUILT IT

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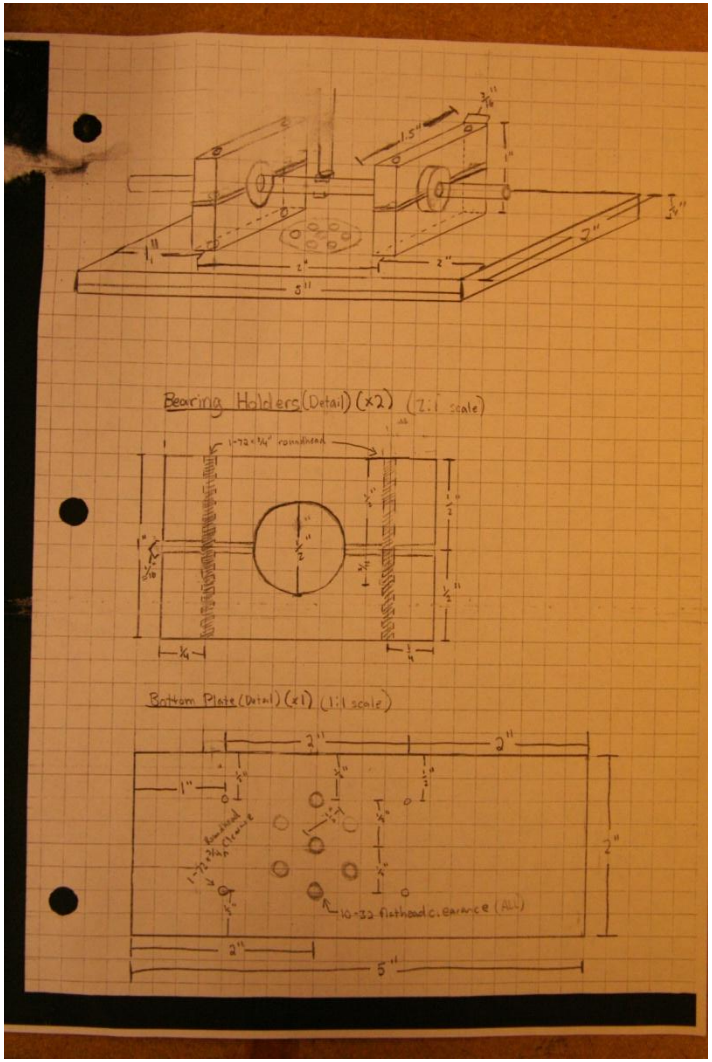

