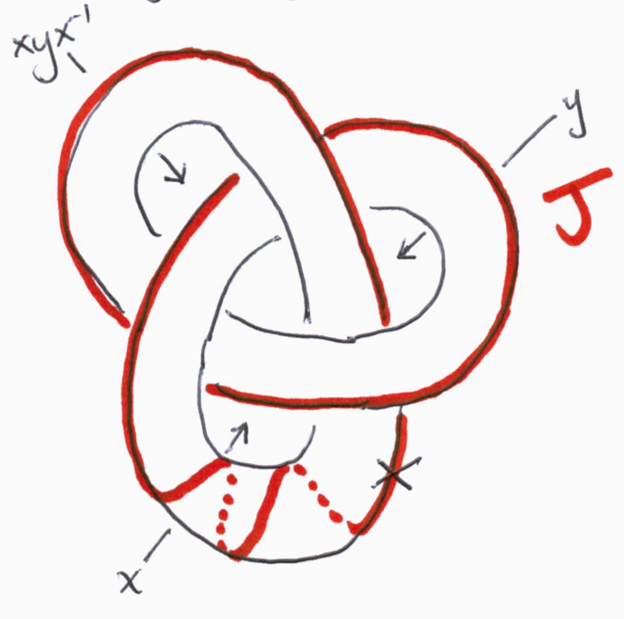
Dehn Surgery

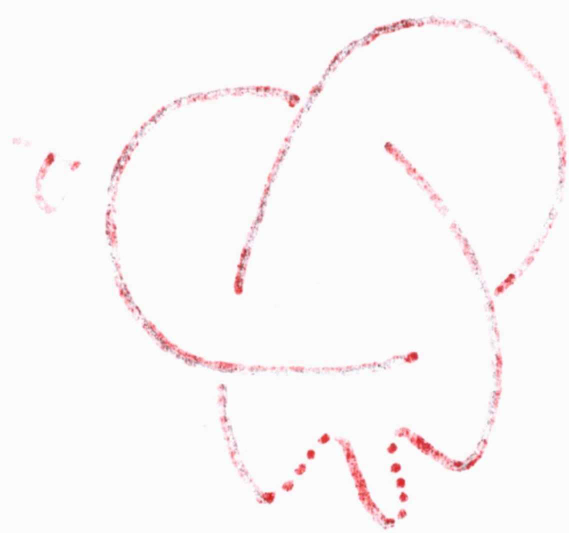
In its original form, the Poincaré conjecture was the following:

<u>Conjecture</u> (False) 1900: A closed, connected, compact 3-manifold with $H_1(M) = 0$ is homeomorphic to the 3-dimensional sphere (Here, just think of $H_1(M)$ as $\pi_1(M) / [\pi_1(M), \pi_1(M)]$ is you have not seen homology).

However Poincaré published a counterexample to this conjecture in 1904, and Max Dehn published infinitely many in 1910. Here is Dehn's construction:

<u>Example</u>: Take a tubular nbhd N of the trefoil in S^3, choose $J \subset \partial N$ to be the curve pictured here:





Now take any homeomorphism

$$h : \partial(S^1 \times D^2) \longrightarrow \partial N$$

with $h(\{pt\} \times \partial D^2) = J$, and form the space

$$M = (S^3 \setminus \mathrm{int}(N)) \cup_h (S^1 \times D^2).$$

Then M is connected, compact and closed, and you can compute $H_1(M;\mathbb{Z})$ using Mayer-Viëtoris, or directly using fundamental groups:

Starting with $S^3 \setminus \mathrm{int}(N)$, we first glue on $\{pt\} \times D^2$, sending its boundary to the curve J. What remains of $S^1 \times D^2$ is $(S^1 \times D^2) \setminus \{pt\} \times D^2$ =  $\cong$ ,

so next we glue in that portion.

Only the first step changes $\pi_1(S^3 \setminus \mathrm{int}(N))$, and that step adds the relation $[J] = 1$.

So we calculate:

$$\pi_1(S^3 \setminus \mathrm{int}(N)) = \langle x, y \mid xyx = yxy \rangle$$

and read off

$$[J] = yx(xyx^{-1})x^{-2} = yx^2yx^{-3}$$

So $\pi_1(M) = \langle x, y \mid xyx = yxy,\ yx^2y = x^3 \rangle$

then set $z = xy$, some algebra gives

$$\pi_1(M) = \langle x, z \mid (zx)^2 = z^3 = x^5 \rangle$$

Now letting all variables commute gives

$$H_1(M) = \langle x, z \mid z^2x^2 = z^3 = x^5 \rangle$$

So $\quad z^2x^2 = z^3 \Rightarrow x^2 = z$

and $\quad z^3 = x^5 \Rightarrow x^6 = x^5 \Rightarrow x = 1$

So $z = 1$, too.

Thus $H_1(M) = 0$ and we've a counterexample to the (original) Poincaré conjecture.

So then it became:

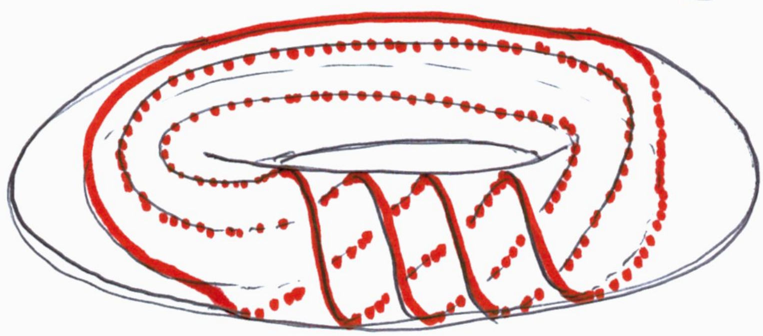
<u>Conjecture</u> (Poincaré, 1904): A closed, connected, compact 3-mfld with $\pi_1(M) = \{1\}$ is homeomorphic to the 3-sphere.

