THE STABILITY OF THE INVERTED PENDULUM

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Outline

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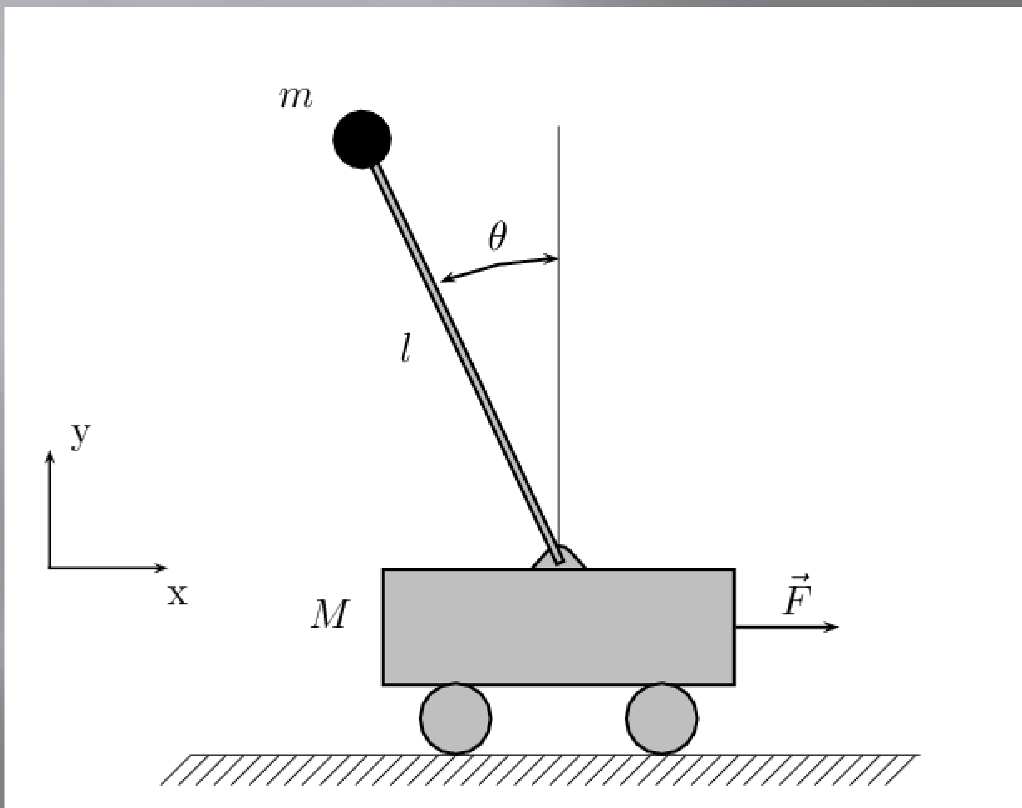
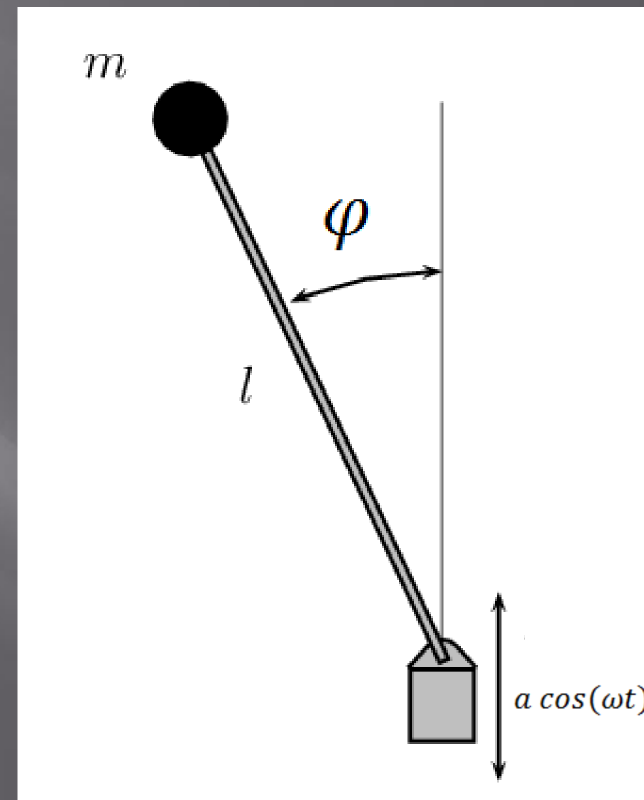
  - Simple Model Results
- Error Analysis

