Name: Solutions

Student Number: ______________

I understand that cheating is a serious offence: ______________
(Signature – *In Ink*)

Please place a check mark beside your section number and instructor:

- ☐ A01 — A. Clay (M/W/F 10:30–11:20, St. John's 118)
- ☐ A02 — M. Sadeghi (M/W/F 9:30–10:20, Drake Centre 343)
- ☐ A03 — M. Virgilio (Tu/Th 8:30–9:45, Armes 208)
- ☐ A04 — Yong Zhang (Tu/Th 11:30–12:45, Armes 200)
- ☐ A05 — R. Borgersen (Tu/Th 1:00–2:15, St. Paul 100)
- ☐ A06 — S. Sankaran (M/W/F 3:30–4:20, Armes 208)
- ☐ D01 — N. Harland (Distance)
- ☐ Challenge

INSTRUCTIONS

I. No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

II. This exam has a title page, 13 pages including this cover page and two blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

III. The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 100 points.

IV. **Answer all questions on the exam paper** in the space provided beneath the question, or clearly indicate that your solution continues on the blank pages at the end of the booklet. Unjustified answers will receive little or no credit.

V. Please do not call or e-mail your instructor to inquire about grades. They will be available shortly after they have been marked.

VI. If the QR codes on your exam paper are deliberately defaced, your exam may not be marked.

Question	Points	Score
1	10	
2	17	
3	23	
4	10	
5	7	
6	7	
7	7	
8	12	
9	7	
Total:	100	

1. Calculate the following limits. Fully justify your answers.

[5] (a) $\lim_{x\to\infty} \ln\left|\frac{2x^2-x+1}{x^2-3x}\right|$

$$= \ln\left|\lim_{x\to\infty} \frac{2x^2-x+1}{x^2-3x}\right|$$

$$= \ln\left|\lim_{x\to\infty} \frac{2-\cancel{1/x}+\cancel{1/x^2}}{1-\cancel{3/x}}\right|$$

$$= \ln\left|\frac{2}{1}\right|$$

$$= \ln(2)$$

[5] (b) $\lim_{x\to 1}\left[(x-1)^3\sin\left(\frac{1}{x-1}\right)\right]$ (Hint: Squeeze Theorem!)

$$-1 \le \sin\left(\frac{1}{x-1}\right) \le 1$$

note abs value!

$$\Rightarrow -|(x-1)^3| \le (x-1)^3\sin\left(\frac{1}{x-1}\right) \le |(x-1)^3|$$

Since $\lim_{x\to 1} |(x-1)^3| = 0$, we get

$$0 \le \lim_{x\to 1} (x-1)^3\sin\left(\frac{1}{x-1}\right) \le 0$$

So $\lim_{x\to 1} (x-1)^3\sin\left(\frac{1}{x-1}\right) = 0$, by the squeeze theorem.

2. Compute the derivatives of the following functions. DO NOT SIMPLIFY YOUR ANSWER!

[4] (a) $g(y) = \ln(1 + 2^{\ln(y+1)})$

$$g'(y) = \frac{1}{1+2^{\ln(y+1)}} \cdot 2^{\ln(y+1)} \cdot \ln(2) \cdot \frac{1}{y+1}$$

[4] (b) $f(x) = 2^{7x} + \log_3(x^4 + 1)$

$$f'(x) = 2^{7x} \cdot \ln(2) \cdot 7 + \frac{1}{(x^4+1)\ln(3)} \cdot 4x^3$$

[4] (c) $j(z) = \int_{z^3}^{5} \sin(t^2)\ln(t)dt$ $= -\int_5^{z^3} \sin(t^2)\ln(t)dt$

