Midterm Test **Solutions**

Q1 a) We write $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$ there exist $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \varepsilon$.

b) We write $\lim_{x \to -\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that $x < M$ implies $|f(x) - L| < \varepsilon$.

c) We say that $\lim_{x \to a^+} f(x)$ does not exist if for every L there exists $\varepsilon > 0$ such that for all $\delta > 0$ $|f(x) - L| > \varepsilon$ for some x with $0 < |x-a| < \delta$.

Q2 a) We say $f(x)$ is continuous at $x=a$ if
$$\lim_{x \to a} f(x) = f(a).$$

b) We say that $f(x)$ is differentiable at $x=a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Q3. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that $0 < |x-2| < \delta$ implies

$$\left| \frac{x-2}{1+x^2} - 0 \right| = \left| \frac{x-2}{1+x^2} \right| = \frac{|x-2|}{|1+x^2|} < \varepsilon.$$

Note that if $\delta \leq 1$ then $\frac{1}{|1+x^2|}$ will be bounded:

For if x is in $(1, 3)$, then $\frac{1}{|1+x^2|}$ is in $\left(\frac{1}{10}, \frac{1}{2}\right)$.

Thus if $\delta \leq 1$ then $\frac{|x-2|}{|1+x^2|} < \frac{|x-2|}{2}$, so choosing $\delta \leq 1$ such that $\frac{\delta}{2} < \varepsilon$ is satisfied will give what we want. Thus we set $\delta = \min\{1, 2\varepsilon\}$.

Then $\delta \leq 1$ implies $\frac{1}{|1+x^2|} < \frac{1}{2}$, and $\delta \leq 2\varepsilon$ implies $|x-2| < 2\varepsilon$. Overall,

$$\left| \frac{x-2}{1+x^2} - 0 \right| = \frac{|x-2|}{|1+x^2|} < \frac{1}{2} \cdot 2\varepsilon = \varepsilon,$$

so we're done.

Q4: a) If $x \to 1^-$, then $x<1$ and in this case

$|x-1| = -(x-1)$, so

$$\lim_{x\to 1^-} \frac{1}{|x-1|} = \lim_{x\to 1^-} \frac{-1}{x-1} = +\infty .$$

b) $$\lim_{x\to\infty} \left(\frac{x^2}{x+1} - \frac{x^2}{x-1} \right)$$

$$= \lim_{x\to\infty} \left(\frac{x^2(x-1) - x^2(x+1)}{(x+1)(x-1)} \right)$$

$$= \lim_{x\to\infty} \frac{\cancel{x^3} - x^2 - \cancel{x^3} - x^2}{x^2-1}$$

$$= \lim_{x\to\infty} \frac{-2x^2}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{-2}{1-\frac{1}{x^2}} = -2 .$$

c) $$\lim_{x\to 0} \frac{\sqrt{4+x} - 2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$= \lim_{x\to 0} \frac{\cancel{4} + x - \cancel{4}}{\cancel{x}(\sqrt{4+x}+2)}$$

$$= \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4} .$$

Q5 From the definition,

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{(1+h)^2} - \frac{1}{1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{1}-(h^2+2h+\cancel{1})}{h}$$

$$= \lim_{h\to 0} \frac{-h^2-2h}{h}$$

$$= \lim_{h\to 0} -h-2 = -2.$$

Q6 Use differentiation rules to calculate:

a) $$y' = \frac{(\sin(x))' \cdot x - \sin(x)\cdot(x)'}{x^2}$$

$$= \frac{\cos(x)\cdot x - \sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2}.$$

$$y'' = \frac{(\cos(x))'\cdot x - \cos(x)\cdot(x)'}{x^2} - \frac{(\sin(x))'\cdot x^2 - \sin(x)\cdot(x^2)'}{x^4}$$

$$= \frac{-\sin(x)\cdot x - \cos(x)}{x^2} - \frac{\cos(x)\cdot x^2 - \sin(x)\cdot 2x}{x^4}.$$

b) $f(x) = \tan\left(\frac{x^2}{x^3-1}\right)$

$$\Rightarrow f'(x) = \sec^2\left(\frac{x^2}{x^3-1}\right) \cdot \frac{2x(x^3-1) - x^2(3x^2)}{(x^3-1)^2}.$$

Q7: The function $f(x) = x^3 - 3x + 1$ is continuous on the interval $[-2, 2]$ since it is a polynomial.

Considering the interval $[-2, -1]$, we get

$f(-2) = -8 + 6 + 1 = -1$

$f(-1) = -1 + 3 + 1 = 3$

So by the IVT, there's c in $[-2, -1]$ with $f(c) = 0$, note c is actually in $(-2, -1)$.

Considering $[-1, 1]$ we get

$f(-1) = 3$

$f(1) = 1 - 3 + 1 = -1$,

so again by the IVT there's a root in $[-1, 1]$, in fact $(-1, 1)$.

Finally considering $[1, 2]$:

$f(1) = -1$

$f(2) = 8 - 6 + 1 = 3$,

so there's a root in $[1, 2]$ actually $(1, 2)$ as well, for a total of 3 roots.