Test 1 50 minutes

Name: Solutions. Student ID:

Q1]... [10 points] Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is not differentiable at $(0,0)$, but that all its directional derivatives exist there.

Consider $\lim_{(x,y)\to(0,0)} f(x,y)$, approaching $(0,0)$ along the curve (t,t^2). Then

$$\lim_{t\to 0} \frac{t^2 \cdot t^2}{t^4+t^4} = \lim_{t\to 0} \frac{t^4}{2t^4} = 1/2,$$

So $f(x,y)$ cannot be continuous at $(0,0)$ since $f(0,0)=0$. This it is not differentiable there, since differentiable $\Rightarrow$ cts.

Now, let $\vec{v}=(v_1,v_2)$ be an arbitrary nonzero vector and consider

$$f'((0,0);\vec{v}) = \lim_{h\to 0} \frac{f((0,0)+h(\vec{v}_1,\vec{v}_2)) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3 v_1^2 v_2}{h(h^4v_1^4 + h^2v_2^2)}$$

$$= \lim_{h\to 0} \frac{v_1^2 v_2}{h^2 v_1^4 + v_2^2} = \frac{v_1^2}{v_2},$$ provided $v_2 \neq 0$.

If $v_2 = 0$, then the limit is

$$\lim_{h\to 0} \frac{0}{h(\cdot\text{etc.})} = 0.$$

So all directional derivatives exist at $(0,0)$.

Q2]... [**10 points**]
Show that the function

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is differentiable everywhere. (You may use without proof that compositions/sums/products/differences of rational functions and trig functions are continuous on their domain).

To show the function is differentiable at $(x,y) \neq (0,0)$, it suffices to show the partials are continuous at (x,y). We compute

$$\frac{\partial f}{\partial x} = 2x\sin\left(\frac{1}{x^2+y^2}\right) + (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)\cdot\frac{-1}{(x^2+y^2)^2}\cdot 2x$$

and

$$\frac{\partial f}{\partial y} = 2y\sin\left(\frac{1}{x^2+y^2}\right) + (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right)\cdot\frac{-1}{(x^2+y^2)^2}\cdot 2y.$$

These functions are continuous on their domains, which is $\mathbb{R}\setminus\{(0,0)\}$. Thus (since their domain is open) this implies that f is differentiable at (x,y), by a theorem from class.

At $(0,0)$, we claim $f'(0,0)$ exists and it is zero. To see this, consider the Taylor formula with $\vec{c}=(0,0)$ and $\vec{v}=(v_1,v_2)$:

$$f((0,0)+(v_1,v_2)) = f(0,0) + \underset{\text{derivative}}{0} + \|(v_1,v_2)\|\, E(\vec{v})$$

$$\Rightarrow (v_1^2+v_2^2)\sin\left(\frac{1}{v_1^2+v_2^2}\right) = \|(v_1,v_2)\|\, E(\vec{v}).$$ We must show that such an $E(\vec{v})$ exists, and $E(\vec{v}) \to 0$ as $\vec{v}\to 0$.

Take $$E(\vec{v}) = \frac{(v_1^2+v_2^2)\sin\left(\frac{1}{v_1^2+v_2^2}\right)}{\|(v_1,v_2)\|} = \frac{(v_1^2+v_2^2)\sin\left(\frac{1}{v_1^2+v_2^2}\right)}{\sqrt{(v_1^2+v_2^2)}}$$

$$= \sqrt{v_1^2+v_2^2}\,\sin\left(\frac{1}{v_1^2+v_2^2}\right)$$

Now $\lim_{\vec{v}\to 0}\sqrt{v_1^2+v_2^2}\,\sin\left(\frac{1}{v_1^2+v_2^2}\right) = 0$, since $\sin\left(\frac{1}{v_1^2+v_2^2}\right)$ is bdd and $\sqrt{v_1^2+v_2^2}\to 0$.

$\Rightarrow f$ is differentiable.

Q3]... **[10 points]** (a) Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f'(x, y) = 0$ for all (x, y) in $S^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. Prove that f is constant on S^1.

Let $g: \mathbb{R} \to \mathbb{R}^2$ be $g(t) = (\cos t, \sin t)$. Then the image of g is S^1. The composition $h = f \circ g$ has derivative

$$h'(t) = f'(\cos t, \sin t) \cdot g'(t)$$
$$= 0 \cdot g'(t) = 0.$$

Thus, as h has derivative zero on $\mathbb{R}$, h is constant.

Thus f must be constant on image of g $= S^1$.

(b) Give an example of an open set $S \subset \mathbb{R}^2$ and a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f'(\mathbf{c}) = 0$ for all $\mathbf{c} \in S$, yet f is not constant on S.

Let $S = \{(x,y) \mid x < 0\} \cup \{(x,y) \mid x > 1\}$.

Let $f(x,y)$ be the function:

$$f(x,y) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1. \end{cases}$$

Then f is continuous, yet on the given open set f' is zero as the function is constant there.

Q4]... **[10 points]** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function whose partial derivatives are differentiable everywhere. For a fixed $\mathbf{v} \in \mathbb{R}^n$, define a new function $f_{\mathbf{v}} : \mathbb{R}^n \to \mathbb{R}$ by $f_{\mathbf{v}}(\mathbf{x}) = f'(\mathbf{x}; \mathbf{v})$. Prove or disprove: $f'_{\mathbf{v}}(\mathbf{x}; \mathbf{u}) = f'_{\mathbf{u}}(\mathbf{x}; \mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

We compute $f_v(\vec{x})$ using the formula for directional derivatives:

$$f_v(\vec{x}) = f'(\vec{x};\vec{v}) = \nabla f(\vec{x}) \cdot \vec{v}$$
$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) v_i, \text{ if } \vec{v} = (v_1, \ldots, v_n).$$

Now to compute $f_v'(\vec{x};\vec{u})$ we apply the directional derivative formula again.

$$f_v'(\vec{x};\vec{u}) = \nabla\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) v_i\right)$$
$$= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\vec{x}) v_i\right) u_j$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_j x_i}(\vec{x}) v_i u_j$$
$$= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i x_j}(\vec{x}) u_j\right) v_i$$

by equality of mixed partials, since the first order partials are differentiable

= by the same steps as above, but in reverse...

$$= f_u'(\vec{x};\vec{v}).$$