By FTC I, $\frac{dj}{dz} = -\sin(z^6)\ln(z^3) \cdot 3z^2$

[5] (d) $f(x) = (x^2 + \sqrt{3})^{\sin(x)}$

$$y = (x^2 + \sqrt{3})^{\sin(x)}$$

$$\Rightarrow \ln(y) = \sin(x)\ln(x^2+\sqrt{3})$$

$$\Rightarrow \frac{1}{y}y' = \cos(x)\ln(x^2+\sqrt{3}) + \sin(x)\frac{1}{x^2+\sqrt{3}} \cdot 2x$$

$$\Rightarrow y' = (x^2+\sqrt{3})^{\sin(x)}\left(\cos(x)\ln(x^2+\sqrt{3}) + \sin(x)\frac{1}{x^2+\sqrt{3}} \cdot 2x\right)$$

3. For each part of this question you must show all your work to receive full credit. Consider the curve given by the function

$$f(x) = \frac{(x+4)(x+8)}{(x+2)^2}.$$

You may use, without checking, that:

$$f'(x) = \frac{-8(5+x)}{(x+2)^3} \qquad \text{and} \qquad f''(x) = \frac{8(13+2x)}{(x+2)^4}.$$

[3] (a) Determine the domain of $f(x)$ and all x- and y-intercepts.

If $x+2=0$ then we get division by zero $\Rightarrow$ domain is $(-\infty,-2) \cup (-2,\infty)$

X-int: $\dfrac{(x+4)(x+8)}{(x+2)^2} = 0 \Rightarrow x = -4, -8$

y-int: $x=0$ gives $\dfrac{4 \cdot 8}{2^2} = 8.$

[4] (b) Find all limits associated with all horizontal asymptote(s) of $f(x)$.

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2+12x+32}{x^2+4x+4}$$

$$= \lim_{x \to \pm\infty} \frac{1 + 12/x + \frac{32}{x^2}}{1 + 4/x + 4/x^2} = \frac{1}{1} = 1.$$

$y = 1$ is a horizontal asymptote.

[2] (c) Find all limits associated with all vertical asymptote(s) of $f(x)$.

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{(x+4)(x+8)}{(x+2)^2} = \frac{\text{positive}}{0^+} = +\infty,$$

so there's a vertical asymptote at $x = -2$.

For convenience, here again are f, f', and f'':

$$f(x) = \frac{(x+4)(x+8)}{(x+2)^2} \qquad f'(x) = \frac{-8(5+x)}{(x+2)^3} \qquad f''(x) = \frac{8(13+2x)}{(x+2)^4}.$$

[2] (d) Find all critical points of $f(x)$ (that is, all critical numbers, together with their y values).

Solve $f'(x)=0 \Rightarrow x=-5$. Then

$$f(-5) = \frac{(-5+4)(-5+8)}{(-5+2)^2} = \frac{(-1)(3)}{(-3)^2} = \frac{-3}{9} = -\frac{1}{3}$$

The only critical point is $(-5, -\frac{1}{3})$.

[3] (e) Find the open intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

function	$(-\infty,-5)$	$(-5,-2)$	$(-2,\infty)$
-8	$-$	$-$	$-$
$5+x$	$-$	$+$	$+$
$(x+2)^3$	$-$	$-$	$+$
$f'(x)$	$-$	$+$	$-$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$

$f(x)$ is increasing on $(-5,-2)$ and decreasing on $(-\infty,-5)$ and $(-2,\infty)$.

[2] (f) Find the coordinates of points at which the local maxima and/or local minima occur.

Local min at $x=-5$ by the first derivative test. Coordinates are $(-5,-\frac{1}{3})$. No local max, since -2 is not in the domain.

For convenience, here again are f, f', and f'':

$$f(x) = \frac{(x+4)(x+8)}{(x+2)^2} \qquad f'(x) = \frac{-8(5+x)}{(x+2)^3} \qquad f''(x) = \frac{8(13+2x)}{(x+2)^4}.$$

[2] (g) Determine the open intervals upon which $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

