Hyperbolic Groups (Presentation Notes)

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1 Motivation

Why are hyperbolic groups interesting?

- From a probabilistic point of view, most[1] finitely generated groups are hyperbolic.
- Hyperbolic groups have solvable word problem[2].
- As a result, "most" finitely generated groups have solvable word problem.

We use geometric techniques to prove the second point.

[1] The underlying statistical model requires that the generating set $\{a_1, \ldots, a_n\}$ and the number of relators be fixed. Next, set an upper bound l on the word length of each relator. Finally, for each relator, choose a word at random (uniformly and independently) from the set of reduced words over $\{a_1, \ldots, a_n\}$ of length at most l.[1]

[2] Given a group $G = \langle S \mid R \rangle$ and an arbitrary word $w \in S^*$, is $w =_G 1$? The word problem is undecidable.

2 Cayley Graphs

Intuitively: A Cayley graph is a graph that models the multiplication of a group G.

- Vertices are labelled by elements of G.
- Edges are labelled by elements of the generating set S of G.

Example. Here are two distinct Cayley graphs for the group $(\mathbb{Z}, +)$.

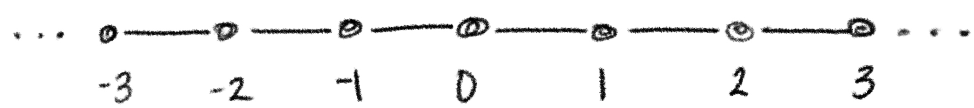


Figure 1: The Cayley graph $\mathrm{Cay}_{\{1\}}(\mathbb{Z})$.

Figure 2: The Cayley graph $\mathrm{Cay}_{\{1,2\}}(\mathbb{Z})$. (Notice how the redundant generator added "extra" edges to the graph.)

Notice that $\mathrm{Cay}_{\{1\}}(\mathbb{Z})$ and $\mathrm{Cay}_{\{1,2\}}(\mathbb{Z})$ are *not* isomorphic as graphs. So, Cayley graphs are not unique.

3 Geodesic Metric Spaces

We want to associate a group G to a metric space. We do this by metrizing $X = \mathrm{Cay}_S(G)$. In particular, it is desirable to turn X into a geodesic metric space.

Definition. Let (X, d) be a metric space and let $L \in \mathbb{R}_{\geq 0}$. A path p between points x, y in X is geodesic if

1. there exists an isometric embedding from $[0, L] \to p$ and
2. if p is the shortest path between x and y.

Definition. A space (X, d) is geodesic if for any pair of points $x, y \in X$, there exists a geodesic path between x and y.

Metrizing $X = \mathrm{Cay}_S(G)$:

- Identify each edge in X with the unit interval $[0, 1]$.
- Define the distance between two vertices x and y to be the length of the shortest edge path connecting x and y.[3]

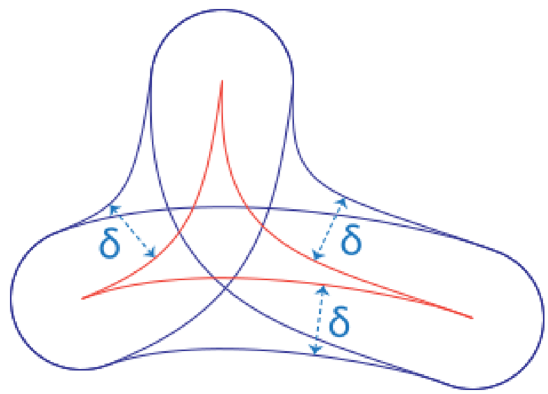
We now have a way to associate a geodesic metric space to any (finitely generated) group G.

[3] This is commonly referred to as the path or word metric.

4 Hyperbolic Groups

Definition. A triangle xyz is geodesic if $\overline{xy}, \overline{yz}, \overline{xz}$ are geodesic paths.

Definition. Let $\delta \in \mathbb{R}_{\geq 0}$. A triangle xyz is $\delta-$slim if the δ nbhd of $\overline{xy}$ and the δ nbhd of $\overline{xz}$ cover all of xyz.



Definition. A geodesic metric space (X, d) is hyperbolic (or $\delta-$hyperbolic) if there exists $\delta \in \mathbb{R}_{\geq 0}$ such that all geodesic triangles in X are $\delta-$slim.

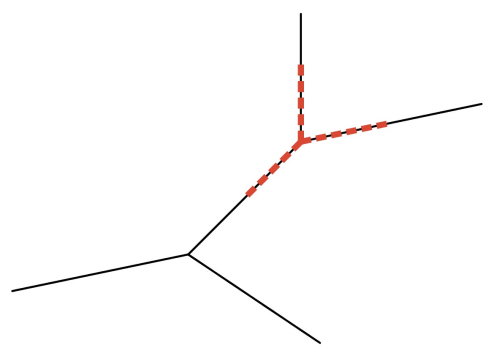


Figure 3: Geodesic triangles form tripods in trees.