At the time this conjecture was made, there were already examples of manifolds M, M' with $\pi_1(M) \cong \pi_1(M')$ but $M \not\cong M'$. Using Dehn's construction, they are:

(Known since 1898, constructed thus way by Dehn).

<u>Example</u>: Let p, q be relatively prime.

Consider the manifold $S^3 \setminus \mathrm{int}(T^2)$, with a curve J on its torus boundary:



Here $p = 4$
$q = 3$.

wrapping p times in one direction, q in the other.

Then glue using $h: S^1 \times D^2 \longrightarrow \partial T$ with

$h(\{pt\} \times \partial D^2) = J$, the resulting space is

$$L(p,q) = (S^3 \setminus \mathrm{int}(T)) \cup_h S^1 \times D^2 \quad \text{and}$$

$$\pi_1(L(p,q)) = \mathbb{Z}/p\mathbb{Z}.$$

So different q-values give the same group, despite.

<u>Thm</u>: (Reidemeister)

$L(p,q)$ and $L(p',q')$ are homeomorphic if and only if

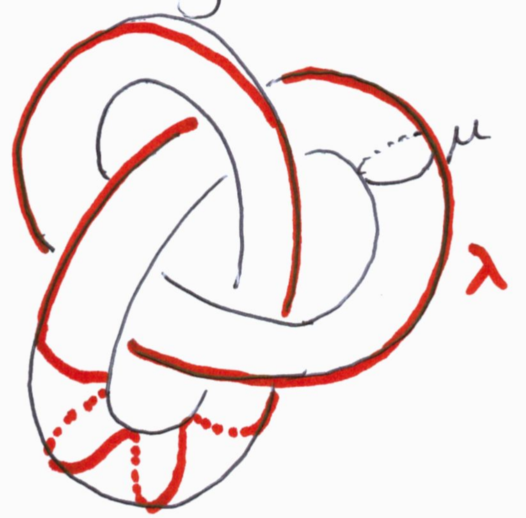
$$\pm q' \equiv q^{\pm 1} \bmod p.$$

So Poincaré's conjecture may have seemed optimistic, but it turns out to be <u>true</u>. In fact, as a consequence of Thurston's Geometrization (proved by Perelman)

These are the <u>only</u> examples of 3-manifolds M, M' with $\pi_1(M) \cong \pi_1(M')$ but $M \not\cong M'$.

Dehn's construction is now known as <u>Dehn</u> <u>surgery</u> on S^3:

Take a knot $K \subset S^3$ and remove a tubular nbhd of it; choose a basis $\{\mu, \lambda\}$ of the fundamental group of the resulting torus boundary:



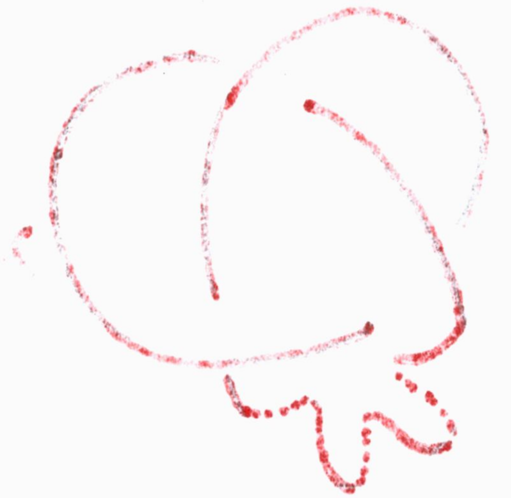
$$\pi_1(\partial(S^3 \setminus \mathrm{int}(N))) \cong \langle \mu, \lambda \rangle \cong \mathbb{Z} \oplus \mathbb{Z}.$$

and for every $J \subset \partial N$, write: $J = \mu^p \lambda^q$.

Gluing $S^1 \times D^2$ to $\partial(S^3 \setminus \mathrm{int}(N))$, sending $\{pt\} \times \partial D^2$ to $\mu^p \lambda^q = J$ gives a manifold:

$$S^3(K; \tfrac{p}{q}) = (S^3 \setminus \mathrm{int}(N)) \underset{J = \mu^p \lambda^q}{\cup} (S^1 \times D^2)$$

"surgery" because we're thinking of "cutting out" and "stitching together" pieces.



We can also generalize by using multiple knots at once, i.e. a link L: Let $V_1, \ldots, V_r \subset S^3$ be solid tori,

$$\partial(S^3 \setminus \{V_1, \ldots, V_r\}) = T_1 \cup T_2 \cup \cdots \cup T_r$$

Then choose a basis $\{\mu_i, \lambda_i\}$ of $\pi_1(T_i)$, use the basis to describe homotopy classes of curves $J_i \subset \partial V_i$. For each $J_i = \mu_i^{p_i} \lambda_i^{q_i}$, glue a solid torus and form

$$S^3\left(L; \frac{p_1}{q_1}, \frac{p_2}{q_2}, \ldots, \frac{p_r}{q_r}\right).$$

E.g. L

Gives instructions to build

$S^3(L; \frac{5}{1}, \frac{5}{2})$ (Weeks manifold)

Just write 5 here instead of $\frac{5}{1}$.

This construction is a fundamental tool in studying 3-manifolds, because

<u>Theorem</u>: (Lickorish 1962, Wallace 1960)

Every closed, connected, compact, orientable 3-manifold can be constructed as

$$S^3(L; \pm 1, \pm 1, \pm 1, \ldots, \pm 1)$$

for some choice of $L \subset S^3$ and signs $\pm$.