Fundamental groups of 3-manifolds.

(Based on lecture notes of Stefan Friedl).

(Our manifolds are compact, connected, orientable, but not closed).

If M is a k-manifold, we would like to study the fundamental group $\pi_1(M)$ in order to say something about M.

__Question__: What kind of groups will we encounter? Do they depend on k?

__Observations__ ①. For $k=1,2$, there is no mystery. The compact k-manifolds in these dimensions are classified, so there is a list of fundamental groups that can arise in these dimensions.

②. Every k-manifold can be written as a CW complex having finitely many 1-cells and one 0-cell, finitely many 2-cells. We can calculate $\pi_1(M)$ from such a decomposition

- Each 1-cell gives a generator
- Each 2-cell gives a relation.

$\Rightarrow$ Any group arising as $\pi_1(M)$ will be __finitely presented__

$$\pi_1(M) = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

Theorem: Let

$$G = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$$

be a finitely presented group and $k \geq 4$. Then there's a k-manifold with $\pi_1(M) \cong G$.

Proof:

Recall the operation "#" which is the connect sum of two k-manifolds: If N, M are k-manifolds then

$$N \# M = (N \setminus \mathrm{int}(B_1)) \cup_f (M \setminus \mathrm{int}(B_2))$$

ie, remove the interior of a k-ball from each and glue along the resulting boundary.

So we start with

$$\underbrace{S^1 \times S^{k-1} \# S^1 \times S^{k-1} \# \ldots \# S^1 \times S^{k-1}}_{n \text{ copies}} = X$$

whose fundamental group is free on generators $x_1, \ldots, x_n$.

We need to modify the fundamental group so that it has relations $r_1, \ldots, r_m$.

Represent each r_i by a curve c_i in X. E.g. if r_1 is $x_2 x_1^2 x_3^{-1}$, then c_1 goes one time around the S^1-factor of the second summand, twice around the S^1-factor of the first summand,

and once in the reverse direction around the third summand (implicit orientations here).

Fatten each c_i so that we can think of it as a "thickened" curve $\underset{\substack{\cong \\ N_i}}{c_i \times D^{k-1}} \hookrightarrow X$.

Now the inclusion map

$i : X \setminus (N_1 \cup \ldots \cup N_m) \longrightarrow X$

actually induces an isomorphism of fundamental groups (this uses $k \geq 4$).

Each N_i has boundary $\partial(c_i \times D^{k-1}) \simeq S^1 \times S^{k-2}$, note that $D^2 \times S^{k-2}$ also has $\partial(D^2 \times S^{k-2}) \simeq S^1 \times S^{k-2}$. Construct a manifold Y from $X \setminus (N_1 \cup \ldots \cup N_m)$ by gluing $D^2 \times S^{k-2}$ onto each of its boundary components.

One can check using Seifert + Van Kampen that gluing $D^2 \times S^{k-2}$ onto ∂N_j has the effect of setting $c_j = 1$. (ie $r_j = 1$).

Thus

$$\pi_1(Y) = \langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$$

<u>Corollary</u>: Since the isomorphism problem for finitely presented groups is unsolvable, we can conclude that there is no algorithm for determining when two k-mflds ($k \geq 4$) are homeomorphic.

For dimension 3, however, the picture is quite different.

<u>Definition</u>: The <u>deficiency</u> of a presentation

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

is the integer $n-m$. The <u>deficiency</u> of a group G is the maximum deficiency of all presentations of G.

<u>Observation</u>: Note that we can always find a presentation with deficiency as low as we please, by adding redundant relators. Thus the challenge is increasing the deficiency, which is why we take a max.

<u>Examples</u>:

① $$\mathbb{Z}^3 \simeq \langle x_1, x_2, x_3 \mid [x_1, x_2], [x_1, x_3], [x_2, x_3] \rangle$$

the presentation has def 0. In fact, one can show that $\operatorname{def}(\mathbb{Z}^3) = 0$ (so the presentation above realizes the max).

② $$\mathbb{Z}^4 \simeq \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_1, x_3], \text{etc} \dots \rangle$$

(all generators commute).

4 choose $2 = 6$

So the deficiency of the presentation is -2.

Indeed one can prove $\operatorname{def}(\mathbb{Z}^4) = -2$, and in general $\operatorname{def}(\mathbb{Z}^k) < 0$ if $k > 3$.

So a first theorem showing fundamental groups of 3-manifolds are special is:

<u>Theorem</u>: If M is a 3-manifold, then $\pi_1(M)$ admits a presentation of deficiency 0.

<u>Corollary</u>: There is no 3-mfld M with $\pi_1(M) = \mathbb{Z}^k$ for $k > 3$. For $k = 3$ on the other hand, we have $\pi_1(S^1 \times S^1 \times S^1) = \mathbb{Z}^3$, and $k = 2$ $\pi_1(S^1 \times S^1 \times [0,1]) \simeq \mathbb{Z}^2$ (allowing boundary) and $k = 1$ gives $\pi_1(S^1 \times S^2) \simeq \mathbb{Z}$.

So, this theorem shows that $k = 3$ is the highest dimension having "special" fundamental groups.

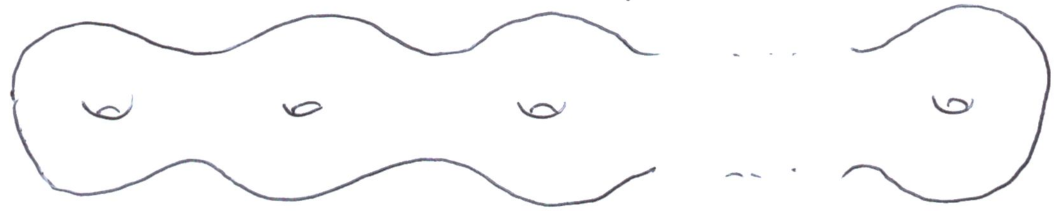
<u>Idea of proof of theorem</u>: (We'll do M compact w/o boundary).

Let M be a 3-manifold. In 3 dimensions, every manifold has a triangulation (this is not possible in dimension ≥ 4! The manifold E_8 in dim 4 has no triangulation, for example).

So, we may think of M as a simplicial complex.

Let H denote a thickened neighbourhood of the 1-skeleton. So H is composed of pieces like

and has the homeomorphism type of

$H \simeq$  } solid.

The complementary region H' has the same homeomorphism type, and H, H' are glued along their common boundary Σ_i, a genus g surfaces.

<u>Now</u>: $\pi_1(H) \simeq \pi_1(H') = F_g$, free group on g generators.

and $\pi_1(\Sigma_i) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$.

Then the Seifert Van-Kampen theorem gives:

$\pi_1(M) \simeq \pi_1(H) * \pi_1(H') /$ relations, one for each a_i, b_i

i.e.

$$\pi_1(M) = \langle x_1, \ldots, x_g, y_1, \ldots, y_g \mid j_1(a_1) j_2(a_1)^{-1}, j_1(b_1) j_2(b_1)^{-1}, \ldots \text{etc} \rangle$$

<u>So 2g</u> generators + $2g$ relations, ie. deficiency zero.

So the crux <u>is</u>: Why triangulable in low dimensions, not in higher?