Two systems



Moving Cart

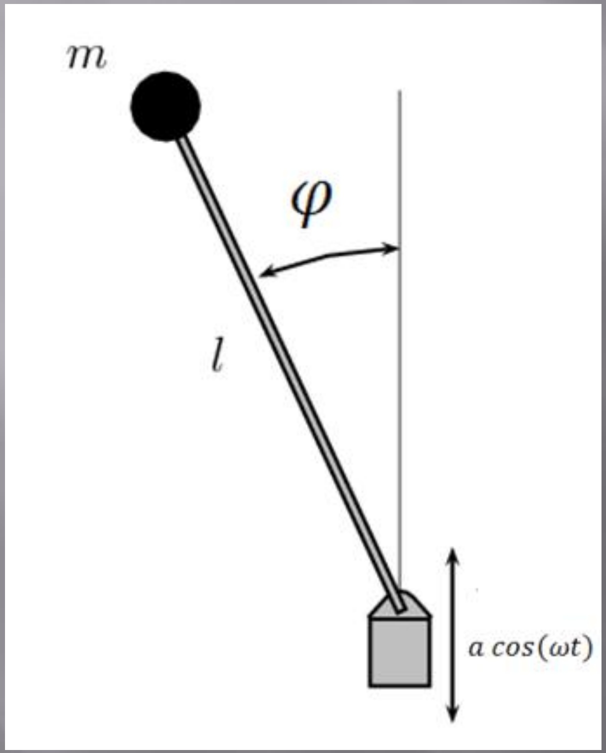


Oscillating Base

Inverted Pendulum in the Real World

- Segway (moving cart)
- Early Seismometers (moving cart)
- Neck and Spine in biomechanics (oscillating)

Simple Theoretical Model



- Oscillating Base
- Treated as a point mass at its center of mass.
- The Lagrangian summarizes the dynamics of the system.
  - Must determine the kinetic and potential energy

Simple Model
Kinetic Energy (1)

- The x and y coordinates of the mass are

  $x = l\sin\varphi$

  $y = l\cos\varphi + a\cos\omega t$

- Differentiating with respect to time,

  $v_x = l\dot{\varphi}\cos\varphi$

  $v_y = -l\dot{\varphi}\sin\varphi - a\omega\sin\omega t$

- Kinetic Energy is

  $$T = \frac{mv^2}{2}$$

Simple Model
Kinetic Energy (2)

$$T = \frac{m}{2}\left[v_x^2 + v_y^2\right]$$

- Substituting in the velocity,

$$T = \frac{m}{2}[l^2\dot{\varphi}^2\cos^2\varphi + \; l^2\dot{\varphi}^2\sin^2\varphi$$
$$+ 2la\omega\dot{\varphi}\sin\varphi\sin\omega t + a^2\omega^2\sin^2\omega t]$$

- But $\sin^2\varphi + \cos^2\varphi$, so

$$T = \frac{m}{2}[l^2\dot{\varphi}^2 + 2la\omega\dot{\varphi}\sin\varphi\sin\omega t + a^2\omega^2\sin^2\omega t]$$

Simple Model
Kinetic Energy (3)

- However,

  $$2la\omega\dot{\varphi}\sin\varphi\sin\omega t =$$
  $$2la\omega^2\cos\omega t\cos\varphi - 2la\omega(\sin\omega t\cos\varphi)$$

- So

  $$T = \frac{ml^2}{2}\dot{\varphi}^2 + la\omega^2 m\cos\omega t\cos\varphi$$

  + (complete derivative) + (function of only t)

Simple Model
Potential Energy

- On Earth, the potential gravitational energy is

  $U = -mgh$

- So the potential energy of the pendulum is

  $U = -mg(l\cos\varphi + a\cos\omega t)$

- Or,

  $U = -mgl\cos\varphi$ + (function of only t)