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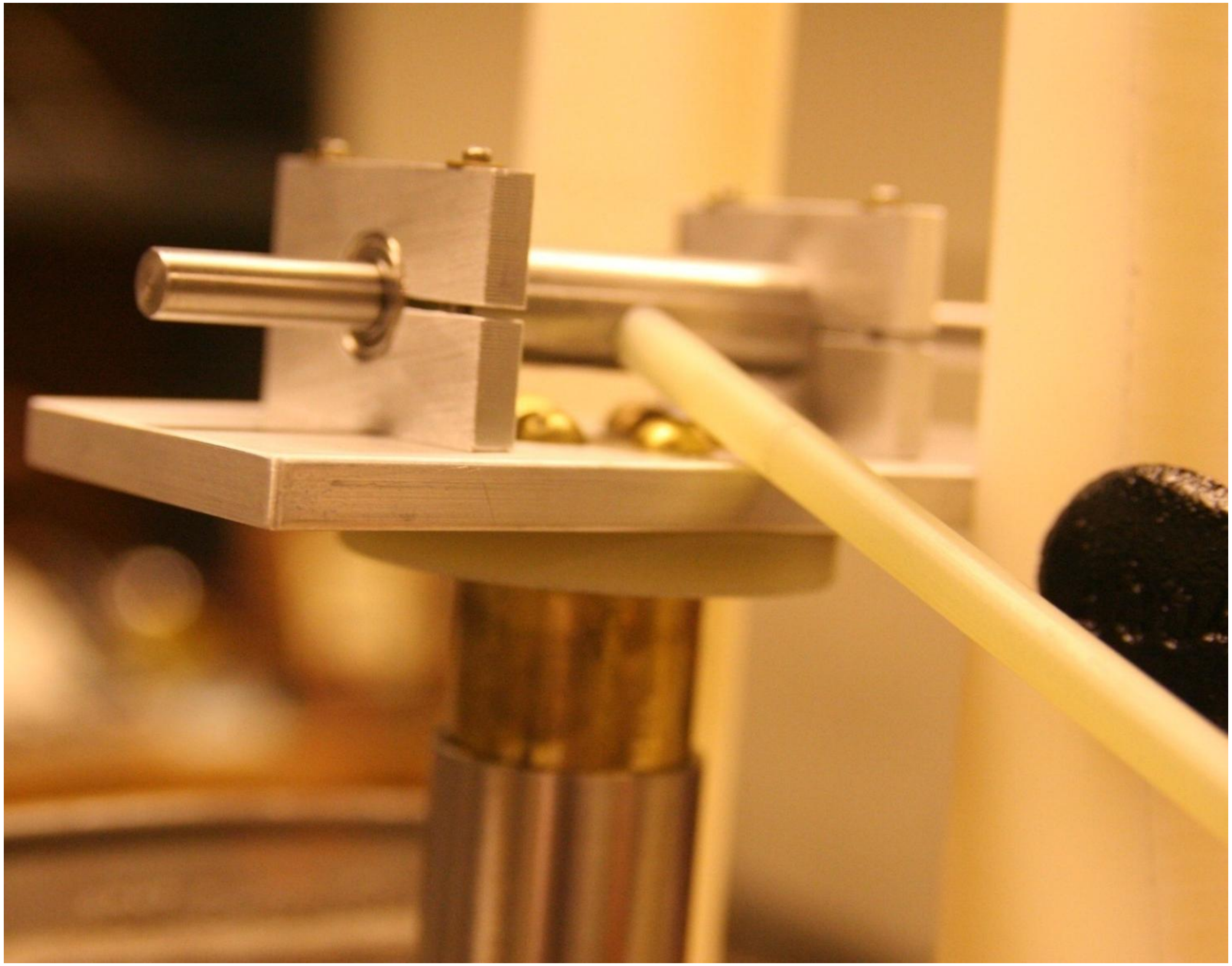