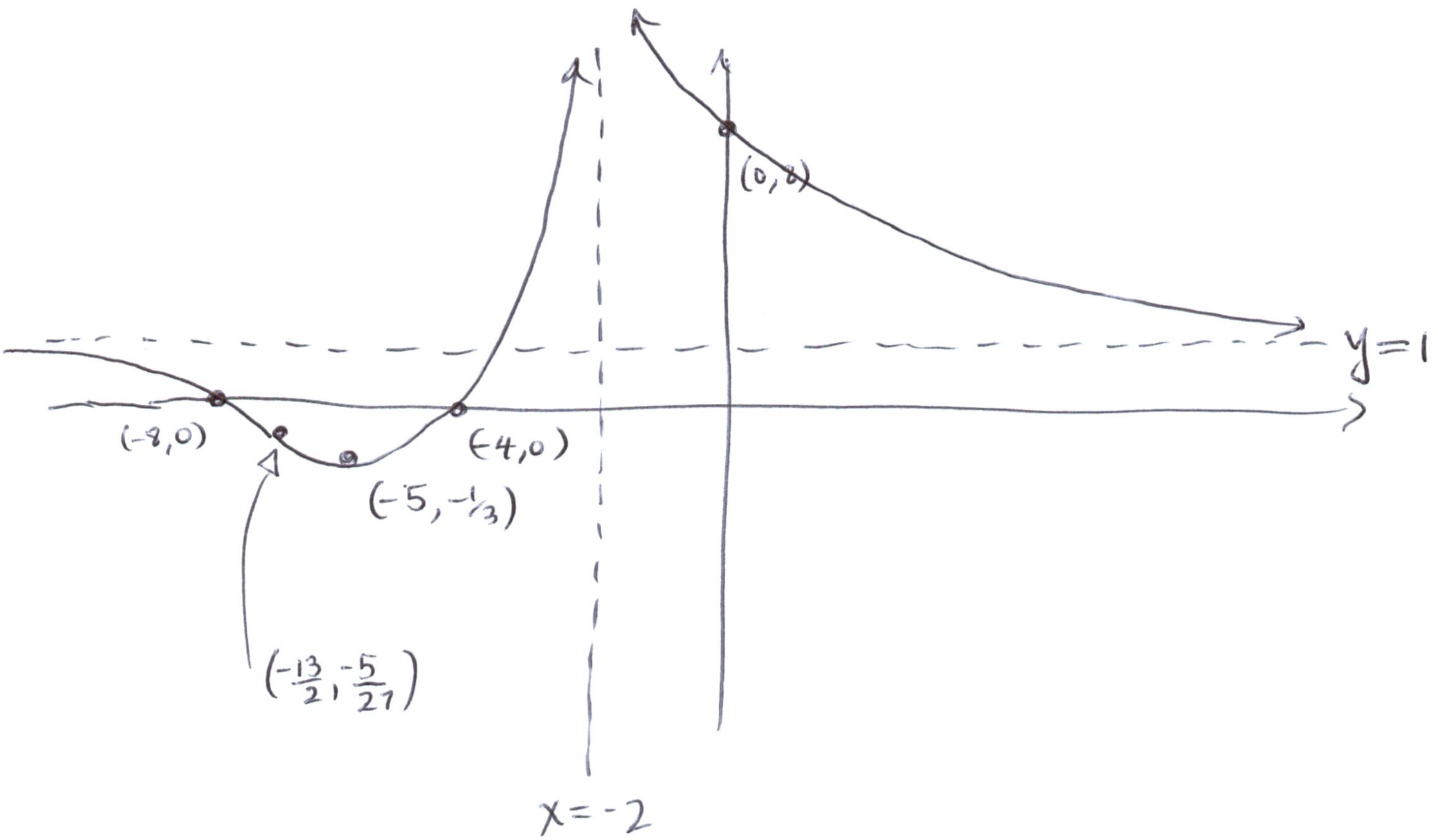
$(x+2)^4$ is always positive, so the sign of $f''(x)$ is determined by the sign of $13+2x$. $\Rightarrow f''(x) > 0$ if x is in (-13/2, -2) or (-2, infinity), so f is concave up. $f''(x) < 0$ if x is in $(-\infty, -\frac{13}{2})$, where f is concave down.