Simple Model Lagrangian

- The Lagrangian is

$$\mathcal{L} = T - U$$

- Or,

$$\mathcal{L} = \frac{ml^2}{2}\dot{\varphi}^2 + la\omega^2 m\cos\omega t\cos\varphi + mgl\cos\varphi$$

+ (complete derivative) + (function of only t)

Simple Model
Equation of Motion

- The equation of motion is obtained by

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = 0$$

$$\ddot{\varphi} + \left(\frac{g}{l} + \frac{a\omega^2}{l}\cos \omega t\right)\sin \varphi = 0$$

Simple Model Stability

- Can be separated into two oscillations:

large small (amplitude)

$$\varphi = \phi(t) + \psi(t)$$

slow fast (frequency)

- The interaction between these oscillations causes the vertical position to become stable.

Simple Model
Interaction of Scales (1)

- $\ddot{\varphi}$ may be written as

$$\ddot{\varphi} = -\frac{\partial u}{\partial \varphi} + f$$

where

$$u = -\frac{g}{l}\cos\varphi \qquad f = -\frac{a\Omega^2}{l}\cos\Omega t \sin\varphi$$

therefore

$$\ddot{\varphi} = \ddot{\phi} + \ddot{\psi} \cong -\frac{du}{d\phi} - \psi\frac{d^2u}{d\phi^2} + f(\phi, t) + \psi\frac{df}{d\phi}$$

Simple Model
Interaction of Scales (2)

- ψ is small, but $\ddot{\psi}$ is large.
- ϕ is large, but it is slow.

- Balancing, we have that

$$\ddot{\psi} = f(\phi, t)$$

- And because ϕ can be treated as a constant,

$$\psi = -\frac{f}{\Omega^2}$$

Simple Model
Interaction of Scales (3)

- We also have

$$\ddot{\phi} = -\frac{du}{d\phi} - \psi\frac{d^2u}{d\phi^2} + \psi\frac{df}{d\phi}$$

- Averaging this, we get

$$\ddot{\phi} = -\frac{du}{d\phi} + \langle\psi\frac{df}{d\phi}\rangle = -\frac{du}{d\phi} - \frac{1}{\Omega^2}\langle f\frac{df}{dt}\rangle$$

$$\ddot{\phi} = -\frac{du_{eff}}{d\phi} \qquad u_{eff} = u + \frac{1}{2\Omega^2}\langle f^2\rangle$$

Simple Model
Interaction of Scales (4)

$$\langle f^2 \rangle = \frac{a^2\Omega^4}{l^2}\frac{1}{T}\sin^2\varphi\int_t^{t+T}\cos^2\Omega\tau\,d\tau$$

$$\langle f^2 \rangle = \frac{a^2\Omega^2}{4l^2}\sin^2\varphi$$

- So we have an equation for the effective potential

$$u_{eff} = \frac{g}{l}\left(-\cos\varphi + \frac{a^2\Omega^2}{4gl}\sin^2\varphi\right)$$

Simple Model
Stability Condition

- The derivative of u_{eff} with respect to φ is

$$\frac{du_{eff}}{d\varphi} = \frac{g}{l}\sin\varphi\left(1 + \frac{a^2\Omega^2}{2gl}\cos\varphi\right)$$

- From this, we see that $\varphi = 0$ and $\varphi = \pi$ are stable.
- The stability condition for $\varphi = \pi$ is

$$\frac{a\Omega}{\sqrt{2gl}} > 1$$

Simple Model
Stability Region

- The potential has two maxima and two minima for $0 \le \varphi < 2\pi$ if the stability condition is met
  - Minimum at $\varphi = 0$ and $\varphi = \pi$
  - Maxima are between 0 and π, and π and 2π
    - Their precise location depends on the parameters.
    - Find by taking the derivative and setting it equal to 0.

- The potential vs. angle graph is shown after the parameters have been determined.

Parameters

- $g = -9.8\ m/s^2$
- $a = 0.009 \pm 0.0005$ m

- Pendulum is a trapezoidal prism
  length = 28cm
  top: 0.9cm x 0.9cm
  bottom: 0.9cm x 0.4cm

l = 0.12m (from center of mass)

Stroboscope (1)

- An instrument used to make a cyclically moving object appear slow-moving or stationary.