Why The Driver Didn't work

$$(2\pi a f)^2 > 2gl$$

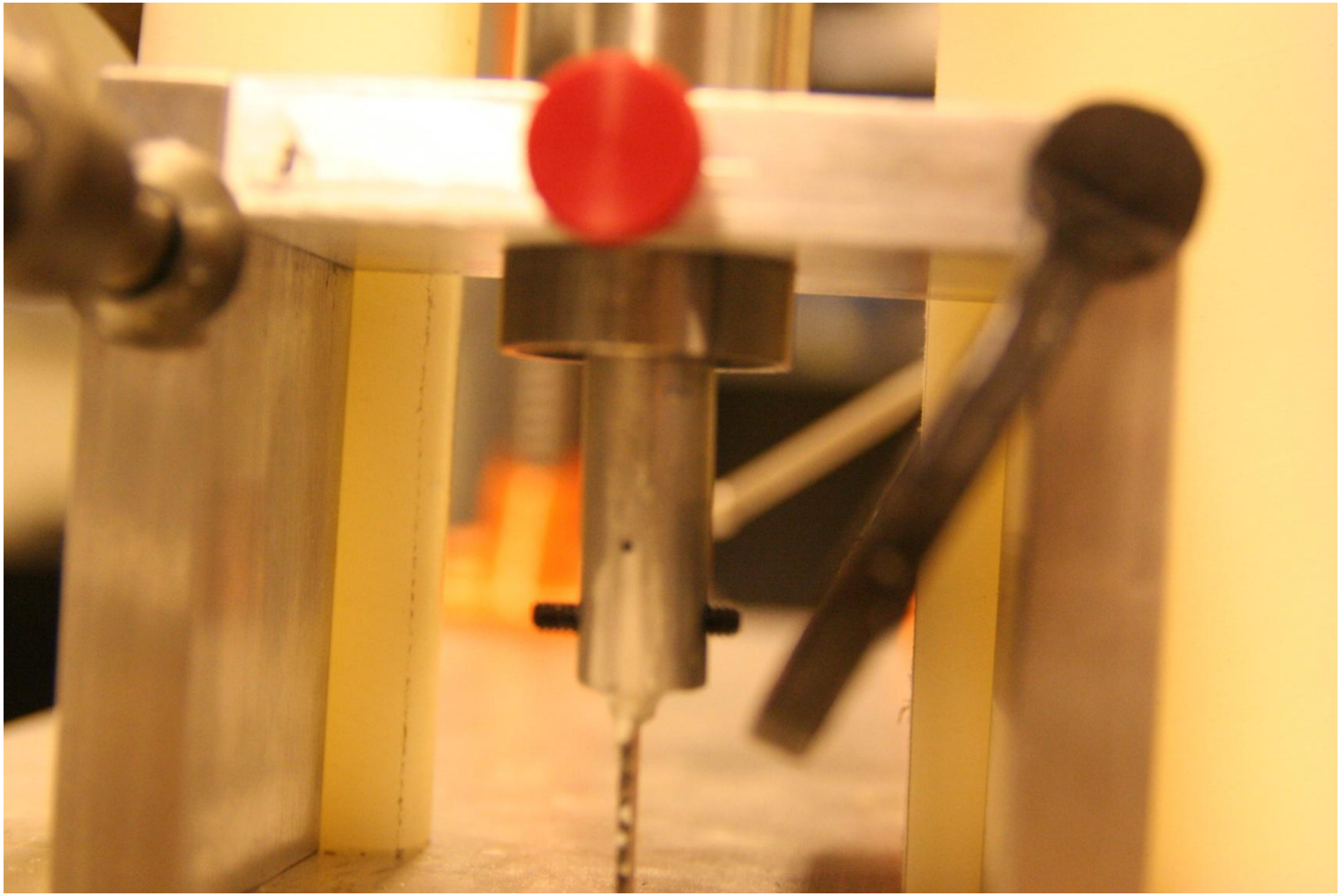
$$a \approx .00635\ m$$

$$f = 20\ Hz$$

$$l_{\max} = \frac{(2\pi a f)^2}{2g} = 3cm$$

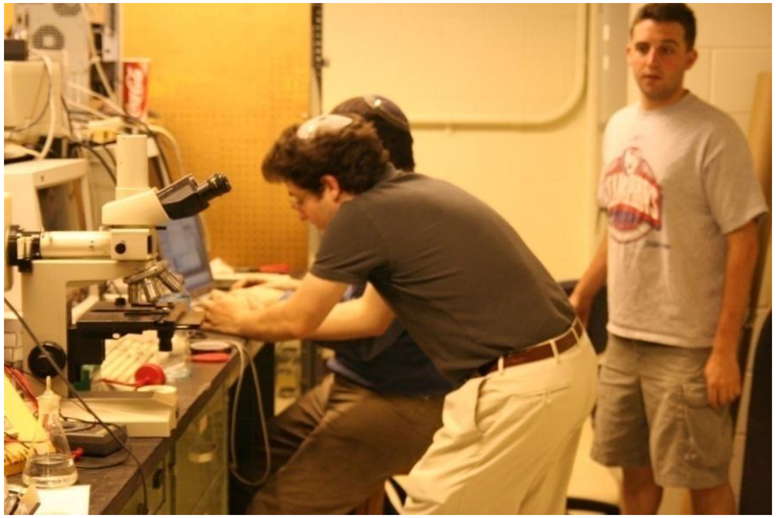