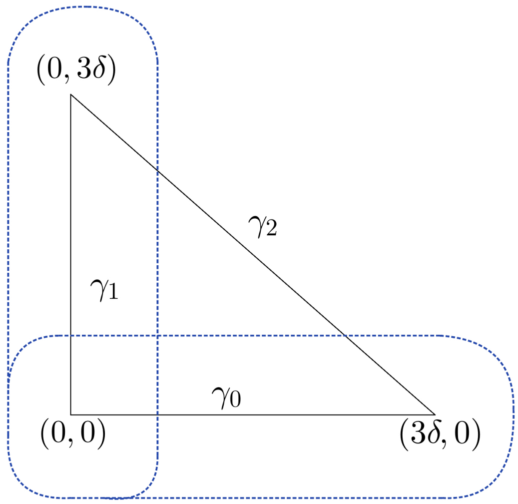


Figure 4: Euclidean space is *not* $\delta-$hyperbolic.

Definition. A group G is hyperbolic if its cayley graph $\text{Cay}_S(G)$ is $\delta-$hyperbolic.

(Note: Implicit in this definition is that G is necessarily finitely generated if it is hyperbolic.)

A PROBLEM WITH THIS DEFINITION:

- We already know that Cayley graphs are *not* unique!
- Given two distinct cayley graphs X and X' of G, what if X is hyperbolic but X' is not?

It turns out that this second point can't happen, so we have a well-defined definition. Let's see why...

5 Quasi-Isometric Embeddings

Definition. Let (M_1, d_1) and (M_2, d_2) be two isometric spaces and let $f : M_1 \to M_2$. Then f is a quasi-isometric embedding if there exist $K, C \in \mathbb{R}^+$ such that

$$\frac{1}{K} d_2(f(x), f(y)) - C \leq d_1(x, y) \leq K d_2(f(x), f(y)) + C$$

for all $x, y \in M_1$.

Quasi-isometric embeddings turn out to be a very important tool in geometric group theory. Essentially, we want to study properties that are invariant under QI embeddings.

Intuitively:

- Quasi-isometric embeddings are like isometric embeddings with some tolerance for "error" on a small enough scale, *i.e.,* quasi-isometries respect large scale geometry.
- If $X \sim_{QI} Y$, then if we zoom out far enough these two spaces should "look" the same.

Examples:

- Recall the two Cayley graphs given earlier in figures 1 and 2.
- Set $M_1 = \text{Cay}_{\{1\}}(\mathbb{Z})$ and $M_2 = \text{Cay}_{\{1,2\}}(\mathbb{Z})$.
- Then taking $f : M_1 \to M_2$ to be an embedding with $K = 1$ and $C = 0$ is a quasi-isometric embedding.
- Or, define $g : M_2 \to M_1$ by mapping black edges to black edges and a blue edge to its neighbouring black edges in the obvious way. Take $K = 2$ and $C = 0$. This is also a QI embedding.

Some other examples:

- All finite graphs (with path metric) are quasi-isometric.
- $\mathbb{R}$ and $\mathbb{Z}$ are quasi-isometric via the map $f : \mathbb{R} \to \mathbb{Z}$ defined by $f : x \mapsto \lfloor x \rfloor$.
- $\mathbb{Z}$ and $\mathbb{Z}^2$ are *not* quasi-isometric (idea of proof: asymptotic argument... show that balls in $\mathbb{Z}^2$ grow much faster than those in $\mathbb{Z}$, so there cannot exist values K, C that would allow you to set a bound on the corresponding distances).

Theorem. [2] Let X and X' be two distinct cayley graphs for a group G. Then $X \sim_{QI} X'$.

Theorem. [2] If $X \sim_{QI} X'$, then X is hyperbolic if and only if X' is hyperbolic.

The proofs for both of these theorems require very careful consideration of $\delta-$slim triangles and QI embeddings. (Very LONG and cumbersome chains of inequalities!)

Now we have a sensible definition for what it means for a group to be hyperbolic. The next goal is to show that hyperbolic groups have Dehn presentation using a geometric argument.

6 Dehn Presentation

Definition. Let S be a finite alphabet and $n \in \mathbb{N}$. The group presentation

$$\langle S \mid u_1 v_1^{-1} = \cdots = u_n v_n^{-1} = 1 \rangle$$

is a Dehn presentation if:

- For all $1 \leq i \leq n$, the length of the word v_i is shorter than the length of the word u_i, and
- Any freely reduced word non empty w over $S \cup S^{-1}$ such that $w =_G 1$ must contain a subword of the form u_i or u_i^{-1}.