- The lamp emits brief and rapid flashes of light.
- The frequency of the flash is adjusted until it is equal to the object's frequency.
  - This makes it appear stationary.
- Example with cpu fan: http://www.youtube.com/watch?v=_eoDVpC67Rc&feature=related

Stroboscope (2)

- Used to measure the frequency of fast oscillations.
  - When the pendulum did not appear to move vertically, frequency can be read off device.
  - Measurements were in units of min^{-1}
- Minimum stable frequency measured as $1970 \pm 49\ min^{-1}$
  - 206 ± 5.1 rad/s
- Stability range measurements at $2097\ min^{-1}$
  - 220.0 rad/s

Simple Model
System of 1st Order Equations

- Necessary for inputting equation of motion into MatLab.

$$y = \dot{\varphi}$$

$$\dot{y} = -\left(\frac{g}{l} + \frac{a\omega^2}{l}\cos(\omega t)\right)\sin\varphi$$

Runge-Kutta

- Family of iterative methods to solve first order ODEs

- Fourth order Runge-Kutta is the standard method

- Error per step: $O(h^5)$; Total Error: $O(h^4)$

Fourth Order Runge-Kutta

- Given an initial value problem

  $$y' = f(t, y) \qquad y(t_0) = y_0$$

- The slope is the weighted average of the slopes:
  - At the beginning of the interval