HOW WE BUILT IT

HOW WE BUILT IT

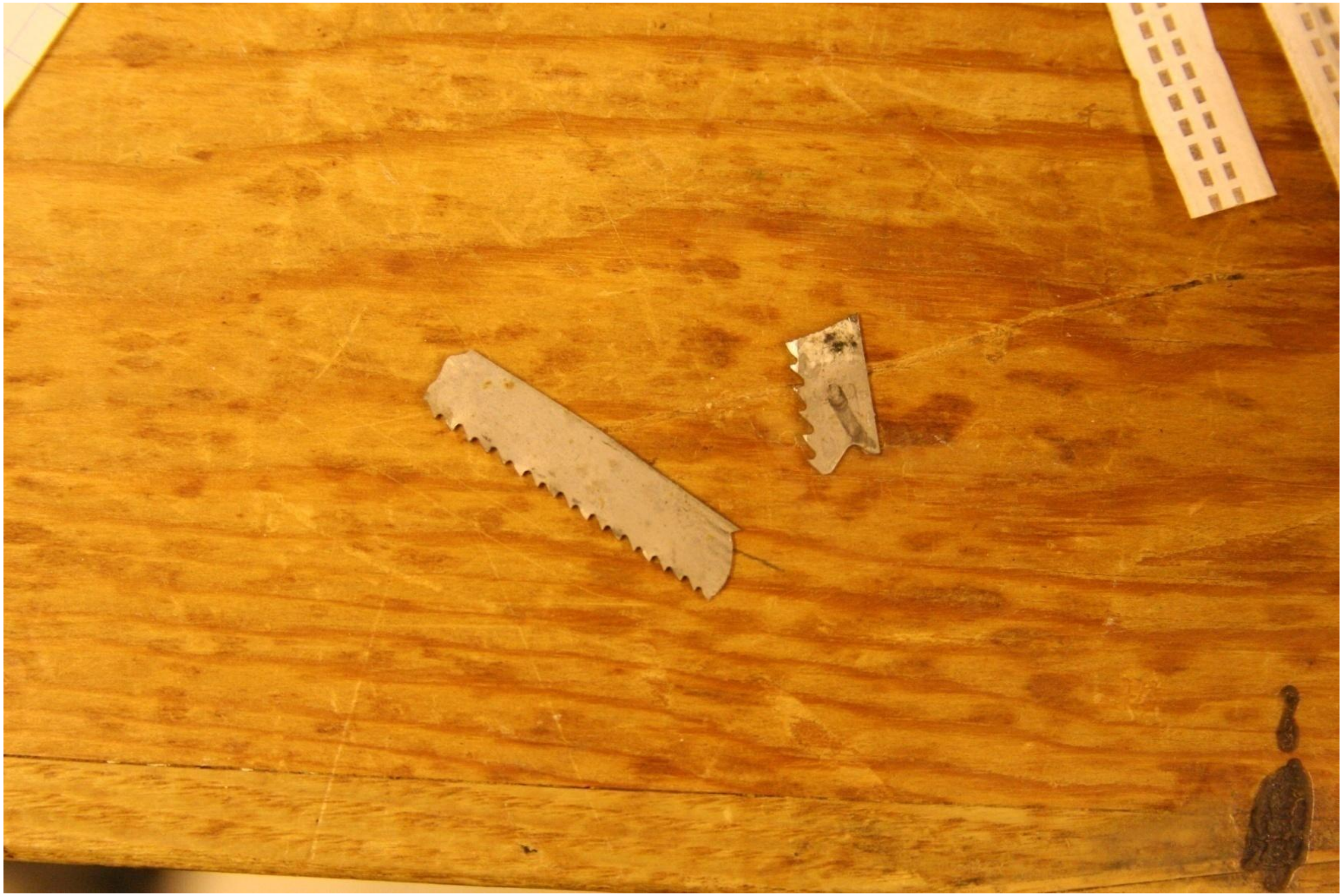


How We Built It

Some of the things we originally used to stabilize the pendulum

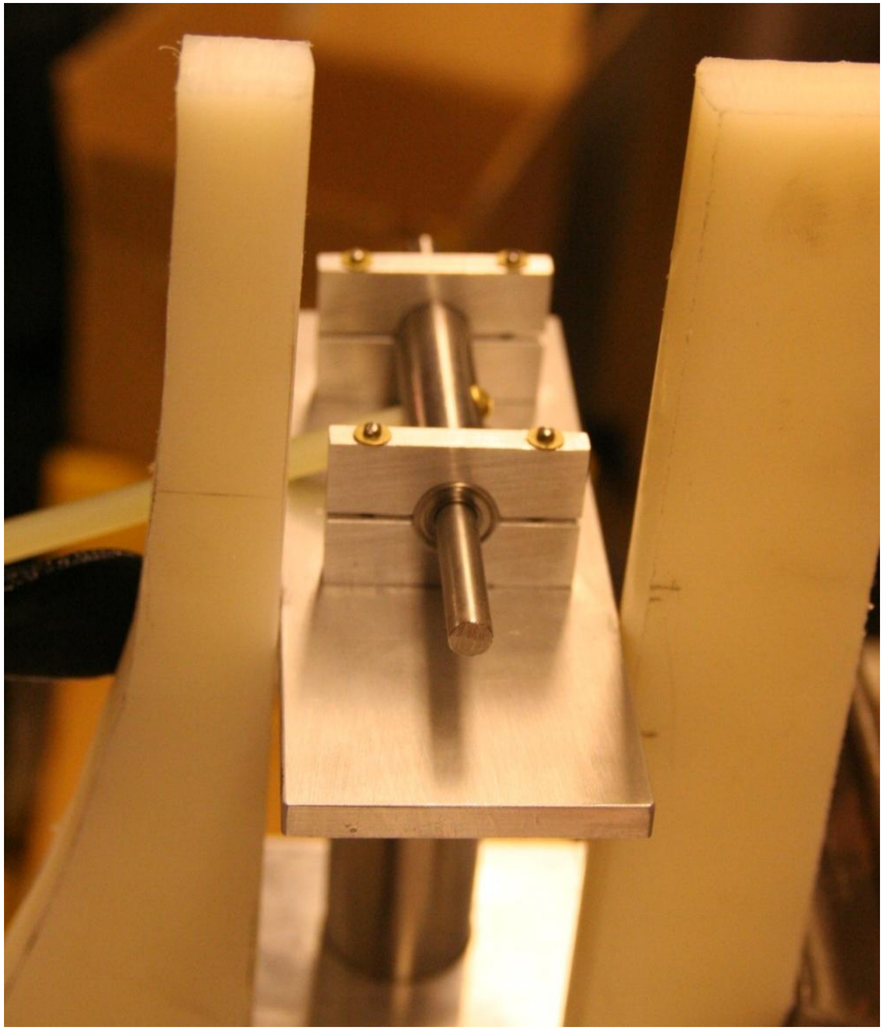


How We Built It