[2] (h) Find the coordinates of the points of inflection.

$$f\left(-\frac{13}{2}\right) = \frac{\left(-\frac{13}{2}+4\right)\left(-\frac{13}{2}+8\right)}{\left(-\frac{13}{2}+2\right)^2} = \frac{\left(-\frac{5}{2}\right)\left(\frac{3}{2}\right)}{\left(-\frac{9}{2}\right)^2} = \frac{-\frac{15}{4}}{\frac{81}{4}} = \frac{-15}{81} = \frac{-5}{27}$$

So the coordinates are $\left(-\frac{13}{2}, -\frac{5}{27}\right)$.

[3] (i) Use the information from parts (a)–(h) to give a **neat** sketch of the graph $y = f(x)$.



[10] 4. A closed cylindrical can (including top and bottom surface) is being made to hold 54π cm^3 of liquid. How should we choose the height and radius to minimize the amount of material needed to build the can?

$\pi r^2 h = 54\pi$

Surface area $= 2\pi r^2 + 2\pi r h$

From $\pi r^2 h = 54\pi$, we get $h = \frac{54\cancel{\pi}}{\cancel{\pi} r^2} = \frac{54}{r^2}$

So surface area $= 2\pi r^2 + 2\pi r\left(\frac{54}{\cancel{r^2}}\right)$

$= 2\pi r^2 + \frac{108\pi}{r}$

Domain: $r > 0$ since it's a distance, no upper limit: $(0, \infty)$.

$S' = 4\pi r + \frac{(-1)\,108\pi}{r^2} = 4\pi r - \frac{108\pi}{r^2}$. Then $S' = 0$

gives $4\pi r - \frac{108\pi}{r^2} = 0 \Rightarrow \frac{108\pi}{r^2} = 4\pi r \Rightarrow r^3 = \frac{108\pi}{4\pi}$

$\Rightarrow r^3 = 27$, so $\underline{\underline{r = 3}}$

Now $S'' = 4\pi + \frac{2(108)\pi}{r^3}$

$\Rightarrow S''(3) = 4\pi + \frac{2(108)\pi}{3^3}$ is positive, so $r = 3$ is a min by the second derivative test.

$\Rightarrow$ The dimensions should be $r = 3$ cm and $h = \frac{54}{9} = 6$ cm.

[7] 5. Find the absolute minimum and the absolute maximum of the function $f(x) = x^2(x-1)^2$ over the interval $[0, 2]$.

$f(x) = x^2(x^2-2x+1) = x^4 - 2x^3 + x^2.$

$\Rightarrow f'(x) = 4x^3 - 6x^2 + 2x.$ So $f'(x)=0$ gives

$x(4x^2-6x+2)=0$

$\Rightarrow 2x(x-1)(2x-1) = 0.$

So $x = 0, 1, \frac{1}{2}.$

We test $x=0, 1, \frac{1}{2}, 2$:

$f(0) = 0$

$f(1) = 0$

$f(\frac{1}{2}) = (\frac{1}{2})^2(-\frac{1}{2})^2 = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$

$f(2) = 4 \cdot 1^2 = 4.$

Therefore the absolute min is $f(x)=0$ at $x=0$ and $x=1$, absolute max is $f(x)=4$ at $x=2$.

[7] 6. Suppose that $f(x)$ is a function on $(0, \infty)$ whose second derivative is $f''(x) = \frac{\pi}{x^2} - \sin(x)$. If $f'(\pi) = f(\pi) = 0$, find $f(x)$.