    $$k_1 = f(t_n, y_n)$$

  - At the midpoint of the interval

    $$k_2 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1\right)$$

    $$k_3 = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2\right)$$

  - At the end of the interval

    $$k_4 = f(t_n + h, y_n + hk_3)$$

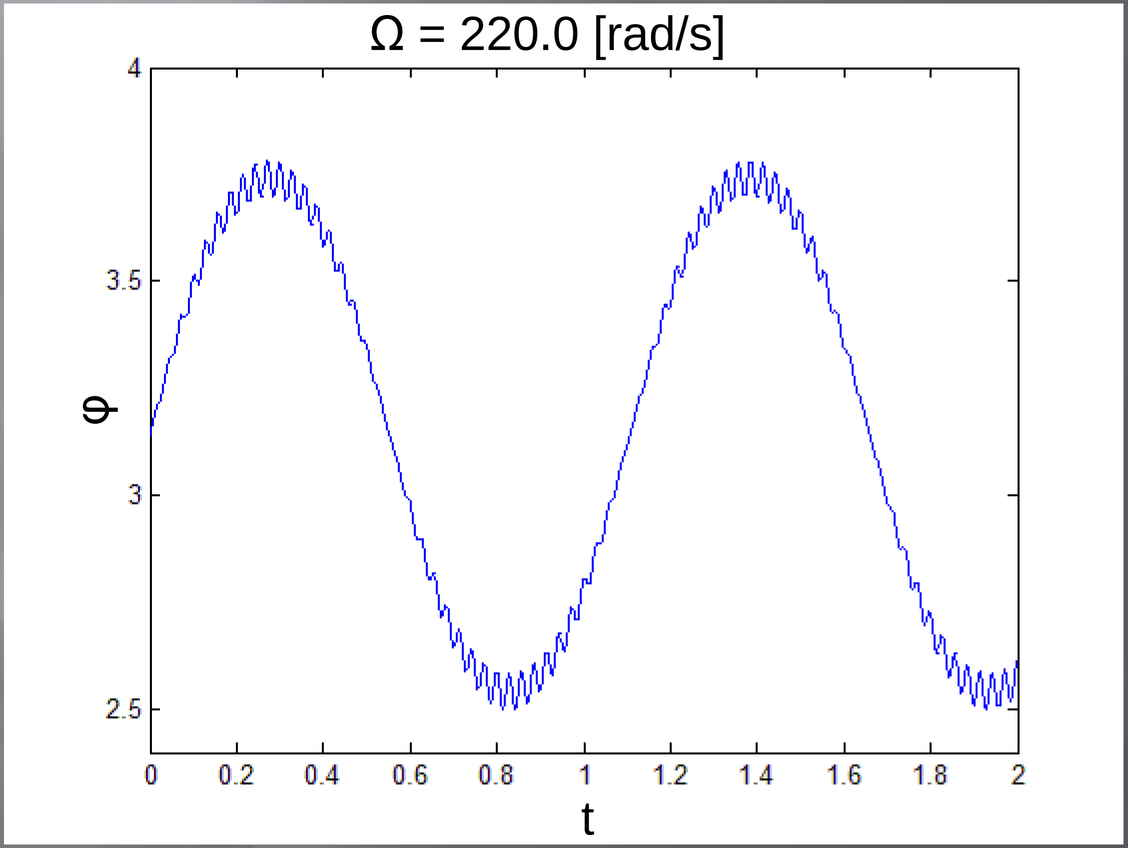
Fourth Order Runge-Kutta

$$slope = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$y_{n+1} = y_n + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