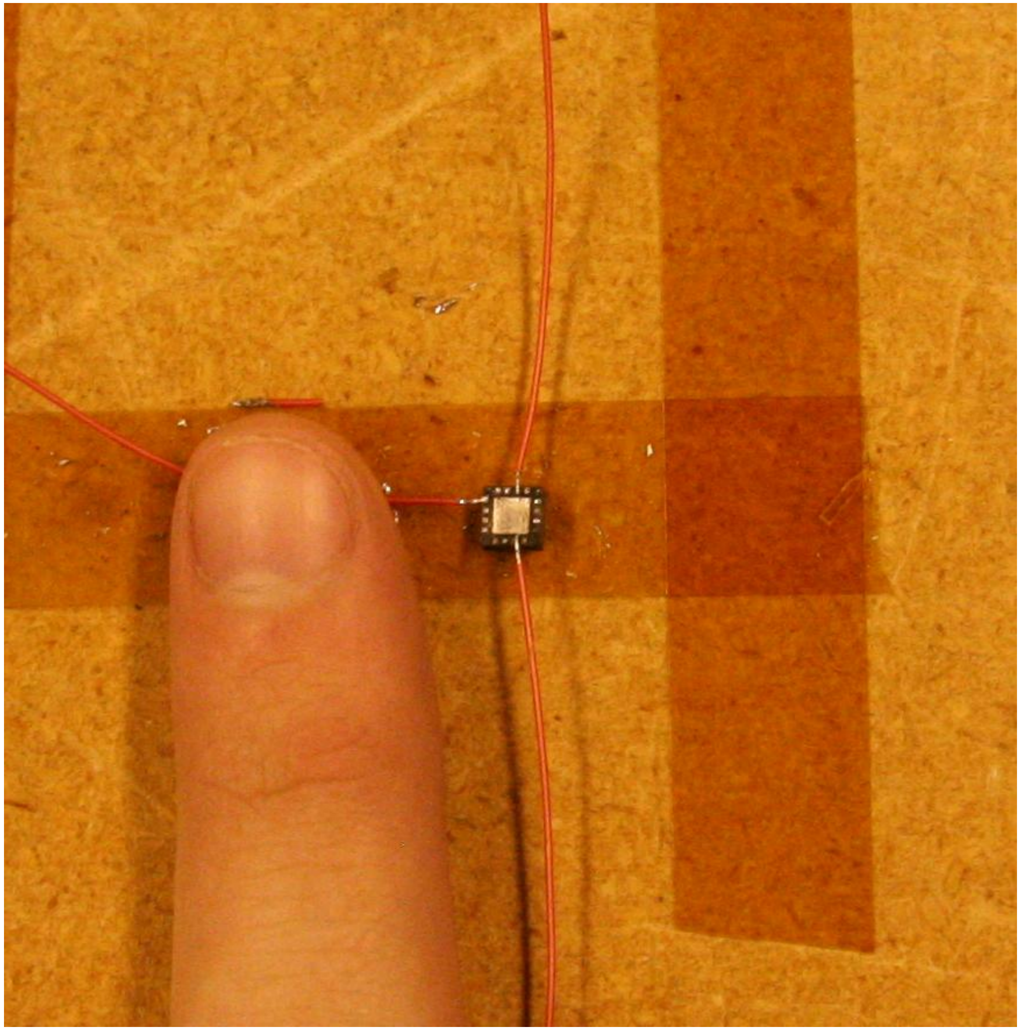


How we built it

HOW WE BUILT IT



DATA AQUISITION



DATA ACQUISITION

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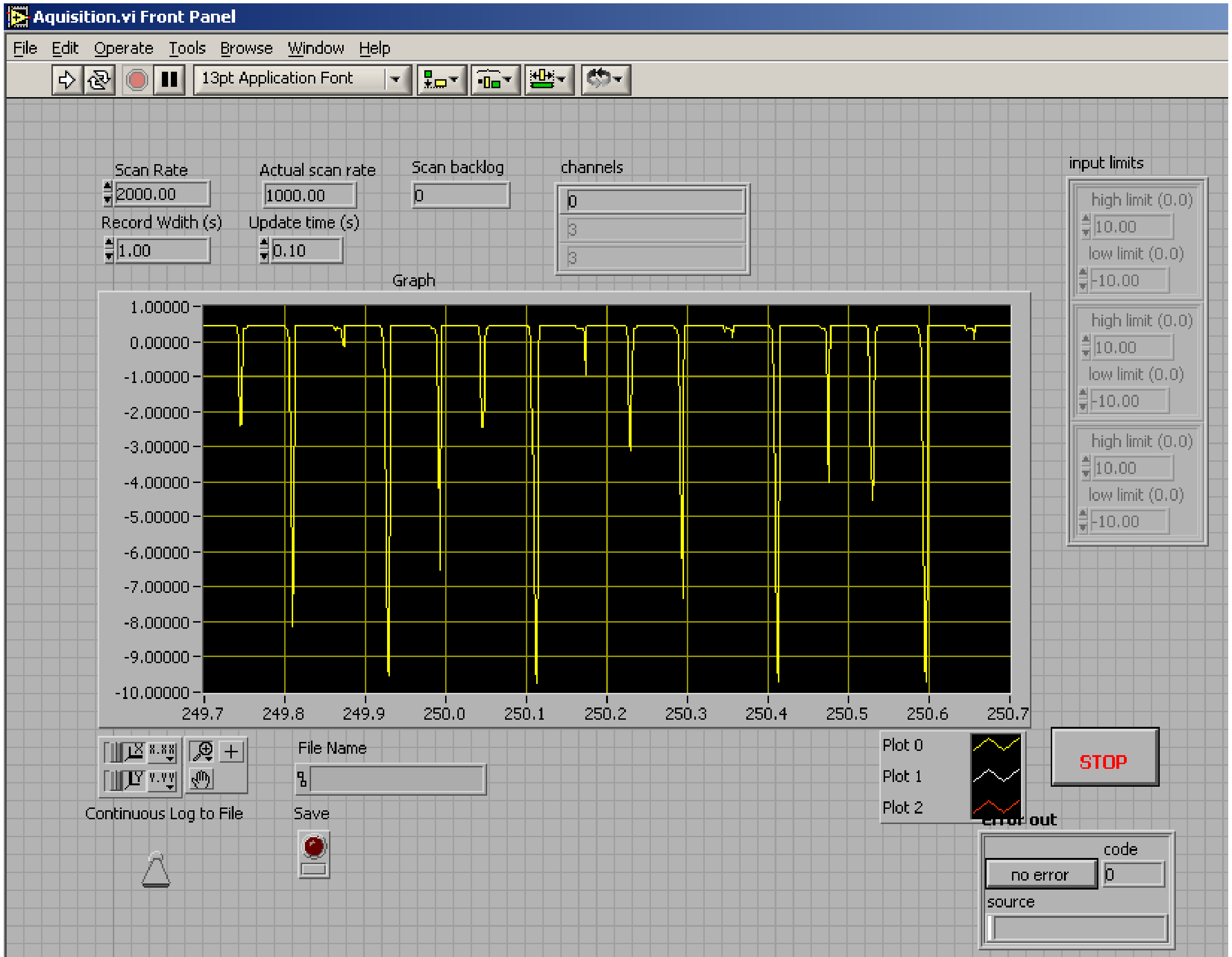
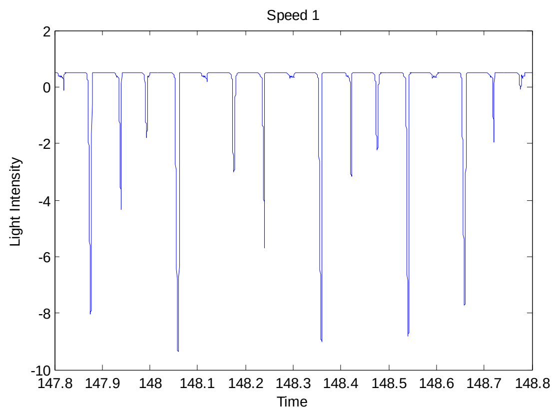