Groups with Dehn presentation have solvable word problem.

Shortening algorithm:

- Choose a word $w \in G$.
- Freely reduce w.
- Check to see if w contains a subword of the form u_i or u_i^{-1}. (This can always be done because w is finite.) If no, then $w \neq_G 1$.
- If yes, replace the occurrence of u_i with v_i. Then $w = w' v_i w''$. Repeat procedure on the word $w' v_i w''$. This always terminates because replacing u_i with v_i reduces the length of the word.

Finally, given a hyperbolic group G we construct a Dehn presentation for G by following non-geodesic paths in $\mathrm{Cay}_S(G)$.

7 Taming Quasi-Geodesics

Think of a "non-geodesic" path as the image of some quasi-isometric embedding $\gamma : [0, L] \to X$. Such a path is commonly referred to as a quasi-geodesic.

We are interested in a special kind of quasi-geodesic path:

Definition. A path p in X is $k-$local geodesic if every subpath q of p with $|q| \leq k$ is geodesic.

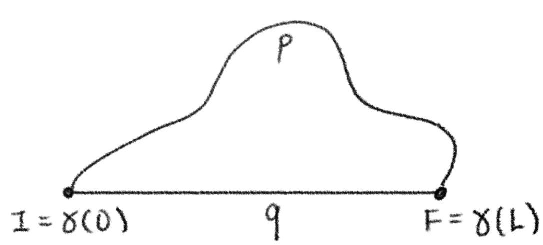
Intuitively: $k-$local geodesic paths are almost geodesic (they are geodesic on a small enough scale).

The following proof offers one example of how the geometry of $\delta-$slim triangles can be used to obtain a "nice" property of hyperbolic spaces.

Lemma 1. [3] Let X be $\delta-$hyperbolic. Set $k > 8\delta$. Suppose p is a $k-$local geodesic path in X. Then there exists a geodesic path q sharing the same end points as p such that p is contained in a 2δ nbhd of q.

In other words, in hyperbolic spaces, $k-$local geodesics stay close to geodesics.

Proof. First, note that the path p can be described by the image of a quasi-isometric embedding $\gamma : [0, L] \to X$.



Choose M to be a point on p having maximal distance from q. Then there exists some $t_M \in [0, L]$ such that $\gamma(t_M) = M$.

Either:

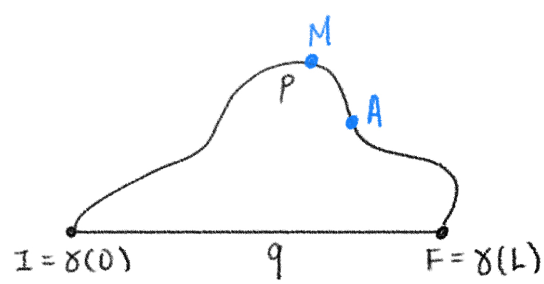
1. $d(0, t_M) < 4\delta$ or

2. $d(0, t_M) > 4\delta$ (and $d(t_M, L) > 4\delta$).

Case 1: Suppose $d(0, t_M) < 4\delta$. Fix a point A on p such that

$$d_X(M, A) > 4\delta,$$

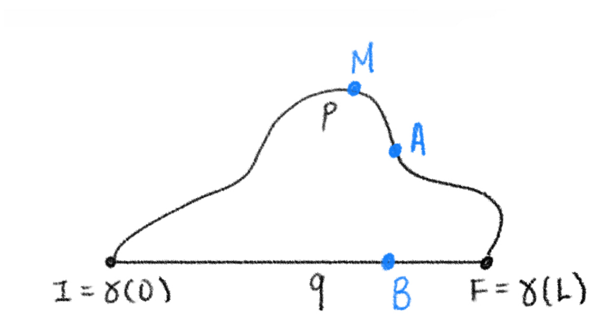
and $p|_{I,A}$ is geodesic.[4]



Note: This is possible because p must have length greater than 8δ since it's a $k-$local geodesic.

Now, choose a point B on q with minimal distance from A.

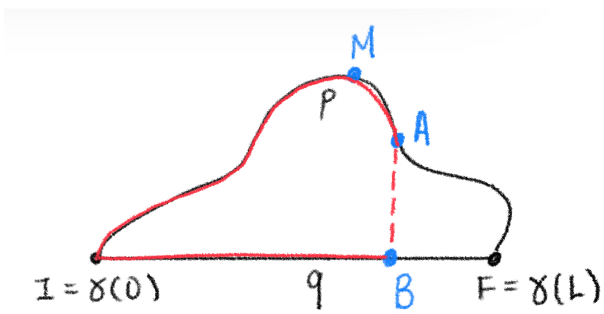
[4]Notation: $p|_{I,A}$ denotes the subarc of p with end points $I = \gamma(0)$ and A.