$$t_{n+1} = t_n + h$$

Simple Model
Time vs. Angle



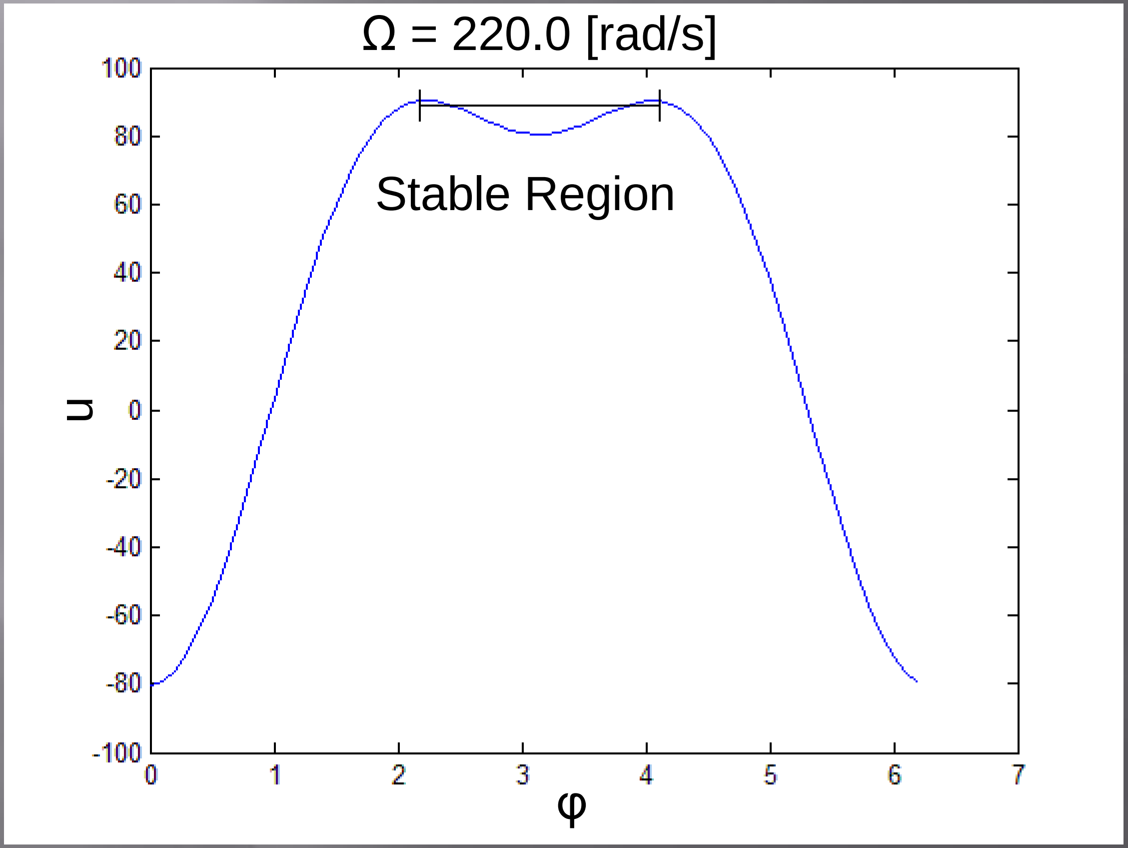
Minimum Frequency of Stability

- Theoretically, the pendulum is stable when

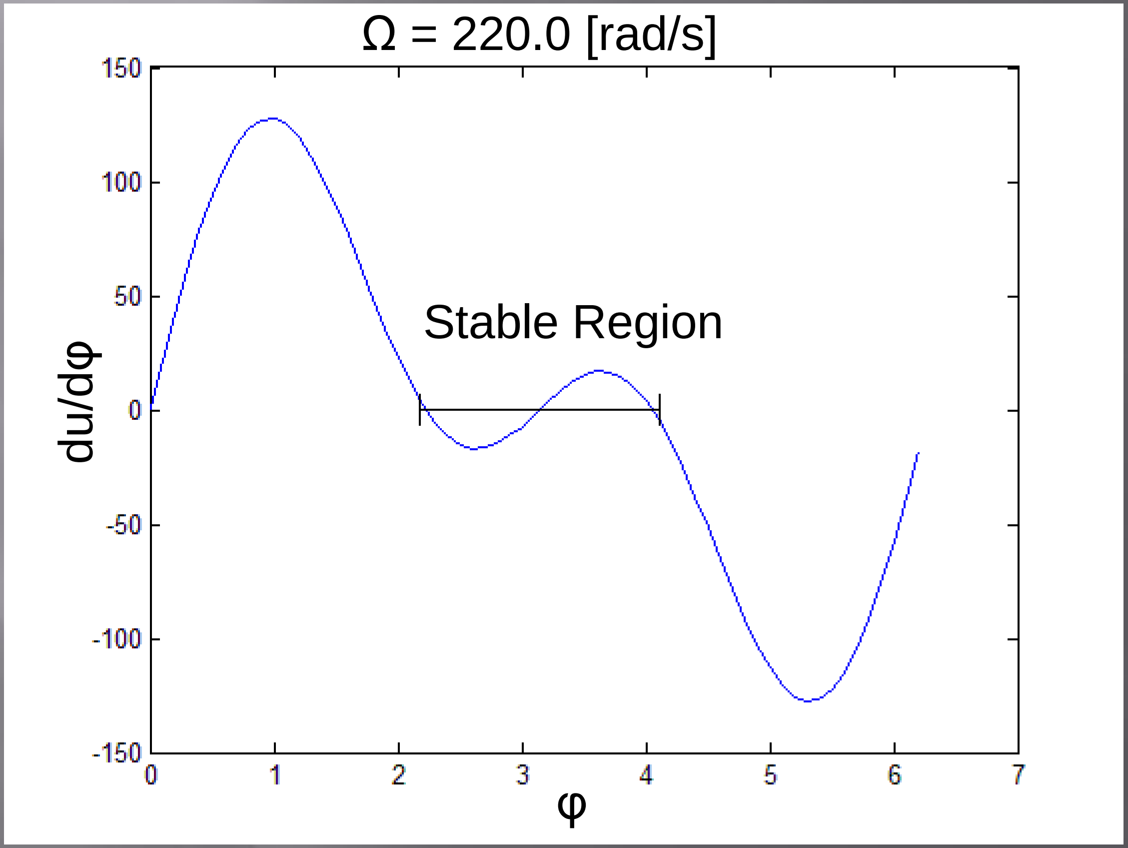
$$\Omega > \frac{\sqrt{2gl}}{a} \qquad \Omega > 164.62 \text{ rad/s}$$

- Experimentally, minimum frequency of stability is 206 ± 5.1 rad/s

Stability Graph



Range of Stability



Stability

- The vertical position is stable between $\varphi = 2.226$ and $\varphi = 4.057$ according to theory.
- Stability range is 0.915 radians, or 52.4° from π

- Experimentally, the pendulum is stable at
- 0.553 ± 0.0088 rad, or 31.7° ± 0.5° from π

Error Analysis

- Stroboscope
  - Difficult to fine tune
  - Resonance problem
- Pendulum set up
  - Interference from guide
  - Pendulum loosely attached to base (wobbling)
  - Tape used to hold constant frequency loosens from the vibrations.

Problems with Simple Model and Future Work

- The pendulum was treated as though it were a point mass at its center of mass.

- This is more appropriate for a sphere at the end of a thin, rigid rod.

- The distributed mass of our pendulum needs to be taken into account.
  - This will be the focus of our future work.

References

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