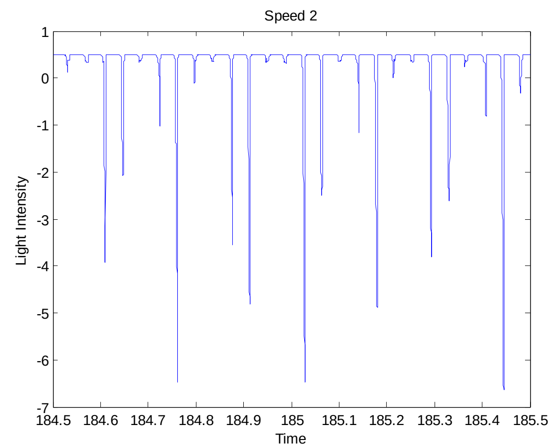


Data

Speed 1 (8 Hz)

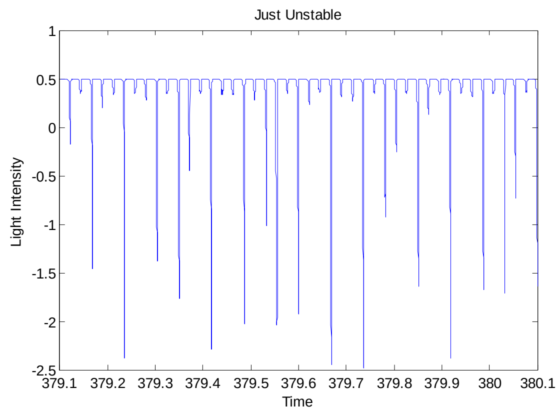


Speed 2 (13 Hz)



Speed 3 (23 Hz)

Speed ~2.8 (22 Hz)



DATA AQUISITION

1415 RPM = 23.58 Hz

Problems we had

- Finding right amplitude and frequency
- Finding right equipment
- Stabilizing apparatus
- Getting apparatus to right frequency
- Soldering

Thank You

- Professor Page
- Wei Chen
- PHY 210 Colleagues
- Sam Cohen
- Mike Peloso