Consider the triangle $\triangle IAB$. This is a geodesic triangle. So, there exists a point x on either $\overline{IB}$ or $\overline{AB}$ such that

$$d_X(M, x) \leq \delta$$

(by the definition of $\delta-$slim triangle).



Suppose $x \in \overline{AB}$, then

$$\begin{aligned} d_X(M, x) - d_X(A, x) = d_X(M, A) & \\ > 4\delta & \qquad \text{by previous assumption} \\ \implies d_X(A, x) > 3\delta & \qquad (\star) \end{aligned}$$

Also,

$$\begin{aligned} d_X(M,B) - d_X(A,B) &\leq (d_X(M,x) + d_X(x,B)) - (d_X(A,x) - d_X(x,B)) \\ &= d_X(M,x) - d_X(A,x) \\ &< \delta - 3\delta && \text{by } (\star) \\ &< 0, \end{aligned}$$

which contradicts the fact that M was chosen to have maximal distance from the path q. So, x must lie on $\overline{IA}$.

Now, for any other point $y \in p$,

$$d_X(y,x) \leq d_X(M,x) \leq \delta.$$

Therefore, p is contained in a $\delta-$nbhd of q.

The proof for case 2 is omitted. For complete proof, see [3] □

Lemma 2. [2] Let X be a $\delta-$hyperbolic space. Any closed loop γ in X contains a subarc p such that $|p| \leq 8\delta$ and p is *not* geodesic.

Proof. By contradiction. Use lemma 1.

Theorem. [4] If G is hyperbolic, then G admits a Dehn presentation.

Proof. We can assume G has finite generating set S and $X = \mathrm{Cay}_S(G)$ is $\delta-$hyperbolic.

Step 1: Define a procedure to construct a set of Dehn relators:

- Fix $k > 8\delta$, and define the set

$$W_k = \{w \in S^* \mid w \text{ is freely reduced and } |w| \leq k\}.$$

- Let $p(w)$ denote the path in X associated to the word w.
- For each $w \in W_k$, decide if $p(w)$ is geodesic or not. (Note: W_k is a finite set so this procedure terminates.)
- If $p(w)$ is not geodesic, set $w = u_i$. There exists a word $w' \in W_k$ such that $p(w')$ is geodesic and shares the same end points as $p(w)$. Set $w' = v_i$.
- As a result, $u_i = v_i$, or equivalently, $u_i v_i^{-1} = 1$.
- When the procedure terminates, we obtain a list of relators

$$R = \{u_1 v_1^{-1}, \ldots, u_n v_n^{-1}\}.$$

Step 2: Verify the presentation $\langle S \mid R \rangle$ is a Dehn presentation and that $G \cong \langle S \mid R \rangle$.

- If $w =_G 1$, then we want to show that $w \in \ll R \gg$.
- Induct on the length of w.
- Base case: $|w| = 0$, *i.e.,* w is the empty word.
- Inductive step: Suppose all words $w =_G 1$ of length at most L are contained in $\ll R \gg$.
- Let $w =_G 1$ have length $L + 1$. Since $w =_G 1$, $p(w)$ forms a cycle or a closed loop in X.
- By lemma 2, $p(w)$ contains a non-geodesic subarc γ of length less than 8δ.

- By construction, the arc γ corresponds to a word u_i.
- So, $w = w'u_iw'' = w'v_iw''$, but $|v_i| < |u_i|$, so $|w'v_iw''| < L + 1$. By inductive hypothesis, $w'v_iw'' \in \ll R \gg$.
- Now, $w'u_iv_i^{-1}w'^{-1} \in \ll R \gg$ (by definition of normal closure). So,

$$\begin{aligned} w'u_iv_i^{-1}w'^{-1} \cdot w'v_iw'' &= w'u_iv_i^{-1}v_iw'' \\ &= w'u_iw'' \in \ll R \gg . \end{aligned}$$

- Notice that, in the induction, we have also verified that if $w =_G 1$, then w contains a subword of the form u_i, so $\langle S \mid R \rangle$ is indeed a Dehn presentation for G.

□

8 Examples of Hyperbolic Groups

- Any finite group.
- Finitely generated free groups.
- Fundamental group of compact negatively-curved Riemannian manifold.
- Virtually cyclic groups, *e.g.,* infinite dihedral group.[5]
- Virtually free groups, *e.g.,* $F \rtimes H$, where F is free and H is finite, or $H * K$, where H and K are finite.

[5]This is a result of the fact: If H is a finite index subgroup of G, then $G \sim_{QI} H$.

References

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