$f''(x) = \pi x^{-2} - \sin(x)$

$\Rightarrow f'(x) = \pi \frac{x^{-1}}{-1} + \cos(x) + C$

Thus $f'(\pi)=0$ gives

$0 = \frac{-\pi}{\pi} + \cos(\pi) + C \Rightarrow C = 2$

Now $f'(x) = \frac{-\pi}{x} + \cos(x) + 2$

$\Rightarrow f(x) = -\pi \ln|x| + \sin(x) + 2x + D$

$f(\pi)=0$ gives

$0 = -\pi\ln\pi + 0 + 2\pi + D \Rightarrow D = \pi\ln\pi - 2\pi.$

So $f(x) = -\pi\ln|x| + \sin(x) + 2x + \pi\ln\pi - 2\pi.$

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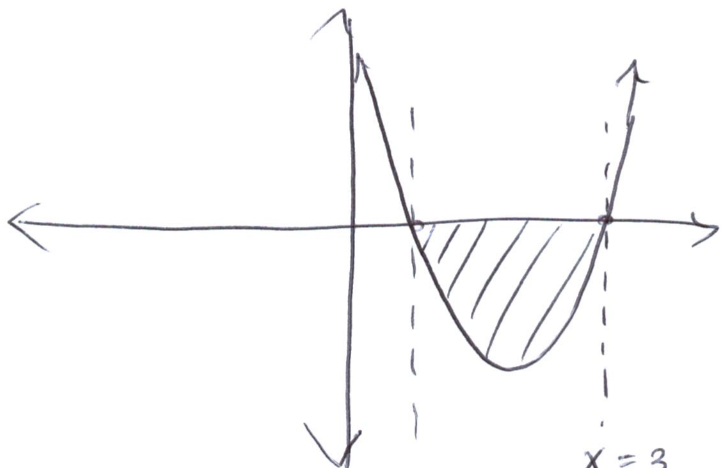
[7] 7. Sketch and find the area of the region lying between the curve $y = x^2 - 4x + 3$ and the x-axis.

$$y = x^2 - 4x + 3 = (x-3)(x-1)$$



$$\text{Area} = -\int_1^3 x^2 - 4x + 3 \, dx$$

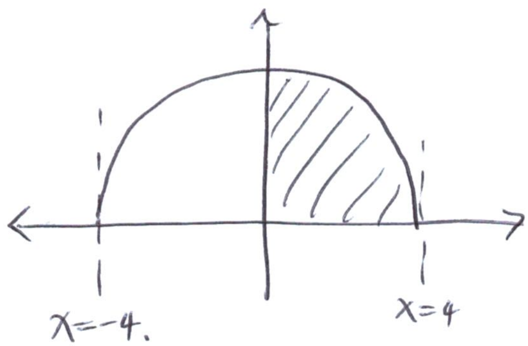
$$= -\left[\frac{x^3}{3} - \frac{4x^2}{2} + 3x\right]_1^3$$

$$= -\left(\left(\frac{3^3}{3} - 2(3)^2 + 3^2\right) - \left(\frac{1}{3} - 2 + 3\right)\right)$$

$$= -(\underbrace{9 - 2(9) + 9}_{0}) + \left(\frac{4}{3}\right)$$

$$= \frac{4}{3}.$$

[5] 8. (a) Evaluate $\int_0^4 \sqrt{16 - x^2}\, dx$ by sketching the area it represents and finding that area.



$$\text{Area} = \frac{1}{4}\pi(4)^2 = 4\pi.$$

[7] (b) Evaluate $\int \left(5x^3 - \frac{(x+1)^2}{x} + \sin(x) - 6e^x\right) dx.$

$\frac{(x+1)^2}{x} = \frac{x^2+2x+1}{x} = x + 2 + \frac{1}{x}$, so the integral gives

$$\frac{5x^4}{4} - \frac{x^2}{2} - 2x - \ln|x| - \cos(x) - 6e^x + C.$$

[7] 9. (a) State the Mean Value Theorem.

If f is a function such that

① f is continuous on $[a,b]$

② f ~~is~~ is differentiable on (a,b)

Then there's a c in (a,b) so that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

(b) Suppose that $f(x)$ is a function that satisfies the hypotheses of the Mean Value Theorem on the interval $[0,2]$. Show that $f(x)$ cannot satisfy $f(0) = -1$ and $f(2) = 4$ if $f'(x) \leq 2$ for all x in $[0,2]$.

By the MVT, there's a c in $(0,2)$ so that

$$f'(c) = \frac{f(2)-f(0)}{2-1} = \frac{4-(-1)}{2} = \frac{5}{2}.$$

This is not possible since $f'(c) \leq 2$. So no function f exists.