MATH 2080

Lecture 1

§ 0.1

$\boxed{\text{This class will be difficult.}}$

Tips and remarks:

- This class will consist entirely of proofs and logical deduction. Practice writing proofs by working together. Improve the clarity of your writing by imposing some basic standards on yourself: only use full sentences, and every sentence you write must have a logical purpose.

- We establish a language for discussion by making definitions. You cannot participate in any discussions (in class or labs) and you cannot complete any assigned work without knowing definitions. Memorize them all.

This does not mean word-for-word memorization of the book, but whatever you write as a definition must be logically equivalent to what is in the book.

- Learn to read and use mathematical symbols: $\forall$, $\exists$, $\in$, $\notin$, $\mathbb{Z}$, $\mathbb{N}$ (written as J in the book), $\mathbb{R}$, $\mathbb{Q}$, etc.

Our notation for sets is:

$$\left\{ \begin{matrix}\text{where the elements} \\ \text{come from}\end{matrix} \;\middle|\; \begin{matrix}\text{condition required to} \\ \text{be an element of the set}\end{matrix} \right\}.$$

E.g.: $\{x \text{ an integer} \mid 0 \le x < 4\}$
$= \{0, 1, 2, 3\}$.

Some definitions (review of MATH 1240).

Sets A and B are equal if they "have the same elements". Write $x \in A$ if x is an element of A, $x \notin A$ if x is not an element of A.

Write $A \subseteq B$ if $x \in A$ implies $x \in B$, and call A a subset of B. We say A is a proper subset if $A \neq B$.

To show two sets are equal, will often show they're contained in one another, i.e.

Theorem: If A and B are sets, then $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proof:
(See text, this should be review from 1240).

There are many sets that will occur again and again in this class, so we name them using special notation:

$\mathbb{N}$ - the set of positive integers (the book uses J)

$\mathbb{Z}$ - the set of integers

$\mathbb{Q}$ - the set of rational numbers

$\mathbb{R}$ - the set of real numbers.

Note

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

and every containment is proper.

$\emptyset$ - the empty set, that is, the set with no elements.

Also recall interval notation: If $a < b$, then

$$[a, b] = \{ x \in \mathbb{R} \mid a \le x \le b \}$$
$$(a, b] = \{ x \in \mathbb{R} \mid a < x \le b \}$$
$$[a, b) = \{ x \in \mathbb{R} \mid a \le x < b \}$$
$$(a, b) = \{ x \in \mathbb{R} \mid a < x < b \}.$$

Recall:

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \} \quad \text{(union)}$$

and

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \} \quad \text{(intersection)}.$$

These operations behave as follows:

Theorem: If A, B, C are sets, then:

(i) $A \cap B = B \cap A$
(ii) $A \cup B = B \cup A$
(iii) $(A \cap B) \cap C = A \cap (B \cap C)$
(iv) $(A \cup B) \cup C = A \cup (B \cup C)$
(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
(vi) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof: We will prove only (v). (i)–(iv) are obvious, and (vi) is obvious.

To show (v), we show
1 $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ and
2 $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$. (This uses our previous theorem).

To show 1, let $x \in A \cap (B \cup C)$. Then $x \in A$ and $x \in B \cup C$, so we find:
If $x \in B$ then $x \in A$ and $x \in B$, hence $x \in A \cap B$, whereas if $x \in C$ then $x \in A$ and $x \in C$, hence $x \in A \cap C$. Overall we arrive at $x \in A \cap B$ or $A \cap C$, meaning $x \in (A \cap B) \cup (A \cap C)$. So

$$A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C).$$

To show 2, let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. Thus $x \in A$ and $x \in B$, or $x \in A$ and $x \in C$. In either case, x is always in A, so $x \in A$ and $x \in B$ or $x \in C$.

Thus $x \in A \cap (B \cup C)$. Therefore ② holds and (v) is proved.

These facts about sets will often be used without explicit mention of these theorems.

Also recall that the complement of A relative to B is

$$B \setminus A = \{x \in B \mid x \notin A\}.$$

or $\{x \in B \mid x \notin A\}$ is sometimes written $B - A$ ("subtraction").

Theorem (De Morgan's laws)

If A, B, C are sets then

(i) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

(ii) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

Proof: We prove only (i), (ii) being a similar exercise.

Let $x \in A \setminus (B \cap C)$. Then $x \in A$ and $x \notin B \cap C$. Since $x \notin B \cap C$, either $x \notin B$ or $x \notin C$. If $x \notin B$, then $x \in A \setminus B$, whereas if $x \notin C$ then $x \in A \setminus C$. Overall, $x \in A \setminus B$ or $x \in A \setminus C$, so $x \in (A \setminus B) \cup (A \setminus C)$.

$$\Rightarrow A \setminus (B \cap C) \subset (A \setminus B) \cup (A \setminus C).$$

On the other hand if $x \in (A \setminus B) \cup (A \setminus C)$, then $x \in A \setminus B$ or $x \in A \setminus C$. So either $x \in A$ but not B, or $x \in A$ but $x \notin C$. In either case $x \in A$, and x cannot be in both B and C. Thus $x \in A$ and $x \notin B \cap C$. So $x \in A \setminus (B \cap C)$.

$$\Rightarrow (A \setminus B) \cup (A \setminus C) \subset A \setminus (B \cap C).$$

This concludes the proof of (i).